Mathematical Models for Social Changes and Criminology

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Granada – BIOMAT 2013

Part 1 – Modelling demography and social dynamics

a) A general introduction

b) Demographic models

c) Models for elementary social dynamics

d) Coupling demography and social dynamics

Introduction: criminology & mathematics

There is a vast literature on the **descriptive** side of criminology, but the use of mathematical models and methods is still largely at its dawn, with the exception of two areas:

a) image analysis (fingerprint identification, crime scene reconstruction, forensic applications)

b) statistical analysis of data

Using empirical statistics

- Looking just for possible correlations (regression) between crime rate and possible explicative variables (such as income, social inequality, % young males, education, probability of arrest, ...) may lead to **contradictory conclusions**

A few examples quoted by Mirta Gordon *(EJAM 21 (2010))*

1) crime rate vs. expected punishment:

Ehrlich (1973, 1975) finds that crime rates are sensitive to the expected size of punishment

Archer and Gartner (1984) find no impact of capital punishment on murders in their cross-national study

2) crime rate vs. average income

Fleisher (1966) and Ehrlich (1973) examined the effect of unemployment rates, income levels, and income disparities: their findings on the effects of average income levels are in contradiction

And, finally …..

3) crime rate vs. education

Tauchen and Witte (1994) find that in a sample of young men, going to work or school tends to reduce the probability of being involved in criminal activities.

Ehrlich (1975) finds a positive relationship between the average number of school years completed by the adult population and property crimes committed across the U.S. in 1960 ('education puzzle').

A better use of mathematics

- Mathematics is not always able to give complete answers but it is always useful to pose the correct questions

- Without (at least) a conceptual model to test and possibly validate, abundance of data may be background noise and not information!

Of course, statistics is important!

Once one has at least a ***conceptual model*** , statistics has to play a major role.

And it is important to analyze data (quite often, huge amounts of data) <u>with some care.</u>

Disaggregate data relevant to different types of crime, take under-reporting into account, using data-mining carefully….

Without good data….

…and without a good use of statistical techniques,

our mathematical models could risk to reduce to interesting exercises and/or to"*nice pieces of mathematics*"

How to get good data?

It is necessary to convince key people (ministries, police, criminologists…) to trust mathematics (and mathematicians)!

And, in any case…

- Police forces usually have an enormous amount of data on crime….. BUT

- too aggregated (mostly at country level)
- systematic biases (under-reporting)
- no agreement on classification of crimes
- difficult to be obtained
- difficult to disentangle causes from effects *(ex: effect of probability of arrest on amount of crime vs. effect of amount of crime on probability of arrest)*

The limits of our approach

- Remember we are just looking for conceptual models
- Thus, our aim is to investigate the structure of possible interactions among observable quantities, state variables, and possible control actions.
- We do not claim that real world is deterministic, but just confine to arguments in terms of mean field approach

Social models in view of criminology

Assume we want to model the "recruitment" of criminals.

According to Marcus FELSON….

two main "state variables" are to be taken into account:

age and **social condition**

(of course there are also other variables to be taken into account, but these are more than enough for a conceptual model)

Age structure: PDE models

"*Density*" n(t,a) such that, for any couple of ages α and β, (0< α < β) the integral

$$\int_{\alpha}^{\beta} n(t,a)\,da$$

represents the number of individuals in the population that have age between α and β at time t.

Age structure: PDE models (2)

It is well known that, if μ(t,a) represents the mortality, we have the equation

$$n_t + n_a + \mu(t,a) = 0$$

that has to be solved with a given initial condition **$n(0,a)=n_0(a)$** and with a boundary condition that is e.g.

$$n(0,t) = \int_0^\infty \lambda(t,a) n(t,a) da$$

where λ(t,a) is the fertility.

Age structure: discrete models

Also this is a very classical topic. One considers m age groups and discretizes the time. Hence for the vector $\underline{n}(t)$ (whose m components are the number of individuals in each group at time t) we have the linear evolution

$$\underline{n}(t+1) = A(t)\underline{n}(t)$$

Where A(t) is the Leslie or Lefkovitch matrix

Leslie or Lefkovitch matrix

$$\mathbf{A} = \begin{pmatrix} r_1 + f_1 & f_2 & f_3 & f_4 & \cdots & f_m \\ s_1 & r_2 & 0 & \cdots & \cdots & 0 \\ 0 & s_2 & r_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & s_{m-1} & r_m \end{pmatrix}$$

where the f_i represent the fertility of each age group, the r_i the individuals that remain in the group in the time step, the s_i the individuals that pass to the next age group (in case of Leslie the r_i are 0 and the age groups are of unit age/time width).

Compartmental models for social structure

Consider a *closed society* and neglect space variability.

The social structure of the society will be identified by a n-vector

$$\underline{U}(t) = (u_1(t), u_2(t), \ldots, u_n(t)),$$

that gives for every time t the number of individuals that belong to each of the n social groups forming the society.

Compartmental models for social structure - 2

For the moment, **disregard criminality** and consider that the n social groups are identified by the average wealth

$$a_1, a_2, \ldots, a_n$$

of its members and assume

$$0 < a_1 < a_2 < \ldots < a_n (< +\infty)$$

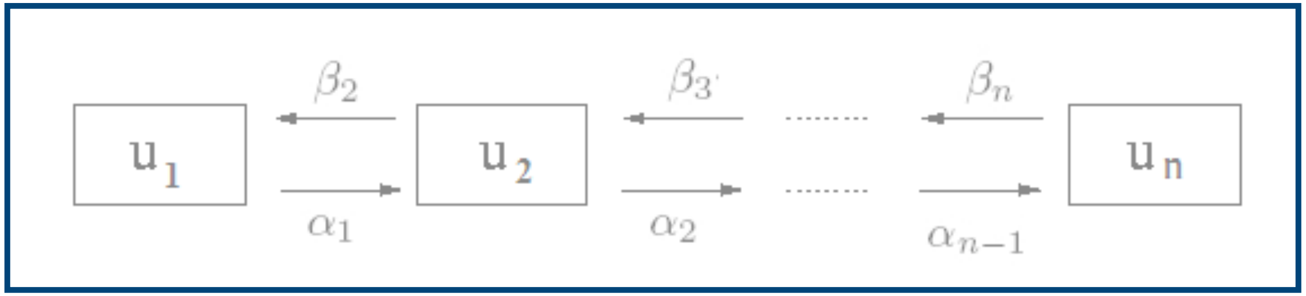
Thus the total wealth of the society is

$$W(t) = \sum_{i=1}^{n} a_i u_i(t)$$

Compartmental models for social structure - 3

To describe the evolution of the society, starting from a given initial situation $\underline{U}(0)$ we have to give the dynamics of transition from/to any of the classes. Assume that:

- transition is possible just from/to adjacent classes
- transition is governed by a linear dynamics



Here α and β represent, for any social class, the rate of **social promotion** and **social relegation**

The associated dynamical system: equilibrium points

$$\dot{u}_i(t) = \alpha_{i-1}u_{i-1}(t) - (\alpha_i + \beta_i)u_i(t) + \beta_{i+1}u_{i+1}(t),$$
$$i = 1,2,\ldots,n, \textit{with}$$
$$\alpha_0 = \alpha_n = \beta_1 = \beta_{n+1} = 0.$$

If the coefficients are positive constants (TOY EXAMPLE) one finds immediately that there exists one unique non-trivial equilibrium that is given by

$$\hat{u}_k = \frac{\alpha_1}{\beta_2}\frac{\alpha_2}{\beta_3}\cdots\frac{\alpha_{k-1}}{\beta_k} \equiv \omega_{k-1}\hat{u}_1, \quad k = 1,2,\ldots,n,$$

(where $\grave{\mathrm{u}}_0 = 1$ by definition). $\quad \textit{Thus, if } \omega = \sum_{k=1}^{n-1} \omega_k, \textit{ we have:}$

$$\hat{u}_1 = N / \omega$$
$$N = \sum_k u_k(t) = \text{constant dimension of the population}$$

Social mobility depending on total wealth

If the coefficients depend on the total wealth, any fixed point of the operator φ defined by

$$\varphi(W) = N\frac{\sum_k a_k \omega_{k-1}(W)}{\omega(W)}$$

gives a non-trivial equilibrium. The converse is true since the coefficients are known once W is given. … the *toy example* is instrumental!
Existence – almost trivial fixed point argument
Uniqueness – additional conditions on **α and β** ensuring $\varphi' < 1$

The case of three social classes

Let u, v, and 1-u-v be the (normalized) dimension of the three classes; let us normalize also wealth so that

$$\mathbf{W= u + av +b(1-u-v).}$$

where $b < a < 1$.

Of course the largest wealth is W=1 (when u=1…*everybody is rich!*) and the smallest is W=b (u=v=0… *everybody is poor!*)

Choose, for instance

$$\alpha_i = W^i; \qquad \beta_{i+1} = 1 - W^{3-i} \qquad \forall i = 1,2$$

so that

$$\alpha_1 = (1-b)\,u + (a-b)\,v + b;\ \alpha_2 = ((1-b)\,u + (a-b)\,v + b)^2;$$

$$\beta_2 = 1 - ((1-b)\,u + (a-b)\,v + b)^2;\ \beta_3 = 1 - b - (1-b)\,u - (a-b)\,v$$

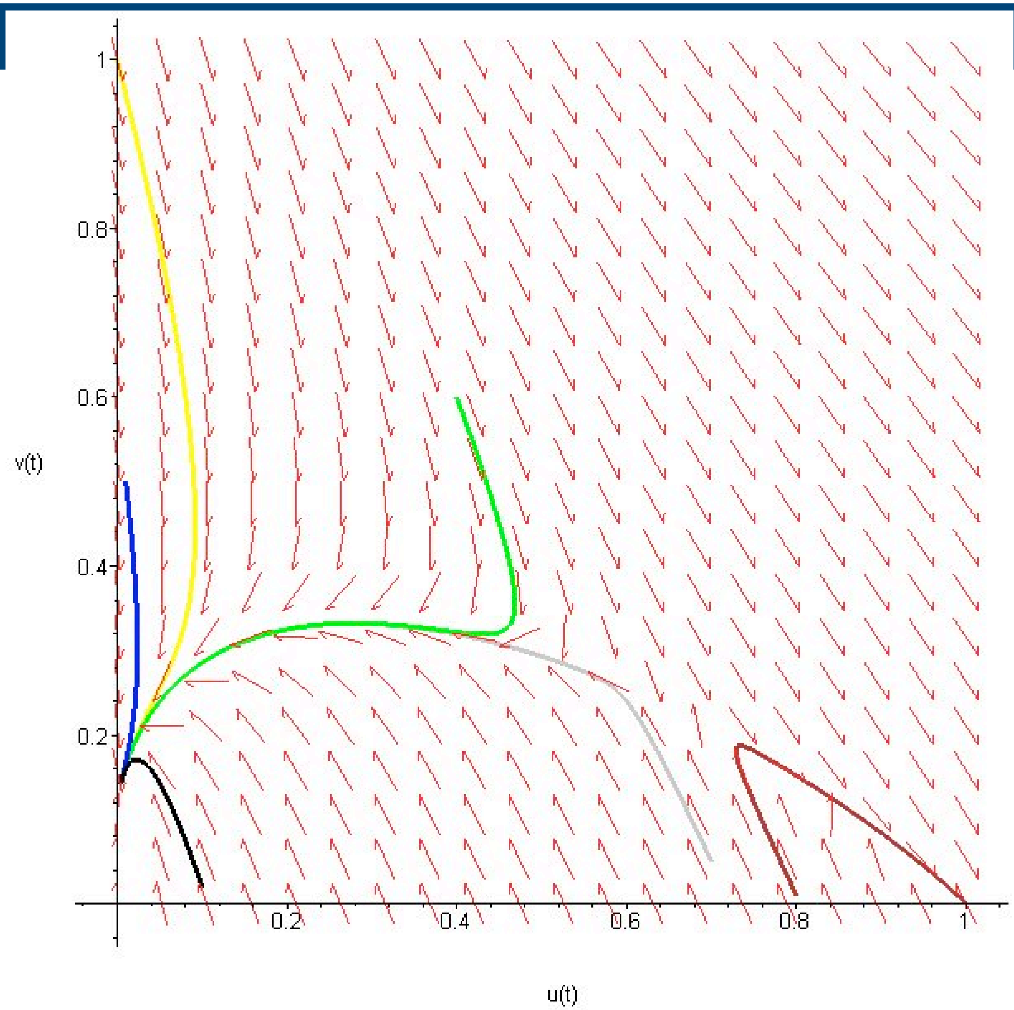
The case of three social classes - 2

The dynamical system becomes

$$\dot{u} = ((1-b)\,u + (a-b)\,v + b)^2\,v - (1 - b - (1-b)\,u - (a-b)\,v)\,u$$
$$\dot{v} = (1 - 2\,(u + av + b(1-u-v)))\,u - \qquad (2.11)$$
$$- (1 + (u + a\,v + b\,(1-u-v)))\,v + u + a\,v + b\,(1-u-v)$$

A particular situation arises when the lowest class does not contribute to the total wealth (b=0). In this case, three fixed points exist: the poorest (u=v=0), the wealthiest (u=1), and a situation corresponding to the coexistence of the three classes (that is a saddle-node).

The corresponding phase portrait



An example of coupling between demography and social dynamics

Two social classes (rich and poor) and two age groups (juveniles and adults).

$$\mathbf{N}(t) = \begin{pmatrix} \mathbf{N}_1(t) \\ \mathbf{N}_2(t) \end{pmatrix} = \begin{pmatrix} \mathbf{PB}_1 & \text{(I-Q)}\mathbf{B}_2 \\ \text{(I-P)}\mathbf{B}_2 & \mathbf{QB}_2 \end{pmatrix} \begin{pmatrix} \mathbf{N}_1(t-1) \\ \mathbf{N}_2(t-1) \end{pmatrix} = \mathbf{AN}(t-1)$$

P and **Q** are the social mobility matrices and the **B**'s are the Lefkovitch matrixes for the two social classes

$$\mathbf{B}_1 = \begin{pmatrix} 0 & f_1 \\ s_1 & s_1 \end{pmatrix} \quad \text{e} \quad \mathbf{B}_2 = \begin{pmatrix} 0 & f_2 \\ s_2 & s_2 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \quad \text{e} \quad \mathbf{Q} = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}$$

A few remarks on the system

$$\begin{pmatrix} n_{11} \\ n_{12} \\ n_{21} \\ n_{22} \end{pmatrix}(t) = \begin{pmatrix} 0 & pf_1 & 0 & (1-q)f_2 \\ ps_1 & ps_1 & (1-q)s_2 & (1-q)s_2 \\ 0 & (1-p)f_1 & 0 & qf_2 \\ (1-p)s_1 & (1-p)s_1 & qs_2 & qs_2 \end{pmatrix} \begin{pmatrix} n_{11} \\ n_{12} \\ n_{21} \\ n_{22} \end{pmatrix}(t-1)$$

The system is extremely rich, even in very specially simple cases. Set e.g. q=0 and consider how the mobility p (assumed constant) affects the equilibria of the system

The linear case

If the spectral radii of the Lefkovitch matrixes for the two social groups are >1 and <1 respectively (a Malthusian character), changing p we can have that both populations tend to extinction or “blow up” etc. (a trivial exercise)

A simple nonlinear case

In a logistic-type case for population1, we find a non-trivial equilibrium (for $p<1$) and p is a bifurcation parameter that has two critical values.

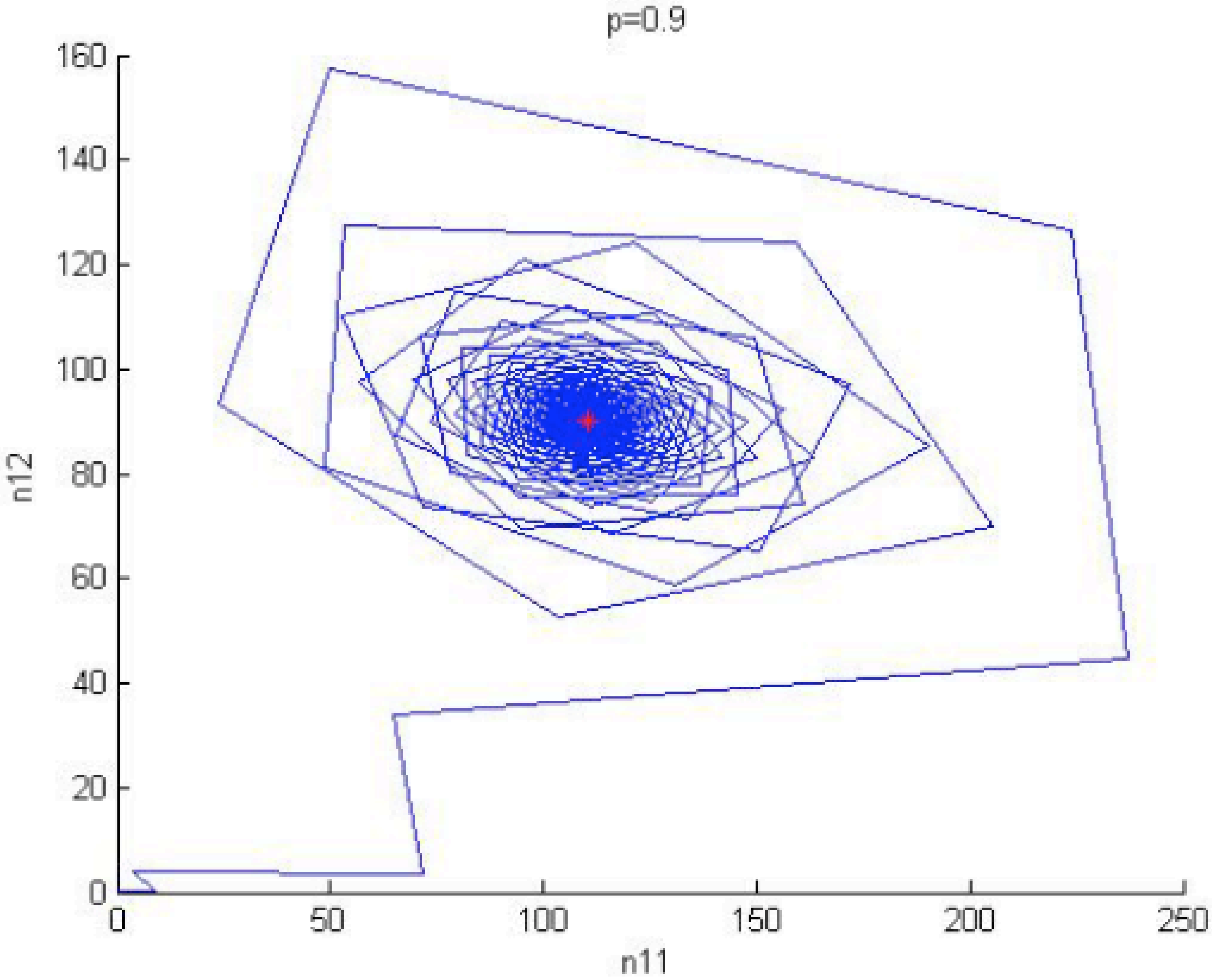


Figura 3.11: Comportamento della coppia di valori n_{11}, n_{12} con $K_1 = 100$, $r = 3.1$, $s_1 = s_2 = 0.5$, $f_2 = 0.8$ e condizioni iniziali $n_{11}(0) = 1$, $n_{12}(0) = n_{21}(0) = n_{22}(0) = 0$

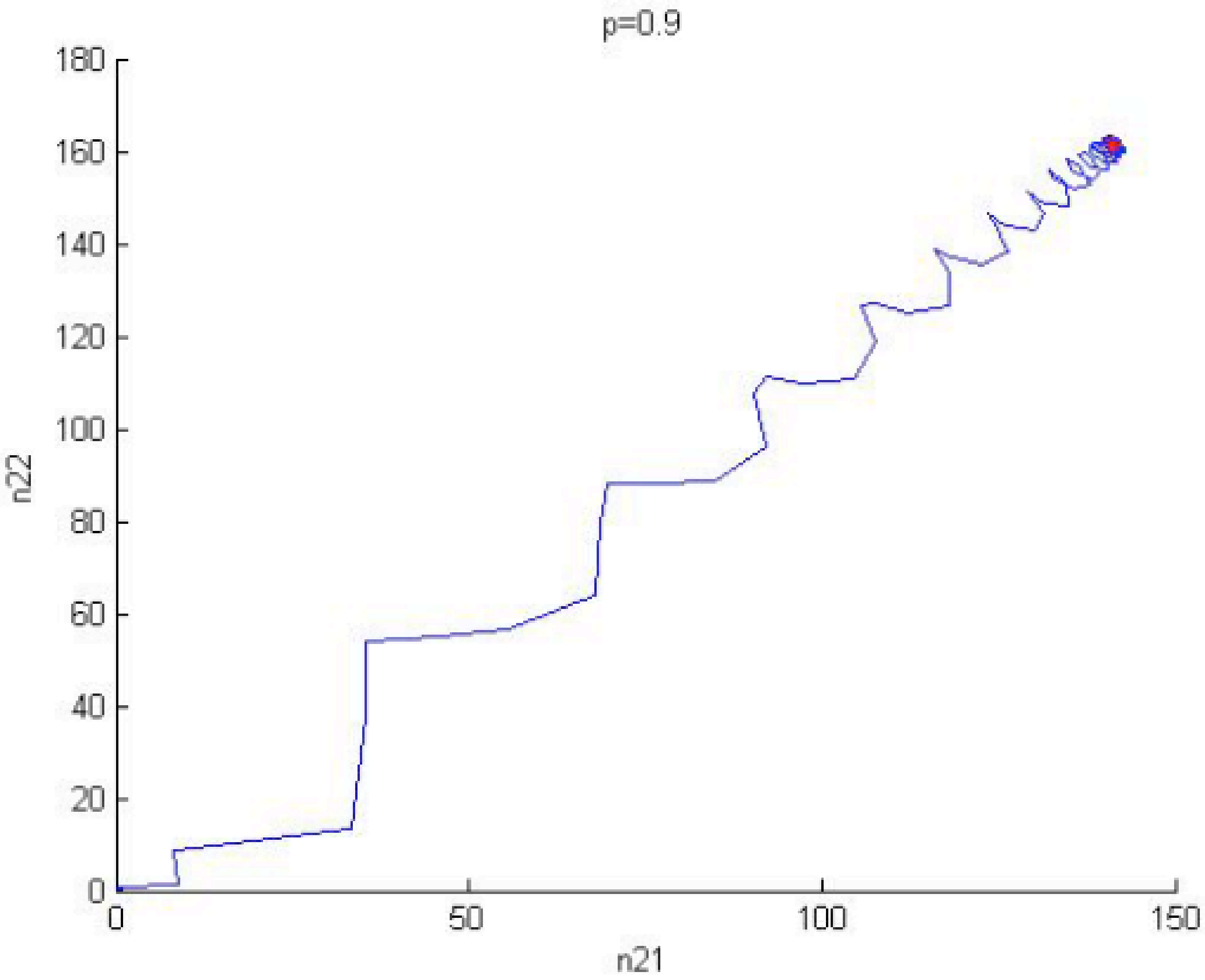


Figura 3.12: Comportamento della coppia di valori n_{21}, n_{22} con $K_1 = 100$, $r = 3.1$, $s_1 = s_2 = 0.5$, $f_2 = 0.8$ e condizioni iniziali $n_{11}(0) = 1$, $n_{12}(0) = n_{21}(0) = n_{22}(0) = 0$

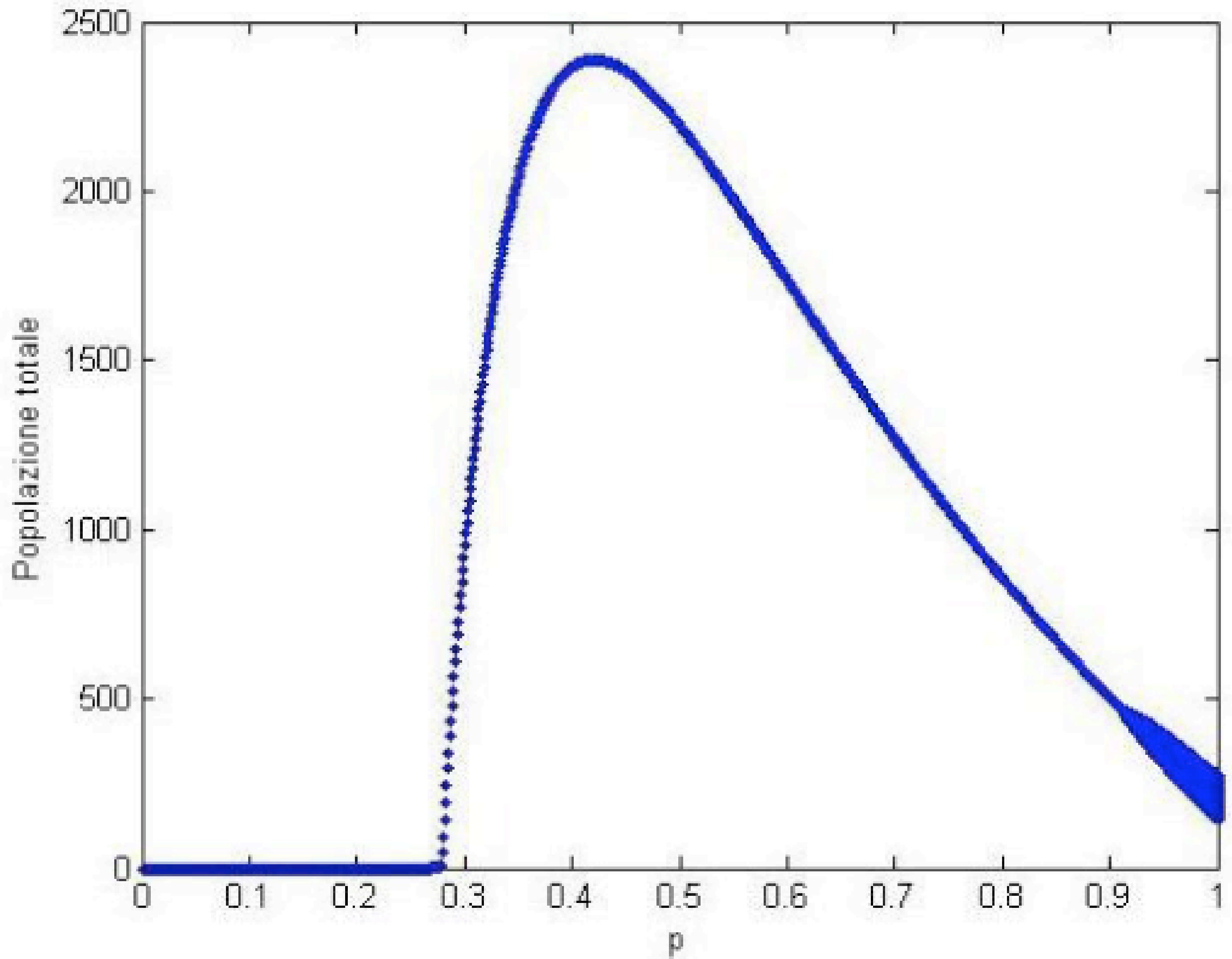


Figura 3.13: Diagramma di biforcazione della popolazione totale in funzione di p