Consensus and Flocking in Self-Alignment Dynamics
Lecture II. Consensus and flocking

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Mathematical considerations

- By L_A we denote the Laplacian matrix associated with $A = \{a_{ij}\}$

$$L_A := D - A, \qquad D_{ii} = \sum_j a_{ij} \quad \longleftarrow \quad \deg_i$$

- Self-alignment is driven by a discrete Laplacian ($\alpha \mapsto 1$):

$$\frac{d}{dt}\mathbf{p}_i(t) = \alpha \sum_j a_{ij}(\mathbf{p}_j - \mathbf{p}_i) \ \mapsto \ \sum_j a_{ij}\mathbf{p}_j - \deg_i \mathbf{p}_i \ \ \leftrightarrow \ \ \frac{d}{dt}\mathbf{p} = -L_A\,\mathbf{p}$$

Observe that $Re\lambda_k(L_A) \geq 0$ and $L_A\mathbf{1} = 0, \ \ \mathbf{1} = (1, 1, \ldots, 1)^\top$.
In particular, if A is symmetric $\lambda(A) \geq 0$:

$\mapsto$ Symmetric A's: $\quad 0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$

$\odot$ The example of the "usual" Laplacian matrix

$$\left\{\ldots \frac{1}{4}, \ldots, \frac{1}{4}, -1, \frac{1}{4}, \ldots, \frac{1}{4}, \ldots\right\}$$

$\odot$ Note the issue of a sign – the Laplacian should be negative:
$(\Delta u, u) = -|\nabla u|^2 \leq 0$ and not positive $\langle L_A u, u\rangle \geq 0$

Mathematical considerations cont'd

- Self-alignment: $\dfrac{d}{dt}\mathbf{p} = -L_A\,\mathbf{p} \;\rightsquigarrow\; \dfrac{1}{2}\dfrac{d}{dt}|\mathbf{p}|^2 = -\langle L_A\,\mathbf{p}, \mathbf{p}\rangle \leq 0$
  which is consistent with convexity ($\rightsquigarrow \mathbf{p}(t) \in \text{conv hull}\{\mathbf{p}_j\}$)
  but lacks coercivity: $-\langle L_A\,\mathbf{p}, \mathbf{p}\rangle \not\lesssim -|\mathbf{p}|^2$
- Operators on graphs: "nodes" $\{p\}_i \in X$ and "edges" $\{\mathbf{u}\}_{ij} \in Y$
  gradient $\nabla : X \mapsto Y$: $(\nabla p)_{ij} := \sqrt{a_{ij}}(p_j - p_i)$
  divergence div : $Y \mapsto X$: $(\text{div}\,\mathbf{u})_i := \sum_j \sqrt{a_{ij}}(u_{ji} - u_{ij})$

$\rightsquigarrow$ with the usual duality: $\langle \nabla p, \mathbf{u}\rangle = \langle p, \text{div}\,\mathbf{u}\rangle$ for symmetric A's

- Self-alignment: $\dfrac{d}{dt}\mathbf{p} = \text{div}\,\nabla\mathbf{p} \;\rightsquigarrow\; \dfrac{1}{2}\dfrac{d}{dt}|\mathbf{p}|^2 = -|\nabla\mathbf{p}|^2$:

$$\frac{1}{2}\frac{d}{dt}\sum_i |\mathbf{p}_i|^2 = -\frac{1}{2}\sum_{ij} a_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2 \leq 0$$

- And since $\sum_i \mathbf{p}_i(t) = \sum_i \mathbf{p}_i(0)$:

$$\frac{1}{2N}\frac{d}{dt}\sum_{ij} |\mathbf{p}_i - \mathbf{p}_j|^2 = -\sum_{ij} a_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2$$

Mathematical considerations cont'd

- Exercise: $\dfrac{d}{dt}\mathbf{p}_i = \sum_j a_{ij}\left(\mathbf{p}_j - \mathbf{p}_i\right)$ with symmetric A's —

$$\rightsquigarrow \quad \frac{1}{2N}\frac{d}{dt}\sum_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2 = -\sum_{ij} a_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2$$

- What about coercivity — RHS $\lesssim -\sum_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2$?

  For example, $-\sum_{ij} a_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2 \le -\min a_{ij}\sum_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2$ implies

$$\frac{d}{dt}\sum_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2 \le -\eta\sum_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2, \quad \eta = 2N\min a_{ij}$$

- Sharp characterization – Courant-Fischer (for symmetric A's):

$$\lambda_2(L_A) = \min_{\sum \mathbf{p}_k = 0}\frac{\langle L_A\,\mathbf{p}, \mathbf{p}\rangle}{|\mathbf{p}|^2} \le N\frac{\sum_{ij} a_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2}{\sum_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2}$$

$$\rightsquigarrow \quad -\sum_{ij} a_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2 \le \frac{1}{N}\lambda_2(L_A)\sum_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2$$

Mathematical considerations – spectral analysis

$$\frac{1}{2}\frac{d}{dt}\sum_{ij}|\mathbf{p}_i-\mathbf{p}_j|^2 \leq -\lambda_2(L_A)\sum_{ij}|\mathbf{p}_i-\mathbf{p}_j|^2$$

$\rightsquigarrow$ Convergence towards flocking/consensus is dictated by the the Fiedler number:

$$\frac{d}{dt}\vee(t) \lesssim -\lambda_2(L_{A(t)})\vee(t), \quad \vee^2_{\mathbf{p}(t)} = \frac{1}{N^2}\sum_{ij}|\mathbf{p}_i(t)-\mathbf{p}_j(t)|^2$$

Exercise: verify the last few steps...

- sharp characterization: $\lambda_2(L_A) \geq N \min_{ij} a_{ij}$

Questions:

- What about the Fiedler number $\lambda_2(L_A)$?
- What about the non-symmetric models of self-alignment?

The symmetric case: spectral analysis

- Convergence towards flocking/consensus is dictated by

$$\frac{d}{dt}\vee(t) \lesssim -\lambda_2(L_{A(t)})\vee(t), \quad \vee^2_{\mathbf{p}(t)} = \frac{2}{N^2}\sum_{ij}|\mathbf{p}_i(t) - \mathbf{p}_j(t)|^2$$

$$\vee^2_{\mathbf{p}(t)} = \frac{1}{N}\sum_{i}|\mathbf{p}_i - \langle\mathbf{p}\rangle|^2, \qquad \langle\mathbf{p}\rangle(t) = \frac{1}{N}\sum_{k}\mathbf{p}_k \equiv \langle\mathbf{p}\rangle(0)$$

$$|\mathbf{p}_i(t) - \langle\mathbf{p}\rangle(0)| \lesssim e^{-\alpha\eta t}\vee_{\mathbf{p}(0)}, \qquad \eta := \min_t \lambda_2(L_A(\mathbf{p}(t)))$$

- Flocking/consensus if $\eta = \min_t \lambda_2(L_A(\mathbf{p}(t))) > 0$ or at least ...

$$\lambda_2(s) \equiv \lambda_2(L_{A(\mathbf{p}(s))}) \int^{\infty} \lambda_2(s)ds = \infty \quad \rightsquigarrow \quad \text{flocking/consensus}$$

- Fiedler number $\lambda_2(L_A)$ quantifies algebraic connectivity: $\mathcal{G}_A = \{\mathbf{p}, A(\mathbf{p})\}$ is uniformly connected if every two agents are connected through a path, $\Gamma_{ij} = \{k_1 = i < k_2 < \ldots < k_r = j\}$

$$\forall \text{ pairs } (\mathbf{p}_i, \mathbf{p}_j) : \ \exists\Gamma_{ij} \ \text{ such that } \ \min_{k_\ell \in \Gamma_{ij}} a_{(k_\ell, k_{\ell+1})} \geq \mu > 0$$

Consensus/flocking by the "energy method"

$$\frac{d\mathbf{p}_i}{dt} = \alpha \sum_{j\neq i} a_{ij}(\mathbf{p}_j - \mathbf{p}_i), \text{ stochastic adjacency matrix } \sum_j a_{ij} = 1$$

- Contraction of diameters: $\mathrm{d}_{\mathbf{p}}(t) := \max_{i,j} |\mathbf{p}_i(t) - \mathbf{p}_j(t)|$

$$\frac{d}{dt}\mathrm{d}_{\mathbf{p}}(t) \leq -\alpha\big(\min_{ij} \eta_{ij}\big)\mathrm{d}_{\mathbf{p}}(t), \quad \eta_{ij} := \sum_k \min(a_{ik}, a_{jk})$$

- If $\eta = \min_{ij} \eta_{ij} > 0$ then $\exists \mathbf{p}^\infty \in \mathrm{conv}(\{\mathbf{p}_i\})$ s.t.

$$|\mathbf{p}_i(t) - \mathbf{p}^\infty| \lesssim e^{-\alpha\eta t}\mathrm{d}_{\mathbf{p}(0)}, \qquad \eta := \min_{ij} \sum_k \min(a_{ik}, a_{jk})$$

DeGroot, Chatterjee, Krause, ... Motsch & ET

- Unconditional flocking using ℓ_1, ℓ_2, ℓ_∞- based approaches: Cucker & Smale, Ha & ET, Carrillo et. al., Ha & Liu, ...
- An ℓ_∞ "energy method – covers non-symmetric models

The energy method vs. spectral analysis

$$|\mathbf{p}_i(t) - \mathbf{p}^\infty| \lesssim e^{-\alpha\eta t} \mathrm{d}_{\mathbf{p}(0)}, \qquad \eta := \min_{ij} \sum_k \min(a_{ik}, a_{jk})$$

- Consensus if $\eta(t) > N \min_{ij} a_{ij} > 0$ – requires global interactions or at least one neighbor connectivity (the symmetric case)
- Spectral analysis of the symmetric case:

$$|\mathbf{p}_i(t) - \mathbf{p}^\infty| \lesssim e^{-\alpha\eta t} \vee_{\mathbf{p}(0)}, \qquad \eta := \min_t \lambda_2(L_A(\mathbf{p}(t)))$$

- If $\eta = \min_t \lambda_2(L_A(\mathbf{p}(t))) > 0$ – requires connectivity of graph

or sufficiently strong influence $\displaystyle\int^\infty \eta(s)ds = \infty \rightsquigarrow$ flocking/consensus

- What is the consensus $\mathbf{p}^\infty$? – by convexity: $\mathbf{p}^\infty \in \mathrm{conv}(\{\mathbf{p}_i\})$
  - ⊙ Symmetric case: $\mathbf{p}^\infty = \langle \mathbf{p} \rangle(0), \quad \langle \mathbf{p} \rangle := \dfrac{1}{N} \sum_i \mathbf{p}_i$
  - ⊙ Non-symmetric case?

Global influence implies unconditional consensus

- Opinion dynamics —

$$\frac{d}{dt}x_i(t) = \alpha \sum_{j\neq i} a_{ij}(x_j - x_i), \quad a_{ij} = \frac{1}{\deg_i}\phi(|x_i - x_j|), \ \deg_i = \begin{cases} N \\ \sum_k \phi_{ik} \end{cases}$$

- $$\eta = N \min_{ij} a_{ij} \ \mapsto N \min \frac{1}{\deg_i}\phi(|x_i - x_j|) \geq \min_{r \leq \mathrm{d}_{\mathbf{x}}(t)} \phi(r):$$

and since $\mathrm{d}_{\mathbf{x}}(t) \leq \mathrm{d}_{\mathbf{x}}(0)$:

$$\frac{d}{dt}\mathrm{d}_{\mathbf{x}}(t) \leq -\alpha\eta \mathrm{d}_{\mathbf{x}}(t), \qquad \eta = \min_{r \leq \mathrm{d}_{\mathbf{x}}(0)} \phi(r) > 0$$

- Set $\mathrm{Supp}\{\phi(\cdot)\} = [0, \rho)$:

$$\rho > \mathrm{d}_{\mathbf{x}}(0) \ \mapsto \ |x_i(t) - x^\infty| \lesssim e^{-\alpha\eta t}, \quad x^\infty \in \mathrm{conv}(\{x_i\})$$

- Global interactions $\mapsto$ unconditional emergence of consensus

Global influence implies unconditional flocking

- Flocking dynamics —

$$\frac{d}{dt}v_i(t) = \alpha \sum_{j\neq i} a_{ij}(v_j - v_i), \quad a_{ij} = \frac{1}{\deg_i}\phi(|x_i - x_j|), \quad \deg_i = \begin{cases} N \\ \sum_k \phi_{ik} \end{cases}$$

- $\phi(0) = 1 \searrow$: $\quad \eta \equiv \eta(t) \geq N \min \frac{1}{\deg_i}\phi(|x_i - x_j|) \geq \phi(\mathrm{d}_{\mathbf{x}}(t))$

$$\frac{d}{dt}\mathrm{d}_{\mathbf{v}}(t) \leq -\alpha\phi(\mathrm{d}_{\mathbf{x}}(t))\mathrm{d}_{\mathbf{v}}(t), \qquad \frac{d}{dt}\mathrm{d}_{\mathbf{x}}(t) \leq \mathrm{d}_{\mathbf{v}}(t)$$

[S.-Y Ha & J.-G Liu] $\quad \mathcal{E}(t) := \mathrm{d}_{\mathbf{v}}(t) + \alpha \int_0^{\mathrm{d}_{\mathbf{x}}(t)} \phi(s)ds \ \downarrow$

- If $\int^{\infty} \phi(s)ds > \mathrm{d}_{\mathbf{v}}(0)$ implies bounded expansion: $\quad \mathrm{d}_{\mathbf{x}}(t) \leq R$

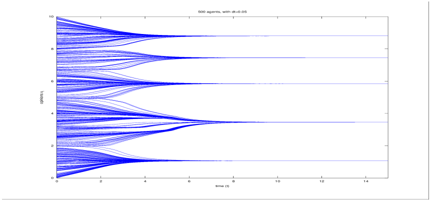
$$\int^{\infty} \phi(s)ds > \mathrm{d}_{\mathbf{v}}(0) \quad \mapsto \quad |v_i(t) - v^{\infty}| \lesssim e^{-\alpha\eta t}, \quad \eta := \phi(R)$$

- Example: $\quad \phi(r) = \frac{1}{1 + r^{2\beta}}$: $\quad$ unconditional flocking if $\beta \leq 1/2$
- Global interactions ($\rho > \mathrm{d}_{\mathbf{x}}(0)$) implies emergence of flocking

Cucker-Smale vs. the new model

Local interactions — the emergence of clusters

- If ϕ is compactly supported: $\rho < \mathrm{d}_{\mathbf{x}}(0) \mapsto$ formation of clusters:



- A cluster $\mathcal{C} \subset \{1, 2, \ldots N\}$: $\begin{cases} \#1. & \max_{i,j\in\mathcal{C}} \quad |x_i - x_j| \leq \rho; \\ \#2. & \min_{i\in\mathcal{C}, j\notin\mathcal{C}} |x_i - x_j| > \rho \end{cases}$

- Self-contained dynamics: $\displaystyle \frac{d}{dt}\mathbf{p}_{i_{i\in\mathcal{C}}} = \sum_{j\in\mathcal{C}} a_{ij}(\mathbf{p}_j - \mathbf{p}_i), \ \sum_{j\in\mathcal{C}} a_{ij} = 1$

- If $\mathbf{p}(t) \in BV$ then $\exists\ \mathbf{p}^\infty$, partitioned into finitely many clusters:

$$\{1, 2, \ldots, N\} = \cup_{k=1}^{K} \mathcal{C}_k : \begin{cases} \text{either} \quad \mathbf{p}_i(t) \to \mathbf{p}^\infty_{\mathcal{C}_k}, & \forall i \in \mathcal{C}_k \\ \text{or} \quad |x_i^\infty - x_j^\infty| > \rho, & i \in \mathcal{C}_k, j \in \mathcal{C}_\ell, k \neq \ell. \end{cases}$$

Question #1 Determine K_∞ based on $\{\mathbf{p}\}(0)$ and ϕ?;

Question #2 $K_\infty = 1$? —that is, consensus of <u>local</u> interactions?

Connectivity is necessary and sufficient for consensus

- $\mathcal{G}_A = \{\mathbf{p}, A(\mathbf{p})\}$ is uniformly connected if every two agents are connected through a path, $\Gamma_{ij} = \{k_1 = i < k_2 < \ldots < k_r = j\}$

$$\forall \mathbf{p}_i, \mathbf{p}_j, \ \exists \Gamma_{ij} \ \text{ such that } \ \min_{k_\ell \in \Gamma_{ij}} a_{(k_\ell, k_{\ell+1})} \geq \mu > 0$$

- Clearly, connectivity is necessary for flocking/consensus ... and it is also sufficient for flocking/consensus:

- Symmetric models: $L_A := I - A$ graph Laplacian of A with e.v. $0 = \lambda_1(L_A) \leq \lambda_2(L_A) \leq \ldots \lambda_N(L_A) \leq 1$

$$V(t) \leq e^{-\alpha\lambda_2 t} V(0), \qquad V^2(t) := \frac{1}{N} \sum_i |\mathbf{p}_i(t) - \langle \mathbf{p} \rangle(0)|^2$$

If $\mathcal{G}_A$ is connected then: $\lambda_2(L_A) \geq \dfrac{\mu}{N^2} \ \mapsto$ consensus/flocking

Connectivity is sufficient for consensus – cont'd

- The symmetric case: Let $diam(\mathcal{G}_A) = \max_{ij} length(\Gamma_{ij})$, then

$$|\mathbf{p}_i - \mathbf{p}_j|^2 \leq diam(\mathcal{G}_A) \sum_{k_\ell \in \Gamma_{ij}} |\mathbf{p}_{k_{\ell+1}} - \mathbf{p}_{k_\ell}|^2;$$

By uniform connectivity

$$\frac{\mu}{diam(\mathcal{G}_A)} |\mathbf{p}_i - \mathbf{p}_j|^2 \leq \sum a_{k_{\ell+1},k_\ell} |\mathbf{p}_{k_{\ell+1}} - \mathbf{p}_{k_\ell}|^2 \leq \sum_{ij} a_{ij} |\mathbf{p}_i - \mathbf{p}_j|^2$$

and hence ("coercivity with vanishing entries")

$$\lambda_2(L_A) = \min_{\mathbf{p}} N \frac{\sum a_{ij} |\mathbf{p}_i - \mathbf{p}_j|^2}{\sum_{ij} |\mathbf{p}_i - \mathbf{p}_j|^2} \geq \frac{\mu}{N \, diam(\mathcal{G}_A)} \geq \frac{\mu}{N^2}$$

Exercise Work out the details.

- If connectivity persists — $\lambda_2(L_{A(\mathbf{p}(s))}) > 0 \rightsquigarrow$ consensus

Question: Trace the propagation of connectivity in time: $\lambda_2(L_{A(\mathbf{p}(t))})$

- If $diam(\mathcal{G}_A) = 1$ then by the ℓ_∞-energymethod: $\lambda_2(L_A) \geq \mu$.

Connectivity is sufficient for consensus – cont'd

- The example of non-symmetric opinion dynamic (e.g., Krause model):

$$\dot{x}_i = \frac{\alpha}{\deg_i}\sum_j \phi_{ij}(x_j - x_i), \quad \deg_i = \sum_j \phi_{ij}, \quad \phi_{ij} \equiv \phi(|x_i - x_j|)$$

- An energy method — the "energy" $\mathcal{E}(t)$ decays:

$$\mathcal{E}(t) := \sum_{ij} \Phi(|x_i - x_j|), \quad \Phi(r) = \int_0^r s\phi(s)ds$$

satisfies $\quad \dfrac{d}{dt}\mathcal{E}(t) \lesssim -\left(\sum_{ij}\phi_{ij}|x_i - x_i|^2\right)^2 \lesssim \mathcal{E}^2(t) \quad$ (coercivity)

If $\mathcal{G}_A$ is connected then: $\mu d_{\mathbf{x}}(t) \lesssim \dfrac{1}{\sqrt{t}} \ \mapsto$ consensus after $t \geq t_0$

On the propagation of connectivity

- Consensus $\mapsto$ connectivity
- Connectivity $\mapsto$ consensus

How the influence function dictates persistence of connectivity?

- Heterophilious dynamics:

  "Similarity breeds connectivity": intuition tells us about ...
  Tendency to align with those that act and think alike:

  Homophilious —- $a_{ij} \sim \dfrac{1}{\deg_i}\phi(|x_i(t) - x_j(t)|)$ with $\phi\downarrow$

  Heterophilious —- "bonding with the different":

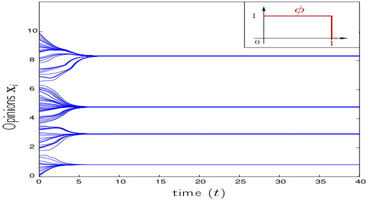
  $$a_{ij} \sim \frac{1}{\deg_i}\phi(|x_i(t) - x_j(t)|) \text{ with } \phi\uparrow$$

- Nearest neighbor dynamics:

  Motivated by the careful observations of Rome group:
  the influencing neighborhood is topological not geometrical

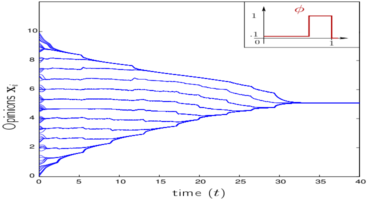
1D Heterophilious dynamics

- Strong influence: $1.0\chi_{\{0<\leq r<1\}}, \quad \phi(r) \searrow$
- Consensus: 100 agents uniformly distributed on $[0, 10]$.



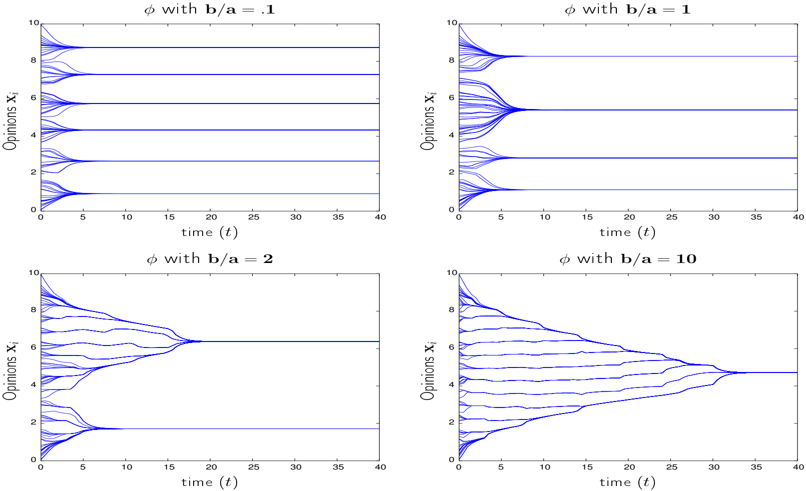
Strong influence $\mapsto$ fragmentation into clusters

- "Weak" influence: $\phi(r) = 0.1\chi_{\{r\leq\frac{1}{\sqrt{2}}\}} + 1.\chi_{\{\frac{1}{\sqrt{2}}\leq r<1\}}, \quad \phi(r) \nearrow$



Heterophily influence $\mapsto$ connectivity and hence consensus!!

More on heterophilious dynamics

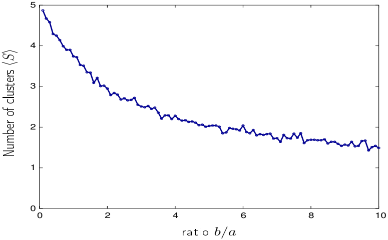


Opinion dynamics with different $\phi = a\chi_{[0,\frac{1}{\sqrt{2}}]} + b\chi_{[\frac{1}{\sqrt{2}},1]}$. Reducing the relative influence of close neighbors (as $b/a \uparrow$), decreases the # of clusters. For $b/a = 10$, the dynamics converges to a consensus.

Heterophilipus dynamics and the decrease of # of clusters

- This was not a coincidence:

Trace the propagation of connectivity in time: # clusters $K(t)$:



Log decay of average number of clusters $\langle K \rangle$ depending on the ratio b/a. For each b/a, we run 100 simulations to estimate $\langle K \rangle$.

2D Heterophilious dynamics

- Local interaction – compactly supported influence function ϕ vs. Local interaction - heterophily dynamics with factor 10:

1D dynamics with nearest neighbor

- Interaction with two nearest neighbors (1D):

$$\frac{d}{dt}x_i(t) = \sum_{|i-j|=1} \phi_{ij}(x_j - x_i), \qquad \phi_{ij} = \phi(|x_i - x_j|)$$

THM. If ϕ is non-decreasing on $d_{\mathbf{x}}(0)$, then the graph remains connected and $|x_i(t) - \overline{x}| \lesssim e^{-\gamma t/N^2}|x_i(0) - \overline{x}|$

- A steeper increase of ϕ enhances the connectivity ...