1 THE OPTIMAL TRANSPORT PROBLEM

- X, Y complete and separable metric spaces
- $c : X \times Y \longrightarrow [0, +\infty]$ Borel
- $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$

(M) $$\inf \left\{ \int_X c(x, T(x))\, d\mu(x) \;\middle|\; T_\# \mu = \nu \right\}$$

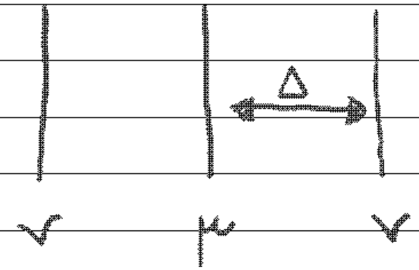
NOTATION: $T_\# : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$, $T_\# \mu(E) = \mu(T^{-1}(E))$

$$(S \circ T)_\# = S_\# \circ T_\# , \quad \int_Y \varphi \, dT_\# \mu = \int_X \varphi \circ T \, d\mu$$

Monge's problem can be ill-posed because:

1) No admissible T exists
2) The infimum is not attained
3) The constraint is not weakly seq. closed

- $\mu = \delta_x$ $\quad \nu = (\delta_y + \delta_z)/2, \quad y \neq z$

- 

$\inf(M) = \Delta$, not attained

$c(x,y) = |y-x|^p, \quad p > 0$

T Lipschitz, 1-1.

$$T_\#(\rho_1 \mathcal{L}^n) = \rho_2 \mathcal{L}^n$$

$$\Updownarrow$$

$$\rho_2(T(x))\,|\det \nabla T(x)| = \rho_1(x) \quad \text{a.e.}$$

Kantorovich's weak formulation (1941):

$$(K) \qquad \inf \left\{ \int_{X \times Y} c(x,y)\, d\gamma(x,y) \;\middle|\; \gamma \in \Gamma(\mu,\nu) \right\}$$

$$\Gamma(\mu,\nu) = \left\{ \gamma \in \mathcal{P}(X \times Y) \;\middle|\; (\pi_X)_\# \gamma = \mu,\ (\pi_Y)_\# \gamma = \nu \right\}$$

Equivalently: $\gamma \in \Gamma(\mu,\nu) \iff \mu(A) = \gamma(A \times Y),\ \nu(B) = \gamma(X \times B)$

$\gamma(A \times B)$ = "The mass in A sent at B"

$\mu \times \nu \in \Gamma(\mu,\nu)$!!! In addition, $\Gamma(\mu,\nu)$ is seq. weakly closed in $\mathcal{P}(X \times Y)$ (w.r.t. duality with $C_b(X \times Y)$)

THEOREM 1. (EXISTENCE OF OPTIMAL PLANS) If c is lower semicontinuous, then (K) has a solution. Moreover, if $c < \infty$, then

$$\inf(M) = \min(K)$$

Idea of the proof: $\gamma \longrightarrow \int_{X \times Y} c \, d\gamma$ is weakly l.s.c.

In addition, by ULAM theorem, $\exists\, C_n, K_n$ compact such that

$$\mu\Big(X \setminus \bigcup_n C_n\Big) = 0 \quad \text{and} \quad \nu\Big(Y \setminus \bigcup_n K_n\Big) = 0.$$

It turns out that

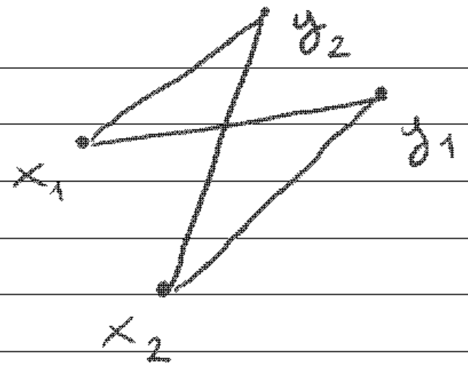
$$\gamma(X \times Y \setminus C_m \times K_m) \leq \mu(X \setminus C_m) + \nu(Y \setminus K_m) \quad \forall \gamma \in \Gamma(\mu, \nu),$$

so that $\Gamma(\mu, \nu)$ is a tight family in $\mathcal{P}(X \times Y)$ and PROKHOROV theorem gives that $\Gamma(\mu, \nu)$ is weakly relatively compact. ∎

Can we recover an optimal T from an optimal γ ??

<u>**Def**</u> (C-MONOTONICITY) $\Gamma \subset X \times Y$ is said to be c-monotone if

$$\sum_{i=1}^{N} c(x_i, y_{\sigma(i)}) \geq \sum_{i=1}^{N} c(x_i, y_i) \quad \forall \underset{1 \leq i \leq N}{(x_i, y_i) \in \Gamma}, \ \sigma \in \Sigma_N$$



THEOREM 2. c l.s.c., γ optimal, $\int c\,d\gamma < \infty$.
Then γ is concentrated on a c-monotone and σ-compact set $\Gamma \subset X \times Y$. The converse holds if

$$(*) \qquad c(x,y) \leq a(x) + b(y), \quad a \in L^1(\mu), \quad b \in L^1(\nu).$$

REMARK Condition (*) is satisfied in many natural examples, but not all: for instance the OT problem in the WIENER space does not fit into this framework.

Theorem 2 leads in a natural way to the analysis of the properties of c-monotone sets, trying to understand how far are they from being graphs.

QUADRATIC COST FUNCTION:

$$X = Y = H, \qquad c(x,y) = \frac{1}{2}|x-y|^2$$

$$\sum_{i=1}^{N} |x_i - y_i|^2 \leq \sum_{i=1}^{N} |x_{\sigma(i)} - y_i|^2 \quad \text{implies}$$

$$\sum_{i=1}^{N} \langle y_i, x_{\sigma(i)} - x_i \rangle \leq 0$$

SUBDIFFERENTIAL of a convex l.s.c. function $F : H \longrightarrow \mathbb{R} \cup \{+\infty\}$

$$\partial F(x) := \{ v \in H \mid F(y) \geq F(x) + \langle y - x, v \rangle \ \forall y \in H \}.$$

Here $x \in D(F) := \{ F < \infty \}$.

THEOREM 3 (ROCKAFELLAR) $\Gamma \subset H \times H$ is contained in the graph of ∂F, for some convex lsc F if, and only if, Γ is monotone

Proof of the "only if" implication: add the inequalities

$$\langle y_i, x_{\sigma(i)} - x_i \rangle \leq F(x_{\sigma(i)}) - F(x_i) \quad , \text{ if } \quad y_i \in \partial F(x_i)$$

to obtain
$$\sum_i \langle y_i, x_{\sigma(i)} - x_i \rangle \leq 0 .$$

Combining theorem 2 and theorem 3, we know that the support of any optimal γ is contained in $\{(x,y) \mid y \in \partial F(x)\}$ for some convex l.s.c. $F : H \longrightarrow \mathbb{R} \cup \{+\infty\}$.

In $H = \mathbb{R}^n$, we know also that

$$\Sigma := \{ x \in \mathbb{R}^n \mid \partial F(x) \text{ contains more than } 1 \text{ point} \}$$

is Lebesgue negligible, and even $(n-1)$-dimensional.

THEOREM 4. $X = Y = \mathbb{R}^n$, $\mu(B) = 0$ $\forall$ $(n-1)$-dimensional set B. Then any optimal plan γ is induced by a transport T, that is the gradient of a convex function F. In addition T is unique.

<u>Proof</u> If T_1, T_2 are optimal and $\mu(\{T_1 \neq T_2\}) > 0$, then

$$\gamma = \frac{1}{2}(\gamma_1 + \gamma_2), \quad \text{with} \quad \gamma_i = (Id \times T_i)_{\#}\mu, \quad i = 0, 1$$

is still optimal, and not induced by a transport. ∎

EXTENSIONS :
- MANIFOLDS (Mc CANN, FATHI-FIGALLI)
- CARNOT GROUPS (A-RIGOT, AGRACHEV-LEE, RIFFORD-FIGALLI)
- WIENER SPACES (FEYEL - USTUNEL)

E Hilbert space, γ Gaussian measure in E

$H = CM(\gamma) = \{ h \in E \mid (\tau_h)_\# \gamma \ll \gamma \}$, a dense subspace of E, with a stronger Hilbert norm $|\cdot|_H$

$$c(x,y) = \begin{cases} \frac{1}{2}|x-y|_H^2 & \text{if } x-y \in H \\ +\infty & \text{otherwise} \end{cases} \qquad \left(\gamma(H) = 0 \, !! \right)$$

DISTANCE, INTERPOLATION AND GEODESICS

(X, d) metric space

$$\mathcal{P}_2(X) = \left\{ \mu \in \mathcal{P}(X) \;\middle|\; \int d^2(x_0, x)\, d\mu(x) < \infty \text{ for some } x_0 \right\}$$

$$W_2^2(\mu, \nu) = \min(K), \quad \text{with } c(x,y) := \frac{1}{2} d^2(x,y)$$

W_2 is a distance: a "formal" proof using transport maps is:

$$2\, W_2^2(\mu, \nu) = \int d^2(T(x), x)\, d\mu(x) \qquad T_\# \mu = \nu$$

$$2\, W_2^2(\nu, \sigma) = \int d^2(S(y), y)\, d\nu(y) \qquad S_\# \nu = \sigma$$

$$2 W_2(\sigma, \mu) \leq \| d(S \circ T, Id) \|_{L^2(\mu)}$$

$$\leq \| d(S \circ T, S) \|_{L^2(\mu)} + \| d(S, Id) \|_{L^2(\mu)}$$

$$= 2 W_2(\mu, \nu) + 2 W_2(\nu, \sigma)$$

In general, either use "inf(M) = min(K)", or introduce the concept of "composition of two plans"

$(\mathcal{P}_2(X), W_2)$ inherits many properties of X:

- complete if X is complete;
- compact if X is compact;

- a length space if X is a length space
- a PC (positively curved) space if X is a PC space.

THEOREM 5. (CONVERGENCE IN $\mathcal{P}_2(X)$) (X, d) complete.

Then $\mu_n \longrightarrow \mu$ in $\mathcal{P}_2(X)$ iff

$$\mu_n \xrightarrow[C_b]{} \mu \quad \text{and} \quad \lim_{n\to\infty} \int d^2(x, x_0)\, d\mu_n(x) = \int d^2(x, x_0)\, d\mu(x).$$

Let us prove that $\mu_n \xrightarrow{W_2} \mu \implies \mu_n \xrightarrow[C_b]{} \mu$

By a monotone approximation argument, it suffices to consider a test function $\varphi \in \mathrm{lip}_b(X)$. Then, with $\gamma \in \Gamma_o(\mu, \mu_n)$, we have:

$$\int \varphi \, d\mu_n = \iint \varphi(y) \, d\gamma_n(x,y)$$

$$= \iint \varphi(x) \, d\gamma_n(x,y) + o(1) = \int \varphi \, d\mu + o(1)$$

because

$$\iint d(x,y) \, d\gamma_n \longrightarrow 0 .$$

THEOREM 6 (GEODESICS IN $\mathcal{P}_2(H)$)

If $\gamma \in \Gamma_0(\mu, \nu)$, then

$$\mu_t := \left((1-t)x + ty\right)_\# \gamma \qquad t \in [0,1]$$

is a constant speed geodesic from μ to ν. Conversely, any constant speed geodesic has this representation for some (unique) γ.

<u>Proof</u> We have to show that $W_2(\mu_s, \mu_t) = (t-s) W_2(\mu, \nu)$

By the triangle inequality, it suffices to show that:

$$W_2(\mu_s, \mu_t) \le (t-s)\, W_2(\mu, \nu), \qquad 0 \le s \le t \le 1.$$

Indeed, $\gamma_{s,t} := \big(\underbrace{(1-s)x + sy}_{z}, \underbrace{(1-t)x + ty}_{w}\big)_{\#} \gamma \in \Gamma(\mu_s, \mu_t)$

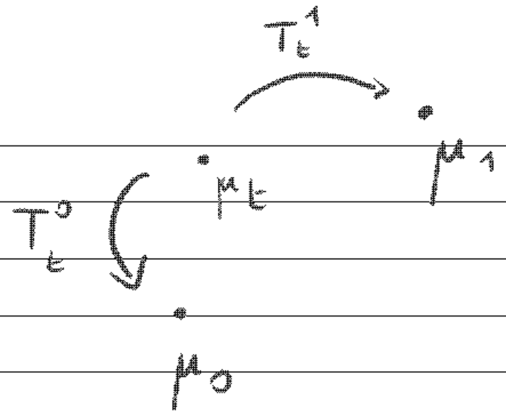
and

$$\int |z-w|^2 \, d\gamma_{st}(z,w) = (t-s)^2 \int |x-y|^2 d\gamma = (t-s)^2 W_2^2(\mu,\nu).$$

$\blacksquare$

REMARK. The proof of the converse implication gives much more information: $\Gamma_0(\mu_t, \mu_0)$ and $\Gamma_0(\mu_t, \mu_1)$, $t \in (0,1)$, are singletons, and induced by maps T_t^0, T_t^1 with $\mathrm{Lip}(T_t^0) \le 1/(1-t)$

$\mathrm{Lip}(T_t^1) \le 1/t$



In the case $\mu_0 << \mathcal{L}^n$, $H = \mathbb{R}^n$, we have

$$\mu_t := \left((1-t)Id + tT\right)_\# \mu_0 \qquad (\text{McCANN})$$

$$T_t^0 = \left((1-t)Id + tT\right)^{-1}$$

The estimate on the lipschitz constant of T_t^0 follows by the monotonicity of $(1-t)Id + tT$, larger than $(1-t)Id$:

$$\langle ((1-t)x + tT(x)) - ((1-t)x' + tT(x')), x - x' \rangle \geq (1-t)|x-x'|^2$$

THEOREM 7. (BEHAVIOUR OF $\mathcal{P}_2^a(\mathbb{R}^n)$ under interpolation)

Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^n)$, $\mu \ll \mathcal{L}^n$. Then $\mu_t \ll \mathcal{L}^n$ $\forall\, t \in [0,1)$ and its density ρ_t is given by:

$$\rho_t = \frac{\rho}{\det \nabla T_t} \circ \left(T_t \Big|_{\Sigma^c} \right)^{-1}$$

Here: ρ is the density of μ;

$T_t = (1-t)\, Id + t\, T$;

Σ is the singular (non differentiability) set of T

As a consequence, we obtain the behaviour of internal energy functionals under geodesic interpolation:

$$\int \mathcal{U}(\rho_t)\,dy = \int \det \nabla T_t \;\; \mathcal{U}\left(\frac{\rho}{\det \nabla T_t}\right) dx\,, \qquad t \in [0,1)$$

$$\text{“}\, y = T_t(x) \,\text{”}$$

The proof that $\mu_t << \mathcal{L}^n$ is simple: indeed,

$$\mathcal{L}^n(A) = 0 \underset{(T_t^{-1} \in \mathrm{Lip})}{\Longrightarrow} \mathcal{L}^n(T_t^{-1}(A)) = 0 \underset{(\mu << \mathcal{L}^n)}{\Longrightarrow} \mu(T_t^{-1}(A)) = 0 \Longrightarrow \mu_t(A) = 0.$$

OPTIMAL TRANSPORTATION FROM THE DIFF. VIEWPOINT

HEURISTICS. Continuity equation

$$\frac{d}{dt}\mu_t + \nabla\cdot(v_t\mu_t) = 0$$

Formally $\delta\mu + \nabla\cdot(v\mu) = 0$, so that

$$T_\mu \mathcal{P}_2(\mathbb{R}^n) \sim \{-\nabla\cdot(v\mu) \mid v \in L^2(\mu,\mathbb{R}^n)\}$$

Metric: $\langle -\nabla\cdot(v\mu), -\nabla\cdot(w\mu)\rangle_\mu := \int_{\mathbb{R}^n} \langle v,w\rangle\, d\mu$

The induced "Riemannian" distance is:

$$\widetilde{W}_2^2(\mu,\nu) := \inf\left\{ \int_0^1 \int |v_t|^2 d\mu_t \ : \quad \mu_0=\mu,\ \mu_1=\nu,\ \frac{d}{dt}\mu_t + \nabla\cdot(v_t\mu_t)=0 \right\}$$

THEOREM 8. (BENAMOU-BRENIER) $W_2 = \widetilde{W}_2$.

$\widetilde{W}_2 \leq W_2$ Let $\mu_t := (T_t)_\# \mu_t$

<u>LEMMA</u> $f \in L^2(\sigma, \mathbb{R}^n) \Rightarrow T_\#(f\sigma) \ll T_\#\sigma$ and its density h satisfies $\|h\|_{L^2(T_\#\sigma)} \leq \|f\|_{L^2(\sigma)}$

Indeed,

$$\langle g, T_\#(f\sigma) \rangle = \langle g \circ T, f\sigma \rangle \leq \|f\|_{L^2(\sigma)} \|g \circ T\|_{L^2(\sigma)}$$

$$g \in C_c(\mathbb{R}^n; \mathbb{R}^n) \qquad = \|f\|_{L^2(\sigma)} \|g\|_{L^2(T_\#\sigma)}$$

By Riesz theorem, this proves the lemma $\blacksquare$

Now, a velocity field compatible with μ_t is given by

$$v_t \mu_t = (T_t)_\# \left((T - Id)\, \mu \right)$$

By the lemma v_t is well defined and

$$\| v_t \|_{L^2(\mu_t)} \leq \| T - Id \|_{L^2(\mu)} = W_2(\mu, \nu)$$

It follows that

$$\tilde{W}_2^2(\mu, \nu) \leq \int_0^1 \| v_t \|^2_{L^2(\mu_t)} \, dt \leq W_2^2(\mu, \nu).$$

Compatibility of v_t:

$$\frac{d}{dt} \int \varphi \, d\mu_t = \frac{d}{dt} \int \varphi(T_t) \, d\mu = \int < \nabla\varphi(T_t), T - Id > d\mu$$

$$= < \nabla\varphi, (T_t)_\# ((T - Id)\mu) > = \int < \nabla\varphi, v_t > d\mu_t .$$

$\widetilde{W}_2 \geq W_2$ Let us assume that v_t is sufficiently regular to have a flow map $X : [0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying

1) $\frac{d}{dt} X(t,x) = v_t(X(t,x))$, $X(0,x) = x$

2) $X(t,\cdot)_\# \mu_0 = \mu_t$.

By 1) $|X(1,x) - x|^2 \leq \int_0^1 |v_t(X(t,x))|^2 dt$

An integration w.r.t. x gives

$W_2^2(\mu_0, \mu_1) \leq \int |X(1,x) - x|^2 d\mu(x) \leq \int_0^1 \int |v_t(X(t,x))|^2 d\mu(x) dt$

$$= \int_0^1 \int |v_t|^2 d\mu_t \, dt.$$

In the general case a regularization argument is needed.

Let $\rho_\varepsilon(x) = (2\pi\varepsilon^2)^{-n/2} \exp(-|x|^2/2\varepsilon^2)$ be the heat kernel and set

$$\mu_t^\varepsilon := \mu_t * \rho_\varepsilon$$

$$v_t^\varepsilon \mu_t^\varepsilon := (v_t \mu_t) * \rho_\varepsilon$$

This regularization preserves the continuity equation. The key fact is that the convexity of the map

$$(z, t) \longrightarrow |z|^2/t \qquad z \in \mathbb{R}^n, \quad t > 0$$

and JENSEN's inequality yield

$$\int |v_t^\varepsilon|^2 d\mu_t^\varepsilon \leq \int |v_t|^2 d\mu_t \qquad \forall\, t \in [0,1], \quad \varepsilon > 0.$$

Thus, together with the local Lipschitz condition

$$\| v_t^\varepsilon \|_{W^{1,\infty}(B_R;\, \mathbb{R}^n)} \in L^\infty(0,1) \qquad \forall\, \varepsilon, R > 0$$

can be used to show that the flow $X^\varepsilon(t,x)$ is globally defined in $[0,1]$ for μ-a.e. x, so that

$$W_2(\mu_0^\varepsilon, \mu_1^\varepsilon) \leq \int_0^1 \int |v_t^\varepsilon|^2 d\mu_t^\varepsilon \, dt \leq \int_0^1 \int |v_t|^2 d\mu_t .$$

letting $\varepsilon \downarrow 0$, the weak lower semicontinuity of

$$(\mu, \nu) \longrightarrow W_2(\mu, \nu)$$

concludes the proof ∎

Now, we are going to make Otto's picture rigorous by relating a metric concept, the notion of ABSOLUTELY CONTINUOUS curve μ_t with values in $\mathcal{P}_2(H)$, to a differential one.

DEFINITION (ABSOLUTE CONTINUITY AND METRIC DERIVATIVE)

$x : [0,1] \longrightarrow E$ is said to be absolutely continuous if

$$d(x(s), x(t)) \leq \int_s^t g(\tau)\, d\tau \qquad \forall\ 0 \leq s \leq t \leq 1$$

for some $g \in L^1(0,1)$. The minimal g with this property is

$$|x'(t)| := \lim_{h \to 0} \frac{1}{|h|} d(x(t+h), x(t)).$$

THEOREM 8. Let $\mu_t : [0,1] \longrightarrow \mathcal{P}_2(H)$ be absolutely continuous.

Then $\exists\ v_t \in L^2(\mu_t ; H)$ such that:

1) the continuity equation $\frac{d}{dt}\mu_t + \nabla\cdot(v_t\mu_t) = 0$ holds;

2) $\|v_t\|_{L^2(\mu_t)} \leq |\mu_t'|$ for a.e. t.

Conversely, if (v_t, μ_t) satisfy (1) and $\|v_t\|_{L^2(\mu_t)} \in L^1(0,1)$,

then μ_t is absolutely continuous, as a $\mathcal{P}_2(H)$-valued map, and

2') $\|v_t\|_{L^2(\mu_t)} \geq |\mu_t'|$ for a.e. t.

By 2'), the velocity field v_t is unique and we call it tangent.

The "constructive" part of Theorem 5 is based on a duality argument, while the "converse" part relies on a smoothing scheme very much similar to the one used in the proof of the BB formula.

The duality argument provides also an alternative characterization of v_t:

1) $\frac{d}{dt}\mu_t + \nabla\cdot(v_t\mu_t) = 0$, with $\|v_t\|_{L^2(\mu_t)} \in L^1(0,1)$

2) $v_t \in \overline{\{\nabla\varphi \mid \varphi \text{ smooth cylindrical}\}}^{L^2(\mu_t;H)}$

DEFINITION. $T_\mu \mathcal{P}_2(H)$

In the case $H = \mathbb{R}^n$ one can take $\varphi \in C_c^\infty(\mathbb{R}^n)$

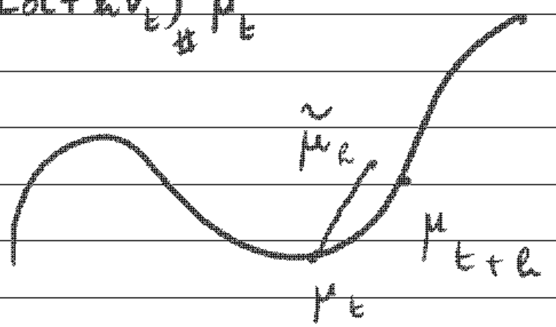
THEOREM 10 (TANGENT FIELD AS LIMIT OF RESCALED OPTIMAL PLANS)

For a.e. t, v_t satisfies: $\forall \gamma_h \in \Gamma_0(\mu_t, \mu_{t+h})$, we have

$$\tilde{\gamma}_h := \left(x, \frac{y-x}{h}\right)_\# \gamma_h \xrightarrow[\mathcal{P}_2(H\times H)]{h\to 0} (Id \times v_t)_\# \mu_t$$

Heuristically, $(Id + h v_t)$ is "almost" optimal as $h \to 0$.

$\tilde{\mu}_h = (Id + h v_t)_\# \mu_t$



$$\frac{1}{h}\left(T_{\mu_t}^{\mu_{t+h}} - Id\right) \xrightarrow[L^2(\mu_t)]{} v_t \quad \text{as } h \to 0.$$

$$W_2(\mu_{t+h}, \tilde{\mu}_h) = o(h)$$