Average conditions for permanence in N-species nonautonomous competitive systems of PDEs.

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Definitions and assumptions

By the nonautonomous competitive system of partial differential equations (PDEs) of Kolmogorov type we mean the system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \mu_i \Delta u_i + f_i(t, x, u_1, \ldots, u_N) u_i, & t > 0,\ x \in \Omega,\ i = 1, \ldots, N \\ \mathcal{B}_i u_i = 0, & t > 0,\ x \in \partial\Omega,\ i = 1, \ldots, N, \end{cases} \tag{R}$$

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- $\mathcal{B}_i$ is the Neumann boundary operator, or the Dirichlet boundary operator or the Robin boundary operator.

Denote by λ_i the *principal eigenvalue* of the elliptic eigenproblem

$$\begin{cases} \Delta\varphi_i + \lambda_i\varphi_i = 0 & \text{on } \Omega, \\ \mathcal{B}_i\varphi_i = 0 & \text{on } \partial\Omega. \end{cases}$$

If we have Dirichlet boundary conditions then $\lambda_i > 0$. By the standard elliptic maximum principles a *principal eigenfunction* φ_i corresponding to λ_i can be chosen so that $\varphi_i(x) > 0$ for all $x \in \Omega$. We assume that φ_i is normalized so that $\max\limits_{x\in\bar{\Omega}} \varphi_i(x) = 1$.

In the Neumann boundary conditions we have $\lambda_i = 0$ and $\varphi_i \equiv 1$.

We deal with the positive solutions.

Definition

The solution $u(t,x) = (u_1(t,x), \ldots, u_N(t,x))$ of (R) is positive if $u_i(t,x) > 0$ for all $i = 1, \ldots, N$, $t \in (0, \tau_{\max})$ and $x \in \Omega$.

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Now we introduce the following assumptions for a function f_i

(A1) $f_i \colon [0,\infty) \times \bar{\Omega} \times [0,\infty)^N \to \mathbb{R}$ $(1 \le i \le N)$, *as well as their first derivatives* $\partial f_i/\partial t$ $(1 \le i \le N)$, $\partial f_i/\partial u_j$ $(1 \le i,j \le N)$, *and* $\partial f_i/\partial x_k$ $(1 \le i \le N,\ 1 \le k \le n)$, *are continuous.*

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(A2) *The functions* $[\,[0,\infty) \times \bar{\Omega} \ni (t,x) \mapsto f_i(t,x,0,\dots,0) \in \mathbb{R}\,]$, $1 \le i \le N$, *are bounded.*

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For $1 \le i \le N$ the function $f_i(t,x,0,\dots,0)$ is called the intrinsic growth rate of the i-th species.

Define

$$\underline{a}_i := \inf\{\, f_i(t, x, 0, \ldots, 0) : t \geq 0,\ \ x \in \bar{\Omega}\,\},$$
$$\overline{a}_i := \sup\{\, f_i(t, x, 0, \ldots, 0) : t \geq 0,\ \ x \in \bar{\Omega}\,\}.$$

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(A3) $(\partial f_i/\partial u_j)(t, x, u) \leq 0$ *for all* $t \geq 0$, $x \in \bar{\Omega}$, $u \in [0, \infty)^N$, $1 \leq i, j \leq N$, $i \neq j$.

$(\partial f_i/\partial u_j)(t, x, u_1, \dots, u_N)$ measures the influence of the j-th species on the growth rate of the i-th species. Systems of type (R) for which (A3) holds we call *competitive*.

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(A4) *There exist* $\underline{b}_{ii} > 0$ *such that* $(\partial f_i/\partial u_i)(t, x, u) \leq -\underline{b}_{ii}$ *for all* $t \geq 0$, $x \in \bar{\Omega}$, $u \in [0, \infty)^N$, $1 \leq i \leq N$.

Fix a positive solution $u(t,x) = (u_1(t,x), \ldots, u_N(t,x))$ of system (R). For each $1 \le i \le N$ let $\xi_i(t)$, $t \in [0,\infty)$, be the positive solution of the following problem

$$\begin{cases} \xi_i' = \big(\max_{x\in\bar\Omega} f_i(t,x,0,\ldots,0) - \underline{b}_{ii}\xi_i\big)\xi_i, \\ \xi_i(0) = \sup_{x\in\bar\Omega} u_i(0,x). \end{cases} \tag{1}$$

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Lemma 1

Assume **(A1)** through **(A4)** and let $\bar{a}_i > 0$. Then for each solution $\xi_i(t)$ of the problem (1) there holds

$$\limsup_{t\to\infty} \xi_i(t) \le \frac{\bar{a}_i}{\underline{b}_{ii}}, \quad 1 \le i \le N.$$

Lemma 2

Assume **(A1)** through **(A4)**. Then for any positive solution $u(t,x) = (u_1(t,x), \dots, u_N(t,x))$ of system (R), and any $1 \leq i \leq N$ there holds

$$u_i(t,x) \leq \xi_i(t), \qquad t \in [0, \tau_{\max}), \ x \in \bar{\Omega},$$

where $\xi_i(t)$ is the solution of the problem (1).

Lemma 3 [dissipativity]

Assume **(A1)** through **(A4)** and $\bar{a}_i > 0$. Then for any maximally defined positive solution $u(t,x) = (u_1(t,x), \ldots, u_N(t,x))$ of system (R) there holds

(i) $\tau_{\max} = \infty$, and

(ii)
$$\limsup_{t\to\infty} u_i(t,x) \le \frac{\bar{a}_i}{\underline{b}_{ii}}, \qquad 1 \le i \le N, \tag{2}$$

uniformly for $x \in \bar{\Omega}$.

(A5) *The derivatives* $\partial f_i / \partial u_j$, $1 \le i, j \le N$, *are bounded and Lipschitz continuous on sets of the form* $[0, \infty) \times \bar{\Omega} \times B$, *where* B *is a bounded subset of* $[0, \infty)^N$.

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Definition

For $1 \le i, j \le N$ and $\varepsilon_0 \ge 0$ we define

$$\overline{b}_{ij}(\varepsilon_0) := \sup\Big\{ -\frac{\partial f_i}{\partial u_j}(t, x, u) : t \ge 0,\ x \in \bar{\Omega},$$
$$u \in \Big[0, \frac{\overline{a}_1}{\underline{b}_{11}} + \varepsilon_0\Big] \times \cdots \times \Big[0, \frac{\overline{a}_N}{\underline{b}_{NN}} + \varepsilon_0\Big] \Big\},$$

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Assumptions (A3) and (A4) imply that $\overline{b}_{ij}(\varepsilon_0) \ge 0$, $1 \le i, j \le N$, and $\overline{b}_{ii}(\varepsilon_0) > 0$, $1 \le i \le N$, whereas it follows from (A5) that $\overline{b}_{ij}(\varepsilon_0) < \infty$, and $\lim_{\varepsilon_0 \to 0^+} \overline{b}_{ij}(\varepsilon_0) = \overline{b}_{ij}$, for $1 \le i, j \le N$.

Averaging

Definition

We define the *lower average* of a function f_i as

$$m[f_i] := \liminf_{t-s\to\infty} \frac{1}{t-s} \int_s^t \min_{x\in\bar{\Omega}} f_i(\tau, x, 0, \ldots, 0)\, d\tau,$$

Definition

We define the *upper average* of a function f_i as

$$M[f_i] := \limsup_{t-s\to\infty} \frac{1}{t-s} \int_s^t \max_{x\in\bar{\Omega}} f_i(\tau, x, 0, \ldots, 0)\, d\tau.$$

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(A6) $m[f_i] > 0$, $1 \le i \le N$.

Definition

System (R) is *permanent*, if there exist positive constants δ_i and R_i such that for each positive solution $u(t,x) = (u_1(t,x), \dots, u_N(t,x))$ of system (R) there exists $T = T(u) > 0$ with the property

$$\delta_i \varphi_i(x) \le u_i(t,x) \le R_i \qquad \text{(permanence)}$$

for all $1 \le i \le N$, $t \ge T$, $x \in \bar{\Omega}$.

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$$m[f_i] > \lambda_i \mu_i + \sum_{\substack{j=1 \\ j \ne i}}^{N} \frac{\overline{b}_{ij} M[f_j]}{\underline{b}_{jj}}, \quad 1 \le i \le N, \qquad \text{(AC)}$$

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- J. Balbus and J. Mierczyński, *Time-averaging and permanence in nonautonomous competitive systems of PDEs via Vance–Coddington estimates*, Discrete and continuous dynamical systems series B, **(17)**, 2012 p. 1407 – 1425.

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The following result will be useful to prove Theorem 1.

Theorem 2 [Vance - Coddington Estimates]

Let $c\colon [t_0, \infty) \to \mathbb{R}$, where $t_0 \geq 0$, be a bounded continuous function, where $c_* > 0$ and $c^* > 0$ are such that $-c_* \leq c(t) \leq c^*$ for all $t \geq t_0$, and let $d > 0$. Assume moreover that there are $L > 0$ and $\beta > 0$ such that

$$\frac{1}{L} \int\limits_{t}^{t+L} c(\tau)\, d\tau \geq \beta$$

for all $t \geq t_0$.

Theorem 2 [Vance - Coddington Estimates] continued

Then for any solution $\zeta(t)$ of the initial value problem

$$\begin{cases} \zeta' = (c(t) - d\zeta)\zeta \\ \zeta(t_0) = \zeta_0, \end{cases}$$

where $\zeta_0 > 0$, there holds

$$\frac{\beta}{d} e^{-L(c_* + \beta)} \leq \liminf_{t\to\infty} \zeta(t) \leq \limsup_{t\to\infty} \zeta(t) \leq \frac{c^*}{d}.$$

(permanence-logistic)

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- R. R. Vance and E. A. Coddington, *A nonautonomous model of population growth*, J. Math. Biol. **27** (1989), no. 5, 491–506.

sketch of the proof of Theorem 1

The right-hand side of the inequality (permanence) is satisfied by Lemma 3 (ii). By assumption (A5) we can choose $\varepsilon_0 > 0$ such that

$$m[f_i] > \lambda_i \mu_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\overline{b}_{ij}(\varepsilon_0) M[f_j]}{\underline{b}_{jj}}$$

for all $1 \leq i \leq N$.
Fix a positive solution $u(t,x) = (u_1(t,x), \ldots, u_N(t,x))$ of system (R). Let $\xi_i(t)$, $1 \leq i \leq N$, $t \geq 0$, be the solutions of (1). Fix $1 \leq i \leq N$.

sketch of the proof of Theorem 1 [continued]

Let $t_0 > 0$ be such a moment that

$$u(t,x) \in \left[0, \frac{\overline{a}_1}{\underline{b}_{11}} + \varepsilon_0\right] \times \cdots \times \left[0, \frac{\overline{a}_N}{\underline{b}_{NN}} + \varepsilon_0\right]$$

for $t > t_0$ and $x \in \bar{\Omega}$.

Let $\eta_i(t)$, $t \geq t_0$, be the positive solution of the following problem

$$\begin{cases} \eta_i' = \big(\min_{x\in\bar{\Omega}} f_i(t,x,0,\ldots,0) - \lambda_i\mu_i - \overline{b}_{ii}(\varepsilon_0)\eta_i - \sum_{\substack{j=1 \\ j\neq i}}^{N} \overline{b}_{ij}(\varepsilon_0)\xi_j(t)\big)\eta_i \\ \eta_i(t_0) = \inf_{x\in\Omega} \frac{u_i(t_0,x)}{\varphi_i(x)}. \end{cases} \quad (3)$$

It is easy to see that $u_i(t,x) \geq \eta_i(t)\varphi_i(x)$ for all $t \geq t_0$ and $x \in \bar{\Omega}$.

sketch of the proof of Theorem 1 [continued]

Now we apply Theorem 2 to (3) where

$$c(t) = \min_{x \in \bar{\Omega}} f_i(t, x, 0, \ldots, 0) - \lambda_i \mu_i - \sum_{\substack{j=1 \\ j \neq i}}^{N} \overline{b}_{ij}(\varepsilon_0) \xi_j(t) \quad \text{i} \quad d = \overline{b}_{ii}(\varepsilon_0).$$

sketch of the proof Theorem 1 [continued]

To prove the permanence of system (R) we show that the parameters in Theorem 2 do not depend on the solution $u(t, x)$, for sufficiently large t. $\square$

Now we replace conditions (AC) with

$$m[f_i] > \lambda_i \mu_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\overline{b}_{ij} \overline{a}_j}{\underline{b}_{jj}} \tag{4}$$

Then we can give the lower estimates on the numbers δ_i (in the definition of permanence) in terms of the parameters of system (R):

$$\delta_i \geq \frac{\beta}{\overline{b}_{ii}} \exp\bigl(-L(m[f_i] - \underline{a}_i)\bigr). \tag{5}$$

A special case of Kolmogorov systems for PDEs is a Kolmogorov systems for ODEs

$$u_i' = f_i(t, u_1, \ldots, u_N)u_i \quad 1 \le i \le N. \tag{K}$$

We can treat such systems as systems of partial differential equations

$$\frac{\partial u_i}{\partial t} = \Delta u_i + f_i(t, u_1, \ldots, u_N)u_i \tag{S}$$

with the Neumann boundary conditions.

Definition

System (K) is *permanent* if there exist positive constants δ_i, R_i such that for any positive solution $u(t) = (u_1(t), \dots, u_N(t))$ of system (K) there exists $T = T(u) > 0$ with the property

$$\delta_i \leq u_i(t) \leq R_i$$

for $1 \leq i \leq N$, $t \geq T$.

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Theorem 3

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Note that again we do not assume that $\underline{a}_i > 0$.

Definition

System (K) is *globally attractive* if any two positive solutions $u(t) = (u_1(t), \dots, u_N(t))$ and $v(t) = (v_1(t), \dots, v_N(t))$ of system (R) satisfy

$$\lim_{t \to \infty} (u_i(t) - v_i(t)) = 0$$

for $1 \le i \le N$.

Sufficient conditions for attractivity in systems of ODEs are not special cases of sufficient conditions for attractivity in systems of PDEs.

Lemma 4 is very technical.

Lemma 4

Assume (A1) through (A5) and $(\mathrm{AC})_{ODE}$. Let $\alpha_1, \ldots, \alpha_N > 0$ be positive constants such that for $i = 1, \ldots, N$ there holds

$$\alpha_i \underline{b}_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{N} \alpha_j \overline{b}_{ji}. \tag{6}$$

Then there exist $Z > 0$ and $\gamma > 0$ such that for each pair of a positive solutions $u(t) = (u_1(t), \ldots, u_N(t))$ and $v(t) = (v_1(t), \ldots, v_N(t))$ of system (K) there exists $t_0 \geq 0$ such that

Lemma 4 [continued]

$$\sum_{i=1}^{N} |u_i(t) - v_i(t)| \leq Z \sum_{i=1}^{N} |u_i(t_0) - v_i(t_0)| \cdot e^{-\gamma(t-t_0)}$$

for $t \geq t_0$.

Conditions (6) are called *column diagonal dominance*.

sketch of the proof

Fix a positive solutions $u(t) = (u_1(t), \dots, u_N(t))$ and $v(t) = (v_1(t), \dots, v_N(t))$ of system (K). We introduce a Lyapunov function

$$\Theta(t) := \sum_{i=1}^{N} \alpha_i \left| \ln \frac{u_i(t)}{v_i(t)} \right|. \tag{7}$$

sketch of the proof continued

We prove that there exists $\varepsilon > 0$ such that

$$D^+\Theta(t) \leq -\varepsilon \sum_{i=1}^{N} |u_i(t) - v_i(t)| \quad \text{dla } t \geq t_0, \tag{8}$$

where D^+ is the upper derivative of Θ.

sketch of the proof continued

Then using the fact that the system (K) is permanent we have that there exist $0 < \delta_* < \delta^* < \infty$ such that $\delta_* \leq u_i(t), v_i(t) < \delta^*$. By mean value theorem it follows that

$$\frac{1}{\delta^*}|u_i(t) - v_i(t)| \leq \left| \ln \frac{u_i(t)}{v_i(t)} \right| \leq \frac{1}{\delta_*}|u_i(t) - v_i(t)|. \tag{9}$$

Using (7), (8) and (9) we see that there exist $Z > 0$, $\gamma > 0$ such that

$$\sum_{i=1}^{N} |u_i(t) - v_i(t)| \leq Z \sum_{i=1}^{N} |u_i(t_0) - v_i(t_0)| \cdot e^{-\gamma(t-t_0)} \tag{10}$$

□

Theorem

Assume (A1) – (A5) and $(AC)_{ODE}$. Then system (K) *is globally attractive.*

Proof.

By [S. Ahmad and A. C. Lazer, *Average conditions for global asymptotic stability in a nonautonomous Lotka-Volterra system*, Nonlinear Anal. Ser A: Theory Methods **40** (2000), no. 1-8, 37-49, Lemma 3.2] it follows that if $(AC)_{ODE}$ holds then there exist constants $\alpha_i > 0$, $i = 1, \ldots, N$, such that

$$\alpha_i \underline{b}_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{N} \alpha_j \overline{b}_{ji} \quad \text{for} \quad i = 1, \ldots, N.$$

Now it suffices to apply Lemma 4. □

Now we consider system (R)

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \mu_i \Delta u_i + f_i(t, x, u_1, \ldots, u_N) u_i, & t > 0,\ x \in \Omega,\ i = 1, \ldots, N \\ \mathcal{B}_i u_i = 0, & t > 0,\ x \in \partial\Omega,\ i = 1, \ldots, N, \end{cases} \tag{R}$$

where $\mathcal{B}_i$ is the **Neumann** boundary conditions.

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where $\mathcal{B}_i$ is the **Neumann** boundary conditions.

Definition

System (R) is *globally attractive* if any two positive solutions $u(t, x) = (u_1(t, x), \dots, u_N(t, x))$ and $v(t, x) = (v_1(t, x), \dots, v_N(t, x))$ of (R) satisfy

$$\lim_{t \to \infty} (u_i(t, x) - v_i(t, x)) = 0$$

for $1 \le i \le N$, uniformly in $x \in \bar{\Omega}$.

Again we have a very technical lemma.

Lemma 5

Assume **(A1)** through **(A5)** and **(AC)**. Let

$$\underline{\delta}\,\underline{b}_{ii} > \sum_{\substack{j=1\\ j\neq i}}^{N} \overline{\delta}\,\overline{b}_{ij}, \quad 1 \le i \le N, \tag{11}$$

where $0 < \underline{\delta} \le \overline{\delta} < \infty$ be such that for any positive solution $u(t,x) = (u_1(t,x), \ldots, u_N(t,x))$ of system (R) there holds

$$\underline{\delta} \le u_i(t,x) \le \overline{\delta}$$

for sufficiently large t and all $x \in \bar{\Omega}$, $1 \le i \le N$.

Lemma 5 [continued]

Then there exist $Z > 0$ and $\gamma > 0$ with the property that for each pair of positive solutions $u(t,x) = (u_1(t,x), \dots, u_N(t,x))$ and $v(t,x) = (v_1(t,x), \dots, v_N(t,x))$ of system (R) there exists $t_0 \geq 0$ such that

$$\sum_{i=1}^{N} \sup_{x\in\bar{\Omega}} |u_i(t,x) - v_i(t,x)| \leq Z \sum_{i=1}^{N} \sup_{x\in\bar{\Omega}} |u_i(t_0,x) - v_i(t_0,x)| \cdot e^{(-\gamma(t-t_0))} \tag{12}$$

for $t \geq t_0$.

In particular, system (R) is globally attractive.

sketch of the proof

Fix positive solutions $u(t,x) = (u_1(t,x), \dots, u_N(t,x))$ and $v(t,x) = (v_1(t,x), \dots, v_N(t,x))$ of system (R). A well known result from the matrix theory states that there exist $\alpha_1, \dots, \alpha_N > 0$ such that

$$\alpha_i \underline{\delta}\, \underline{b}_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{N} \alpha_j \bar{\delta}\, \bar{b}_{ji}(\varepsilon), \quad 1 \leq i \leq N. \tag{13}$$

We introduce a Lyapunov functional

$$\Theta(t) = \sum_{i=1}^{N} \alpha_i \Theta_i(t), \text{ where } \Theta_i(t) := \sup_{x \in \bar{\Omega}} \left| \ln \frac{u_i(t,x)}{v_i(t,x)} \right|. \tag{14}$$

Fix $1 \leq i \leq N$.

sketch of the proof continued

We prove that

$$D^+\Theta_i(t) \le -\underline{\delta}\, \underline{b}_{ii}\Theta_i(t) + \overline{\delta} \sum_{\substack{j=1 \\ j\ne i}}^{N} \overline{b}_{ij}(\varepsilon)\Theta_j(t), \qquad t \ge t_0. \tag{15}$$

Note that by (13)

$$\sum_{i=1}^{N} \alpha_i \left(-\underline{\delta}\, \underline{b}_{ii}\Theta_i(t) + \overline{\delta} \sum_{\substack{j=1 \\ j\ne i}}^{N} \overline{b}_{ij}(\varepsilon)\Theta_j(t) \right) \le -\epsilon \sum_{i=1}^{N} \Theta_i(t), \tag{16}$$

sketch of the proof continued

Hence by (15), (16) and the definition of Θ we have that

$$\sum_{i=1}^{N} \sup_{x\in\bar{\Omega}} |u_i(t,x) - v_i(t,x)| \le Z \sum_{i=1}^{N} \sup_{x\in\bar{\Omega}} |u_i(t_0,x) - v_i(t_0,x)| \cdot e^{(-\gamma(t-t_0))} \tag{17}$$

□