Mathematical Models for Social Changes and Criminology - 3

- Criminality modelled by population dynamics (eco-epidemiological)
- Coupling social and criminal dynamics

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Granada – BIOMAT 2013

A special “triangle model”
(both epidemiological and ecological)

$$\dot{X}(t) = r(1-X)(k-X) - AXY - BZ$$

$$\dot{Y}(t) = fAXY - H\frac{YZ}{D+Y} - FY - GY^2$$

$$\dot{Z}(t) = gAXY - hH\frac{YZ}{D+Y} - CZ$$

The dynamics of the catching of criminals is modelled according to a Holling-type formula.
H measures the efficiency of the police.
In a sense, both Y and Z are “predators” w.r.t. X and Y is a “prey” of Z.

A first special case

If X >> (Y+Z), we can take the subpopulation of "targets" as being constant and the system becomes

$$\frac{dY}{dt} = f\alpha Y - Y(F+GY) - \frac{HY}{D+Y}Z$$
$$\frac{dZ}{dt} = g\alpha Y - h\frac{HY}{D+Y}Z - CZ$$

where

$$\alpha = \frac{kX_0}{E+X_0}$$

We take H (efficiency of security forces) as a bifurcation parameter and set the other constants as follows:

$$\alpha = 10^2; \quad d = 10^{-1}; \quad g = 10^{-2}; \quad f = 10^{-1}; \quad h = 0; \quad C = 1, \quad F = 10^{-1}, \quad G = 1;$$

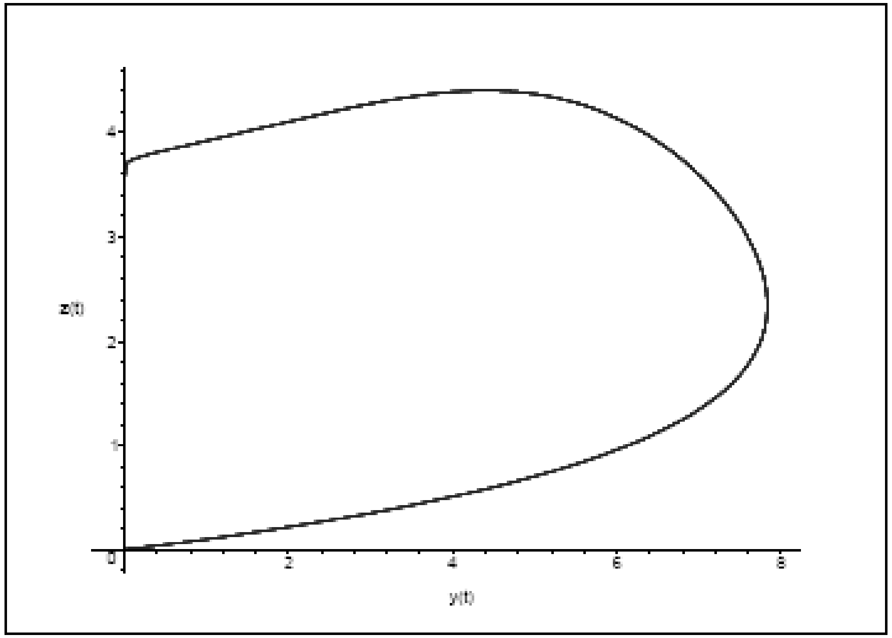
… as H grows …

… the non-trivial equilibrium changes ist nature!

H	Type of Equilibria
$(0, \bar{H}_1)$	STABLE NODE
$(\bar{H}_1, \bar{H}_2)$	STABLE FOCUS
$\bar{H}_2$	CENTER
$(\bar{H}_2, \bar{H}_3)$	UNSTABLE FOCUS
$(\bar{H}_3, \bar{H}_4))$	UNSTABLE NODE
$(\bar{H}_4, \bar{H}_5))$	UNSTABLE FOCUS
$\bar{H}_5$	CENTER
$(\bar{H}_5, \infty)$	STABLE FOCUS

Moreover, the equilibrium point tends to the origin as H increases

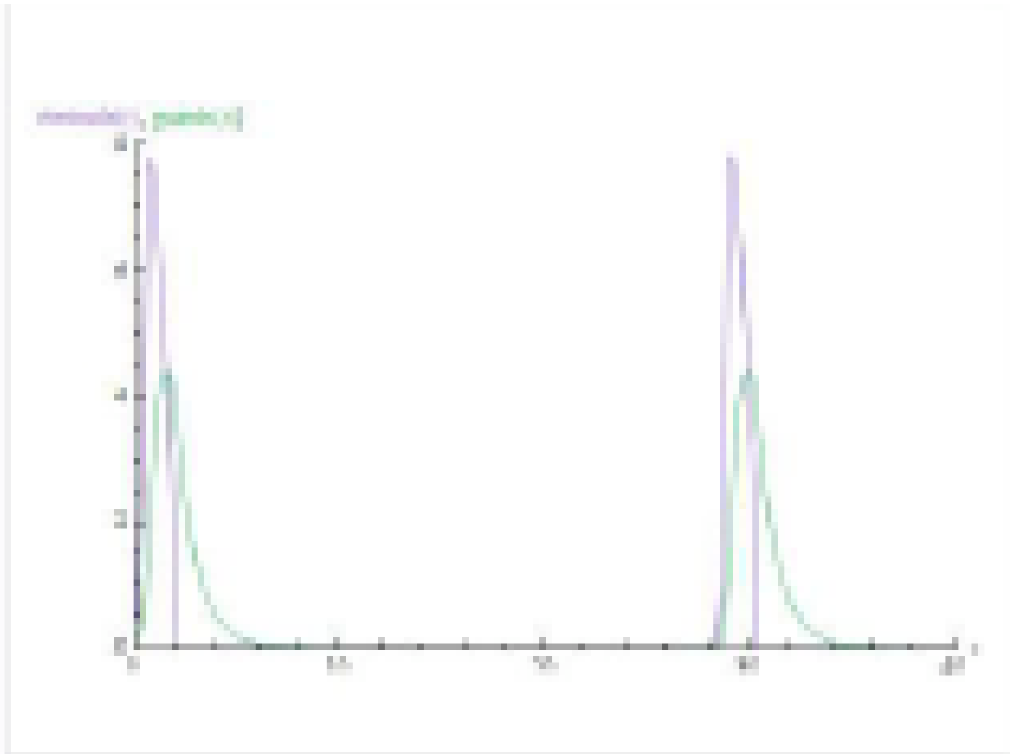
Some simulations for this case are shown in next slide



H =7

The limit cycle

Oscillatory behaviour (two time scales)

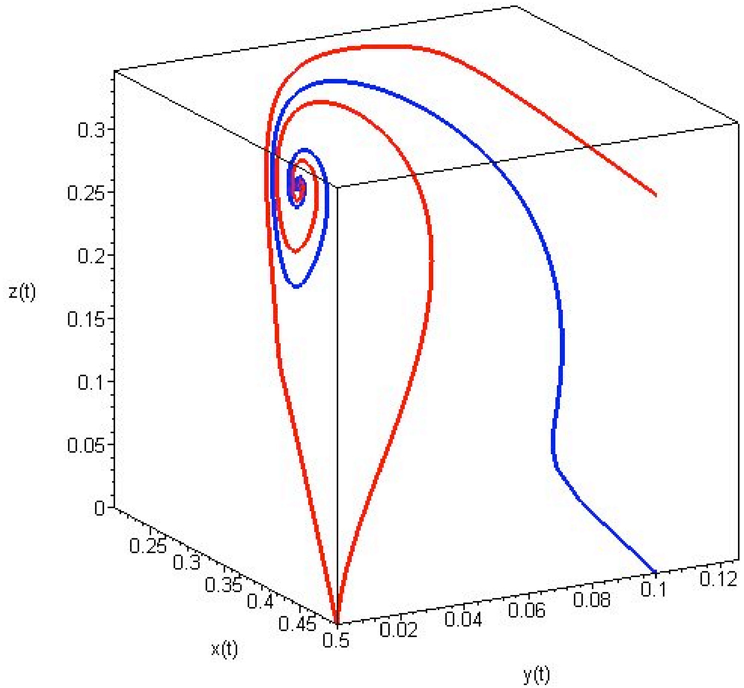


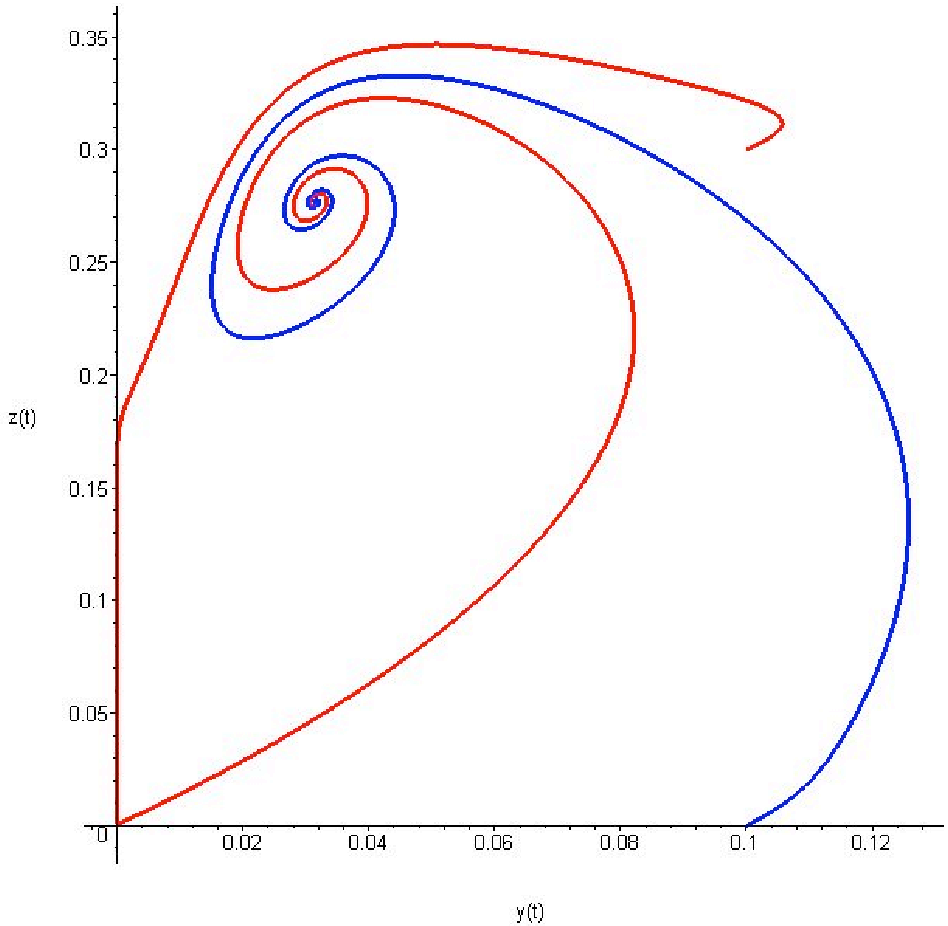
General case: summary of the bifurcation analysis

- As H grows form 0 to ∞ we have the following behaviour:
  - Global fixed point attractor
  - Generalized Hopf bifurcation (saddle-node)
  - Bistability of cycles
  - Subcritical Hopf bifurcation
  - Only one asymptotically stable limit cycle
  - Hopf bifurcation
  - Stable focus

General case

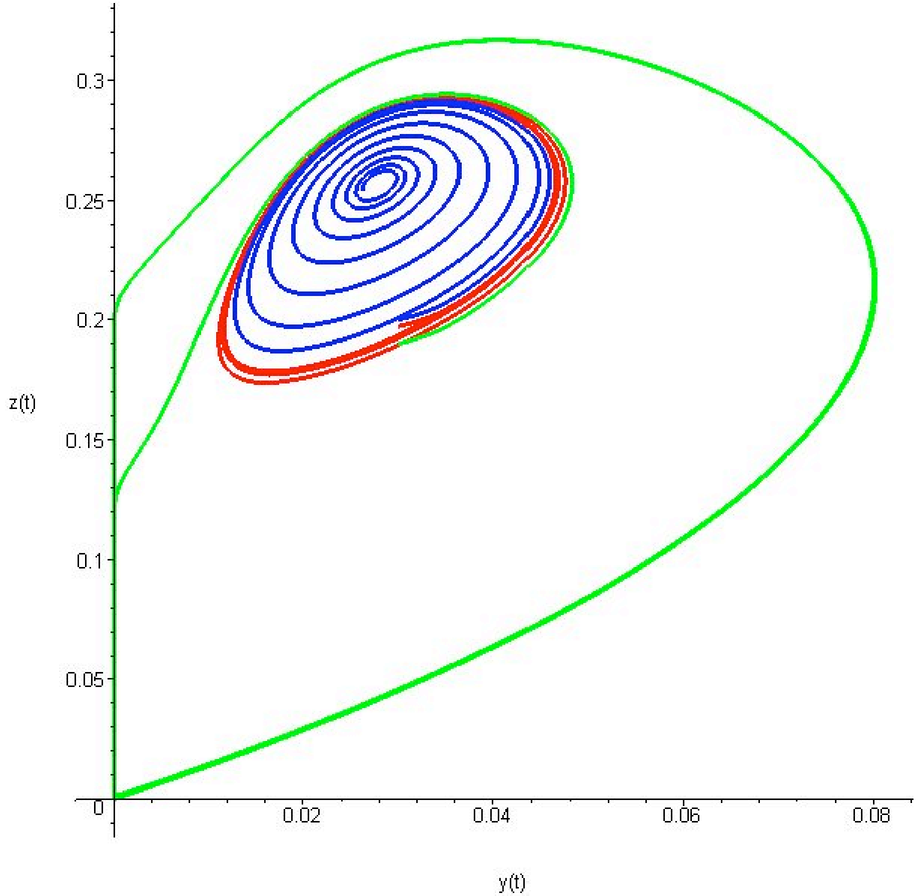
H = 0.0033





H= 0.00355

Stable equilibrium surrounded by two limit cycles.



Including some social dynamics

Haiyun Zhao, Zhilan Feng, Carlos Castillo-Chávez: *The dynamics of poverty and crime*, 2002

$$\begin{cases} \frac{dN}{dt} = \mu T - (\sigma + \mu)N \\ \frac{dS}{dt} = \sigma N - \beta S \frac{C}{T} - (\gamma + \mu)S \\ \frac{dC}{dt} = \beta S \frac{C}{T} + \phi \beta R \frac{C}{T} - (\rho + \mu)C \\ \frac{dP}{dt} = \rho C - (\delta + \mu)P \\ \frac{dR}{dt} = \gamma S + \delta P - \phi \beta R \frac{C}{T} - \mu R \\ T = N + S + C + P + R \end{cases}$$

The subpopulation R represents the “recovered”

The "reproduction number"

Following the methods of epidemiology, we define number R playing the role of a reproduction number

R = A + B

$$A = \frac{\sigma}{\sigma+\mu}\left(1 - \frac{\gamma}{\gamma+\mu}\right)\frac{\beta}{\rho+\mu}$$

$$B = \frac{\sigma}{\sigma+\mu}\frac{\gamma}{\gamma+\mu}\frac{\phi\beta}{\rho+\mu}$$

Crime-free equilibrium F and non-zero equilibrium E

We just refer to the case $\gamma << \mu$, for sake of brevity

If $R<1$, then F is asymptotically stable and E does not exist

If $R>1$, then F is unstable, E is unique and stable.

Social dynamics and criminality

We start by recalling the compartmental model for social dynamics

$$\dot{u}_i(t) = \alpha_{i-1} u_{i-1}(t) - (\alpha_i + \beta_i) u_i(t) + \beta_{i+1} u_{i+1}(t),$$
$$i = 1, 2, \ldots, n, \; with$$
$$\alpha_0 = \alpha_n = \beta_1 = \beta_{n+1} = 0.$$

and we add three more subpopulations: criminals, prisoners and guards

Criminals, prisoners, and guards

- Criminals Y(t):
  - recruitment rate R(t) *(in principle from each social class)*
  - crime rate K(t) *(in principle from each social class)*
  - arrest rate A(t)
  - "spontaneous" decay D(t)
- Prisoners J(t)
  - release rate F(t)
- Guards G(t)
  - hiring rate H(t)
  - "induced" decay L(t)
  - "spontaneous" decay P(t)

Specification of rates

$$R(t) = k_1 u_1(t) Y(t), \qquad k_1 = k(W(t)),$$

$$K(t) = \sum_k \vartheta_k u_k(t) Y(t), \qquad \vartheta_k = m_k / (l_k + u_k(t)).$$

$$A(t) = mY(t)G(t)/(l + Y(t))$$

$$D(t) = -\rho Y(t) - \nu Y^2(t)$$

$$F(t) = -\tau J(t)$$

$$H(t) = hK(t)$$

$$L(t) = -\delta A(t)$$

$$D(t) = -qG(t)$$

recruitment rate R(t)
crime rate K(t)
arrest rate A(t)
"spontaneous" decay D(t)
release rate F(t)
hiring rate H(t)
"induced" decay L(t)
"spontaneous" decay P(t)

Some assumptions

- Criminals are recruited just from the poorest class.
- Efficiency of “contagion” is proportional to wealth.
- Individuals leaving classes Y and G “spontaneously” leave the population permanently

The resulting system

$$\dot{Y} = \kappa W u_1 Y - \varrho Y - \nu Y^2 - m Y G/(\ell + Y)$$

$$\dot{G} = -q G + h Y \sum_k m_k u_k/(\ell_k + u_k) - \delta m Y G/(\ell + Y)$$

$$\dot{J} = m Y G/(\ell + Y) - \tau J$$

$$\dot{u}_1 = -\alpha_1 u_1 + \beta_2 u_2 + \tau J - \kappa W u_1 Y$$

$$\dot{u}_i = \alpha_{i-1} u_{i-1} - (\alpha_i + \beta_i) u_i + \beta_{i+1} u_{i+1}, \quad i = 2 \dots n-1$$

$$\dot{u}_n = \alpha_{n-1} u_{n-1} - \beta_n u_n$$

… and the equation for W(t)

$$\dot{W} = \sum_k a_k u_k - \lambda W Y \sum_k m_k u_k / (\ell_k + u_k) - g(W) G$$

It can be proved that the corresponding problem (once the initial conditions have been prescribed and the promotion and relegation coefficients satisfy very reasonable assumptions) has one unique global solution and all its components are non-negative (!)

Police measures vs. social action

We describe a particular example in which:

- Only two social classes exist
- Police size is taken as constant
- Population in jail not taken into account (***time scale intermediate between average time in jail and time required to change the policy of hiring police forces***)

Social measures vs. police action

$$\dot{u}_1 = -\alpha_1 u_1 + \beta_2 u_2 - kWu_1Y + \rho Y + \nu Y^2 + mYG/(l+Y)$$
$$\dot{u}_2 = \alpha_1 u_1 - \beta_2 u_2$$
$$\dot{Y} = kWu_1Y - \rho Y - \nu Y^2 - mYG/(l+Y)$$

- Moreover, the equation for W is:

$$\dot{W} = a_1u_1 + a_2u_2 - (\theta + s)W - \lambda YWc_2u_2/(l_2 + u_2) - gWG$$

Some qualitative results

- A criminal-free equilibrium point always exists. The parameter k can be seen as a bifurcation parameter. When k crosses a critical value, the criminal-free equilibrium becomes unstable and a new stable equilibrium point is found.
- And, of course, this critical value depends on the (constant) level of guards. For some values of the parameters

$$k_{cr} = 0.12 + 1.2G + 0.12G^2$$

Some qualitative results-2

- For fixed *k*, we consider alpha as the control parameter (social promotion). In correspondence to the crime-free equilibrium, the Jacobian matrix has two negative eigenvalues and a third eigenvalue whose sign (for the same numerical values of the other parameters) depends on alpha and on G.

How and how much?

$$\dot{W} = a_1 u_1 + a_2 u_2 - (\theta + s)W - \lambda Y W c_2 u_2 / (l_2 + u_2) - gWG$$

- Assume the social promotion for the “poor” class is increasing with sW (yearly rate of social expenses)
- The quantity (s+gG)W denotes the rate of expenditures to contrast criminality (**how much**). The factor s (or the ratio gG/s) measures **how** the society decides to spend.
- Optimize (i.e maximize wealth, minimize criminality) the choice with a given time horizon.

How and how much?

Of course, we have to give a precise definition of the quantity we want to optimize.

More precisely, we have to define the cost functional.

This means define some kind of "metrics" in the domain (W,Y), decide if the control is continuous etc.

We will give some more detail in a model that takes space dependence into account.

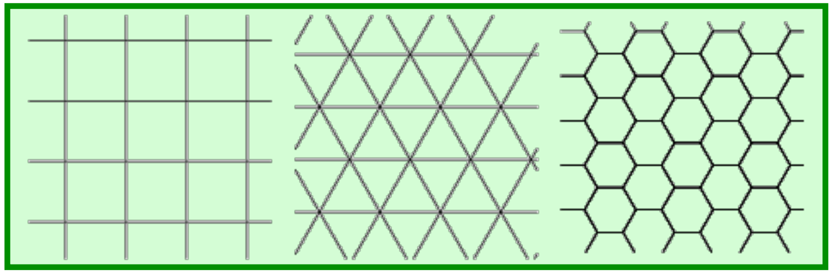
Criminology and cellular automata

- Of course compartmental models do not take <u>space variation</u> into account.
- Nevertheless, they can be used to **tune** space-dependent models, in particular models based on **cellular automata.**

A basic introduction

- A cellular automaton is a collection of "colored" cells on a grid of specified shape that evolves through a number of discrete time steps according to a set of rules based on the states of neighboring cells. The rules are then applied iteratively for as many time steps as desired.

The number k of colors (or distinct states) a cellular automaton may assume must also be specified



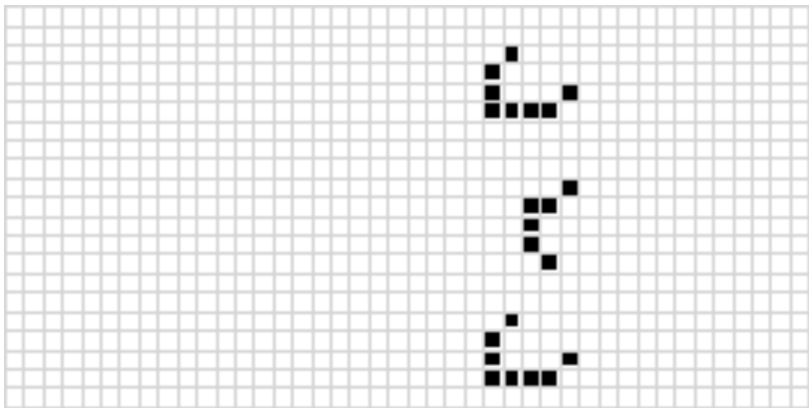
In the sequel we will deal with the simplest case: k=2

M.Gardner's "game of life"

- Two states: black (B) or white (W). Rectangular grid.
- For each cell, one checks all eight of the surrounding. Any cells that are B are counted, and this number δ is then used to determine what will happen to the cell.
- For example

1. Cell is **B:** if $\delta=2$ or $\delta=3$ it remains **B** otherwise it turns to **W**
2. Cell is **W:** iff $\delta=3$ it becomes **B**

Running the CA from a given initial situation



A toy model for tax evasion

- The simplest case: *two possible states* for each of the 1600 cells of a square grid:
- X = law-abiding citizens; Y= cheaters (e.g.tax evaders)

Evolution rules: we assume that a citizen (cell) changes his/her attitude according to

(i) **the influence of the behaviour of the "neighbors"**

(ii) **factors that are independent of the behaviour of the others (global field)**

The influence of neighbors

The probability of transition from one state to another is given by

$$P_{X \to Y}^{LOC} = l \frac{N_{LY}}{N_L},$$

$$P_{Y \to X}^{LOC} = k \frac{N_L - N_{LY}}{N_L},$$

(l and k are in [0,1]).

Here N_L is the number of cells of a neighborhood $m \times m$, and N_{LY} is the number of cheaters in the same neighborhood.

The “global” field

It is simply the probability of changing state, indpendently of the behaviour of neighbors

$$P_{X\to Y}^{NONLOC} = \tau, \qquad 0 < \tau < 1$$

$$P_{Y\to X}^{NONLOC} = \alpha, \qquad 0 < \alpha < 1.$$

How large is the “neighborhood”?

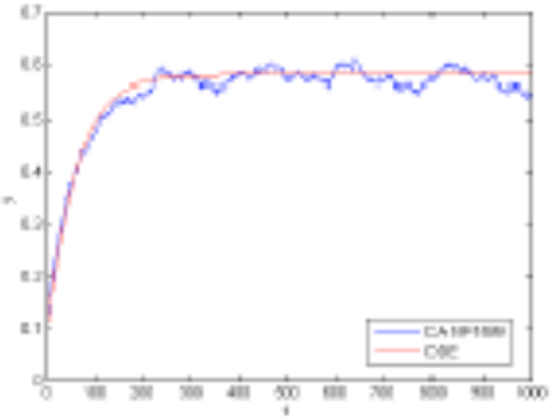
- The value of N_L accounts for the structure of the society (how much it is “interconnected”).
- Of course if $N_L = 1599$ (all the society influences the attitude of each citizen), then the CA is the stochastic approach to the finite difference approximation of the ODE

$$dY/dt = \tau^* (N - Y) - \alpha^* Y + (I^*/N)(N - Y) - (k^*/N) Y (N - Y)$$

The corresponding ODE model

- After normalization and setting $d = l - k$, we have

$$\begin{aligned} \dot{y}(t) &= \tau\,(1-y) - \alpha\,y + d\,y\,(1-y) \\ y(0) &= y_0 \end{aligned}$$



Time evolution of the cheater population y of the ODE and CA with maximum neighborhood ($N_L = 1599$). The parameter setup is $\alpha_0 = 0.01$ $\tau_0 = 0.008$, $l_0 = 0.31$ $k_0 = 0.30$. The initial condition is $y_0 = 0.1$. The CA curve is an average over 10 simulations.