Morita invariance in Hopf-(co)cyclic (co)homology.

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Motivations, overviews.

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In Poisson geometry

[Ginzburg, Weinstein, Lu, Xu,...]

Two Morita equivalent Poisson manifolds, they have isomorphic Poisson homology.

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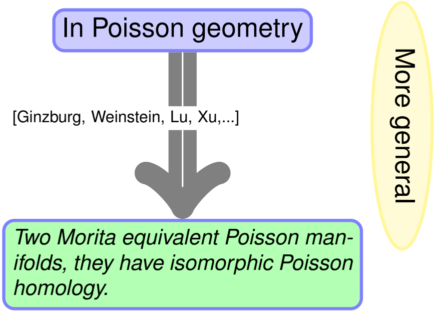

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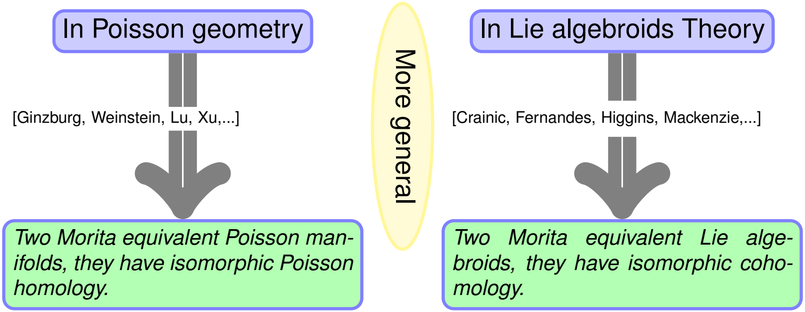

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A *Lie algebroid* is a vector bundle $\mathcal{E} \to \mathcal{M}$ over a smooth manifold, together with a map of vector bundles $\omega : \mathcal{E} \to T\mathcal{M}$ and Lie structure $[-,-]$ on the vector space $\Gamma(\mathcal{E})$ of global smooth sections of $\mathcal{E}$, such that the induced map $\Gamma(\omega) : \Gamma(\mathcal{E}) \to \Gamma(T\mathcal{M})$ is a Lie algebra map which satisfy: for all $X, Y \in \Gamma(\mathcal{E})$ and any $f \in \mathcal{C}^\infty(\mathcal{M})$ one has

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Two Lie algebroids $(\mathcal{E}_i, \mathcal{M}_i)$, $i = 1, 2$, are said to be *Morita equivalent* provided that there exist surjective submersions $\varphi_i : Q \to \mathcal{M}_i$ with <u>simply connected fibers</u> such that:

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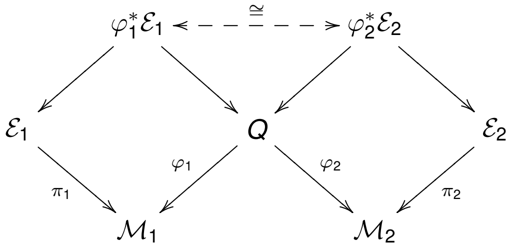

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We know that to each Lie algebroid $(\mathcal{E}, \mathcal{M})$ we can associated a *left Hopf algebroid* $(\mathcal{C}^\infty(\mathcal{M}), \mathcal{V}\Gamma(\mathcal{E}))$. In fact there is a functor

$$\text{Lie-Algd}_{\mathcal{M}} \longrightarrow \text{(left)Hopf-Algd}_{\mathcal{C}^\infty(\mathcal{M})}$$

such that the (co)homology of $(\mathcal{E}, \mathcal{M})$ coincides, up to isomorphism, with the (co)cyclic (co)homolgy of $(\mathcal{C}^\infty(\mathcal{M}), \mathcal{V}\Gamma(\mathcal{E}))$.

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Questions:

(1) Do Morita equivalent Lie algebroids have Morita equivalent associated (left) Hopf algebroids?

(2) Are two Morita equivalent (left) Hopf algebroids $(R, U) \overset{M}{\sim} (S, V)$, have isomorphic Hochschild (co)homology (co)cyclic (co)homology? i.e., *Morita invaraiance* of $HH_{\bullet}, HH^{\bullet}, HC_{\bullet}, HC^{\bullet}$?

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As we will see this approach is not far from some geometric application. To this end, we construct a Morita base change left Hopf algebroid over noncommutative 2-torus (with rational parameter) and show that its cyclic homology can be computed by means of the homology of the Lie algebroid of vector fields on the classical 2-torus.

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Based on the paper:

L. El Kaoutit and N. Kowalzig, *Morita base change in Hopf-cyclic (co)homology.* To appear in Lett. Math. Phys.

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En general, $\mathbf{Mod}_U$ need not to be monoidal, and there is no such characterisation using comodules.

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The *Hopf-Galois map* is defined by

$$\beta : U_{1\otimes R^o} \otimes_{R^o} {}_{1\otimes R^o}U \to {}_{1\otimes R^o}U \otimes_R {}_{R\otimes 1^o}U, \quad u \otimes_{R^o} v \mapsto u_{(1)} \otimes_R u_{(2)}v.$$

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We use Sweedler-type notation

$$u_+ \otimes_{R^o} u_- := \beta^{-1}(u \otimes_R 1), \quad \text{for all } u \in U,$$

for *the translation map*

$$\beta^{-1}(- \otimes_R 1) : U \to U_{1\otimes R^o} \otimes_{R^o} {}_{1\otimes R^o}U,$$

which plays the rôle of the antipode as in the classical case.

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$$t_n(m \otimes_{R^{\mathrm{o}}} x) = (m_{(0)}u^1_+) \otimes_{R^{\mathrm{o}}} u^2_+ \otimes_{R^{\mathrm{o}}} \cdots \otimes_{R^{\mathrm{o}}} u^n_+ \otimes_{R^{\mathrm{o}}} (u^n_- \cdots u^1_- m_{(-1)})$$

where $x := u^1 \otimes_{R^{\mathrm{o}}} \cdots \otimes_{R^{\mathrm{o}}} u^n$.

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with structure maps in degree n given by

$$\delta_i(z \otimes_R m) = \begin{cases} 1 \otimes_R u^1 \otimes_R \cdots \otimes_R u^n \otimes_R m, & \text{if } i = 0 \\ u^1 \otimes_R \cdots \otimes_R \Delta(u^i) \otimes_R \cdots \otimes_R u^n \otimes_R m, & \text{if } 1 \leq i \leq n \\ u^1 \otimes_R \cdots \otimes_R u^n \otimes_R m_{(-1)} \otimes_R m_{(0)}, & \text{if } i = n+1. \end{cases}$$

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where we abbreviate $z := u^1 \otimes_R \cdots \otimes_R u^n$.

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Let R and S two rings together with two bimodules ${}_SP_R$ and ${}_RQ_S$ with isomorphisms of bimodules:

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Thus (R, S, P, Q, ϕ, ψ) can be considered as a *Morita context*.

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Thus (R, S, P, Q, ϕ, ψ) can be considered as a *Morita context*.

This can be extended to a Morita context $(R^{\mathrm{e}}, S^{\mathrm{e}}, P^{\mathrm{e}}, Q^{\mathrm{e}}, \phi^{\mathrm{e}}, \psi^{\mathrm{e}})$, where $P^{\mathrm{e}} := P \otimes_{\Bbbk} Q^{\mathrm{o}}$, $Q^{\mathrm{e}} := Q \otimes_{\Bbbk} P^{\mathrm{o}}$ and $\phi^{\mathrm{e}}, \psi^{\mathrm{e}}$ are obvious.

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By Schauenburg's result, starting with a left Hopf algebroid (R, U) we can endow the S^{e}-ring

$$\tilde{U} := P^{\mathrm{e}} \otimes_{R^{\mathrm{e}}} U \otimes_{R^{\mathrm{e}}} Q^{\mathrm{e}}$$

with a structure of left S-Hopf algebroid. The pair $(S, \tilde{U})$ is called *the Morita base change (left) Hopf algebroid* of (R, U).

Morita invariance for Hopf-(co)cyclic (co)homology.

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We can construct a quasi-isomorphisms between the chain complexes $C_\bullet(U, M)$ and $C_\bullet(\tilde{U}, \tilde{M})$ (resp., between the cochain complexes $C^\bullet(U, M)$ and $C^\bullet(\tilde{U}, \tilde{M})$). The following is our main result.

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Theorem

Let (R, U) be a left Hopf algebroid, M a left U-comodule right U-module which is SaYD, and (R, S, P, Q, ϕ, ψ) a Morita context. Consider the Morita base change left S-Hopf algebroid and the image of M

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$$\tilde{U} := P^{\mathrm{e}} \otimes_{R^{\mathrm{e}}} U \otimes_{R^{\mathrm{e}}} Q^{\mathrm{e}}, \quad \tilde{M} := P \otimes_R M \otimes_R Q.$$

We then have the following natural $\mathbb{k}$-module isomorphisms:

$$HH_\bullet(U, M) \cong HH_\bullet(\tilde{U}, \tilde{M}), \qquad HH^\bullet(U, M) \cong HH^\bullet(\tilde{U}, \tilde{M})$$

$$HC_\bullet(U, M) \cong HC_\bullet(\tilde{U}, \tilde{M}), \qquad HC^\bullet(U, M) \cong HC^\bullet(\tilde{U}, \tilde{M})$$

between the Hochschild (co)homologies and (co)cyclic (co)homologies of the (co)cyclic objects $C_\bullet(U, M)$ and $C_\bullet(\tilde{U}, \tilde{M})$ (resp.,$C^\bullet(U, M)$ and $C^\bullet(\tilde{U}, \tilde{M})$) .

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Following Rinehart, the pair (R, L) is called *Lie-Rinehart algebra* with *anchor* map ω, provided that for all $X, Y \in L$ and $a, b \in R$, we have

$$(aX)(b) = a(X(b)), \qquad [X, aY] = a[X, Y] + X(a)Y$$

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Example

Here are the basic examples which will be handled, and which in fact stimulate the above general definition.

(i) The pair $(R, \mathrm{Der}_{\mathbb{k}}(R))$ admits trivially a structure of Lie-Rinehart algebra.

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(iii) A smooth manifold $\mathcal{M}$ is a *Poisson manifold* if, and only if its cotangte bundle $(T^*\mathcal{M}, \mathcal{M})$ is a Lie algebroid. Hence $(\mathcal{C}^{\infty}(\mathcal{M}), \Gamma(T^*\mathcal{M}))$ is a Lie-Rinehart algebra for a Poisson manifold $\mathcal{M}$.

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$$\Pi : R\#U(L) \longrightarrow \mathcal{V}L := \frac{R\#U(L)}{\mathcal{J}_L}, \; \mathcal{J}_L := \langle a\#X - 1\#aX \rangle_{a \in R,\, X \in L}.$$

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The usual $\Bbbk$-bialgebra structure of $U(L)$ can be lifted to a structure of R-bialgebroid on $\mathcal{V}L$, which gives in fact a structure of left Hopf R-algebroid. The comultiplication and counit are obvious, the translation map is given on generator by

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Obviously there are two morphisms $\iota_R : R \longrightarrow \mathcal{V}L, \iota_L : L \longrightarrow \mathcal{V}L$ of $\mathbb{k}$-algebras and $\mathbb{k}$-Lie algebras, respectively, which satisfy

$$\iota_R(a)\iota_L(X) \,=\, \iota_L(aX), \;\; \iota_L(X)\iota_R(a) - \iota_R(a)\iota_L(X) \,=\, \iota_R(X(a)), \, \forall\, a \in R,\, X \in L.$$

By construction, the algebra $\mathcal{V}L$ and the maps ι_R, ι_L form an universal object subject to these equations.

Application to noncommutative 2-torus.

Vector bundles versus $\sqrt{\text{Morita}}$ theories.

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`Vector bundles versus` $\sqrt{\texttt{Morita}}$ `theories.`

Assume we are given a finitely generated and projective module P_R which is faithful, in the sense that any equation of the form $Pa = 0$, for some $a \in R$, implies $a = 0$. Then it is well known that R is Morita equivalent to the endomorphisms ring $\mathrm{End}(P_R)$. This is because R is a commutative ring.

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Now assume that we are given a (complex) smooth vector bundle $\pi : \mathcal{P} \to \mathcal{M}$ of constant rank ≥ 1, then it is well know that the global smooth sections $\Gamma(\mathcal{P})$ form a finitely generated and projective $\mathcal{C}^{\infty}(\mathcal{M})$-module of constant rank over the complex valued smooth functions algebra on $\mathcal{M}$. This module is in fact always faithful (this can also follows directly from an argument on maximal ideals associated to points).

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In this way $\mathcal{C}^\infty(\mathcal{M})$ is Morita equivalent to the endomorphisms algebra $\mathrm{End}(P_{\mathcal{C}^\infty(\mathcal{M})}) \cong \Gamma(\mathrm{End}(\mathcal{P}))$, where $\mathrm{End}(\mathcal{P})$ is the complex endomorphism algebra bundle.

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Noncommutative torus revisited.

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Consider the Lie group $\mathbb{S}^1 = \{z \in \mathbb{C} \setminus \{0\} \,|\, |z| = 1\}$ as a real 1-dimensional torus by identifying it with the additive quotient $\mathbb{R}/2\pi\mathbb{Z}$. The real d-dimensional torus $\mathbb{T}^d := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is by the same way identified with $\mathbb{R}^d/2\pi\mathbb{Z}^d$.

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Following [Dubois-Violette et al 2001; Khalkhali 2009], fix an element $q \in \mathbb{S}^1$ whose argument is rational modulo 2π, and take $N \in \mathbb{N}$ to be the smallest natural number such that $q^N = 1$.

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Consider the semidirect product group $\mathcal{G} := \mathbb{Z}_N^2 \ltimes \mathbb{S}^1$ where $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, and operation

$$(m, n, \theta)\,(m', n', \theta') = (m + m', n + n', \theta\theta'\mathrm{q}^{mn'}),$$

for every pair of elements $(m, n, \theta), (m', n', \theta') \in \mathcal{G}$.

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Then there is a right action of the group $\mathcal{G}$ on the torus $\mathbb{T}^3$ given as follows:

$$(\mathrm{x}, \mathrm{y}, \mathrm{z})(m, n, \theta) = (\mathrm{q}^m\mathrm{x}, \mathrm{q}^n\mathrm{y}, \theta\mathrm{z}\mathrm{y}^m), \quad (\mathrm{x}, \mathrm{y}, \mathrm{z}) \in \mathbb{T}^3, \text{ and } (m, n, \theta) \in \mathcal{G}.$$

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Now, by extending the $\mathcal{G}$-action on $\mathbb{T}^3$ to the trivial bundle $\mathbb{T}^3 \times \mathbb{C}^N \to \mathbb{T}^3$, we can construct as follows the *associated vector bundle*.

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$$\begin{aligned} \mathcal{G} &\longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N) \\ (m, n, \theta) &\longmapsto \left[\omega \mapsto \theta U_0^n V_0^{-m} \omega\right], \end{aligned}$$

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where U_0, V_0 are the following $(N \times N)$-matrices

$$U_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad V_0 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & q & 0 & \cdots & 0 \\ 0 & 0 & q^2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & q^{N-1} \end{pmatrix},$$

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which satisfy the relations

$$U_0 V_0 = \mathrm{q} V_0 U_0, \qquad U_0^N = V_0^N = \mathbb{I}_N.$$

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$$= \Big((\mathrm{q}^m\mathrm{x},\mathrm{q}^n\mathrm{y},\theta\mathrm{z}\mathrm{y}^m);\ \theta^{-1}U_0^{-n}V_0^m\omega\Big).$$

We are now able to associate a non trivial vector bundle to the trivial bundle $\mathbb{T}^3 \times \mathbb{C}^N \to \mathbb{T}^3$. That is, we can claim that there is a morphism of vector bundles

$$\begin{array}{ccc} \mathbb{T}^3 \times \mathbb{C}^N & \longrightarrow & \mathbb{T}^3 \times_{\mathcal{G}} \mathbb{C}^N := \mathcal{E}_\mathrm{q} \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \overline{\mathbf{p}}} \\ \mathbb{T}^3 & \xrightarrow[\mathbf{p}]{} & \mathbb{T}^2 \end{array}$$

($\mathcal{E}_\mathrm{q}$ is the orbits space of $\mathbb{T}^3 \times \mathbb{C}^N$)

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$$
\begin{aligned}
((\mathrm{x},\mathrm{y},\mathrm{z});\omega)\,(m,n,\theta) &= \Big((\mathrm{x},\mathrm{y},\mathrm{z})(m,n,\theta);\ (m,n,\theta)^{-1}\omega\Big) \\
&= \Big((\mathrm{q}^m\mathrm{x},\mathrm{q}^n\mathrm{y},\theta\mathrm{z}\mathrm{y}^m);\ \theta^{-1}U_0^{-n}V_0^m\omega\Big).
\end{aligned}
$$

We are now able to associate a non trivial vector bundle to the trivial bundle $\mathbb{T}^3 \times \mathbb{C}^N \to \mathbb{T}^3$. That is, we can claim that there is a morphism of vector bundles

$$
\begin{array}{ccc}
\mathbb{T}^3 \times \mathbb{C}^N & \longrightarrow & \mathbb{T}^3 \times_{\mathcal{G}} \mathbb{C}^N := \mathcal{E}_\mathrm{q} \\
{\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \bar{\mathbf{p}}} \\
\mathbb{T}^3 & \underset{\mathbf{p}}{\longrightarrow} & \mathbb{T}^2
\end{array}
$$

($\mathcal{E}_\mathrm{q}$ is the orbits space of $\mathbb{T}^3 \times \mathbb{C}^N$)

As a conclusion, the algebra $\mathcal{C}^\infty(\mathbb{T}^2)$ of all smooth complex valued functions on $\mathbb{T}^2$ is Morita equivalent to the endomorphisms algebra of global smooth sections $\mathrm{End}(\Gamma(\mathcal{E}_\mathrm{q})) \cong \Gamma\big(\mathrm{End}(\mathcal{E}_\mathrm{q})\big)$.

Application to noncommutative 2-torus.

Noncommutative torus revisited.

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Let $u = e^{i2\pi t}$ and $v = e^{i2\pi s}$ be the coordinate functions on the torus $\mathbb{T}^2$. There is a $\mathbb{C}$-algebra isomorphism

$$\Gamma(\mathrm{End}(\mathcal{E}_\mathrm{q})) \cong \mathcal{C}^\infty(\mathbb{T}^2_\mathrm{q}) \quad \text{sending } (uU_0) \mapsto U, \quad (vV_0) \mapsto V.$$

where the algebra $\mathcal{C}^\infty(\mathbb{T}^2_\mathrm{q})$ is the complex noncommutative 2-torus whose elements are formal power Laurent series in U, V with rapidly decreasing sequence of coefficients, subject to the relation

$$UV = \mathrm{q}VU.$$

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In conclusion we have the following Morita context

$$(\mathcal{C}^\infty(\mathbb{T}^2), \mathcal{C}^\infty(\mathbb{T}^2_{\mathrm{q}}), \Gamma(\mathcal{E}_{\mathrm{q}}), \Gamma(\mathcal{E}_{\mathrm{q}})^*)$$

Application to noncommutative 2-torus.

Corollary

Let $\mathrm{q} \in \mathbb{S}^1$ be a root of unity (with rational argument mod 2π), and consider the Morita context

$$(\mathcal{C}^\infty(\mathbb{T}^2), \mathcal{C}^\infty(\mathbb{T}^2_{\mathrm{q}}), \Gamma(\mathcal{E}_{\mathrm{q}}), \Gamma(\mathcal{E}_{\mathrm{q}})^*, \phi, \psi).$$

Consider the Lie-Rinehart algebra $\big(R = \mathcal{C}^\infty(\mathbb{T}^2), K = \mathrm{Der}_{\mathbb{C}}(\mathcal{C}^\infty(\mathbb{T}^2))\big)$ of the Lie algebroid of vector fields on the complex torus $\mathbb{T}^2$ and its associated left Hopf algebroid $(R, \mathcal{V}K)$. Let M be a right $\mathcal{V}K$-module and left $\mathcal{V}K$-comodule. We then have the following natural $\mathbb{C}$-module isomorphisms

$$\begin{aligned} H_\bullet(\mathcal{V}K, M) &\simeq H_\bullet(\widetilde{\mathcal{V}K}, \tilde{M}), & HC_\bullet(\mathcal{V}K, M) &\simeq HC_\bullet(\widetilde{\mathcal{V}K}, \tilde{M}), \\ H^\bullet(\mathcal{V}K, M) &\simeq H^\bullet(\widetilde{\mathcal{V}K}, \tilde{M}), & HC^\bullet(\mathcal{V}K, M) &\simeq HC^\bullet(\widetilde{\mathcal{V}K}, \tilde{M}), \end{aligned}$$

where $\widetilde{\mathcal{V}K}$ is the Morita base change left Hopf algebroid over the noncommutative torus $\mathcal{C}^\infty(\mathbb{T}^2_{\mathrm{q}})$.

Application to noncommutative 2-torus.

Furthermore, assume that M be R-flat. Then we have that

$$H_\bullet(\widetilde{\mathcal{V}K}, \tilde{M}) \simeq H_\bullet(K, M), \qquad HC_\bullet(\widetilde{\mathcal{V}K}, \tilde{M}) \simeq \bigoplus_{i \geq 0} H_{\bullet - 2i}(K, M),$$
$$H^\bullet(\widetilde{\mathcal{V}K}, \tilde{M}) \simeq M \otimes_R \bigwedge\nolimits_R^\bullet K, \qquad HP^\bullet(\widetilde{\mathcal{V}K}, \tilde{M}) \simeq \bigoplus_{i \equiv \bullet \,\mathrm{mod} 2} H_i(K, M)$$

are natural $\mathbb{C}$-module isomorphisms, where $H_\bullet(K, M) := \mathrm{Tor}_\bullet^{\mathcal{V}K}(M, R)$, and where $HP^\bullet$ denotes periodic cyclic cohomology.

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