Mathematical Models for Social Changes and Criminology - 4

- A toy model fot tax evasion
- Tuning the parameters
- Fighting cheaters: how much to spend
- Fighting cheaters: where to spend

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A toy model for tax evasion

- The simplest case: *two possible states* for each of the 1600 cells of a square grid:

- X = law-abiding citizens; Y= cheaters (e.g.tax evaders)

Evolution rules: we assume that a citizen (cell) changes his/her attitude according to

(i) **the influence of the behaviour of the "neighbors"**

(ii) **factors that are independent of the behaviour of the others (global field)**

The influence of neighbors

The probability of transition from one state to another is given by

$$P_{X \to Y}^{LOC} = l \, \frac{N_{LY}}{N_L},$$

$$P_{Y \to X}^{LOC} = k \, \frac{N_L - N_{LY}}{N_L},$$

(l and k are in [0,1]).

Here N_L is the number of cells of a neighborhood $m \times m$, and N_{LY} is the number of cheaters in the same neighborhood.

The “global” field

It is simply the probability of changing state, indpendently of the behaviour of neighbors

$$P_{X \to Y}^{NONLOC} = \tau, \qquad 0 < \tau < 1$$

$$P_{Y \to X}^{NONLOC} = \alpha, \qquad 0 < \alpha < 1.$$

How large is the "neighborhood"?

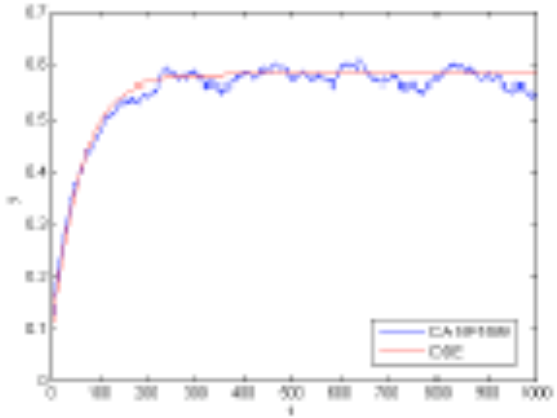
- The value of N_L accounts for the structure of the society (how much it is "interconnected").
- Of course if $N_L = 1599$ (all the society influences the attitude of each citizen), then the CA is the stochastic approach to the finite difference approximation of the ODE

$$dY/dt = \tau^* (N - Y) - \alpha^* Y + (l^*/N)(N - Y) - (k^*/N) Y (N - Y)$$

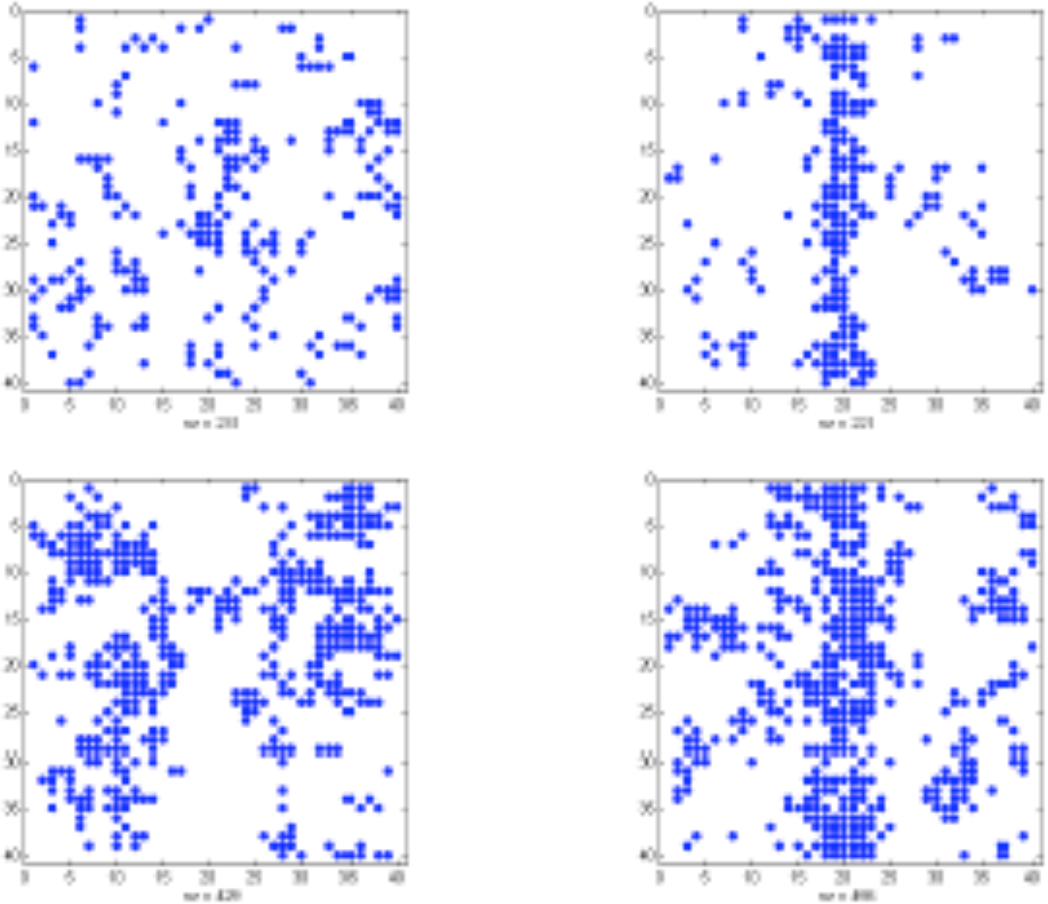
The mean-field approximation

- After normalization and setting $d = l - k$, we have

$$\begin{aligned} \dot{y}(t) &= \tau(1-y) - \alpha y + d\,y(1-y) \\ y(0) &= y_0 \end{aligned}$$

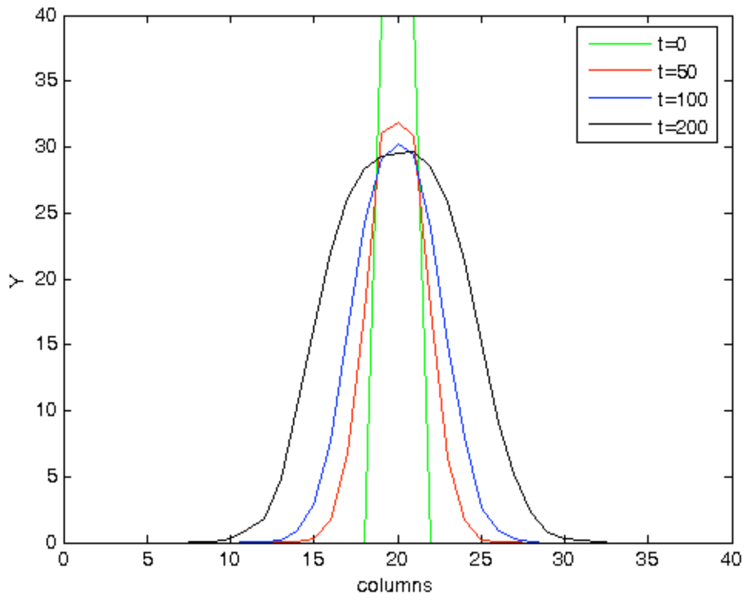


Time evolution of the cheater population y of the ODE and CA with maximum neighborhood ($N_L = 1599$). The parameter setup is $\alpha_0 = 0.01$ $\tau_0 = 0.008$, $l_0 = 0.31$ $k_0 = 0.30$. The initial condition is $y_0 = 0.1$. The CA curve is an average over 10 simulations.

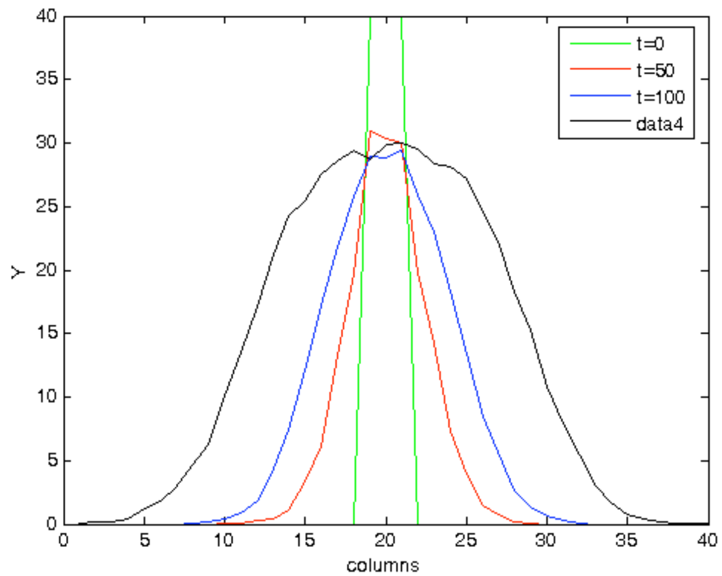


: Screenshots of the time evolution of the CA at time $t = 10$ [figures (A) and (B)] and $t = 50$ [figures (C) and (D)] for two initial spatial distributions of the cheaters: random, figures (A) and (C) and, fully distributed in the three central columns of the grid, figures (B) and (D). White cells are occupied by tax payers, whereas blue cells are occupied by cheaters. The initial condition is $Y_0 = 120$. The parameter setup is: $\alpha_0 = 0.005$ $\tau_0 = 0.004$ $l_0 = 0.31$ $m_0 = 0.30$. The neighborhood size is $N_L = 8$.

Space dependence



With a 3 x 3 neighborhhod



With a 5 x 5 neighborhood

In sociological terms "*how much the society is interconnected*"

Fighting tax evasion

- The (normalized) wealth obeys

$$\dot{w} = 1 - y - \theta(1 + \varphi) w$$

Here **theta** represents the budgetary policy and **phi** is the fraction devoted to fighting tax evasion.

Accordingly, we assume that *alpha and k increase* (whereas *tau and l decrease*) as **φw** increases

Effectiveness of the measures

- Just to give an example we postulate a single "sensitivity" parameter p such that

$$\alpha = \alpha_0 (2 - e^{-p\varphi w})$$

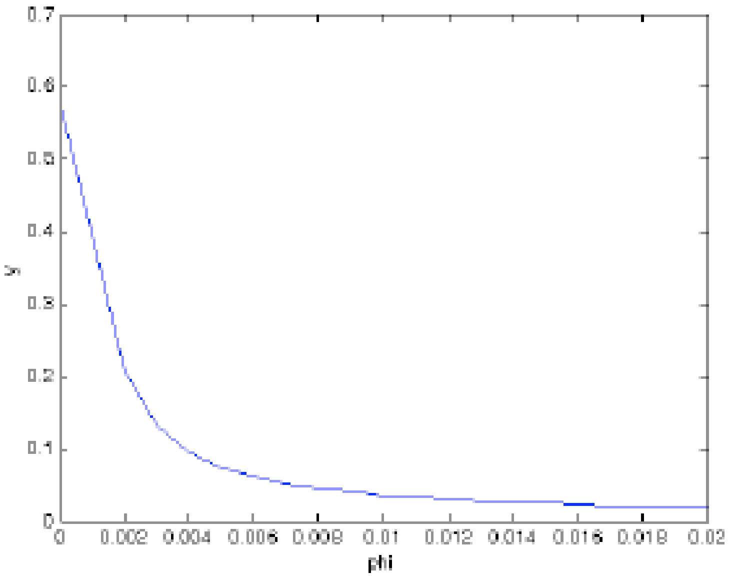
$$k = k_0 (2 - e^{-p\varphi w})$$

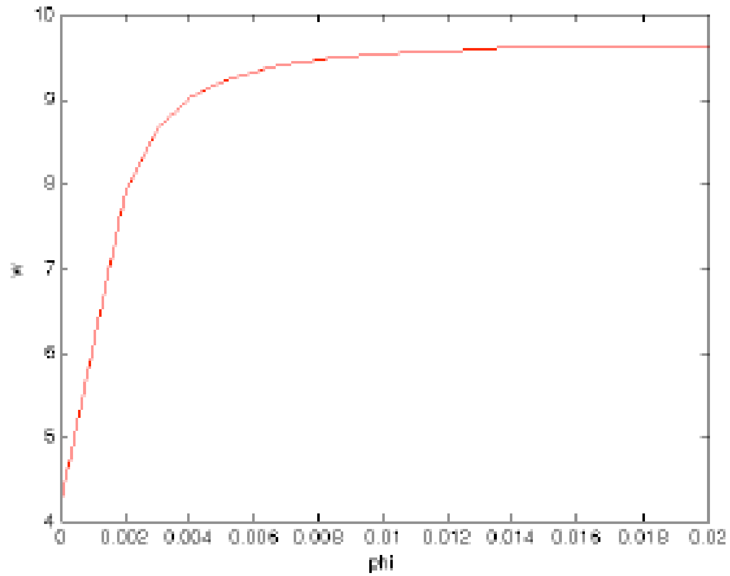
$$\tau = \tau_0 (1 + e^{-p\varphi w})$$

$$l = l_0 (1 + e^{-p\varphi w})$$

A simulation

Adjourning CA and the discretized equation for the wealth simultaneously, we have the following asymptotic values for y (fraction of evaders) and w (wealth), for increasing values of φ.





Does an optimal value of φ exist?

The "anarchic" state of the society is characterized by the asymptotic values of Y and W (Y_0 and W_0) corresponding to $\varphi = 0$.

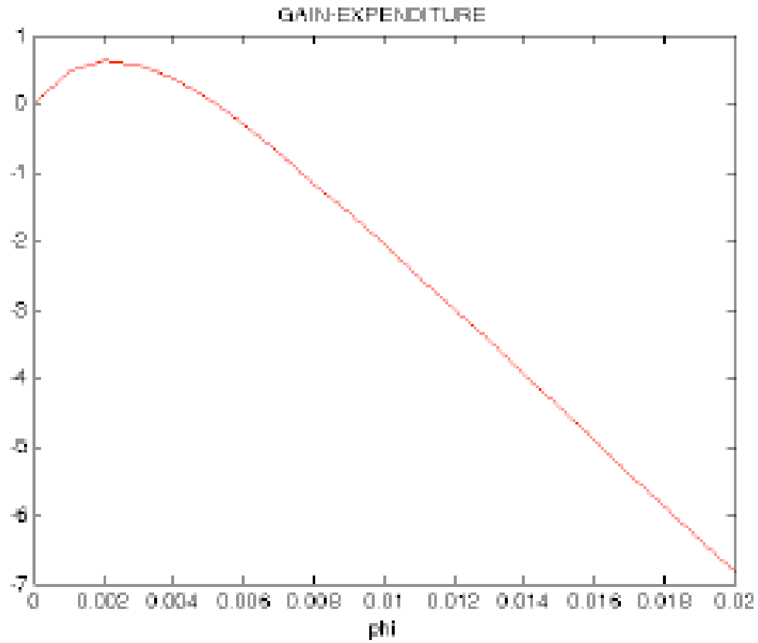
We start from this state and consider a time horizon say of 5 years over which we adopt a given policy (i.e. we fix a value of φ).

We compute the final gain G in terms of wealth and the total (additional) expenses E that are given by

$$G(\varphi) = w_\infty(\varphi) - w_0$$

$$E(\varphi) = \sum_{j=1}^{4} \varphi w_j$$

An optimal value of φ exists



Gain – expenditure over a time period of 5 years

Space-dependent contrast policy

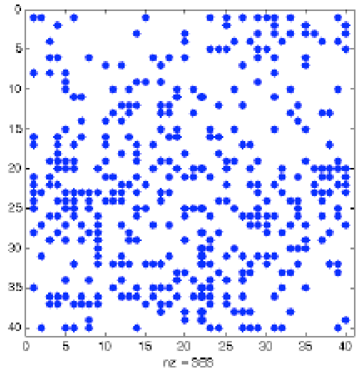
- Consider a grid (with the same "natural" parameters) and with an average value φ of intensity of effort to fight criminality.
- But assume a "chessboard" scheme with regions with φ=0 and φ=constant.
- Simulate the asymptotic value of the fraction of criminals, starting from the same initial situation

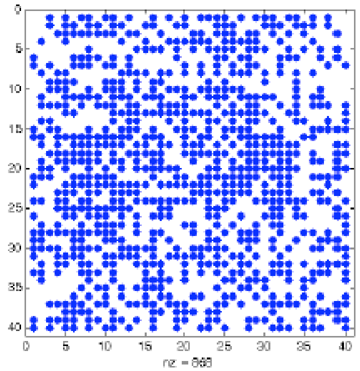
Case	Solution
2 regions	$y = 0.27$
4 regions	$y = 0.25$
16 regions	$y = 0.23$
Homogeneous	$y = 0.20$

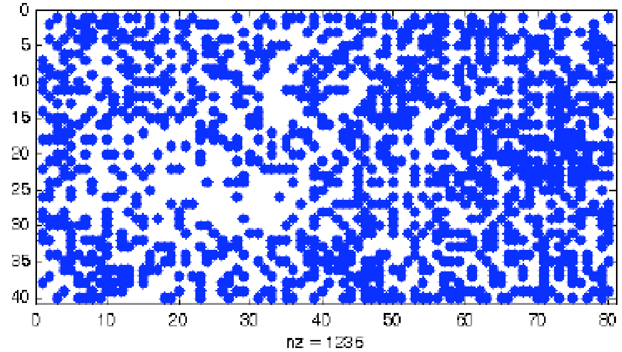
"Contagion" across the borders

- Consider TWO ajacent square grids and assume they correspond to different "fiscal cultures".
- Assume there is no fighting against evasion ("natural" or "anarchic" situation).
- Question: What is the difference between the case of isolated worlds and the case of "permeable boundaries"?

	First World	Second World
Isolated	$y_\infty = 0.27$	$y_\infty = 0.60$
In contact $N_l = 120$	$y = 0.39$	$y = 0.54$
In contact $N_l = 24$	$y = 0.31$	$y = 0.55$
In contact $N_l = 8$	$y = 0.27$	$y = 0.56$







Advantages of cooperation

- Assume that the two societies have the same **"fiscal culture"**, but just one spends a given amount of its budget to **contrast criminality.**

$$\tau_0^{(1)} = 0.01;\ \alpha_0^{(1)} = 0.05;\ \zeta_0^{(1)} = 0.5;\ k_0^{(1)} = 0.3;\ p = 5;\ N_L = 120$$

$$\tau_0^{(2)} = 0.01;\ \alpha_0^{(2)} = 0.05;\ \zeta_0^{(2)} = 0.5;\ k_0^{(2)} = 0.3;\ p = 5;\ N_L = 120$$

Advantages of cooperation (2)

- We look for the asymptotic situation of the two societies assuming that country 1 spends a fraction φ1 of its wealth w1 to fight cheaters in the same country and fraction φs of it to fight cheaters in the other country
- On the contrary, country 2 does (or cannot….) spend money to fight cheaters.

- Consider two different situations:
- All the money is spent in the country where it is produced ($\varphi_s = 0$)
- Part of it is spent in the other country ($\varphi_s > 0$)

Advantages of cooperation

	Setup	Results	
First test	$\varphi_1 = 0.035$ $\varphi_S = 0$ $\varphi_2 = 0$	$y_1 = 0.05$ $y_2 = 0.75$	$w_1 = 9.16$ $w_2 = 2.51$
Second test	$\varphi_1 = 0.025$ $\varphi_S = 0.010$ $\varphi_2 = 0$	$y_1 = 0.04$ $y_2 = 0.32$	$w_1 = 9.26$ $w_2 = 6.75$

Cooperation can reduce the level of crime in both regions!

And increases the wealth in both countries as well!

Evolution on networks

The topology of contagion is not related to spatial proximity alone.

In particular, referring to special kinds of “illegal behaviours” or to diffusion of innovation, of fashion, even of political opinion, *social networks* are more and more important

The mean-field approximation

Essentially, the approach based on an O.D.E. is equivalent to the assumption of homogeneous distribution of number of contacts for every node in the whole network and to a properly weighted coefficient for the contagion.

But, due to the extreme variability of the types of networks, the dependence of the dynamics on the structure of the network is an open problem.

An approach with PDE's

Is it possible to introduce a model in terms of differential equations that mimics the behaviour of the CA?

The main point is how to deal with the "local" terms, i.e. how to incorporate space dependence in the law that expresses the evolution of the density of cheaters. *We confine to 1 dimension for simplicity.*

The "contagion" term should say that the time derivative of y(x,t) is given by the number of law abiding citizens ("susceptive") multiplied by the number of cheaters ("infective") in a neighborhood

A possible form for the contagion term

$$\frac{\partial y(x,t)}{\partial t} = d\int_a^b K(x-\xi)\left[1-y(x,t)\right]y(\xi,t)\,d\xi$$

K is a kernel whose support expresses the "interconnection" of the society.

We approximate y to the second order in the space variable

$$y(\xi,t) \cong y(x,t) + y_x(x,t)(\xi - x) + y_{xx}(x,t)(\xi - x)^2 / 2$$

Assumption on the kernel

Concerning K we can reasonably assume that it is an even function and that its support is "small" w.r.t. b-a.

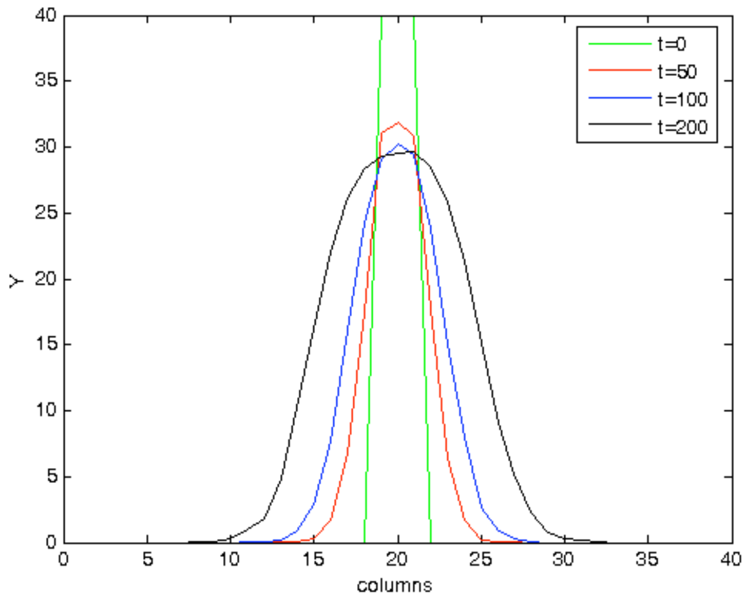
But this means that the *first order momentum* of K vanishes, so that the resulting PDE is nothing else than a Fisher equation

$$\frac{\partial y(x,t)}{\partial t} = dI_0 y(x,t)[1 - y(x,t)] + (I_2/2)\frac{\partial y(x,t)}{\partial x^2}$$

The PDE approximation

The equivalent diffusivity is given by the order 2 momentum of the kernel (i.e. by the dimension of the neighborhood), as we have already seen in the simulations

Space dependence



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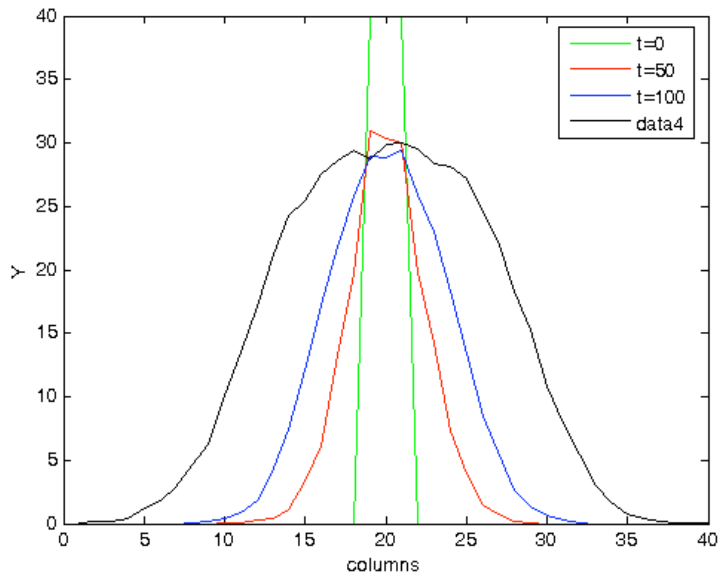

With a 5 x 5 neighborhood

In sociological terms "*how much the society is interconnected*"

The PDE approximation

Of course this is a bi-stable equation that has solutions in terms of travelling waves.

One could introduce additional fighting mechanisms in terms of direct actions (not just by influencing the transition from the state of cheater to the state of law-abiding citizen.

Effects like in the Fitzhugh-Nagumo eq. Could be introduced….

Thank you very much for your attention/patience!