# A kinetic theory approach to the modeling of crowd dynamics

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• **M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale, and V. Zdravkovic**, Interaction ruling animal collective behavior depends on topological rather than metric distance: evidence from a field study, *Proc. Nat. Acad. Sci.*, 105(4) (2008), 1232-1237.

• **N. Bellomo and C. Dogbè**, On the modelling of traffic and crowds - a survey of models, speculations, and perspectives, *SIAM Review*, 53(3) (2011), 409–463.

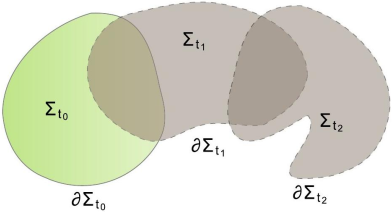
• **N. Bellomo, and A. Bellouquid**, On The Modeling of Crowd Dynamics: Looking at the Beautiful Shapes of Swarms, *Netw. Heter. Media.*, 6 (2011), 383–399.

• **N. Bellomo and J. Soler**, On the mathematical theory of the dynamics of swarms viewed as complex systems, *Math. Models Methods Appl. Sci.*, 22, Suppl. 1: 1140006, (2012).

• **N. Bellomo, A. Bellouquid, and D. Knopoff**, From the micro-scale to collective crowd dynamics, arXiv:1301.55/4v1 [math-ph] 23 Jan 2013.

# Crowd dynamics in a spatial domain

# What about swarms?



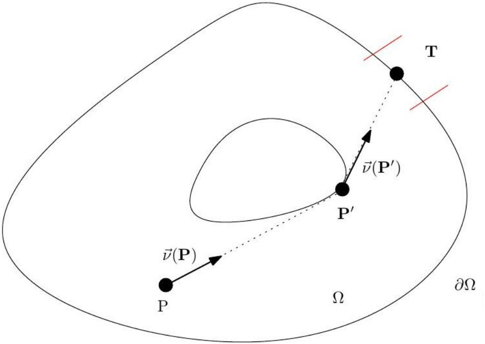
# Hallmarks of the kinetic theory of active particles

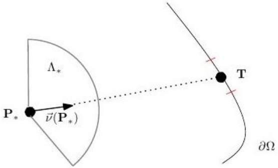
- The overall system is subdivided into *functional subsystems* constituted by entities, called *active particles*, whose individual state is called *activity*;
- The state of each functional subsystem is defined by a suitable, time dependent, *distribution function over the microscopic state*;
- Interactions are modeled by games, more precisely stochastic games, where the state of the interacting particles and the output of the interactions are known in probability;
- Interactions are delocalized and nonlinearly additive;
- The evolution of the distribution function is obtained by a balance of particles within elementary volumes of the space of the microscopic states, where the dynamics of inflow and outflow of particles is related to interactions at the microscopic scale.

# Toward a kinetic theory of active particles

| | **Crowd dynamics** |
|---|---|
| **Active particles** | Pedestrians |
| **Microscopic state** | Position<br>Velocity<br>Activity |
| **Functional subsystems** | Different abilities<br>Individuals pursuing different targets<br>etc. |

# Crowds in bounded domains with obstacles



Polar coordinates with discrete values are used for the velocity variable $\mathbf{v} = \{v, \theta\}$:

$$I_\theta = \{\theta_1 = 0, \dots, \theta_i, \dots, \theta_n = \frac{n-1}{n} 2\pi\}, \quad I_v = \{v_1 = 0, \dots, v_j, \dots, v_m = 1\}.$$

$$f(t, \mathbf{x}, \mathbf{v}, u) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}(t, \mathbf{x}, u) \, \delta(\theta - \theta_i) \otimes \delta(v - v_j) \, .$$

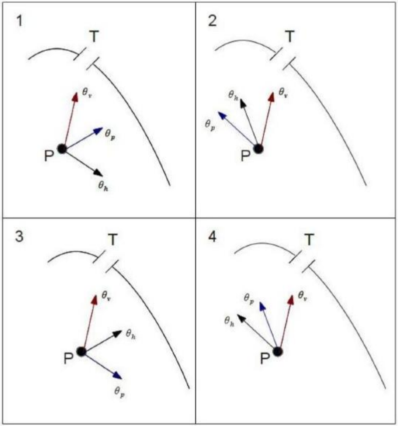
Some specific cases can be considered. For instance the case of two different groups, labeled with the superscript $\sigma = 1, 2$, which move towards two different targets.

$$f^\sigma(t, \mathbf{x}, \mathbf{v}, u) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}^\sigma(t, \mathbf{x}) \, \delta(\theta - \theta_i) \otimes \delta(v - v_j) \otimes \delta(u - u_0) \, ,$$

Local density:

$$\rho(t, \mathbf{x}) = \sum_{\sigma=1}^{2} \rho^\sigma(t, \mathbf{x}) = \sum_{\sigma=1}^{2} \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}^\sigma(t, \mathbf{x}) \, ,$$

# Interactions in the table of games




Particle in P moves to a direction $\theta_h$ (black arrow) and interacts with a field particle moving to $\theta_p$ (blue arrow), the direction to the target is $\theta_v$ (red arrow).

# Interaction functions

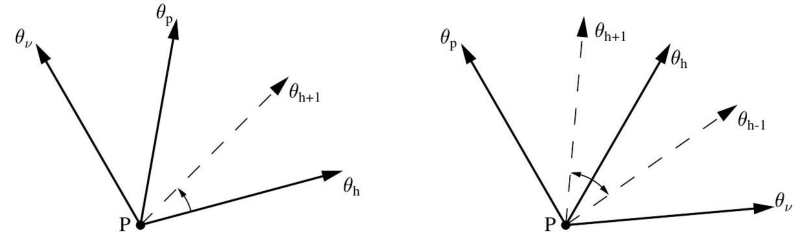
- **Interaction rate:**

$$\eta(\rho(t,\mathbf{x})) = \eta^0(1+\rho(t,\mathbf{x})).$$

- **Transition probability density:** The approach proposed here is based on the assumption that particles are subject to three different influences, namely the *trend to the exit point*, the *influence of the stream* induced by the other pedestrians, and the selection of the path with minimal density gradient. A simplified interpretation of the phenomenological behavior is obtained by assuming the factorization of the two probability densities modeling the modifications of the velocity direction and modulus:

$$\mathcal{A}^{\sigma}_{hk,pq}(ij) = \mathcal{B}^{\sigma}_{hp}(i)\big(\theta_h \to \theta_i | \rho(t,\mathbf{x})\big) \times \mathcal{C}^{\sigma}_{kq}(j)\big(v_k \to v_j | \rho(t,\mathbf{x})\big).$$

# Interactions in the table of games




(a) (b)

A particle can change its velocity direction, in probability, only to an **adjacent state**. (a) A candidate particle with direction $\theta_h$ interacts with an upper stream with direction $\theta_p$ and target directions $\theta_\nu$ and decides to change its direction to $\theta_{h+1}$. (b) A candidate particle interacts with an upper stream and lower target directions, and decides to change its direction either to $\theta_{h+1}$ or $\theta_{h-1}$.

# Interaction functions

– *Interaction with a upper stream and target directions, namely* $\theta_p > \theta_h, \quad \theta_\nu > \theta_h$*:*

$$\begin{aligned} \mathcal{B}_{hp}^{\sigma}(i) &= u_0(1-\rho) + u_0\,\rho \quad \text{if} \quad i = h+1\,, \\ \mathcal{B}_{hp}^{\sigma}(i) &= 1 - u_0(1-\rho) - u_0\,\rho \quad \text{if} \quad i = h\,, \\ \mathcal{B}_{hp}^{\sigma}(i) &= 0 \quad \text{if} \quad i = h-1\,. \end{aligned}$$

# Mathematical structures

$$\boxed{\begin{array}{l}\text{Variation rate of}\\ \text{the number of}\\ \text{active particles}\end{array}} = \boxed{\begin{array}{l}\text{Inlet flux rate}\\ \text{caused by}\\ \text{conservative interactions}\end{array}} - \boxed{\begin{array}{l}\text{Outlet flux rate}\\ \text{caused by}\\ \text{conservative interactions}\end{array}}$$

$$\begin{aligned}(\partial_t \quad &+ \quad \mathbf{v}_{ij} \cdot \partial_{\mathbf{x}}) f_{ij}^{\sigma}(t, \mathbf{x}) = J[\mathbf{f}](t, \mathbf{x}) \\ &= \sum_{h,p=1}^{n} \sum_{k,q=1}^{m} \int_{\Lambda} \eta[\rho(t, \mathbf{x}^*)] \mathcal{A}_{hk,pq}^{\sigma}(ij)[\rho(t, \mathbf{x}^*)] f_{hk}^{\sigma}(t, \mathbf{x}) f_{pq}^{\sigma}(t, \mathbf{x}^*)\, d\mathbf{x}^* \\ &- \quad f_{ij}^{\sigma}(t, \mathbf{x}) \sum_{p=1}^{n} \sum_{q=1}^{m} \int_{\Lambda} \eta[\rho(t, \mathbf{x}^*)] f_{pq}^{\sigma}(t, \mathbf{x}^*)\, d\mathbf{x}^*, \qquad (1)\end{aligned}$$

where $\mathbf{f} = \{f_{ij}\}$.

# Mild form of the initial value problem

The initial value problem consists in solving Eqs. (1) with initial conditions given by

$$f_{ij}^{\sigma}(0,\mathbf{x}) = \phi_{ij}^{\sigma}(\mathbf{x}).$$

Let us introduce the mild form obtained by integrating along the characteristics:

$$\begin{aligned}\widehat{f_{ij}^{\sigma}}(t,\mathbf{x}) \quad &= \quad \phi_{ij}^{\sigma}(\mathbf{x}) + \int_0^t \left( \widehat{\Gamma_{ij}^{\sigma}}[\mathbf{f},\mathbf{f}](s,\mathbf{x}) - \widehat{f_{ij}^{\sigma}}(s,\,\mathbf{x})\widehat{\L[\mathbf{f}]}(s,\,\mathbf{x}) \right) ds, \\ & \qquad i \in \{1,\ldots,n\}, \quad j \in \{1,\ldots,m\}, \quad \sigma \in \{1,2\},\end{aligned}$$

where the following notation has been used for any given vector $f(t,\mathbf{x})$: $\widehat{f_{ij}^{\sigma}}(t,\mathbf{x}) = f_{ij}^{\sigma}(t, x + v_j\cos(\theta_i)t, y + v_j\sin(\theta_i)t)$.

# Existence theory

**H.1.** For all positive $R$, there exists a constant $C_\eta > 0$ so that $0 < \eta(\rho) \leq C_\eta\,, \quad \text{whenever} \quad 0 \leq \rho \leq R$.

**H.2.** Both the encounter rate $\eta[\rho]$ and the transition probability $\mathcal{A}^{\sigma}_{hk,pq}(ij)[\rho]$ are Lipschitz continuous functions of the macroscopic density $\rho$, i.e., that there exist constants $L_\eta, L_{\mathcal{A}}$ is such that

$$| \, \eta[\rho_1] - \eta[\rho_2] \, | \leq L_\eta \, | \, \rho_1 - \rho_2 \, | ,$$

$$| \, \mathcal{A}^{\sigma}_{hk,pq}(ij)[\rho_1] - \mathcal{A}^{\sigma}_{hk,pq}(ij)[\rho_2] \, | \leq L_{\mathcal{A}} \, | \, \rho_1 - \rho_2 \, | ,$$

whenever $0 \leq \rho_1 \leq R$, $0 \leq \rho_2 \leq R$, and all $i, h, p = 1, .., n$ and $j, k, q = 1, .., m$.

• Let $\phi_{ij}^{\sigma} \in L^{\infty} \cap L^1$, $\phi_{ij}^{\sigma} \geq 0$, then there exists $\phi^0$ so that, if $\| \phi \|_1 \leq \phi^0$, there exist $T$, $a_0$, and $R$ so that a unique non-negative solution to the initial value problem exists and satisfies:

$$f \in X_T, \qquad \sup_{t \in [0,T]} \| f(t) \|_1 \leq a_0 \| \phi \|_1,$$

$$\rho(t, \mathbf{x}) \leq R, \quad \forall t \in [0,T], \quad \mathbf{x} \in \Omega.$$

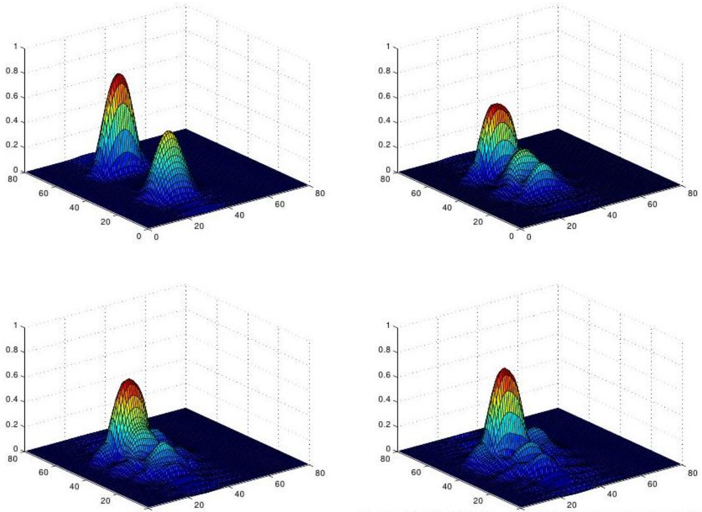
Moreover, if $\sum_{\sigma=1}^{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \| \phi_{ij}^{\sigma} \|_{\infty} \leq 1$, and $\| \phi \|_1$ is small, one has $\rho(t, \mathbf{x}) \leq 1, \quad \forall t \in [0,T], \quad \mathbf{x} \in \Omega.$

• There exist $\phi^r$, $(r = 1, \ldots, p-1)$ such that if $\| \phi \|_1 \leq \phi^r$, there exists $a_r$ so that it is possible to find a unique non-negative solution to the initial value problem satisfying for any $r \leq p-1$ the following $f(t) \in X[0, (p-1)T]$,

$$\sup_{t \in [0,T]} \| f(t + (r-1)T) \|_1 \leq a_{r-1} \| \phi \|_1,$$

and $\rho(t + (r-1)T, \mathbf{x}) \leq R, \quad \forall t \in [0,T], \quad \mathbf{x} \in \Omega.$ Moreover, $\rho(t + (r-1)T, \mathbf{x}) \leq 1, \quad \forall t \in [0,T], \quad \mathbf{x} \in \Omega.$

# Case-study 1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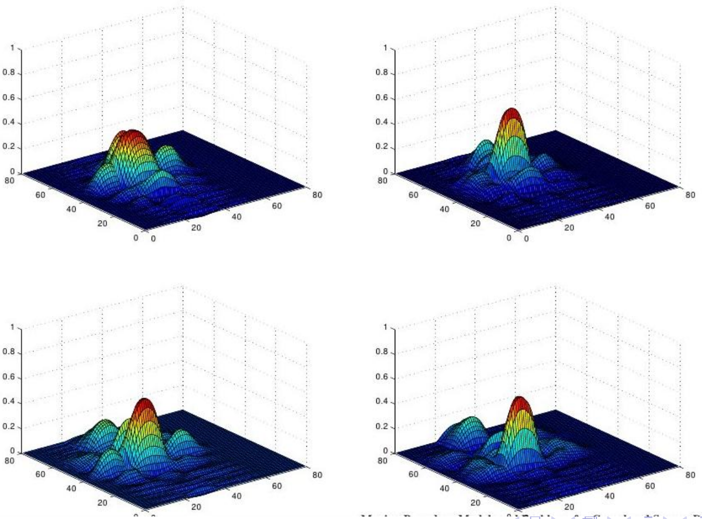
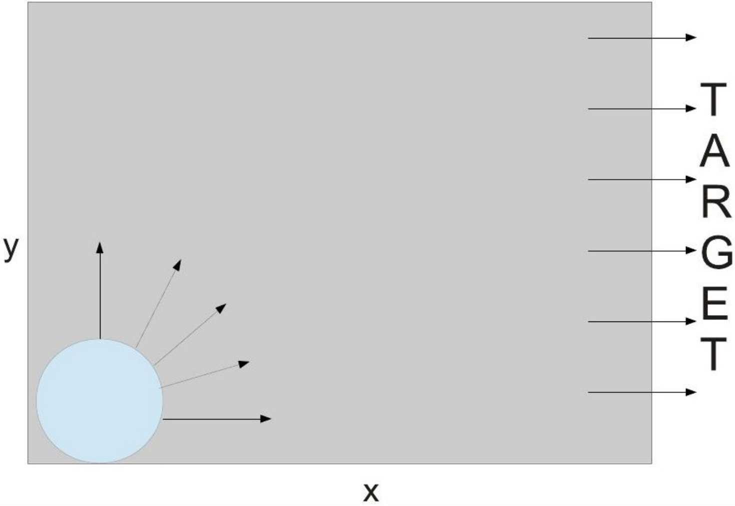
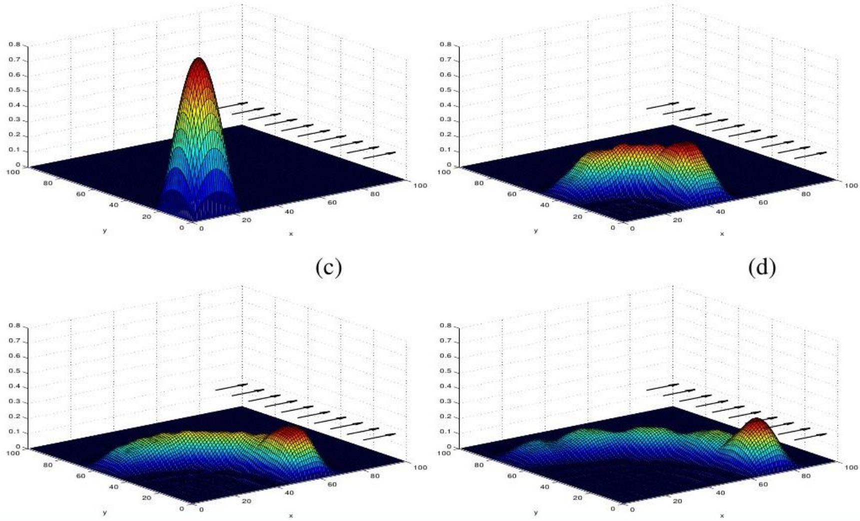
# Case-study 1



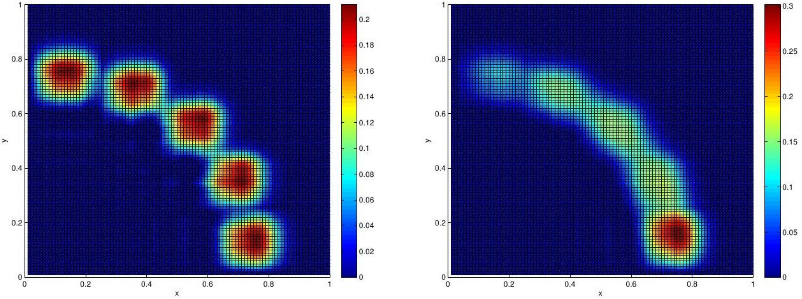
# Case-study 2



# Case-study 2



# Case-study 2 - Top view



# Questions?