Regularity of solutions for equations with integral diffusion and the quasi geostrophic equation

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Quasi-Geostrophic flow equation

The Q-G equation is a 2-D "Navier Stokes" type equation. In 2-D, Navier Stokes simplifies considerably since

a) Incompressibility, $\operatorname{div} \vec{v} = 0$, implies that $(-v_2, v_1)$ is a gradient: $(-v_2, v_1) = \nabla\varphi$.

b) $\operatorname{curl} \vec{v}$ is a scalar, $\theta = \operatorname{curl} \vec{v} = \Delta\varphi$.

Navier Stokes equation thus becomes a system:

i) $\theta_t + \vec{v}\nabla\theta = \Delta\theta$

ii) $\operatorname{curl}\vec{v} = \theta$

For the Q-G equation, we still have that

$$(-v_2, v_1) = \nabla\varphi \ .$$

But the potential φ is related to vorticity by $\theta = -\Delta^{1/2}\varphi$. That is, the final system becomes

$$\theta_t + \vec{v}\nabla\theta = (\Delta^{1/2})\theta$$

$(-v_2, v_1) = R_1\theta, R_2\theta$ where R_i are the Riesz transforms of θ.

Riesz transforms and the dependence of v on θ

More precisely, we can deduce this relation through Fourier transform:

$$\hat{\theta} = |\zeta|\hat{\varphi} \qquad \text{and} \qquad \hat{v} = (-\zeta_2\hat{\varphi}, \zeta_1\hat{\varphi}) \ .$$

In particular

$$\hat{v} = \left(-\frac{\zeta_2}{|\zeta|}\hat{\theta}, \frac{\zeta_1}{|\zeta|}\hat{\theta}\right)$$

The multipliers $\dfrac{\xi_i}{|\xi|}$ are classical operators, called Riesz transforms that correspond in physical space x, to convolution with kernels

$$R_i(x) = \frac{x_i}{|x|^{n+1}}$$

i.e.

$$v_i^1(x) = \int R_i(x-y)\theta(y)\,dy$$

Note that on one hand

$$\|v\|_{L^2(R^n)} = \|\hat{v}\|_{L^2(R^n)} \leq \|\hat{\theta}\|_{L^2(R^n)}$$

that is, the Riesz transforms are bounded operators from L^2 to L^2. On the other hand, R is not integrable neither at zero nor at infinity. It is a remarkable theorem that because of the spherical cancellation on R (mean value zero and smoothness) we have:
The operator $R * \theta = v$ is a bounded operator from L^p to L^p for any $1 < p < \infty$ (Calderon-Zygmund). Unfortunately, it is easy to show that *singular integral operators* are not bounded from L^∞ to L^∞. They are bounded, though, from *BMO* to *BMO*.

BMO spaces

What is BMO? It is the space of functions with bounded mean oscillation.
That is, in any cube Q the "average of u minus its average" is bounded by a constant C

$$\frac{1}{|Q|}\int_Q \left| u(x) - \frac{1}{Q}\int_\theta u(y)\,dy \right| dx \leq C$$

The smallest C good for all cubes defines a seminorm (it does not distinguishes constant that we may factor out). The space of functions u in BMO is smaller than any L^p $(p < \infty)$ but not included in L^∞ $((\log|x|)^-$ is a typical example).
In fact functions u in BMO have "exponential" integrability

$$\int_{Q_1} e^{C|u|} \leq \infty$$

Proof of regularity

The regularity theory for the Quasi Geostrophic Equation is based on two linear transport regularity theorems:

Theorem 1

Let θ be a (weak) solution of

$$\theta_t + v\nabla\theta = \Lambda^{1/2}\theta \quad in \quad R^n \times [0, \infty)$$

for some incompressible vector field v (with no apriori bounds) and initial data θ_0 in L^2.
Then

$$\|\theta(X, 1)\|_{L^\infty} \leq C\|\theta(x, 0)\|_{L^2} \ .$$

Having shown boundness, for the QG equation, our stituation is now the following: We have a solution θ that satisfies the energy bound:

$$\begin{cases} \sup_t \|\theta(t)\|^2_{L^2(\mathbb{R}^n)} + \|D_{1/2}\theta\|^2_{\mathbb{R}^{n+1}_+} \leq C \\ \text{and also } \ \|\theta\|_{L^\infty(X,t)} \leq 1 \\ \text{We want to prove that } \theta \text{ is Hölder continuous.} \end{cases} \tag{\#}$$

To do that we need to reproduce the local in space De Giorgi method. Of the velocity field, we may assume now (being the Riesz transform of θ, that

$$\sup_t \left(\|v\|^2_{L^2(\mathbb{R}^n)} + |v|_{BMO(\mathbb{R}^n)} \right) \leq C \,. \tag{$*$}$$

We decouple v from θ, and will prove a linear theorem, where for v satisfying $(*)$ and θ satisfying $(\#)$ and the equation

$$\theta_t + v\nabla\theta = \Delta^{1/2}\theta \,,$$

we have that θ is locally C^α.

A review of De Giorgi's Theorem

The proof we will present is strongly based in the ideas that De Giorgi developed to solve the 19th Hilbert problem.
The problem consisted in showing the regularity of minimizers w_0 to Lagrangians

$$\int_\Omega F(\nabla w)\,dx$$

Such a minimizer satisfies

$$\operatorname{div} F_j(\nabla w) = 0$$

but, when solving the minimization problem we only know that $\nabla w \in L^2$ (if F is strictly convex).

If we knew that ∇w was continuous, Shauder theory will apply and solve the problem. First derivatives, $u_e = D_e u_0$, satisfy the equation

$$D_i F_{ij}(\nabla w_0) D_j (w_0)_e = 0 \, .$$

De Giorgi studied then solutions u of

$$D_i a_{ij}(x) D_j u = 0$$

with no assumption on a_{ij}, except uniform ellipticity:

$$I \leq a_{ij} \leq \Lambda I$$

and showed that u is C^α.
Applying this theorem to $(w_0)_e$, he solved the Hilbert problem.

Theorem 2

Let u be a solution of $D_i a_{ij} D_j u = 0$ in B_1 of $\mathbb{R}^N$ with $0 < \lambda I \leq a_{ij}(x) \leq \Lambda I$ (i.e., a_{ij} is uniformly elliptic). Then $u \in C^\alpha(B_{1/2})$ with

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\|u\|_{L^2(B_1)}$$

$(\alpha = \alpha(\lambda, \Lambda, n))$.

Proof.

The proof is based on the interplay between Sobolev inequality, that says that $\|u\|_{L^{2+\varepsilon}}$ is controlled by $\|\nabla u\|_{L^2}$ and the energy inequality, that says that in turn, *u being a solution of the equation*

$\|\nabla u_\theta\|_{L^2}$ is controlled by $\|u_\theta\|_{L^2}$ for every truncation θ : $\quad u_\theta = (u - \theta)^+$.

□

We recall Sobolev and energy inequalities:

Sobolev:

If v is supported in B_1, then

$$\|v\|_{L^p(B_1)} \leq C\|\nabla v\|_{L^2(B_1)}$$

for some $p(n) > 2$.
If we are not too picky we can prove it by representing

$$v(x_0) = \int_{B_1} \frac{\nabla v(x) \cdot x_0 - x}{|x - x_0|^n}\, dx = \nabla v * G\ .$$

Since G belongs "almost" to $L^{\frac{n}{n-1}}$, any $p < \frac{2n}{n-2}$ would do.
$p = \frac{2n}{n-2}$ requires another proof.

Energy inequality:

If $u \geq 0$, $D_i a_{ij} D_j u \geq 0$ and $\varphi \in C_0^\infty(B_1)$ then

$$\int_{B_1} (\nabla \varphi u)^2 \, dx \leq C \sup |\nabla \varphi|^2 \int_{B_1 \cap \operatorname{supp} \varphi} u^2 \, .$$

(Note that there is a loss going from one term to the other: $\nabla \varphi u$ versus u.)

Proof.

We multiply Lu by $\varphi^2 u$. Since everything is positive we get

$$-\int \nabla^T (\varphi^2 u) A \nabla u \geq 0 \, .$$

We have to transfer a φ from the left ∇ to the right ∇.
We use that whenever we have a term of the form

$$\int \nabla^T \varphi u A u (\nabla \varphi) \leq \varepsilon \int \nabla^T (\varphi u) A \nabla (\varphi u) + \frac{1}{\varepsilon} \int |\nabla \varphi|^2 u^2 \|A\| \, .$$

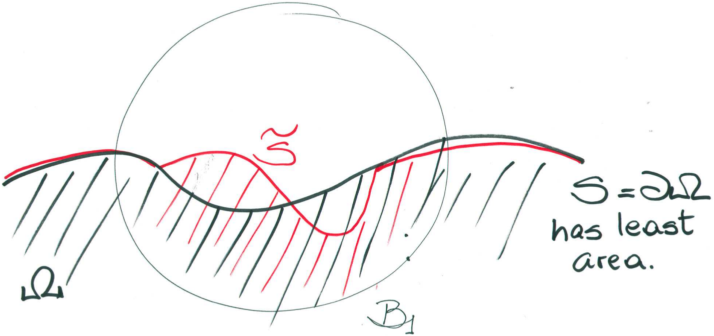
The proof of the theorem is split in two parts:

Lemma 3

(From an L^2 to an L^∞ bound) *If* $\|u^+\|_{L^2(B_1)}$ *is small enough* $(< \delta_0(n, \lambda, \Lambda)$, *then*

$$\sup_{B_{1/2}} u^+ \leq 1 \, .$$

Before going into the proof, let me give a simple, geometric analogy to Lemma 1 with a simple proof.
Suppose that Ω is a domain in $\mathbb{R}^n$ and $\partial\Omega$ is a minimal surface when restricted to B_1, in the sense that the boundary of any perturbation inside B_1 will have larger area.

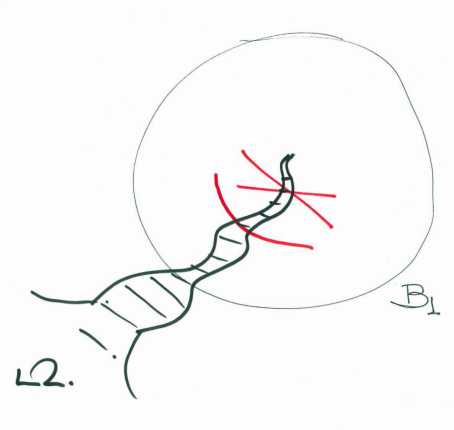


$\text{Area}\, \tilde{S} \geq \text{Area}\, S.$

We want to prove that

Lemma

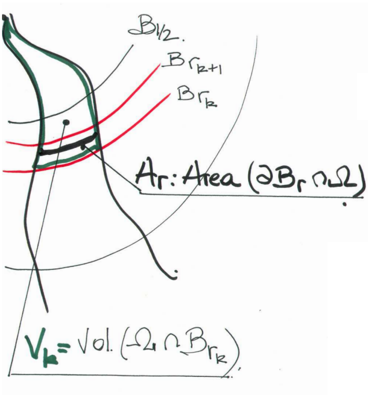
If volume$(\Omega \cap B_1) \leq \varepsilon_0$*, a small enough constant, then* $\mathrm{Vol}(\Omega \cap B_{1/2}) = 0$.



For that purpose we will take diadic balls, $B_{r_k} = B_{\frac{1}{2}+2^{-k}}$, converging to $B_{1/2}$ and rings $R_{r_k} = B_{r_{(k-1)}} - B_{r_k}$; and we will find a *non linear* recurrence relation for $V_k = \text{Vol}(\Omega \cap B_{r_k})$ for k even, that will imply that V_k goes to zero, in particular, Ω never reaches $B_{1/2}$.

In this analogy Volume replaces the square of the L^2 norm of u and Area the energy $\int(\nabla u)^2$.

The argument is based on the interplay of area and volume.



A_r controls V_k for $r \geq r_k$:

$$V_k \leq V_r \leq [\text{Area}(\partial\text{“}V_r\text{”})]^{n/n-1} =$$

$$\text{(two parts)} = [A_r + \text{Area } \partial\Omega \cap B_r]^{n/n-1}$$

By minimality $\leq [2A_r]^{n/n-2}$
(this is the “energy inequality”)

V_k controls A_r for some r in R_{k+1}:

$$\text{Vol}(\text{“}V_k\text{”} \setminus \text{“}V_{k+1}\text{”}) \sim \int_{r_{k+1}}^{r_k} A_r \geq 2^{-k} \inf_{r_{k+1} \leq r \leq r_k} A_r$$

If we combine both estimates we get (notice the different exponents)

$$V_{k+1} \leq \underset{r_{k+1} \leq r \leq r_k}{(2A_r)^{n/n-1}} \leq 2^{(\frac{n}{n-1})k} V_k \quad \boxed{n/(n-1)} \; !!$$

If $V_0 < \varepsilon_0$, the build up in the exponent as we iterate, beats the large geometric coefficient, and V_k goes to zero.

We now pass to the proof of the lemma. The origin becomes now plus infinity, $\|u\|_{L^2}$ plays the role of volume, and $\|\nabla u\|_{L^2}$, that of area. We have the added complication of having to truncate in space.

Proof.

We will consider a sequence of truncations

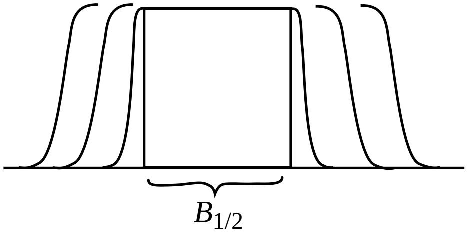
$$\varphi_k u_k$$

where φ_k is a sequence of shrinking cut off functions converging to $\chi_{B_{1/2}}$.
More precisely:

$$\varphi_k = \begin{cases} \equiv 1 & \text{for } \; x \leq 1 + 2^{-(k+1)} \\ \equiv 0 & \text{for } \; x \geq 1 + 2^{-k} \end{cases}$$

$$|\nabla \varphi_k| \leq C\, 2^k \,.$$

Note that $\varphi_k \equiv 1$ on supp φ_{k+1}



While u_k is a sequence of monotone truncations converging to $(u-1)^+$:

$$u_k = [u(1-2^{-k})]^+ \ .$$

Note that where $u_{k+1} > 0$, $u_k > 2^{-(k+1)}$. Therefore *if* $(\varphi_{k+1} u_{k+1}) > 0$, *then*, $(\varphi_k u_k) > 2^{-(k+1)}$.

We will now show that, if $\|u\|_{L^2(B_1)} = A_0$ is small enough then

$$A_k = \int (\varphi_k u_k)^2 \to 0 \ .$$

In particular $(u-1)^+\big|_{B_{1/2}} = 0$ a.e., that is, in $B_{1/2}$ u never goes above 1.

This is done through a (*non linear*!!) recurrence relation for A_k.

We have

Sobolev inequality:

$$\left[\int (\varphi_{k+1} u_{k+1})^p\right]^{2/p} \leq C \int (\nabla \varphi_{k+1} u_{k+1})^2 .$$

But, from Hölder

$$\int (\varphi_{k+1} u_{k+1})^2 \leq \left[\int (\varphi_{k+1} u_{k+1})^p\right]^{2/p} \cdot |\{\varphi_{k+1} u_{k+1} > 0\}|^{\varepsilon}$$

so we get

$$A_{k+1} \leq C \int [\nabla(\varphi_{k+1} u_{k+1})]^2 \cdot |\{\varphi_{k+1} u_{k+1} > 0\}|^{\varepsilon} .$$

We now control the RHS by A_k through the energy inequality:
From energy we get

$$\int \nabla(\varphi_{k+1} u_{k+1})^2 \leq C\, 2^{2k} \int_{\operatorname{supp} \varphi_{k+1}} u_{k+1}^2 \ .$$

(But $\varphi_k \equiv 1$ on $\operatorname{supp} \varphi_{k+1}$)

$$\leq C\, 2^{2k} \int (\varphi_k u_k)^2 = C\, 2^{2k} A_k \ .$$

To control the last term, we observe that

$$|\{\varphi_{k+1} u_{k+1} > 0\}|^{\varepsilon} \leq |\{\varphi_k u_k > 2^{-k}\}|^{\varepsilon} \ .$$

And by Chebyshev:

$$\leq 2^{2k\varepsilon}\left(\int(\varphi_k u_k)\right)^{\varepsilon} .$$

So we get

$$A_{k+1} \leq C\, 2^{4k}(A_k)^{1+\varepsilon} .$$

Then, for $A_0 = \delta$ small enough $A_k \to 0$ (prove it). The buildup of the exponent in A_k, forces A_k to go to zero. In fact, A_k has faster than geometric decay, i.e., for any $M > 0$, $A_k < M^{-k}$ if $A_0(M)$ is small enough.

Corollary 4

If u is a solution of $Lu = 0$ in B_1, then

$$\|u\|_{L^\infty(B_{1/k})} \leq C\|u\|_{L^2(B_1)} .$$

Step 2.

Oscillation decay: Let $\operatorname{osc}_D u = \sup_D u - \inf_D u$.

Theorem 5

If u is a solution of $Lu = 0$ in B_1 then there exists $\sigma(\lambda, \Lambda, m) < 1$ such that

$$\operatorname*{osc}_{B_{1/2}} u \leq \sigma \operatorname*{osc}_{B_1} u \ .$$

Lemma 6

Let $0 \leq v \leq 1$, $Lv \geq 0$ *in* B_1. *Assume that* $|B_{1/2} \cap \{v = 0\}| = \mu$ $(\mu > 0)$. *Then* $\sup_{B_{1/4}} v \leq 1 - \sigma(\mu)$.

In other words, if v^+ is a subsolution of Lv, smaller than one in B_1, and is "far from 1" in a set of non trivial measure, it cannot get too close to 1 in $B_{1/2}$.

The proof is based on the following idea: suppose that, in B_1, $|u| \leq 1$, i.e., osc $u \leq 2$. Then u is positive or negative, at least half of the time. Say it is negative, i.e.,

$$|\{u^+ = 0\}| \geq \frac{1}{2}|B_1|\,.$$

Then, on $B_{1/2}$, u should not be able to be too close to one.
For u harmonic, for instance, this just follows from the mean value theorem. If we know that

$$|\{u^+ = 0\}| \geq \left(1 - \frac{\delta}{2}\right)|B_1|\,,$$

then

$$\|u^+\|^2_{L^2(B_1)} \leq \delta/2$$

and the previous lemma would tell us that $u^+|_{B_{1/2}} \leq 1/2$.
So we must bridge the gap between $|\{u^+ = 0\}| \geq \frac{1}{2}|B_1|$ and $|\{u^+ = 0\}| \geq (1 - \frac{\delta}{2})B_1$.

A main tool is the De Giorgi isoperimetric inequality that establishes that a function, u, with finite Dirichlet energy, needs “some room in between” to go from a value (say zero) to another one (say one).
It may be considered a quantitative version of the fact that a function with a jump discontinuity cannot be in H^1.

Sublemma

Let $0 \leq w \leq 1$.

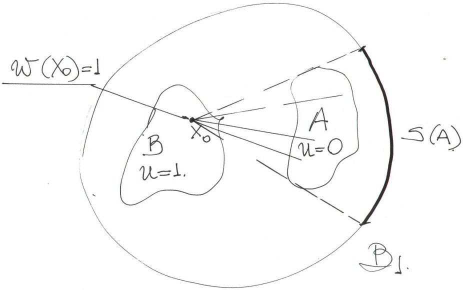
$$|A| = |\{w = 0\} \cap B_{1/2}|$$
$$|C| = |\{w = 1\} \cap B_{1/2}|$$
$$|D| = |\{0 < w < 1\} \cap B_{1/2}|$$

Then if $\int |\nabla w|^2 \leq C_0^2$

$$C_0|D| \geq C_1(|A|\,|C|)^2 \; .$$

Proof

For X_0 in B we reconstruct w by integrating along any of the rays that go from X_0 to a point in A



$$1 = w(x_0) = \int w_r\, dr \quad \text{or}$$

$$|A| \leq \text{area } S(A) < \int_D \frac{|\nabla w(y)|\, dy}{|X_0 - Y|^{n-1}}$$

$$\left(w_r\, dv\, d\sigma \leq \frac{|\nabla w| r^{n-1} dv\, d\sigma}{r^{n-1}} \right)$$

Integrating X_0 on C

$$|A|\ |C| \le \int_D |\nabla w(Y)| \left(\int_C \frac{dX_0}{|X_0 - Y|^{n-1}} \right) dy$$

Among all C with the same measure $|C|$ the integral in X_0 is maximized by the ball of radius $|C|^{1/n}$, centered at X_0

$$\int_C \cdots \le |C|^{1/n} \,.$$

So

$$|A|\ |C| \le |C|^{1/n} \left(\int_D |\nabla w|^2 \right)^{1/2} |D|^{1/2} \,.$$

Since $\int |\nabla w|^2 \le C_0^2$ the proof is complete.

With this sublemma, we go to the proof:

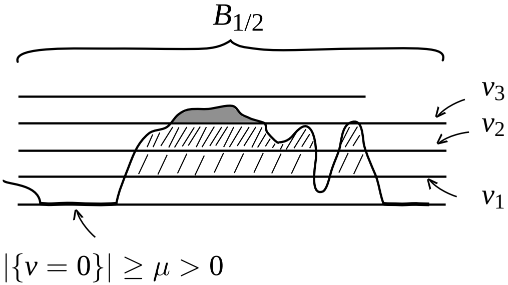
Idea of the proof

We will consider a diadic sequence of truncations approaching one

$$v_k = [v - (1 - 2^{-k})]^+$$

and their renormalizations

$$w_k = 2^k v_k$$

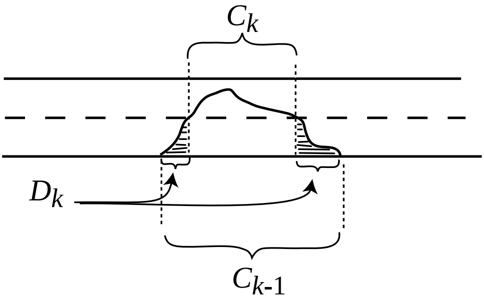


From the isoperimetric inequality, each time we truncate we expect the measure of the support to decay.

After a finite number of steps, the measure of the support of w_{k_0} will fall below the critical value $\delta/2$, and w_k will only be able to reach halfway towards one, i.e.,

$$v|_{B_{1/2}} \leq 1 - 2^{-k_0}$$

We will be interested in the set $C_k = \{v_k > 2^{-(k+1)}\} = \{w_k > 1/2\}$.
Its complement $A_k = \{v_k = 0\}$ and the transition: $D_k = [C_k - C_{k-1}]$



We will show that by applying the isoperimetric inequality and the previous lemma in a finite number of steps, k_0, $k_0(\lambda, \Lambda, \mu)$,

$$|C_{k_0}| = 0 \ .$$

Then $\sigma(\mu) = 2^{-k_0}$.

Note that

a) $A_0 = \mu$, ($\mu = 1/2$ will do for our case)

b) By the energy inequality, since $|w_k|_{B_1} \leq 1$,

$$\int_{B_{1/2}} |\nabla w_k|^2 \leq C$$

c) If C_k gets small enough

$$4 \int (w_k)^2 \leq |C_k| < \delta \ ,$$

we apply the first lemma to $2w_k$ and $2w_k|_{B_{1/4}} \leq 1$, *done*.

Proof of the theorem

We interate this argument with $2\min(w_k, \frac{1}{2}) = w$. If C_k stays bigger than δ after a finite number of steps $k_0 = k(\delta, \mu)$, we get $\sum |D_k| \geq |B_{1/2}|$ *impossible* so for some $k < k_0$, $|C_k| \leq \delta$ that makes $|C_{k+1}| = 0$ from the first part of the proof.

Corollary 7

$\operatorname{osc}_{B_{2^{-k}}} u \leq \lambda^k \operatorname{osc}_{B_1} u$

Corollary 8

$u \in C^\alpha(B_{1/2})$ with $\lambda = 2^{-\alpha}$ (defines α).

Corollary 9

If $\|u\|_{L^\infty(R^n)} \leq C \Rightarrow u$ is constant.

Note.

The argument in Lemma 1 is very useful (and powerful) when two quantities of different homogeneity compete with each other: area and volume (in a minimal surface) or area and harmonic measure, or harmonic measure and volume as in free boundary problems.

0.1. The fractional Laplacian and harmonic extensions

The fractional Laplacian, $\Delta^\alpha \theta$, can be defined as convolution with a singular kernel $(0 < \alpha < 1)$

$$\Delta^\alpha \theta(X_0) = C(\alpha) \int \frac{[\theta(X) - \theta(X_0)]}{|X - X_0|^{n+2\alpha}}\, dx$$

or through Fourier transform

$$\widehat{-\Delta^\alpha \theta}(\xi) = |\xi|^{2\alpha} \hat{\theta}(\xi)$$

0.2.

Note that the kernel

$$K = C(\alpha)|X|^{-(n+2\alpha)}$$

is singular near zero, so, in principle, some cancellation in u is expected for the integral to converge. For instance θ bounded and in C^2 near X_0 suffices. Also, $C(\alpha) \sim (1-\alpha)$ guarantees that as $\alpha \to 1$, $\Delta^\alpha \theta$ converges to $\Delta\theta$.

A particularly interesting case is the case $\alpha = 1/2$, since in this case $-(\Delta^{1/2})u$ coincides with the Dirichlet to Neuman map. More precisely, given $\theta(X)$ in $\mathbb{R}^n$, we extend it to $\theta^*(x, y)$ in $(\mathbb{R}^{n+1})^+$ by convolving with the Poisson kernel:

$$P_y(X) = \frac{C\,y}{(y^2 + |X|^2)^{\frac{n+1}{2}}} = y^{-n} P_1(X/y)$$

Then $\theta^*(x,y)$ satisfies

$$\Delta_{x,y}\theta^* = 0$$

and it can be checked that $\Delta^{1/2}\theta(X_0) = D_y\theta^*(X_0,0)$ in two ways:

a) Represent $\theta^*(X_0,h)$ as

$$\theta^*(X_0,h) = [P_h * \theta](X_0)$$

and take limit on the increment quotient

$$D_y\theta^*(X_0,0) = \lim_{h\to 0} \frac{\theta^*(X_0,h) - \theta^*(X_0,0)}{h}$$

or

b) Fourier-transform in X:

$$\widehat{\theta^*}(\xi,y) \ \text{ satisfies } \ |\xi|^2\widehat{\theta^*} = D_{yy}\widehat{\theta^*}$$

Thus

$$\widehat{\theta^*}(\xi, y) = \widehat{\theta}(\xi)e^{-y|\xi|}$$

In particular

$$D_y\widehat{\theta}(\xi, 0) = -\widehat{\theta}(\xi)|\xi| = (\widehat{\Delta^{1/2}\theta})(\xi)$$

In particular, we can make sense of the Green's and "energy" formula for the half Laplacian.
Let $\sigma(x), \theta(x)$ be two "nice, decaying" functions defined in $\mathbb{R}^n$, and $\bar{\sigma}(x, y)$, $\bar{\theta}(x, y)$ decaying extensions into $(\mathbb{R}^{n+1})^+$.
Then, we have

$$\int_{\mathbb{R}^n} \sigma(\bar{\theta})_\nu = \int_{(\mathbb{R}^{n+1})^+} \nabla_{x,y}\bar{\sigma}\nabla_{(x,y)}\bar{\theta} + \int_{(\mathbb{R}^{n+1})^+} \bar{\sigma}\Delta_{x,y}\bar{\theta}$$

If we choose $\bar{\theta}(x, y)$, the harmonic extension, θ^*, the term $\bar{\theta}_\nu(x, 0)$ becomes $-\Delta^{1/2}\theta$, and $\Delta\theta^* \equiv 0$, giving us

$$\int_{\mathbb{R}^n} \sigma(-\Delta^{1/2})\theta = \int_{(\mathbb{R}^{n+1})^+} \nabla\sigma\nabla\theta^*$$

Further, if we choose

$$\sigma = (\theta - \lambda)^+ \quad \text{and} \quad \bar{\sigma} = (\theta^* - \lambda)^+$$

(i.e., the *truncation* of the *extension* of θ) we get

$$\int_{\mathbb{R}^n} (\theta - \lambda)^+(-\Delta^{1/2}\theta) = \int_{(\mathbb{R}^{n+1})^+} [\nabla(\theta^* - \lambda)^+]^2 \, dx \, dy$$

To complete our discussion, we point out that the harmonic extension θ^* of θ, is the one that minimizes Dirichlet energy

$$E(\theta^*) = \int (\nabla \theta^*)^2$$

and that this minimum defines the $H^{1/2}$ norm of θ. In particular, we obtain

$$\int_{\mathbb{R}^n} (\theta - \lambda)^+ (-\Delta^{1/2})\theta = \iint [\nabla(\theta^* - \lambda)]^2 \, dx \, dy$$

$$\geq \iint [\nabla(\theta - \lambda)^*]^2 \, dx \, dy = \|(\theta - \lambda)\|^2_{H^{1/2}}$$

(since the *harmonic extension of the truncation* has *less energy* than the *truncation of the harmonic extension*).

To recapitulate:

The operator $\Delta^{1/2}$ is interesting because:

a) It can be understood as a "surface diffusion" process.
b) It is the "Euler Lagrange equation" of the $H^{1/2}$ energy.
c) Being of "order one", diffusion competes with transport.

In fact, the derivatives of θ:

$$D_{X_i}\theta = R_i(\Delta^{1/2}\theta) \quad \text{and} \quad \Delta^{1/2}(\theta) = \sum R_i(D_{X_i}\theta)$$

where R_i, the Riez transform, is the singular integral operator with symbol $\xi_i/|\xi|$.

Quasi-Geostrophic flow equation

The Q-G equation is a 2-D “Navier Stokes” type equation. In 2-D, Navier Stokes simplifies considerably since

a) Incompressibility, $\operatorname{div} \vec{v} = 0$, implies that $(-v_2, v_1)$ is a gradient: $(-v_2, v_1) = \nabla\varphi$.

b) $\operatorname{curl} \vec{v}$ is a scalar, $\theta = \operatorname{curl} \vec{v} = \Delta\varphi$.

Navier Stokes equation thus becomes a system:

i) $\theta_t + \vec{v}\nabla\theta = \Delta\theta$

ii) $\operatorname{curl} \vec{v} = \theta$

For the Q-G equation, we still have that

$$(-v_2, v_1) = \nabla\varphi \, .$$

But the potential φ is related to vorticity by $\theta = -\Delta^{1/2}\varphi$. That is, the final system becomes

$$\theta_t + \vec{v}\nabla\theta = (\Delta^{1/2})\theta$$

$(-v_2, v_1) = R_1\theta, R_2\theta$ where R_i are the Riesz transforms of θ.

Riesz transforms and the dependence of v on θ

More precisely, we can deduce this relation through Fourier transform:

$$\hat{\theta} = |\zeta|\hat{\varphi} \qquad \text{and} \qquad \hat{v} = \left(-\zeta_2\hat{\varphi}, \zeta_1\hat{\varphi}\right).$$

In particular

$$\hat{v} = \left(-\frac{\zeta_2}{|\zeta|}\hat{\theta}, \frac{\zeta_1}{|\zeta|}\hat{\theta}\right)$$

The multipliers $\dfrac{\xi_i}{|\xi|}$ are classical operators, called Riesz transforms that correspond in physical space x, to convolution with kernels

$$R_i(x) = \frac{x_i}{|x|^{n+1}}$$

i.e.

$$v_i^1(x) = \int R_i(x-y)\theta(y)\,dy$$

Note that on one hand

$$\|v\|_{L^2(R^n)} = \|\hat{v}\|_{L^2(R^n)} \leq \|\hat{\theta}\|_{L^2(R^n)}$$

that is, the Riesz transforms are bounded operators from L^2 to L^2. On the other hand, R is not integrable neither at zero nor at infinity. It is a remarkable theorem that because of the spherical cancellation on R (mean value zero and smoothness) we have:
The operator $R * \theta = v$ is a bounded operator from L^p to L^p for any $1 < p < \infty$ (Calderon-Zygmund). Unfortunately, it is easy to show that *singular integral operators* are not bounded from L^∞ to L^∞. They are bounded, though, from *BMO* to *BMO*.

BMO spaces

What is BMO? It is the space of functions with bounded mean oscillation.
That is, in any cube Q the “average of u minus its average” is bounded by a constant C

$$\frac{1}{|Q|}\int_Q \left| u(x) - \frac{1}{Q}\int_\theta u(y)\,dy \right| dx \leq C$$

The smallest C good for all cubes defines a seminorm (it does not distinguishes constant that we may factor out). The space of functions u in BMO is smaller than any L^p $(p < \infty)$ but not included in L^∞ $((\log|x|)^-$ is a typical example).
In fact functions u in BMO have “exponential” integrability

$$\int_{Q_1} e^{C|u|} \leq \infty$$

Proof of regularity

The regularity theory for the Quasi Geostrophic Equation is based on two linear transport regularity theorems:

Theorem 10

Let θ be a (weak) solution of

$$\theta_t + v\nabla\theta = \Lambda^{1/2}\theta \quad in \quad R^n \times [0,\infty)$$

for some incompressible vector field v (with no apriori bounds) and initial data θ_0 in L^2.
Then

$$\|\theta(X,1)\|_{L^\infty} \leq C\|\theta(x,0)\|_{L^2} \ .$$

Remarks

i) All we ask from v is that the energy inequality makes sense for any function $h(\theta)$ with linear growth:
Formally, if we multiply and integrate, we may write

$$\int_{T_1}^{T_2} \int_{R^n} h(\theta) v \nabla \theta = \int\int v \nabla H(\theta) = \int\int \operatorname{div} v H(\theta) = 0$$

$(H'(\theta) = h(\theta)$

Therefore the contribution of the transport term in the energy inequality vanishes.

In the case of the Q-G equation this can be attained by rigorously constructing θ in a particular way, for instance as a limit of solutions in increasing balls, B_K.

ii) From the scaling of the equation: For any λ

$$\theta_\lambda = \frac{1}{\lambda}\theta(\lambda x, \lambda t)$$

is again a solution (with a different v), we obtain

$$\|\theta(x,t_0)\|_{L^\infty_x} = t_0\|\theta_{t_0}(x,1)\| = t_0\|\theta_{t_0}(x,0)\| = t_0^{-n/2}\|\theta_0\|_{L^2}$$

That is uniform decay for large times.

The proof of Theorem 1 is a baby version of the DeGiorgi theorem based in the interplay between the

- Energy Inequality (that controls the derivatives of θ by θ itself).

and the

- Sobolev Inequality (that controls θ by its derivatives)

Baby version because, as in the minimal surface example, no cut off in space is necessary.

The Energy Inequality is attained, as usual, by multiplying the equation with a truncation of θ,

$$(\theta_\lambda) = (\theta - \lambda)^+$$

and integrating in $R^n \times [T_1, T_2]$.

As we pointed out before, the term corresponding to transport vanishes, and we get:

$$\int (\theta_\lambda)^2(y, T_2) - (\theta_\lambda)^2(y, T_1)\, dy + 0 = \iint_{R^n \times [T_1, T_2]} \theta_\lambda \Lambda^{1/2} \theta \, dy \, dt$$

The last term corresponds, for the harmonic extension $\theta^*(x,z)$ to

$$\int_{T_1}^{T_2} dt \left(\int (\theta^*)_\lambda(y,0,t) D_z(\theta^*)(y,0,t)\, dy \right)$$

$$= - \int_{T_1}^{T_2} \iint_{R_+^{n+1}} \nabla(\theta^*)_\lambda(y,z,t) \nabla\theta^*(y,z,t)\, dy\, dz$$

$$= - \int_{T_1}^{T_2} dt \iint_{R_+^{n+1}} [\nabla\theta_\lambda^*]^2 \; dy\, dz$$

Note that $(\theta^*)_\lambda$ is *not* the harmonic extension of θ_λ, but the truncation of the extension of θ, i.e., $(\theta^* - \lambda)^+$

Nevertheless, it is an extension of θ_λ and as such,

$$\|\theta_\lambda^*\|_{H^1(R_+^{n+1})} \geq \|\theta_\lambda\|_{H^{1/2}(R^n)}$$

Therefore we end up with the following energy inequality

$$\|\theta_\lambda(\cdot, T_2)\|_{L^2}^2 + \int_{T_1}^{T_2} \|\theta_\lambda\|_{H^{1/2}}^2 \, dt \le \|\theta_\lambda(T_1)\|_{L^2}^2$$

We will denote by A the term on the left (A_{T_1,T_2}) and B (B_{T_1}) the one on the right.
Therefore B_{T_1} controls in particular (from Sobolev inequality) all of the future:

$$\sup_{t \ge T_1} \|\theta(t)\|_{L^2}^2 + \int_{T_1}^{\infty} \|\theta(t)\|_{L^p}^2 \le B_{T_1}$$

This combination, in turn, actually controls

$$\|\theta\|_{L^q(R^n \times [T_1,\infty)}^2$$

for some q, with $2 < q < p$ the following way:

Every such q is a convex combination

$$q = \alpha 2 + (1-\alpha)p = \frac{1}{r}2 + \frac{1}{s}p$$

for r, s appropriate conjugate exponents.

Therefore, fixing such a q, we have for each time, t:

$$\int \theta^q \leq \left(\int \theta^2\right)^{1/r} \cdot \left(\int \theta^p\right)^{1/s}$$

We choose $s = p/2$ (> 1) and integrate in t: For the corresponding q, we get

$$\|\theta\|^q_{L^q(R^n\times[t_1,\infty)} \leq \sup_t \|\theta\|^{2/r}_{L^2(R^n)} \cdot \int_{T_1}^{\infty} \|\theta\|^2_{L^p} \leq (B_{T_1})^{1+\frac{1}{r}} = (B_{T_1})^{q/2}$$

We call $C_{T_1} = \|\theta\|^{2/q}_{L^q(R^n\times[t,\infty]}$
i.e. $C_{T_1} \leq B_{T_1}$

We are ready to prove the L^{∞} bound. For that purpose, we will find a recurrence relation for the constants

$$C_{T_k}(\theta_k)$$

of a sequence of increasing cut-offs $\lambda_k = 1 - 2^{-k}$ of θ (i.e., $\theta_k = \theta_{\lambda_k}$) and cut-offs in time $t_k = 1 - 2^{-k}$, that will imply that $\theta_{\infty} = (\theta - 1)^{+} \equiv 0$ for $t > 1$.

Indeed, on one hand, from Sobolev:

$$C_{T_k}(\theta_k) \leq B_{T_k}(\theta_k) \ .$$

We now invert the relation. For $I = [T_{k-1}, T_k] \times R^n$ we have

$$\iint_I (\theta_k)^2 \leq \left[\iint_I \theta_k^q \right]^{2/q} |\{\theta_k > 0\} \cap I|^{1/\bar{q}} = \alpha \cdot \beta$$

(by Hölder with θ^2 and $\chi_{\theta_k > 0}$) with $\bar{q}$ the conjugate exponent to $q/2$). In turn $\alpha \leq C_{T_{k-1}}(\theta_{k-1})$ and by going from k to $k-1$, we can estimate: (should sound familiar by now)

$$\beta = \left| \{\theta_{k-1} > 2^{-k}\} \cap I \right|^{1/\bar{q}} \leq \left[2^{qk} \iint_I (\theta_{k-1})^q \right]^{1/\bar{q}} \quad \text{(by Chebichef)}$$

(since $\theta_k \leq \theta_{k-1}$ and further $\theta_k > 0$ implies $\theta_{k-1} > 2^{-k}$).

That is $\beta \leq 2^{Ck}\left[C_{T_{k-1}}(\theta_{k-1})\right]^{\varepsilon}$ and putting together the estimates for α and β:

$$\iint_I (\theta_k)^2 \leq 2^{Ck}\left[C_{T_{k-1}}(\theta_{k-1})\right]^{1+\varepsilon}$$

But then,

$$\inf_{[T_{k-1}<t<T_k]} B_t(\theta_k) \leq 2^k 2^{Ck} \cdot \left[C_{T_{k-1}}(\theta_{k-1})\right]^{1+\varepsilon}$$

We obtain the recurrence relation

$$C_{T_k}(\theta_k) \leq 2^{\bar{C}k}\left[C_{T_{k-1}}(\theta_{k-1})\right]^{1+\varepsilon}$$

Due to the $1+\varepsilon$ nonlinearity, $C_{T_k}(\theta_k) \to 0$ if $C_0(\theta^+)$ was small enough, i.e., if $\|\theta_0\|_{L^2} \leq \delta_0$ then $\|\theta(x,t)\|_{L^\infty} \leq 1$, for $t \geq 1$.

Having shown boundness, for the QG equation, our stituation is now the following: We have a solution θ that satisfies the energy bound:

$$\begin{cases} \sup_t \|\theta(t)\|^2_{L^2(\mathbb{R}^n)} + \|D_{1/2}\theta\|^2_{\mathbb{R}^{n+1}_+} \leq C \\ \text{and also } \ \|\theta\|_{L^\infty(X,t)} \leq 1 \\ \text{We want to prove that } \theta \text{ is Hölder continuous.} \end{cases} \tag{\#}$$

To do that we need to reproduce the local in space De Giorgi method. Of the velocity field, we may assume now (being the Riesz transform of θ, that

$$\sup_t \left(\|v\|^2_{L^2(\mathbb{R}^n)} + |v|_{BMO(\mathbb{R}^n)} \right) \leq C \, . \tag{$*$}$$

We decouple v from θ, and will prove a linear theorem, where for v satisfying $(*)$ and θ satisfying $(\#)$ and the equation

$$\theta_t + v\nabla\theta = \Delta^{1/2}\theta$$

Theorem: *θ is locally C^α.*

To simplify the notation we will assume that θ exists for $t \geq -4$ and will focus on the point $(X, t) = (0, 0)$. The Hölder continuity will be proven through an oscillation Lemma, i.e., we will prove that on a geometric sequence of cylinders

$$\Gamma_k = B_{4^{-k}} \times [4^{-k}, 0]$$

the oscillation of θ

$$\omega_k = [\sup_{\Gamma_k} \theta - \inf_{\Gamma_k} \theta]$$

decreases geometrically, i.e.,

$$\omega_{k+1} \leq \mu \omega_k \ \text{ for } \ \mu < 1$$

This is proved in several steps, following the L^2 to L^∞ and the oscillation Lemmas, discussed before.
The underlying idea is the following:

Suppose that on the cylinder $\Gamma_0 = B_1 \times [-1, 0]$, θ lies between -1 and 1.

Then at least half of the time it will be below or above zero.

Let us say that it is below zero. Then, because of the diffusion process, by the time we are at the top of the cylinder, and near to zero, θ should have gone uniformly, strictly below one, so now $-1 \leq \theta \leq 1 - \delta$ and the oscillation ω has been reduced.

If we achieve this result, we renormalize and repeat. How do we achive this oscillation reduction? For the heat equation, this will just follow from simple properties of the fundamental solution.

Here, following DeGiorgi, we proceed in two steps. First, we show that if θ is “most of the time negative” or very tiny in $B_1 \times [-1, 0]$, then indeed it cannot stick to one close to the top of the cylinder and it goes strictly below one in say $B_{1/4} \times [-1/4, 0]$.

Next we have to close the gap between “being negative most of the time” and “being negative half of the time”, since this last statement is what we can verify at each step

This takes a finite sequence of cut-offs and renormalizations, exploiting the fact that for θ to go from a level (say zero), to another (say one), some minimal amount of energy is necessary (the De Giorgi isoperimetric inequality). Finally, once this has been reached, we can iterate.

In our case, the arguments are complicated by the global character of the diffusion that may cancel the local effect we described above. Luckily we may encode the global effect locally into the harmonic extension, but this requires some careful treatment.

The first technical complication is that we must now truncate not only in θ and t but also in X, but this does not have the effect of fully localizing the energy inequality.

In the light of the iterative interaction between Sobolev and energy inequality, let's explore a little bit what kind of energy formulas we may expect after a cut-off in space.

Let us start with a cut-off in x and z, for $\theta(x,t)$ and its harmonic extension $\theta^*(x,z,t)$.

That is, η is a smooth nonnegative function of x,z, with support in B_4^*, and as usual we multiply the equation by $\eta^2\theta_\lambda^*$ (that coincides with θ_λ for $z=0$) and integrate.

We get the following terms:

$$\int_{T_1}^{T_2}\int \eta^2\theta_\lambda\theta_t\,dx\,dt \equiv \int \eta^2(\theta_\lambda)^2(T_2)\,dx - \int \eta^2(\theta_\lambda)^2(T_1)\,dx \qquad \text{(I)}$$

Next we have the transport term, an extra term not usually present in the energy inequality.

$$\begin{aligned} \iint \eta^2 \theta_\lambda v \nabla\theta \, dx\, dt &= \iint \eta^2 \operatorname{div}[v(\theta_\lambda)^2] \, dx\, dt \\ &= \iint 2\eta\nabla\eta \, [v(\theta_\lambda)^2] \, dx\, dt \end{aligned} \tag{II}$$

We split the term in the two factors $(\nabla\eta)v\,\theta_\lambda$ and $\eta\theta_\lambda$, the logic being that v is almost bounded and thus the first term is almost like the standard right hand side in the energy inequality while the second would be absorbed by the energy.

For each fixed t, we get

$$|II| \leq \int_{T_1}^{T_2} \|\eta\theta_\lambda\|_{L^{2n/n-1}} \|\nabla\eta[v\theta_\lambda]\|_{L^{2n/n+1}}$$

$$\leq \int_{T_1}^{T_2} \varepsilon\|\eta\theta_\lambda\|^2_{L^{2n/n-1}} + \frac{1}{\varepsilon}\|\nabla\eta[v\theta_\lambda]\|_{L^{2n/n+1}}$$

But $\frac{2n}{n+1} < 2$, so we can split by Hölder $[\nabla\eta]\theta$ in L^2 and v in a (large) L^p, more precisely L^{2m} since we that that v is in every L^p.
That is

$$II \leq \varepsilon \int_{T_1}^{T_2} \|\eta\theta_\lambda\|^2_{L^{2n/n-1}} + \frac{1}{\varepsilon}\int_{T_1}^{T_2} \|v\|^2_{L^{2n}(B_2)} \|[\nabla\eta]\theta_\lambda\|^2_{L^2} \, .$$

(Remember that by hypothesis $\|v\|_{L^{2n}(B_2)} \leq C$ for every t.)
I.e.

$$II \leq \varepsilon C \int_{T_1}^{T_2} \|\eta\theta_\lambda\|^2_{L^{2n/n-1}} + \frac{1}{\varepsilon} C \int_{T_1}^{T_2} \|[\nabla\eta]\theta_\lambda\|^2_{L^2}$$

Finally, III is our energy term, i.e.,

$$III = \iint \eta^2 \theta_\lambda \Delta^{1/2} \theta \, .$$

Using the harmonic extension θ^* we get that

$$III = -\iint \eta^2 \theta_\lambda \theta^*_\nu \, dx \, dt = -\iiint \nabla_{x,z}(\eta^2 \theta^*_\lambda) \nabla \theta^* \, dx \, dz \, dt \, .$$

By the standard energy inequality computation, we get that

$$III \leq -\iiint [\nabla_{x,z} \eta \theta^*_\lambda]^2 \, dx \, dz \, dt + \iiint (\nabla \eta)^2 (\theta^*_\lambda)^2 \, .$$

We may choose η to be a cut-off in x and z or to integrate to infinity in z, if we have control of θ^*_λ in z.

(Note: If for some reason we know that $(\theta_\lambda^*) \equiv 0$ in $B_1 \times \{z_0\}$ for some z_0, we may cut-off only in x and still stop the integration at z_0.)

Putting together I, II and III we get

$$\sup_{T_1 \le t \le T_2} \|\eta\theta_\lambda\|_{L^2}^2 + \int_{T_1}^{T_2} \|\nabla(\eta\theta_\lambda^*)\|_{L^2}^2$$

$$\le \|\eta\theta_\lambda(T_1)\|_{L^2}^2 + \varepsilon \int_{T_1}^{T_2} \|\eta\theta_\lambda\|_{L^{2n/n-1}}^2$$

$$+ \frac{1}{\varepsilon} C \int_{T_1}^{T_2} \|(\nabla\eta)\theta_\lambda\|_{L^2}^2 + \int_{T_1}^{T_2} \|(\nabla\eta)\theta_\lambda^*\|_{L^2_{x,z}}^2 \, dt$$

Notice that $\eta\theta_\lambda^*$ is one extension of $\eta\theta_\lambda$ and therefore the term in the left

$$\int_{T_1}^{T_2} \|\nabla\eta\theta_\lambda^*\|^2_{L^2(x,z)} \quad \text{controls} \quad \int_{T_1}^{T_2} \|\eta\theta_\lambda\|^2_{H^{1/2}} \ ,$$

so the left hand side controls, by Sobolev inequality, the term

$$\int_{T_1}^{T_2} \|\eta\theta_\lambda\|^2_{L^{2n/n-1}} \, dt$$

and absorbs the ε term on the right.

In fact, if it weren't because of the term on the right

$$\int_{T_1}^{T_2} \|(\nabla\eta)\theta_\lambda^*\|_{L^2_{x,z}}^2$$

involving the extra variable z, everything would reduce to $\mathbb{R}^n$, and we have the usual interplay between the Sobolev and energy inequality, as in the global case, and a straightforward adaptation of the second order case would work.

In light of this obstruction let's reassess the situation. As we mentioned before our first lemma would be (following the iterative scheme) to show that if, say the L^2 norm of θ^+ is very small (and $\theta^+ \leq 2$), in $B_4 \times [-4, 0]$, then θ^+ is strictly less than two in $B_1 \times [-1, 0]$; ($\theta^+ \leq \gamma_0 < 2$ for some γ_0).

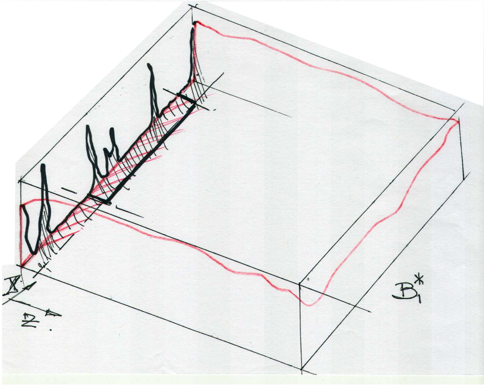
Let us see how that can work:

A geometric description of the argument

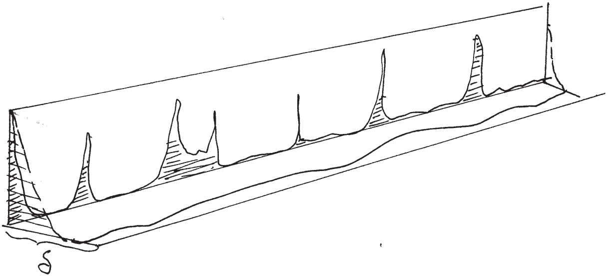
Starting up: We decompose θ^* in two parts

θ_1^* goes to zero linearly as $z \to 0$ for $|x| < 1/2$

θ_2^* has a very small trace in L^2, so it becomes very small in L^∞ as z grows. Given δ, we may assume that $\theta^*(X, \delta) \leq C\delta$ since we can choose $\|\theta\|_{L^2}$ as small as we please



Therefore, the first truncation $\theta^*_{\lambda_0}$ is controlled by its trace, with very small L^2 norm, and the very narrow sides (size δ), whose influence decays exponentially moving inwards in X:



We will try now to perpetuate, in our inductive scheme, this configuration.

The idea of the inductive scheme is then as follows:

1) In X, we will cut diadically (as in De Giorgi) converging to $\chi_{B_{1/2}}$
2) In θ, also diadically converging to λ, $(1 < \lambda < 2)$
3) In Z, though, we will cut at a very fast geometric rate, going to zero (δ^k).

The reason we may hope to maintain this configuration, is because inherent to the De Giorgi argument is the very fast, (faster than geometric decay) of the L^2 norm of the truncation θ_k.

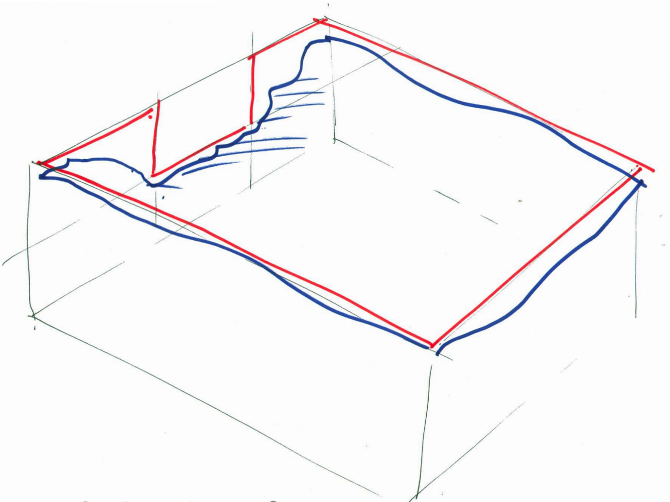
The idea is that, on one hand the fast cut off in Z, δ^k will make the influence of the tiny sides decay so much in X that at the level of the next cut off (in X) it will be wiped out by the diadic cut off in θ.

While the contribution of the trace θ_k will decay so fast (faster than M^{-k} if we choose c_0 very small) that

$$\theta_k^*(X, \delta^k) \leq \theta_k \star P_{\delta^k} \leq \|\theta_k\|_{L^2} \|P_{\delta^k}\|_{L^2}$$

will also be wiped out by the consecutive truncation.

At the end of the process at time t_0 we have only information on the trace θ, but we can go inwards by harmonicity.



End of iteration: final time $t_0 \geq -1$

Proof of the first step: In this first step, we prove that a solution, θ, between zero and two, with very small L^2 norm, separates from $\theta = 2$ in a smaller cylinder.

Lemma 11

We assume that $\|v\|_{L^\infty(-4,0;BMO(\mathbb{R}^N))} + \sup_{-4\le t\le 0} \left| \int_{B_4} v(t,x)\, dx \right| \le C_0.$

Then, there exists $\varepsilon_0 > 0$*, and* $\lambda > 0$ *such that for every* θ *solution to* (1) *the following property holds true.*
If we have:

$$\theta^* \le 2 \qquad \text{in } [-4,0] \times B_4^*,$$

and

$$\int_{-4}^{0} \int_{B_4^*} (\theta^*)_+^2 \, dx\, dz\, ds + \int_{-4}^{0} \int_{B_4} (\theta)_+^2 \, dx\, ds \le \varepsilon_0,$$

then:

$$(\theta)_+ \le 2 - \lambda \qquad \text{on } [-1,0] \times B_1.$$

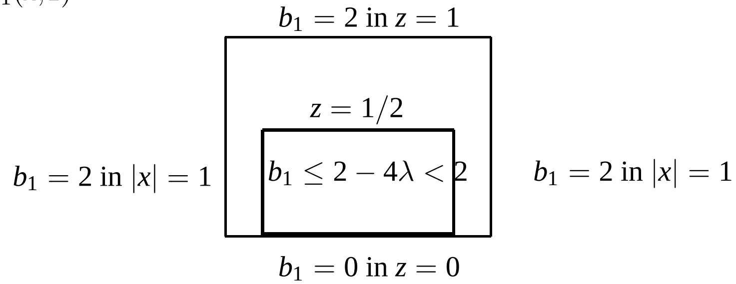
Proof

The proof follows the strategy discussed above.

First, some previous tools.

Since the method was based on the control of θ^* by two harmonic functions, before starting the proof we build two useful barriers:

$b_1(x,z)$



If • b_1 is harmonic in B_1^*

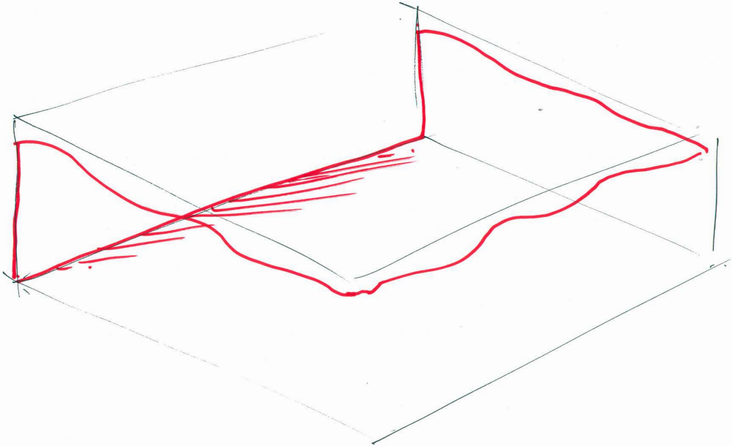
• $b_1 = 2$ in ∂B_1^* except $z = 0$

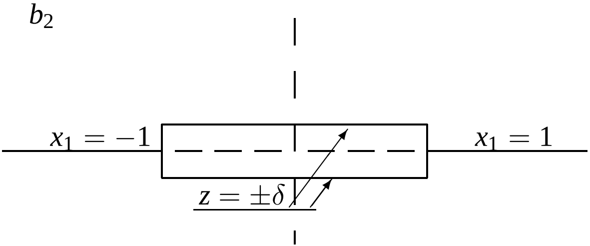
• $b_1 = 0$ in $\partial B_1^* \cap \{z = 0\}$

Then for some $0 < \lambda$

$$b_1 \le 2 - 4\lambda < 2 \ \text{ in } \ B_{1/2}^*$$

Θ_1 has linear decay as $y \to 0$.





If • b_2 is harmonic in D

$b_2 = 0$ for $z \pm \delta$

$b_2 = 1$ for $x_1 = 1$, $\quad b_2 = 0$ for $x_1 = -1$

Then $$b_2 \leq \bar{C} \cos \frac{z}{\delta} e^{-(1-x_1)/\delta}$$

in particular, if $1 - x_1 = h \gg \delta$

$$b_2 \leq \mathcal{C} e^{-h/\delta} \qquad \mathcal{C} = (\cos 1)^{-1}$$

Remark

Exponential decay also holds for $D_i a_{ij} D_j$ by applying Harnack inequality to the intervals $I_k = \{k \leq x_1 \leq k+1\}$.

Now we are ready to set the main inductive steps as discussed above.

When we do so, we will realize that we have to start the process for some advanced value k of the step.

So we will go back and do a first large step to cover the starting of the process.

Setting of the constants

We recall that $\lambda > 0$ is defined by the fact that the barrier function

- $b_1 < 2 - 4\lambda$ in $B^*_{1/2}$
- $\bar{C} = (\cos 1)^{-1}$ is the constant in the bound for the barrier function b_2
- C_0 the smallness constant in the hypothesis of Lemma 10 will be chosen later as $c_0(\lambda, M)$.

We need to fix constants M for the rate of decay of the L^2 norm of the truncation θ_k and δ for the rate of decay of the support, in z, of θ_k^*.

We require:

i) $n\bar{C}e^{-(2\delta)^{-k}} \leq \lambda 2^{-k-2}$ (δ small so the side contribution is absorbed by the cut off)

ii) $\delta^n (M\delta^n)^{-k} \|P(1)\|_{L^2} \leq \lambda 2^{-k-2}$ $(M(\delta)$ large to keep the support of the trunction in the δ^k strip)

iii) $M^{-k} \geq c_0^k M^{-(k-3)(n+1/n)}$ for $k \geq 12n$ (so that the inductive decay gives us the fast geometric decay)

$P(1)$ denotes the restriction of the Poisson kernel, $P(x,z)$ to $z \equiv 1$.

The proof is easy. We construct first δ to verify the first inequality in the following way. If $\delta < 1/4$, the inequality is true for $k > k_0$ due to the exponential decay. If necessary, we then choose δ smaller to make the inequality also valid for $k < k_0$. Now that δ has been fixed, we have to choose M large to satisfy the remaining inequalities. Note that the second inequality is equivalent to:

$$\left(\frac{2}{\delta^n M}\right) \leq \frac{\lambda \delta^n}{4\|P(1)\|_{L^2}} .$$

It is so sufficient to take:

$$M \geq \sup\left(\frac{2}{\delta^n}, \frac{8\|P(1)\|_{L^2}}{\lambda \delta^n}\right) .$$

The third inequality is equivalent to:

$$\left(\frac{M}{C_0^N}\right)^{k/N} \geq M^{3(1+1/N)} .$$

For this case it is sufficient to take $M \geq \sup(1, C_0^{2N})$. Indeed, this ensures $M^2/C_0^{2N} \geq M$ and so:

$$\left(\frac{M}{C_0^N}\right)^{k/N} \geq M^{k/(2N)} \geq M^6$$

The main inductive step will be the following:

Step 3. Induction: We set

$$\theta_k = (\theta - C_k)_+ \ ,$$

with $C_k = 2 - \lambda(1 + 2^{-k})$. We consider a cut-off function in x only such that:

$$\mathbf{1}_{\{B_{1+2^{-k-1}}\}} \le \eta_k \le \mathbf{1}_{\{B_{1+2^{-k}}\}} \ , \qquad\qquad |\nabla \eta_k| \le C2^k \ ,$$

and we denote:

$$A_k = 2\int_{-1-2^{-k}}^{0} \int_0^{\delta^k}\!\!\int_{\mathbb{R}^N} |\nabla(\eta_k \theta_k^*)|^2 \, dx \, dz \, dt + \sup_{[-1-2^{-k},1]} \int_{\mathbb{R}^N} (\eta_k \theta_k)^2 \, dx \, dt \ .$$

We want to prove that for every $k \ge 0$:

$$A_k \le M^{-k} \tag{1}$$

$$\eta_k \theta_k^* \ \text{ is supported in } \ 0 \le z \le \delta^k \ . \tag{2}$$

We first prove inductively (2). Suppose it is true for k, we want to prove it for $k+1$.

Since $\theta_{k+1}^* \leq \theta_k^* - \lambda 2^{-(k+1)}$, we do it by estimating θ_k^* in the "flat rectangle" $B_{1+2^{-(k)}} \times [0, \delta^k]$ where η is supported.

On the top, $z = \delta^{-k}$, $\theta_k^* = 0$ by induction.

On the bottom, $\theta_{k+1}^* \leq \eta_k \theta_k^*$ and hence the contribution from θ_{k+1}^* is smaller than its harmonic extension

$$\eta_k \theta_k^* * P(z) \qquad \text{(the Poisson kernel)}$$

The influence from each of the narrow lateral sides is bounded by the rescaling of b_2 that shrinks the interval $[0,1]$ in z into $[0,\delta^k]$.

Therefore, from the edge $x_j = 1 + 2^{-k}$ to $x_j = 1 + 2^{-(k+1)}$, b_2 decays by $e^{-\frac{1}{2}(\frac{2^{-k+1}}{\delta^k}}$.

Overall on $B_{1+2^{-k+1}}$, the support of $\eta_{k+1}\theta_{k+1}$, we obtain for θ_k^* the estimate

$$\theta_k^* \leq \eta_k\theta_k * P_z + \bar{C}\, e^{-\frac{1}{4}\frac{2^{-k}}{\delta^k}}$$

The term

$$\eta_k \theta_k * P(\delta_{k+1})$$

we bound by

$$\|\eta_k \theta_k\|_{L^2} \|P(\delta_{k+1})\|_{L^2} \leq M^{-k/2} \delta^{-n(k+1)} \|P(0)\|_{L^2}$$

To force $\theta^*_{k+1} = 0$ for $z = \delta^k$, we then need

$$M^{-k/2} \delta^{-(k+1)} \|P(1)\|_{L^2} + N\bar{C}\, e^{-\frac{1}{4}\frac{2^{-k}}{\delta^k}} \leq \lambda 2^{-(k+1)}$$

For this to happen, we need, for instance, $M\delta > 4$, k large (say bigger than k_0) and (for instance) $2\delta < 1/2$.

Second technical lemma

In the first technical lemma, we have established that if $0 \leq \theta_+ \leq 2$ and its energy or norm is very small, in B_4^*, then, $\theta_+ \leq 2 - \lambda$ in B_j, i.e., the oscillation of θ actually decays.

We want now to get rid of the "very small" hypothesis.

This second lemma proves that if $\theta_+ \leq 0$ "half of the time and it needs very little room, δ, to go from $\{\theta_+ \leq 0\}$ to $\{\theta \geq 1\}$, it is because $(\theta - 1)^+$ has very small norm to start with. This produces a dichotomy: or the support of θ decreases substantially, or θ becomes small anyway.

Lemma 12

For every $\varepsilon_1 > 0$*, there exists a constant* $\delta_1 > 0$ *with the following property:*
For every solution θ *to* (1) *with* v *verifying* (2) *and:*

$$\theta^* \leq 2 \qquad \text{in } \; Q_4^*,$$
$$|\{(x,z,t) \in Q_4^*;\ \theta^*(x,z,t) \leq 0\}| \geq \frac{|Q_4^*|}{2},$$

we have the following implication:
If

$$|(x,z,t) \in Q_4^*;\ 0 < \{\theta^*(x,z,t) < 1\}| \leq \delta_1$$

then:

$$\int_{Q_1} (\theta - 1)_+^2 \, dx \, dt + \int_{Q_1^*} (\theta^* - 1)_+^2 \, dx \, dz \, dt \leq C\sqrt{\varepsilon_1}.$$

This lemma is, of course, the adapted version of the De Giorgi's isoperimetric inequality.
The idea of the proof is the following:
We first throw away a small set of times, for which
$I_t = \int_{B_1^*} (\nabla u^*)^2 \, dx \, dz$ is very large:

$$I_t \geq \frac{K^2}{\varepsilon_1^2}$$

This is a tiny set of times

$$|S| \leq C \, \varepsilon^2 / k^2$$

since

$$\iint (\nabla u^*)^2 \, dx \, dz \, dt \leq C_0$$

Outside of S, for each time t, the isoperimetric inequality is valid

$$|A| \, |B| \leq |D| \, K / \varepsilon_1$$

But for some t, say $t < \frac{-1}{64}$, we may choose a slice where $|A| > \frac{1}{64}$ and $|D| \leq \delta$.

Then $|B| \leq (64)^2 \delta \, K/\varepsilon_1 \leq K\varepsilon_1$ if $\delta \sim \varepsilon^2$

In particular,$(\theta - 1)^+$ has very small L^2 norm for that t: $\| \quad \| \leq k\varepsilon_1$

But the energy inequality then controls the L^2 norm of $(\theta - 1)^+$ into the future

The same iteration as De Giorgi's completes the proof.

Proposition 13

There exists $\lambda^* > 0$ *such that for every solution* θ *of* (1) *with* v *verifying* (2)*, if:*

$$\theta^* \leq 2 \qquad \text{in } \; Q_1^*$$
$$|\{(t,x,z) \in Q_1^*;\; \theta^* \leq 0\}| \geq \frac{1}{2},$$

then:

$$\theta^* \leq 2 - \lambda^* \qquad \text{in } \; Q_{1/16}^*.$$

Up to here, the proof did not distinguish the $1/2$-power of the Laplacian from any other power. We could have replaced it by Δ^σ for $\sigma > 0$, and used the extension in Caffarelli-Silvestre (Arxiv.org).
It is in the iteration process that σ becomes critical:
Indeed, to iterate, we rescale

$$\theta_k = \frac{1}{\lambda}\theta_{k-1}\left(\frac{1}{2}x, \frac{1}{2}t\right)$$

that satisfies the same (linear) equation with a rescaled vector field v.
(Here we use that $\nabla\theta$ and $\Delta^{1/2}\theta$ have the same homogeneity.)
The only detail is that, to keep $\|v\|_{L^{2m}}$ bounded by the $\|v\|_{BMO}$, we have to make sure that $⨍ v \equiv 0$.
For that we make the change of variables $\bar{x} = x - \vec{\varphi}(t)$, where $(\vec{\varphi})' = ⨍ v$
This produces finite distortion (in fact vanishing distortion) since $⨍_{Q_r} v$ grows very slowly due to its exponential integrability.