# INTRODUCTION

Titolo nota 07/04/2008

EVOLUTION PROBLEMS (OF GRADIENT TYPE) IN SPACES OF PROBABILITY MEASURES

**Part I :** An introduction to optimal transport theory ( optimal maps, geodesics, "differentiable" structure of $\mathcal{P}_2(\mathbb{R}^n)$)

**Part II :** Gradient flows ( Implicit Euler scheme, weak solutions based on energy dissipation rate (EDI), and evolution variational inequalities (EVI), error estimates, stability)

## PART III : Applications to evolution problems in $\mathcal{P}_2(\mathbb{R}^n)$

$$\frac{d}{dt}\rho_t + \operatorname{div}(v_t \rho_t) = 0 , \quad \text{with} \quad v_t = -\nabla^W \mathcal{E}(\rho_t)$$

$$\mu_t = \rho_t \mathcal{L}^n \qquad\qquad \mathcal{E} : L^1(\mathbb{R}^n) \longrightarrow \mathbb{R} \cup \{+\infty\}$$

**BASIC EXAMPLE** FOKKER-PLANCK EQUATION

$$\frac{d}{dt}\rho_t = \operatorname{div}(\nabla \rho_t + \rho_t \nabla V)$$

$$\mathcal{E}(\rho) = \int (\rho \ln \rho + \rho V)\, dx \qquad \left(\text{relative entropy w.r.t. } e^{-V}\mathcal{L}^n\right)$$

$$\nabla^W \mathcal{E}(\rho) = \left(\frac{\nabla \rho}{\rho} + \nabla V\right)$$

$$\frac{d}{dt}\rho_t = \mathrm{div}\left( \nabla \rho_t + \rho_t \nabla V + \rho_t \nabla (W * \rho_t) \right)$$

$$\mathcal{E}(\rho) = \int (\rho \ln \rho + \rho V)\, dx + \int W(x-y)\, \rho(x)\, \rho(y)\, dx\, dy$$

$+$ (Non linear variants....)

ADVANTAGES

- "Geometric" description of the evolution problem with no reference measure
- Strong stability properties

($V, W$ singular, $\mathbb{R}^n \to H$)

$$\begin{array}{ccc} & \mathbb{R}^n & \\ \swarrow & \downarrow & \\ \text{Manifolds} & \text{PC spaces} & \end{array}$$

**LIMITATIONS** Uniqueness and stability depend on convexity assumptions and on an Hilbertian (RIEMANNIAN) structure

**EXAMPLE** $B$ Banach space, with uniformly convex norm $\|\cdot\|$

$F : B \to \mathbb{R}$ $\langle dF(x), v \rangle = \|v\|_B^2 = \|dF(x)\|_{B'}^2$

differentiable, convex, lip. $v := \nabla F(x)$

$$\begin{cases} u' = -\nabla F(u) \\ u(0) = x \end{cases}$$

Existence is known, not uniqueness!

EVANS PDE and MK mass transfer problems, 1999

VILLANI AMS '03, SANTA FLOEUR lectures '08

A-GIGLI-SAVARÉ BIRKHÄUSER '05

- Nonlinear interpolation of measures (McCANN 1997)
- Time-dependent optimal transport problem (BENAMOU-BRENIER '00)
- Differentiable and "Riemannian" structure of $\mathcal{P}_2(\mathbb{R}^n)$ (OTTO 1998-00)