Consensus and Flocking in Self-Alignment Dynamics
Lecture I. Introduction - self-organized dynamics

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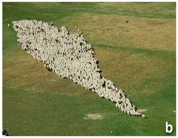
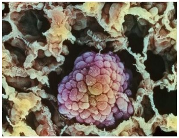
Observed behavior of self-organized dynamics

- Examples of dynamics which is self-organized $\mapsto$
  - ⊙ flocks of birds; schools of fish;
  - ⊙ herds of sheep; swarming of bacteria; colonies of ants, ...







- Not limited to biological examples:
  - ⊙ Human crowd; traffic jam; "opinion dynamics" $\mapsto$ consensus


- Large number of interacting "agents" $\mapsto$
- Emerging spatio-temporal structures:
  swarms, colonies, parties, ..., flocks, clusters, consensus, ...

Example #1: Krause model for opinion dynamics

- Environmental averaging: state space of "opinions" $\{x_i(t)\}_i$

$$x_i(t+1) = \sum_j a_{ij}(\mathbf{x}) x_j(t), \quad \sum_j a_{ij} = 1$$

- Time-step $1 \mapsto \Delta t$ and frequency $1 \mapsto \alpha$

$$\frac{x_i(t+\Delta t) - x_i(t)}{\Delta t} = \alpha \sum_j a_{ij}(\mathbf{x}) \cdot (x_j(t) - x_i(t)), \quad \sum_j a_{ij} = 1$$

- A symmetric model with global influence: $a_{ij}(\mathbf{x}) = \frac{1}{N} \phi(|x_i - x_j|)$
- Krause model (1977) — local influence model:

$$a_{ij}(\mathbf{x}(t)) = \left\{ \begin{array}{ll} \frac{1}{N_i}, & \text{if } |x_i(t) - x_j(t)| \leq \rho, \\ 0, & \text{if } |x_i(t) - x_j(t)| > \rho \end{array} \right\} \text{ non-symmetric}$$

$\mapsto$ Each agent "i" averages the opinion of its N_i neighbors

Models of opinion dynamics

$$\frac{x_i(t+\Delta t) - x_i(t)}{\Delta t} = \alpha \sum_j a_{ij}(\mathbf{x}) \cdot (x_j(t) - x_i(t)), \quad \sum_j a_{ij} = 1$$

$$a_{ij}(\mathbf{x}(t)) = \frac{1}{\deg_i}\phi(|x_i - x_j|), \quad \deg_i = \text{degree of influence of agent "}i\text{"}$$

- Global models involves all agents: $\deg_i = N$
- Local model: $\phi(r) = 1_{[0,\rho)}(r)$ involves $\deg_i = N_i$ neighbors

- Ben-Naim, Blondel, canuto, Deffuant, Fagnani, Hegselmann, Kruase, Kurz, Rambau, Toscani, Tsitsiklis, Weisbuch...

Questions:

- Formation of K "parties": $x_i(t) \longrightarrow x^{\infty}_{\mathcal{C}_k}, k \leq K$ as $t \to \infty$?
- When do we reach a "consensus": $x_i(t) \longrightarrow x^{\infty}$ as $t \to \infty$?

Example #2: Vicsek model for flocking

- Vicsek et. al.[1] (1995): averaging of orientations: $\overline{v}_i \in \mathbb{S}^1$

self-alignment : $\quad \overline{v}_i(t+\Delta t) \overset{\text{normalized}}{\longleftarrow} \frac{1}{N_i} \sum_{j:|x_j - x_i| \leq \rho} \overline{v}_j(t)$

$$\frac{\overline{v}_i(t+\Delta t) - \overline{v}_i(t)}{\Delta t} \rightsquigarrow \alpha \sum_j a_{ij}(\mathbf{x})(\overline{v}_j(t) - \overline{v}_i(t)), \quad \text{frequency } \alpha$$

$$\begin{cases} a_{ij}(\mathbf{x}(t)) = \left\{ \begin{array}{ll} \frac{1}{N_i}, & \text{if } |x_i(t) - x_j(t)| \leq \rho, \\ 0, & \text{if } |x_i(t) - x_j(t)| > \rho \end{array} \right\} \text{ non-symmetric} \\ \\ x_i(t+\Delta t) = x_i(t) + \left(\overline{v}_i(t+\Delta t) + \mathrm{noise}_{[-\eta,\eta]}\right) s\Delta t. \end{cases}$$

- A second-order model for "flocking"
- Aoki, Bellomo, Bertozzi, Carrillo, Degond, D'Orsogna, Frouvelle, J.-G. Liu, Motsch, Panferov, Rosado, Trelat, ...

[1] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, O. Shochet

Example #3: collective behavior of animal groups

• Couzin et. al.[2] model (2002)
phase space: $(x_i(t), \overline{v}_i(t)) \in \mathbb{R}^3 \times \mathbb{S}^3, \quad i = 1, 2, \ldots N$

$$\begin{cases} x_i(t+\Delta t) = x_i(t) + \left(\overline{v}_i(t+\Delta t)_{[-\theta\Delta t, \theta\Delta t]} + \text{noise}_{[-\eta,\eta]}\right) s\Delta t \\ \overline{v}_i(t+\Delta t) = \begin{cases} -\dfrac{1}{N_i} \displaystyle\sum_{j:|x_j - x_i| \le \delta} \dfrac{x_j(t) - x_i(t)}{|x_j(t) - x_i(t)|} & \text{repulsion} \\ \dfrac{1}{2N_i} \left[\displaystyle\sum_{k:\delta \le |x_k - x_i| \le \rho} \dfrac{x_k(t) - x_i(t)}{|x_k(t) - x_i(t)|} + \sum_k \dfrac{v_k(t)}{|v_k(t)|} \right] \end{cases} \end{cases}$$

• non-isotropic dynamics
• Groups of ants (Bruckstein, Couzin), locust (Erban & Haskovec), fish (Barbaro, Birnir, Hemelrijk & Hildenbrandt), birds (Starflag - G. Parisi et. al.), ... ,

[2] I. D. Couzin, J. Krause, R. James, G.D. Ruxton, & N.R. Franks

Example #4: Cucker-Smale model of flocking

- Cucker-Smale (2007): A semi-discrete model for self-alignment

⊙ State space of velocities: $\{v_i(t)\} \in \mathbb{R}^d, \quad i = 1, 2, \ldots N$:

$$\frac{d}{dt} v_i(t) = \alpha \sum_{1 \le j \le N} a_{ij} \left(v_j(t) - v_i(t) \right)$$

⊙ A second-order, semi-discrete model :

$$\left\{ \quad a_{ij}(\mathbf{x}) := \frac{1}{\deg_i} \phi\left(|x_i(t) - x_j(t)| \right), \qquad \frac{d}{dt} x_i(t) = v_i(t) \right.$$

- Search of coherent structures — flocks ($\mapsto$ "consensus"):

  ⋆ $\max_{ij} |x_i(t) - x_j(t)| \le \mathrm{d}_{\mathbf{x}}$ finite diameter in state space

  ⋆⋆ $v_i(t) \stackrel{t \to \infty}{\longrightarrow} v^\infty$ $\rightsquigarrow$ shrinking diameter in velocity space

Cucker-Smale model of flocking

$$\frac{d}{dt}v_i(t) = \alpha \sum_{1\leq j\leq N} a_{ij}\left(v_j(t) - v_i(t)\right), \quad a_{ij} = \frac{1}{\deg_i}\phi(|x_i - x_j|)$$

- Search of coherent structures — flocks ($\mapsto$ "consensus"):

  $\star$ $\max_{ij}|x_i(t) - x_j(t)| \leq \mathrm{d}_{\mathbf{x}}$ finite diameter in state space

  $\star\star$ $v_i(t) \overset{t\to\infty}{\longrightarrow} v^\infty$ $\rightsquigarrow$ shrinking diameter d_v in velocity space

$\odot$ Depending on the choice of influence function ϕ (decay rate β):

$$\begin{cases} \text{global interactions:}(\deg_i = N) & \phi(|x_i - x_j|) = \dfrac{1}{1 + |x_i - x_j|^{2\beta}} > 0 \\ \text{local interactions:}(\deg_i = N_i) & \phi(|x_i - x_j|) = 1_{[0,\rho)}(|x_i - x_j|) \geq 0 \end{cases}$$

- Carrillo, Fornasier, S.-Y. Ha, Illner, Karper, J.-G. Liu, Panferov, Piccoli, Rosado, J. Shen, Trelat, ... ,

Limitations C-S model: state space is not homogeneous

- What can go wrong with a global C-S model?

- Local alignment: $N_1 = \#G_1,\ G_1 = \{j :\ |x_j - x_i| \lesssim \rho\}$:

$$\frac{d}{dt} v_i(t) = \frac{\alpha}{N_1} \sum_{1 \le j \le N_1} \phi_{ij} \left(v_j(t) - v_i(t) \right)$$

$$\phi_{ij} := \phi(|x_i - x_j|)$$

- Global alignment: $N_2 = \#G_2,\ G_2 = \{j :\ |x_j - x_i| \gg \rho\}$:

$$\frac{d}{dt} v_i(t) = \frac{\alpha}{N_1 + N_2} \left[\sum_{1 \le j \le N_1} \phi_{ij} \left(v_j(t) - v_i(t) \right) + \sum_{1 \le j \le N_2} \phi_{ij} \left(v_j(t) - v_i(t) \right) \right]$$

- If $N_2 \gg N_1$, flock G_2 will bring to a halt the dynamics of G_1

Example #5: a new model (S. Motsch & ET)

Alignment is weighted by the relative distance: $\phi_{ij} := \phi(|x_i - x_j|)$

$$\frac{d}{dt}v_i(t) = \alpha \sum_{j \neq i} a_{ij}\left(v_j(t) - v_i(t)\right), \quad a_{ij} := \frac{1}{\deg_i}\phi_{ij}, \ \deg_i = \sum_k \phi_{ik}$$

- If all agents are "clustered" $\mapsto$ recover the C-S dynamics:

$$\phi_{ik} \approx \phi_0 \rightsquigarrow \deg_i \approx N\phi_0 \rightsquigarrow \ \frac{d}{dt}v_i(t) = \frac{\alpha}{N\phi_0}\sum_j \phi_{ij}\left(v_j(t) - v_i(t)\right)$$

- If not: the dynamics of agents "i" in G_1 is "divorced" from G_2:

$$\phi_{ik} \approx \begin{cases} \phi_0, & \text{if } k \in G_1 \\ 0, & \text{if } k \in G_2 \end{cases} \quad : \quad \frac{d}{dt}v_i(t) = \frac{\alpha}{\underbrace{N_1\phi_0}_{\deg_i}}\sum_j \phi_{ij}\left(v_j(t) - v_i(t)\right)$$

- non-symmetric interactions: $a_{ij} \neq a_{ji}$

Self- alignment $\mapsto$ clusters, consensus, connectivity, ...

- A general class of N agents identified w/ “position” $\mathbf{p}_i \equiv \mathbf{p}_i(t)$:

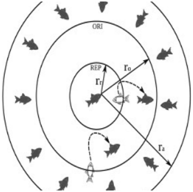
$$\frac{d}{dt}\mathbf{p}_i(t) = \alpha \sum_{j \neq i} a_{ij}(\mathbf{p}_j - \mathbf{p}_i), \qquad \sum_{j=1}^{N} a_{ij} \leq 1 \;\mapsto\; \sum_{j=1}^{N} a_{ij} = 1$$

- Agents do not react to positions, but to relative gradient

- Examples of
  First-order models “opinion dynamics”: $\mathbf{p}_i \mapsto x_i$ (or $\mathbf{x}_i \ldots$)
  Second-order models “flocking”: $\mathbf{p}_i \mapsto \overline{\mathbf{v}}_i, \ \mathbf{v}_i$

- Three zones (AAAs) models: Craig Reynolds (1987)

  Repulsion (Avoidance);
  Cohesion (Attraction)
  Alignment: $a_{ij} \geq 0$;



Self- alignment $\mapsto$ clusters, consensus, connectivity, ...

$$\frac{d}{dt}\mathbf{p}_i(t) = \alpha \sum_j a_{ij}(\mathbf{p}_j - \mathbf{p}_i) = \alpha \left(\sum_j a_{ij}\mathbf{p}_j \right) - \mathbf{p}_i, \quad \sum_{j=1}^{N} a_{ij} = 1$$

- Two classes of self-organized models
  - ⊙ Global models: $a_{ij} \geq \eta > 0$;

    unconditional consensus : $\mathbf{p}_i(t) \stackrel{t\to\infty}{\longrightarrow} \mathbf{p}^\infty$
  - ⊙ Local models: $a_{ij} \geq 0$; formation of clusters

    $\{1, 2, \ldots, N\} = \cup \mathcal{C}_{k=1}^K$: $\mathbf{p}_i(t) \stackrel{t\to\infty}{\longrightarrow} \mathbf{p}_{\mathcal{C}}^\infty, \qquad i \in \mathcal{C}$

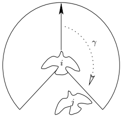
Questions:

- What "rules of engagement" influence the emergence of clusters?
- $K = 1$ — the emergence of one cluster — consensus:
- Propagation of connectivity of $a_{ij} = a_{ij}(\mathbf{p}(t))$;

Additional aspects in group dynamics

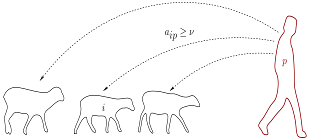
- Vision: agent "i" is influenced only by those that it can "see":

$$\mathcal{N}_i := \{ j \;\mid \langle x_j - x_i, v_i \rangle \geq -\theta |x_j - x_i| \cdot |v_i| \} :$$



$$\frac{d}{dt} v_i(t) = \alpha \sum_{j \in \mathcal{N}_i} a_{ij} \left(v_j(t) - v_i(t) \right), \quad a_{ij} := \frac{1}{N_i} \phi(|x_i - x_j|)$$

- "Leaders": $\exists$ "ℓ": $a_{i\ell} \geq \beta \phi(|x_i - x_\ell|)$



- Control: $\dot{\mathbf{v}}(t) = \alpha \bar{\mathbf{v}}(t) - \mathbf{v}(t) + \mathbf{u}(t)$, $\quad \bar{\mathbf{v}}_i := \sum a_{ij} \mathbf{v}_j$
- Noise and stochastic aspects
- Stability