PART II : GRADIENT FLOWS

Titolo nota 28/03/2008

1. HILBERTIAN THEORY
2. WEAK FORMULATIONS OF G.F. AND IMPLICIT EULER SCHEME
3. EXISTENCE, UNIQUENESS, STABILITY OF G.F. AND REGULARIZING EFFECTS

$\longleftrightarrow$

$F : \mathcal{M} \longrightarrow \mathbb{R}$

Ingredients:
- Energy F
- metric (distance) $dF_x(v) = \langle \nabla F(x), v \rangle$

$$\dot{x}(t) = -\nabla F(x(t)) \qquad (GF)$$

1 HILBERTIAN THEORY

$\lambda \geq 0$

We shall consider convex (or λ-convex) functions F:

$$F((1-t)x + ty) \leq (1-t)F(x) + t F(y) - \frac{\lambda}{2} t(1-t)|x-y|^2$$

$$\langle \xi - \eta, x - y \rangle \geq \lambda |x-y|^2 \qquad \text{(MONOTONICITY INEQUALITY)}$$

$\xi \in \partial F(x)$, $\eta \in \partial F(y)$

$$\partial F(x) := \left\{ v \;\middle|\; F(y) \geq F(x) + \langle v, y-x \rangle + \frac{\lambda}{2}|y-x|^2 \quad \forall y \in H \right\}$$

$\nabla F(x) :=$ the element with minimal norm of $\partial F(x)$

$D(F) := \{F < +\infty\}$

For convex functions in Hilbert spaces, a much more flexible formulation of (GF) is :

$$(*) \quad \begin{cases} x'(t) \in -\partial F(x(t)) & \text{a.e. } t>0 \\ x \in AC^2_{loc}((0,+\infty);H) \\ \lim_{t \downarrow 0} x(t) = \bar{x}. \end{cases}$$

THEOREM 1. Let $F: H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be convex and l.s.c.

i) (Existence and uniqueness) $\forall\, \bar{x} \in \overline{D(F)}$ (*) has a unique solution;

ii) (Minimal selection and energy identity) for a.e. t, $x'(t) = -\nabla F(x(t))$, so that also the (GF) equation has a unique solution.

Moreover $F(x(t)) \in AC_{loc}((0,+\infty))$ and

$$F(x(s)) - F(x(t)) = \int_s^t |\nabla F|^2(x(r))\,dr \qquad \forall\ 0 < s \leq t < \infty$$

$(s = 0$ if $\bar{x} \in D(F))$

iii) (Regularizing effects) $\quad x'_+(t) = -\nabla F(x(t))$, $\ (F \circ x)'_+(t) = -|\nabla F|^2(x(t))$

for all $t > 0$, and

$$F(x(t)) \leq \inf_{v \in D(F)} \left\{ F(v) + \frac{1}{2t} |v - \bar{x}|^2 \right\},$$

$$|\nabla F|^2(x(t)) \leq \inf_{v \in D(\partial F)} \left\{ |\nabla F|^2(v) + \frac{1}{t^2} |v - \bar{x}|^2 \right\}.$$

iv) (Asymptotic behaviour)

$$F(x(t)) - F(x_{min}) \leq \left(F(\bar{x}) - F(x_{min})\right) e^{-2\lambda t}$$

In particular, if $\lambda > 0$, the ENERGY INEQUALITY $F(x) - F(x_{min}) \geq \frac{\lambda}{2}|x - x_{min}|^2$ gives

$$|x(t) - x_{min}| \leq \sqrt{\frac{2}{\lambda}\left(F(\bar{x}) - F(x_{min})\right)}\, e^{-\lambda t}.$$

2 WEAK FORMULATIONS OF (GF) : (EDI) and (EVI)

We encode both the system (GF) and the energy identity in a single inequality:

$$\frac{d}{dt} F(x(t)) \leq -\frac{1}{2} |\nabla F(x(t))|^2 - \frac{1}{2} |x'(t)|^2 .$$

Indeed, along any curve $y(t)$, we have

$$\frac{d}{dt} F(y(t)) = \langle \nabla F(y(t)), y'(t) \rangle$$

$$\geq - |\nabla F(y(t))| \, |y'(t)| \qquad \left(= \text{ iff } -y'(t) \parallel \nabla F(y(t)) \right)$$

$$\geq -\frac{1}{2} |\nabla F|^2 (y(t)) - \frac{1}{2} |y'(t)|^2$$

$$= \text{ iff } \quad |\nabla F(y(t))| = |y'(t)|$$

It is technically more convenient to consider the inequality in an integral form, namely:

$$\frac{1}{2}\int_0^t |x'(r)|^2 dr + \frac{1}{2}\int_0^t |\nabla F(x(r))|^2 dr \leq F(\bar{x}) - F(x(t)) \qquad \text{(EDI)}$$

We will see that:

1) (EDI) makes sense also in metric spaces;

2) (EDI) has a discrete counterpart.

Let us consider now the (EVI) formulation. This one relies very much on two ingredients: • A global ENERGY INEQUALITY (and therefore convexity!)

• The DERIVATIVE OF SQUARED DISTANCE

In Hilbert spaces both ingredients are easy: if u solves (GF) we have

$$\frac{d}{dt}\frac{1}{2}|u(t)-v|^2 = \langle u'(t), u(t)-v \rangle$$

$$= \langle -u'(t), v-u(t) \rangle$$

$$\leq F(v) - F(u(t)) - \frac{\lambda}{2}|v-u(t)|^2$$

DEFINITION. In a metric space (E, d), an absolutely continuous curve $u(t)$ is said to be an (EVI) solution to (GF) if

$$\frac{d}{dt}\frac{1}{2}d^2(u(t), v) \leq F(v) - F(u(t)) - \frac{\lambda}{2}d^2(v, u(t)) \quad \text{for a.e. } t>0, \forall v \in D(F) \quad \text{(EVI)}$$

Under suitable assumptions on (d, F), (EVI) has a discrete version as well.

The (EVI) formulation is very strong, and it leads easily to stability (for instance with the respect to Γ-convergence of the energy F) and to contractivity.

THEOREM 2. (CONTRACTIVITY) Let x, y be solutions to (EVI). Then $d(x(t), y(t)) \leq d(\bar{x}, \bar{y})\, e^{-\lambda t}$

PROOF. Insert $v = y(t)$ in

$$\frac{d}{dt} d^2(x(t), v) \leq 2F(v) - 2F(x(t)) - \lambda d^2(x(t), v)$$

and $w = x(t)$ in

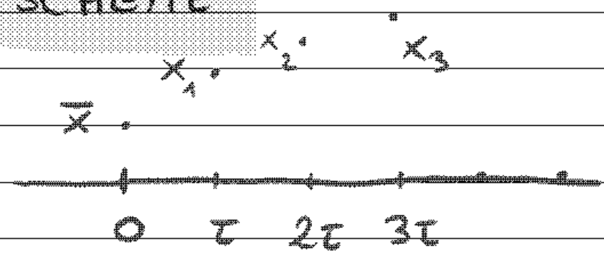
$$\frac{d}{dt} d^2(y(t), w) \leq 2F(w) - 2F(y(t)) - \lambda d^2(y(t), w)$$

to obtain the differential inequality

$$\frac{d}{dt} d^2(x(t), y(t)) \leq -2\lambda\, d^2(x(t), y(t)).$$

This argument can be made rigorous using, for instance, KRUZKHOV method of doubling of variables.

THE IMPLICIT EULER SCHEME



$\tau > 0$ time step

$X_\tau(t) := x_{n+1} \qquad t \in (n\tau, (n+1)\tau]\ , \quad n \geq 0$

$X_\tau(0) = \bar{x}$

The sequence (x_n) is built recursively: $\quad x_0 = \bar{x}$ and

$$x_{n+1} \quad \text{minimizes} \quad y \to F(y) + \frac{1}{2\tau} d^2(y, x_n)$$

THE HILBERT, CONVEX CASE

The minimality of x_{k+1} gives the discrete Euler equation

$$(*) \quad \frac{x_{k+1} - x_k}{\tau} \in -\partial F(x_{k+1}),$$

so that $x_{k+1} = (Id + \tau \partial F)^{-1}(x_k)$. In terms of the piecewise affine interpolant $\tilde{x}_\tau$, $(*)$ reads

$[s]$ = integer part

$$\tilde{x}_\tau'(t) \in -\partial F\left((Id + \tau\partial F)^{-1}\left(\tilde{x}_\tau(\tau[t/\tau])\right)\right).$$

This is the explicit time discretization scheme for the ODE

$$y'(t) = -(\partial F)_\tau (y(t)), \quad \text{where}$$

$$(\partial F)_\tau := \frac{Id - (Id + \tau \partial F)^{-1}}{\tau} = \partial F \circ (Id + \tau \partial F)^{-1}.$$

This is the (ODE) used in the classical existence proofs by approximation: $(\partial F)_\tau \in lip(H, H)$, with $lip((\partial F)_\tau) \leq 2/\tau$.

In order to show that x_τ converges as $\tau \to 0$ to a continuous solution of (GF) we have to read (EDI) and (EVI) inside the Euler scheme.

HOW (EDI) CAN BE READ IN THE EULER SCHEME

DEFINITION (SLOPE) Let $F : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $x \in D(F)$.

We set
$$|\partial F|(x) = \limsup_{y \to x} \frac{[F(x) - F(y)]^+}{d(x,y)}$$

Equivalently, $|\partial F|(x)$ is the smallest $C \geq 0$ satisfying

$$F(y) \geq F(x) - C\, d(x,y) + o(d(x,y)).$$

With this characterization, a simple application of Hahn-Banach theorem gives

$$|\partial F|(x) = |\nabla F|(x) \quad , \text{ for } E = H, \ F \text{ convex, l.s.c.}$$

Using the concept of slope we can give a meaning also to absolutely continuous maps $x : [0,+\infty) \longrightarrow E$:

$$\frac{1}{2}\int_0^t |x'(r)|^2 \, dr + \frac{1}{2}\int_0^t |\partial F(x(r))|^2 \, dr \leq F(\bar{x}) - F(x(t)), \quad t>0$$

metric derivative

LEMMA 3. (FIRST DISCRETE EULER EQUATION) let $y \in E$ such that

$$F(y) + \frac{1}{2\tau} d^2(y,x) = \min_{v} F(v) + \frac{1}{2\tau} d^2(v,x).$$

Then $|\partial F|(y) \le d(x,y)/\tau$.

<u>PROOF.</u> $F(y) - F(\tilde{y}) \le \frac{1}{2\tau}\left\{ d^2(\tilde{y},x) - d^2(y,x) \right\} \le \frac{d(y,\tilde{y})}{2\tau}\left(d(y,x) + d(\tilde{y},x) \right)$

$$|\partial F(y)| \le \lim_{\tilde{y} \to y} \frac{1}{2\tau}\left(d(\tilde{y},x) + d(y,x) \right) = d(x,y)/\tau \quad \blacksquare$$

Now we can interpolate between x and y in a VARIATIONAL way, as follows.

For $\sigma \in (0, \tau)$ we choose y_σ among the minimizers of $\tilde{y} \longrightarrow F(\tilde{y}) + \frac{1}{2\sigma} d^2(\tilde{y}, x)$.

LEMMA 4. $g(\sigma) := F(y_\sigma) + \frac{1}{2\sigma} d^2(y_\sigma, x) \in \mathrm{lip}_{loc}((0, \tau])$ and

$$g'(\sigma) = -\frac{1}{\sigma^2} d^2(y_\sigma, x)$$

Since $g(0_+) = F(x)$, by integration from 0 to τ the lemma gives

$$F(x) - F(y) = \frac{d^2(x, y)}{2\tau} + \int_0^\tau d^2(y_\sigma, x)/2\sigma^2 \, d\sigma$$

$$\geq \tau \frac{d^2(x,y)}{2\tau^2} + \frac{1}{2}\int_0^\tau |\partial F|^2 (y_\sigma)\, d\sigma$$

$$\sim \frac{\tau}{2}|x'|^2 \qquad \sim \tau/2 \, |\partial F|^2$$

Adding all inequalities with $(x, y) = (x_k, x_{k+1})$ one obtains an approximate version of (EDI) for $\hat{X}_\tau$ (the variational interpolation) that provides, in the limit as $\tau \to 0$, solutions to (EDI).

PROOF OF LEMMA 4

$$g(\sigma+h) - g(\sigma) \leq \left\{ \left(F(y_\sigma) + \frac{1}{2(\sigma+h)} d^2(y_\sigma, x) \right) - \left(F(y_\sigma) + \frac{1}{2\sigma} d^2(y_\sigma, x) \right) \right\}$$

$$= -\frac{h}{2\sigma^2} d^2(y_\sigma, x) + o(h)$$

At any differentiability point σ, we get $g'(\sigma) = -\frac{1}{2\sigma^2} d^2(y_\sigma, x)$.

HOW (EVI) CAN BE READ IN THE EULER SCHEME

CONVEXITY IN LENGTH METRIC SPACES. $F : E \to \mathbb{R} \cup \{+\infty\}$ is said to be λ-convex if $t \to F(\gamma(t))$ is $\lambda d^2(x,y)$-convex along all constant speed geodesics $\gamma : [0,1] \to E$, $\gamma(0)=x$, $\gamma(1)=y$.

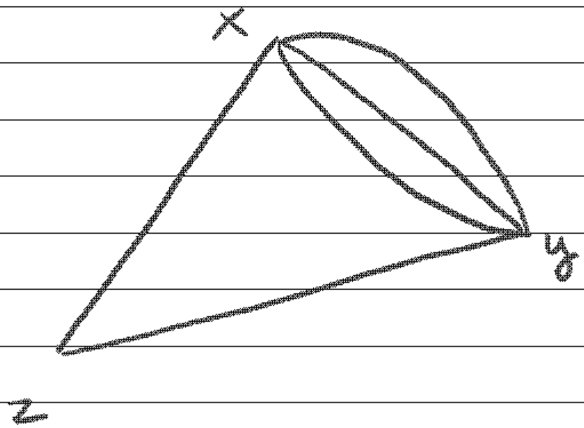
Equivalently

$$F(\gamma(t)) \leq (1-t)F(x) + t\,F(y) - \frac{1}{2}\lambda t(1-t)\,d^2(x,y)$$

NON POSITIVELY CURVED METRIC SPACES. A length space (E,d) is said to be NPC if $\frac{1}{2}d^2(\cdot, z)$ is 1-convex for all $z \in E$.

The definition, due to ALEXANDROV, is motivated by the Hilbertian identity

$$\frac{1}{2}\left|(1-t)x+ty-z\right|^2 = (1-t)\,\frac{1}{2}|x-z|^2 + t\,\frac{1}{2}|y-z|^2 - \frac{1}{2}\,t(1-t)\,|y-z|^2$$



NPC $K<0$

PC $K>0$

The theory of gradient flows works well for convex functionals in PC spaces. Unfortunately, we will see that $\mathcal{P}_2(H)$ is PC !

DEFINITION. We say that F and d are COMPATIBLE if $\forall x, y, z \in E$ there exists a continuous curve $\gamma : [0,1] \to E$ with $\gamma(0) = x$, $\gamma(1) = y$ and

i) $F(\gamma(t)) \leq (1-t) F(x) + t F(y) - \frac{\lambda}{2} t(1-t) d^2(x,y)$;

ii) $\frac{1}{2} d^2(\gamma(t), z) \leq (1-t) \frac{1}{2} d^2(x,z) + t \frac{1}{2} d^2(y,z) - \frac{t(1-t)}{2} d^2(x,y)$.

Of course, if E is NPC and F is λ-convex, then F and d are compatible.

LEMMA 4. (SECOND DISCRETE EULER EQUATION) If (F, d) are compatible, and $F(y) + \frac{1}{2\tau} d^2(y,x) = \min_{z \in E} \left\{ F(z) + \frac{1}{2\tau} d^2(z,x) \right\}$, then

$$\frac{1}{2\tau}\left(d^2(y,z) - d^2(x,z)\right) \leq F(z) - F(y) \qquad \forall z \in E.$$

$$\sim \frac{d}{dt} d^2(x(t), z)$$

$y_0 = y$

$y_1 = z$

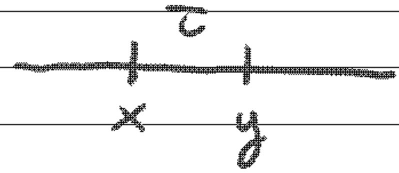
PROOF. ($\lambda = 0$) $\quad F(y) + \frac{1}{2\tau} d^2(y,x) \leq F(y_t) + \frac{1}{2\tau} d^2(y_t, x)$

(conv. of F and $d^2(\cdot, x)$) $\quad \leq (1-t)F(y) + tF(z) + \frac{(1-t)}{2\tau} d^2(y,x) + \frac{t}{2\tau} d^2(z,x)$

$$- \frac{t(1-t)}{2\tau} d^2(y,z)$$

$$F(y) - F(z) \leq \frac{1}{2\tau} d^2(x,z) - \frac{(1-t)}{2\tau} d^2(y,z) \quad . \text{ let } t \downarrow 0 \quad \blacksquare$$

Now, using the discrete (EVI) property



$$\frac{1}{\tau}\left(d^2(y,z) - d^2(x,z)\right) \leq F(z) - F(y) \quad \forall z$$

we can recover precisely the Hilbertian theory.

<u>ASSUMPTIONS</u>

- (E, d) complete metric space ;
- $F \geq 0$, l.s.c. compatible with d ;
- The discrete semigroup $S_\tau \bar{x}(t)$ exists.

not really necessary

THEOREM 5 (EXISTENCE AND UNIQUENESS) For all $\bar{x} \in \overline{D(F)}$ there exists a (unique) solution $S\bar{x}(t)$ of (EVI) starting from $\bar{x}$. If $\bar{x} \in D(F)$ it satisfies the a priori estimate

$$\sup_{t \geq 0} d\left(S_\tau \bar{x}(t), S\bar{x}(t)\right) \leq 8\sqrt{\tau}\sqrt{F(\bar{x})}.$$

THEOREM 6 (REGULARIZING EFFECTS AND POINTWISE FORMULATIONS) $x(t) = S\bar{x}(t)$

1) the right metric derivative $|x'(t_+)|$ exists $\forall t > 0$ and

$$|x'(t_+)| = |\partial F|(x(t));$$

2) $t \to F(x(t))$ is locally AC in $(0, +\infty)$, right differentiable and $$\frac{d}{dt_+} F(x(t)) = -|\partial F|^2(x(t));$$

3) $$F(x(t)) \leq \inf_v \left\{ F(v) + \frac{1}{2t} d^2(v, \bar{x}) \right\}$$

$$|\partial F|^2(x(t)) \leq \inf_v \left\{ |\partial F|^2(v) + \frac{1}{t^2} d^2(v, \bar{x}) \right\}.$$

Let us prove convergence of S_τ and the a priori error estimates when $\bar{x} \in D(F)$.

Strategy: Compare S_τ to $S_{\tau/2}$, using $(EVI)_\tau$.

LEMMA 7. For $t = n\tau$, $n \geq 1$, we have

$$d^2(S_\tau \bar{x}(t), S_{\tau/2} \bar{y}(t)) - d^2(\bar{x}, \bar{y}) \leq 2\tau F(\bar{x})$$

PROOF (Step 1) We show the inequality for $t = \tau$, precisely

$$d^2\big(S_\tau \bar{x}(\tau), S_{\tau/2}\bar{y}(\tau)\big) - d^2(\bar{x}, \bar{y}) \leq 2\tau \left\{ F(\bar{x}) - F(S_\tau \bar{x}(\tau)) \right\} \qquad (1)$$

By $(EVI)_{\tau/2}$ we get

$$d^2\big(S_{\tau/2}\bar{y}(\tau/2), z\big) - d^2(\bar{y}, z) \leq \tau \left\{ F(z) - F(S_{\tau/2}\bar{y}(\tau/2)) \right\} \qquad (2)$$

$+$

$$d^2\big(S_{\tau/2}\bar{y}(\tau), z\big) - d^2\big(S_{\tau/2}\bar{y}(\tau/2), z\big) \leq \tau \left\{ F(z) - F(S_{\tau/2}\bar{y}(\tau)) \right\} \qquad (3)$$

$$d^2\big(S_{\tau/2}\bar{y}(\tau), z\big) - d^2(\bar{y}, z) \leq 2\tau \left\{ F(z) - F(S_{\tau/2}\bar{y}(\tau)) \right\} \qquad (4)$$

$$d^2\big(S_\tau \bar{x}(\tau), z\big) - d^2(\bar{x}, z) \leq 2\tau \left\{ F(z) - F(S_\tau \bar{x}(\tau)) \right\} \qquad (5)$$

Now, set $z = \overline{x}$ in (4) and $z = S_{\tau/2}\overline{y}(\tau)$ in (5) and add, to get

$$d^2\left(S_\tau \overline{x}(\tau), S_\tau \overline{y}(\tau)\right) - d^2(\overline{x}, \overline{y}) \leq 2\tau \left\{ F(\overline{x}) - F(S_\tau \overline{x}(\tau)) \right\}. \quad (6)$$

<u>Step 2</u> From (6) we get

$$d^2\left(S_\tau \overline{x}(2\tau), S_\tau \overline{y}(2\tau)\right) - d^2\left(S_\tau \overline{x}(\tau), S_\tau \overline{y}(\tau)\right) \leq 2\tau \left\{ F(S_\tau \overline{x}(\tau)) - F(S_\tau \overline{x}(2\tau)) \right\}$$

Add, to get

$$d^2\left(S_\tau \overline{x}(2\tau), S_\tau \overline{y}(2\tau)\right) - d^2(\overline{x}, \overline{y}) \leq 2\tau \left\{ F(\overline{x}) - F(S_\tau \overline{x}(2\tau)) \right\}.$$

. . . . $\blacksquare$

The lemma with $\bar{x} = \bar{y}$ gives

$$d\left(S_{\tau/2^i} \bar{x}(t), S_{\tau/2^{i+1}} \bar{x}(t) \right) \leq 2^{(1-i)/2} \sqrt{\tau} \sqrt{F(\bar{x})}$$

Therefore $S_{\tau/2^i} \longrightarrow S^\tau$ as $i \to \infty$, with

$$d(S_\tau \bar{x}(t), S^\tau(t)) \leq \sum_{i \geq 0} 2^{(1-i)/2} \sqrt{\tau} \sqrt{F(\bar{x})}.$$

We conclude showing that S^τ solves (EVI):

Indeed, we can read $(EVI)_\tau$ in the sense of distributions as

$$\frac{d}{dt}\frac{1}{2}d^2\left(S_\tau \bar{x}(t), z\right) \leq \sum_{n=0}^{\infty} \tau\Big(F(z) - F\big(S_\tau \bar{x}(\underbrace{(n+1)\tau}_{=\,\tau[t/\tau]+\tau})\big)\Big)\,\delta_{n\tau}$$

Since $\displaystyle\limsup_{\tau\downarrow 0} -F\big(S_\tau \bar{x}(\tau+\tau[t/\tau])\big) \leq -F\big(S^\tau \bar{x}(t)\big)$

we get

$$\frac{d}{dt}\,\frac{1}{2}\,d^2\left(S^\tau \bar{x}(t), z\right) \leq \big(F(z) - F(S^\tau x(t))\big)\,\mathcal{L}^1 . \qquad \blacksquare$$

Proof of regularizing effects.

Integrate from 0 to t (EVI) and use monotonicity of $t \to F(x(t))$ to get

$$\frac{1}{2}\left(d^2(x(t), v) - d^2(\bar{x}, v)\right) \leq \int_0^t F(v) - F(x(s))\, ds$$

$$\leq t\left(F(v) - F(x(t))\right), \quad \text{i.e.}$$

$$F(x(t)) \leq F(v) + \frac{1}{2t} d^2(v, \bar{x})$$

To prove the regularization of $|\partial F|(x(t))$ we use the slope estimate

$$\frac{F(u)-F(v)}{d(u,v)} \leq |\partial F(u)|$$

and

$$\lim_{t\to 0} t\,F(x(t)) = 0 :$$

$$\frac{t^2}{2}\,|\partial F|^2(x(t)) \leq \int_0^t s\,|\partial F|^2(x(s))\,ds$$

$$= -\int_0^t s\,\big(F(x(s))\big)'\,ds$$

$$= \int_0^t F(x(s))\,ds - t\,F(x(t))$$

(EVI) $$\leq t\,F(v) + \frac{1}{2}\left(d^2(\bar{x}, v) - d^2(x(t), v)\right) - t\,F(x(t))$$

$$\leq t\,|\partial F(v)|\,d(v, x(t)) + \frac{1}{2} d^2(\bar{x}, v) - \frac{1}{2} d^2(x(t), v)$$

$$\leq \frac{t^2}{2} |\partial F(v)|^2 + \frac{1}{2} d^2(\bar{x}, v)$$

$$|\partial F|^2(x(t)) \leq |\partial F(v)|^2 + \frac{1}{t^2} d^2(v, \bar{x}).$$

STABILITY OF (EVI):

1) With respect to $\bar{x}$;

2) With respect to F

(SEQUENTIAL) Γ-CONVERGENCE

$F_h, F : X \longrightarrow \overline{\mathbb{R}}$

(X, σ)

$$\Gamma(\sigma)-\limsup_{h\to\infty} F_h(x) = \inf\left\{ \limsup_{h\to\infty} F_h(x_h) \;\middle|\; x_h \xrightarrow{\sigma} x \right\}$$

$$\Gamma(\sigma)-\liminf_{h\to\infty} F_h(x) = \inf\left\{ \liminf_{h\to\infty} F_h(x_h) \;\middle|\; x_h \xrightarrow{\sigma} x \right\}$$

$$F = \Gamma(\sigma)\text{-}\lim_{h\to\infty} F_h$$ if the two Γ_σ-limits are equal, i.e.

$$\Gamma_\sigma\text{-}\liminf_h F_h \geq F \geq \Gamma_\sigma\text{-}\limsup_h F_h$$

Equivalently

$$\forall\ x_h \xrightarrow{\sigma} x \qquad \liminf_{h\to\infty} F_h(x_h) \geq F(x) \qquad (1)$$

$$\forall \varepsilon>0\ \exists\ x_h \xrightarrow{\sigma} x \qquad \limsup_{h\to\infty} F_h(x_h) \leq F(x)+\varepsilon. \qquad (2)$$

It is often useful to consider topologies in (1) and (2), with the first weaker than the second.

THEOREM 8 (STABILITY) (E, d) metric, $F_h, F : E \to \overline{\mathbb{R}}$.

1) (F_h, d), $h \in \mathbb{N}$ and (F, d) are compatible;

2) $\Gamma(d)-\limsup_{h\to\infty} F_h \leq F$;

3) $\Gamma(\sigma)-\liminf_{h\to\infty} F_h \geq F$ for some topology σ for which d is sequentially lower semicontinuous

4) F_h are equi-coercive in bounded sets of E for the topology σ.

Then, if $\overline{x}_h \in D(F_h)$ satisfy $d(\overline{x}_h, \overline{x}) \to 0$, $F_h(\overline{x}_h) \to F(\overline{x}) \in \mathbb{R}$, the corresponding (EVI) solutions converge locally uniformly in $[0, +\infty)$.

<u>SKETCH OF PROOF</u> By the universal error estimate

$$d\left(S\bar{x}(t),\ S_\tau\bar{x}(t)\right) \leq 8\sqrt{\tau F(\bar{x})}$$

we need only to show pointwise convergence of the discrete semigroups, i.e.

$x_h \longrightarrow x$, y_h minimizes $\quad z \longrightarrow F_h(z) + \dfrac{1}{2\tau} d^2(z, x_h)$

$\Longrightarrow \quad y_h \longrightarrow y$, where y minimizes

$$z \longrightarrow F(z) + \frac{1}{2\tau} d^2(z, x)$$

The assumptions we made on F_h imply, whenever $x_h \to x$,

$$\Gamma(\sigma) - \lim_{h \to \infty} F_h(\cdot) + \frac{1}{2\tau} d^2(\cdot, x_h) = F(\cdot) + \frac{1}{2\tau} d^2(\cdot, x)$$

and therefore convergence of minimizers to minimizers.