BOCSES over Small Linear Categories and Corings

Laiachi El Kaoutit

Abstract This note does not claim anything new, since the material exposed here is somehow folkloric. We provide the main steps in showing the equivalence of categories between the category of BOCSES over a small linear category and the category of corings over the associated ring with enough orthogonal idempotents.

Keywords BOCSES · Small linear categories · Monoidal categories · Corings · Unital modules

1 Introduction

The term BOCS, was used as a terminology for a certain object that have been introduced by Roiter (see for instance [20]) with the aim of systematizing the study of a wide class of *matrix problems* that often appears in representation theory. Inconspicuously, the notion of BOCS is not far from that of cotriple (dual notion of that introduced in [7]). Namely, as it was corroborated by Bautista et al. [4], BOCSES can be realized as a special kind of cotriples (or comonads) over certain functor categories. In this way, the category of BOCSES representations over a full subcategory of all vector spaces turns out to be equivalent to some full subcategory of the Kleisli category attached to the corresponding comonad (see [15] for the precise definition of this category). We refer to [4] for more details on this equivalence of categories.

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L. El Kaoutit (✉)
Departamento de Álgebra and IEMath-Granada, Universidad de Granada, Granada, Spain
e-mail: kaoutit@ugr.es
URL: http://www.ugr.es/~kaoutit/

Facultad de Educación, Econonía y Tecnología de Ceuta, Cortadura del Valle s/n, E-51001 Ceuta, Spain

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A more general equivalence between comonads and corings was provided by the author in [8]. Specifically, in [8] we established a bi-equivalence of bicategories between the bicategory of corings over rings with local units and the bicategory of comonads in (right) unital modules with continuous underlying functors (i.e., functors which are right exact and preserve direct sums). Roughly speaking, corings are comonoids in a suitable category of bimodules (either over ring with unit or without). Apparently they appeared for the first time in the literature, under this name, in Sweedler's work [21] about Jacobson–Bourbaki–Hochschild's Theorem (a kind of Galois correspondence first Theorem for division rings. Specifically, a bijection between the set of all coideals of a Sweedler's canonical coring of a division rings extension and the set of all intermediate division rings). Apart from the interest that corings generated in the late sixties and eighties [12–14, 17], in the last decades these objects proved to be crucial in the study of relative modules (representation theories over entwined structures, or distributive laws, see [5, 6]).

In this note we give the main steps to show that for a given small linear category (linear over a commutative base ring), the category of BOCSES over this category is equivalent to the category of corings over the associated ring with enough orthogonal idempotents. As a final conclusion, we can claim that both notions of BOCS (over small linear categories) and comonad (with continuous functor over unital modules), are equivalent to the notion of coring (over ring with enough orthogonal idempotents).

Notations and Conventions: We work over a commutative base ring with 1 (or $1_{\Bbbk}$) denoted by $\Bbbk$. The category of (central) $\Bbbk$-modules is denoted by $\mathrm{Mod}_{\Bbbk}$. A morphism in this category is referred to as a $\Bbbk$-*linear map*. A hom-sets category $\mathcal{C}$ is said to be *small* if its class of objects is actually a set. The notation $c \in \mathcal{C}$ stands for: c is an object of $\mathcal{C}$. The set of all morphisms from an object $c \in \mathcal{C}$ to another one $c' \in \mathcal{C}$, is denoted by $\mathrm{Hom}_{\mathcal{C}}\big(c,\ c'\big)$. The category $\mathcal{C}$ is said to be $\Bbbk$-*linear* if it is additive [16, pp. 192], each of its hom-sets is a $\Bbbk$-module and the composition law consists of $\Bbbk$-bilinear maps. For instance, the category of modules $\mathrm{Mod}_{\Bbbk}$ is a $\Bbbk$-linear category (not necessarily small). A covariant functor $F : \mathcal{C} \to \mathcal{D}$ between two $\Bbbk$-linear categories is said to be $\Bbbk$-*linear* provided that the maps $\mathrm{Hom}_{\mathcal{C}}\big(c,\ c'\big) \to \mathrm{Hom}_{\mathcal{D}}\big(F(c),\ F(c')\big)$ are $\Bbbk$-linear for every pair of objects $(c, c') \in \mathcal{C} \times \mathcal{C}$. Given two $\Bbbk$-linear functors $F, F' : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are small $\Bbbk$-linear categories, we denote by $\mathrm{Nat}(F, F')$ the set of all natural transformations from F to F'. An element α of this set is a family of morphisms $\{\alpha_c\}_{c\in\mathcal{C}}$, where $\alpha_c \in \mathrm{Hom}_{\mathcal{D}}\big(F(c),\ F'(c)\big)$ are such that the following diagrams commute

$$\begin{tikzcd} F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"'] & F'(c) \arrow[d, "F'(f)"] \\ F(c') \arrow[r, "\alpha_{c'}"] & F'(c'), \end{tikzcd}$$

for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}\big(c,\ c'\big)$. For more basic notions on categories, functors and adjunctions, we refer to the first chapters of Mac Lane's book [16].

In this paper, we shall consider rings without identity element. Nevertheless, we will consider a class of rings (or $\Bbbk$-algebras) which have enough orthogonal idempotents, in the sense of [10, 11], and that are mainly constructed from small categories. Specifically, given any small, hom-sets and $\mathbb{k}$-linear category $\mathcal{D}$, we can consider the *path algebra*, or the *Gabriel's ring, of* $\mathcal{D}$: Its underlying $\mathbb{k}$-module is the direct sum $R = \bigoplus_{x,\, x' \in \mathcal{D}} \mathrm{Hom}_{\mathcal{D}}\big(x,\ x'\big)$ of $\mathbb{k}$-modules. The multiplication of this ring is given by the composition of $\mathcal{D}$. This means that, for any two homogeneous generic elements $r \in \mathrm{Hom}_{\mathcal{D}}\big(x,\ x'\big)$, $r' \in \mathrm{Hom}_{\mathcal{D}}\big(y,\ y'\big)$, the multiplication $(1_{\mathbb{k}}r)\,.\,(1_{\mathbb{k}}r')$ is defined by the rule:

$$(1_{\mathbb{k}}r)\,.\,(1_{\mathbb{k}}r') \;=\; 1_{\mathbb{k}}(rr'),$$

the image of the composition of r and r', when they are composable, otherwise we set $(1_{\mathbb{k}}r)\,.\,(1_{\mathbb{k}}r') = 0$, see [11, pp. 346]. For any $x \in \mathcal{D}$, we denote by 1_x the image of the identity arrow of x in the $\mathbb{k}$-module R.

In general, the ring R has no unity, unless the set of objects of $\mathcal{D}$ is finite. Instead of that, it has a set of local units (for the precise definition see for instance [1–3] or [8]). Namely, the local units are given by the set of idempotents elements:

$$\Big\{1_{x_1} \dotplus \cdots \dotplus 1_{x_n} \in R \,|\; x_i \in \mathcal{D},\ i = 1, \cdots, n,\ \text{ and } n \in \mathbb{N} \setminus \{0\}\Big\}.$$

For example, if we have a discrete category $\mathcal{D}$ with X as a set of objects, that is, the only morphisms are the identities ones, then $R = \mathbb{k}^{(X)}$ is the ring defined as the direct sum of X-copies of the base ring $\mathbb{k}$.

An *unital right* R*-module* is a right R-module M such $MR = M$; *left unital modules* are similarly defined (see [11, pp. 347]). For instance, the previous ring R attached to a given small hom-set $\mathbb{k}$-linear category $\mathcal{D}$ decomposes as a direct sum of left, and also of right, unital R-modules:

$$R \;=\; \bigoplus_{x \in \mathcal{D}} R\, 1_x \;=\; \bigoplus_{x \in \mathcal{D}} 1_x\, R.$$

Following [10], a ring which satisfies these two equalities is referred to as a *ring with enough orthogonal idempotents*, whose complete set of idempotents is the set $\{1_x\}_{x \in \mathcal{D}}$. An *unital bimodule* is a bimodule which is unital on both sides.

2 The Monoidal Structure of the Category of Functors on Small Categories

Let $\mathcal{A}$ be a $\mathbb{k}$-linear small category; we shall use the letters $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}, \mathfrak{p}_1, \mathfrak{q}_2, ...$ to denote the objects of $\mathcal{A}$. The identity morphism of $\mathfrak{p} \in \mathcal{A}$ will be denoted by $1_{\mathfrak{p}}$. The opposite category of $\mathcal{A}$ is denoted as usual by $\mathcal{A}^{op}$, and its objects by $\mathfrak{p}^o, \mathfrak{q}^o, \mathfrak{r}^o, \mathfrak{s}^o, \mathfrak{p}_1^o, \mathfrak{q}_2^o, ...$ The object set of $\mathcal{A}^{op}$ is that of $\mathcal{A}$, while morphisms of $\mathcal{A}^{op}$ are the reversed ones in $\mathcal{A}$, that is, a morphism $f^o : \mathfrak{p}^o \to \mathfrak{q}^o$ in $\mathcal{A}^{op}$ stands for a morphism $f : \mathfrak{q} \to \mathfrak{p}$ in $\mathcal{A}$.

We will work with the categories of $\Bbbk$-module valued functors. These are the categories of all $\Bbbk$-linear covariant functors $\mathsf{Funct}\Big(\mathcal{A},\ \mathrm{Mod}_{\Bbbk}\Big)$ and $\mathsf{Funct}\Big(\mathcal{A}^{op},\ \mathrm{Mod}_{\Bbbk}\Big)$. Functors from the product category $\mathcal{A}^{op} \times \mathcal{A}$ to $\mathrm{Mod}_{\Bbbk}$ are also invoked, for instance, the functor: $(\mathfrak{p}^o, \mathfrak{q}) \to \mathrm{Hom}_{\mathcal{A}}\big(\mathfrak{p},\ \mathfrak{q}\big)$.

It is well known from Freyd Theorem's [18, pp. 99 and 109] that the set $\big\{\mathrm{Hom}_{\mathcal{A}}\big(\mathfrak{p},\ -\big)\big\}_{\mathfrak{p} \in \mathcal{A}}$ forms a generating set of small objects in the category $\mathsf{Funct}\Big(\mathcal{A},\ \mathrm{Mod}_{\Bbbk}\Big)$. Analogously, the set $\{\mathrm{Hom}_{\mathcal{A}^{op}}\big(-,\ \mathfrak{q}^o\big)\}_{\mathfrak{q}^o \in \mathcal{A}^{op}}$ is a generating set of small objects in the category $\mathsf{Funct}\Big(\mathcal{A}^{op},\ \mathrm{Mod}_{\Bbbk}\Big)$.

For simplicity, we shall denote by $H(-, \mathfrak{p}) := \mathrm{Hom}_{\mathcal{A}}\big(-,\ \mathfrak{p}\big)$ and by $H(\mathfrak{q}, -) := \mathrm{Hom}_{\mathcal{A}}\big(\mathfrak{q},\ -\big)$, for every pair of objects $\mathfrak{p}, \mathfrak{q} \in \mathcal{A}$. The forthcoming lemma, known in the literature as Yoneda lemma, is a well known fact in functor categories, whose proof is based up on the following observation: For every $\mathfrak{p}, \mathfrak{q} \in \mathcal{A}$ and $\alpha \in \mathrm{Nat}\Big(H(-, \mathfrak{p}),\ H(-, \mathfrak{q})\Big)$, we have that

$$\alpha_{\mathfrak{p}'}(f) = \alpha_{\mathfrak{p}}(1_{\mathfrak{p}}) \circ f, \quad \text{for every } f \in \mathrm{Hom}_{\mathcal{A}}\big(\mathfrak{p}',\ \mathfrak{p}\big). \tag{1}$$

Lemma 1 *Let $\mathfrak{p}$ and $\mathfrak{q}$ be two objects in the category $\mathcal{A}$. Then, there is a bijective map*

$$\begin{array}{ccc} \mathrm{Nat}\Big(H(-, \mathfrak{p}),\ H(-, \mathfrak{q})\Big) & \xrightarrow{\ \zeta_{\mathfrak{p},\mathfrak{q}}\ } & \mathrm{Hom}_{\mathcal{A}}\big(\mathfrak{p},\ \mathfrak{q}\big) \\ \alpha_{-} & \longmapsto & \alpha_{\mathfrak{p}}(1_{\mathfrak{p}}), \end{array}$$

which is a natural isomorphism on the category $\mathcal{A}^{op} \times \mathcal{A}$.

Proof Straightforward. □

Let $F : \mathcal{A} \to \mathrm{Mod}_{\Bbbk}$ be an object in the category $\mathsf{Funct}\Big(\mathcal{A},\ \mathrm{Mod}_{\Bbbk}\Big)$. Define the following functor

$$\begin{array}{rcl} \overline{F}:\ \mathrm{Mod}_{\Bbbk} & \longrightarrow & \mathsf{Funct}\Big(\mathcal{A}^{op},\ \mathrm{Mod}_{\Bbbk}\Big) \\ M & \longmapsto & \left[\begin{array}{c} \mathcal{A}^{op} \longrightarrow \mathrm{Mod}_{\Bbbk} \\ \mathfrak{p}^o \mapsto \mathrm{Hom}_{\mathrm{Mod}_{\Bbbk}}\big(F(\mathfrak{p}),\ M\big) \end{array}\right] \end{array}$$

Next we will check that $\overline{F}$ has a left adjoint functor, which we denote by:

$$F^* : \mathsf{Funct}\Big(\mathcal{A}^{op},\ \mathrm{Mod}_{\Bbbk}\Big) \longrightarrow \mathrm{Mod}_{\Bbbk}.$$

First we define as follows the action of F^* over the full subcategory whose objects set is given by $\Big\{H(-, \mathfrak{q})\Big\}_{\mathfrak{q} \in \mathcal{A}}$, that is, over the set of small generators. This is given by the following assignments:

- $F^*\Big(H(-,\mathfrak{q})\Big) = F(\mathfrak{q})$, for every object $\mathfrak{q} \in \mathcal{A}$;
- $F^*(\alpha_-) = F(\alpha_{\mathfrak{p}}(1_{\mathfrak{p}}))$, for every morphism $\alpha_- : H(-,\mathfrak{p}) \to H(-,\mathfrak{q})$ (i.e., natural transformation).

Using Eq. (1), one can easily show that F^* is a well defined functor over the aforementioned subcategory. Applying now Mitchell's Theorem [18, Theorem 4.5.2], one shows that F^* extends uniquely (up to natural isomorphisms) to the whole category. We denote this extension functor by F^* too, thus, we have a functor $F^* : \mathsf{Funct}\Big(\mathcal{A}^{op}, \mathrm{Mod}_{\Bbbk}\Big) \longrightarrow \mathrm{Mod}_{\Bbbk}$.

Let M be any $\Bbbk$-module and $\mathfrak{q}$ any object of $\mathcal{A}$, we consider the mutually inverse maps

$$
\begin{array}{ccc}
\mathrm{Hom}_{\Bbbk}\big(F(\mathfrak{q}), M\big) & \xrightarrow{\Phi_{H(-,\mathfrak{q}),\, M}} & \mathrm{Hom}_{\mathsf{Funct}(\mathcal{A}^{op},\mathrm{Mod}_{\Bbbk})}\Big(\mathrm{Hom}_{\mathcal{A}}(-,\mathfrak{q}),\ \mathrm{Hom}_{\Bbbk}\big(F(-), M\big)\Big) \\
f & \xrightarrow{\widehat{\ \ }} & \left[\begin{array}{c} \widehat{f}_{\mathfrak{p}} : \mathrm{Hom}_{\mathcal{A}}(\mathfrak{p},\mathfrak{q}) \longrightarrow \mathrm{Hom}_{\Bbbk}\big(F(\mathfrak{p}), M\big) \\ \gamma \longmapsto f \circ F(\gamma) \end{array}\right] \\
\sigma_{\mathfrak{q}}(1_{\mathfrak{q}}) & \xleftarrow{\ \widetilde{\ \ }\ } & \Big[\sigma_- : \mathrm{Hom}_{\mathcal{A}}(-,\mathfrak{q}) \longrightarrow \mathrm{Hom}_{\Bbbk}\big(F(-), M\big)\Big]
\end{array}
$$

By Lemma 1, $\Phi_{-,-}$ is natural over the class of objects of the form $(H(-,\mathfrak{q}),\ M) \in \mathsf{Funct}\Big(\mathcal{A}^{op}, \mathrm{Mod}_{\Bbbk}\Big) \times \mathrm{Mod}_{\Bbbk}$. Again by Mitchell's Theorem [18, Theorem 4.5.2], the natural transformation $\Phi_{-,-}$ extends to a natural isomorphism on the whole category $\mathsf{Funct}\Big(\mathcal{A}^{op}, \mathrm{Mod}_{\Bbbk}\Big) \times \mathrm{Mod}_{\Bbbk}$:

$$
\mathrm{Hom}_{\Bbbk}\big(F^*(-), -\big) \xrightarrow{\ \Phi_{-,-}\ } \mathrm{Hom}_{\mathsf{Funct}(\mathcal{A}^{op},\mathrm{Mod}_{\Bbbk})}\Big(-,\ \overline{F}(-)\Big), \tag{2}
$$

which proves the claimed adjunction.

Now, we define the following two variable correspondence (see also [9, Chap. 5, Exercise I] and [19, pp. 26]):

$$
\begin{array}{rcl}
-\underset{\mathcal{A}}{\otimes}- : \mathsf{Funct}\Big(\mathcal{A}^{op}, \mathrm{Mod}_{\Bbbk}\Big) \times \mathsf{Funct}\Big(\mathcal{A}, \mathrm{Mod}_{\Bbbk}\Big) & \longrightarrow & \mathrm{Mod}_{\Bbbk} \\
(T, F) & \longrightarrow & T \underset{\mathcal{A}}{\otimes} F = F^*(T) \\
\Big([\alpha : H(-,\mathfrak{q}_1) \to H(-,\mathfrak{q}_2)],\, [\beta : F_1 \to F_2]\Big) & \longrightarrow & \alpha \underset{\mathcal{A}}{\otimes} \beta = F_2(\alpha_{\mathfrak{q}_1}(1_{\mathfrak{q}_1})) \circ \beta_{\mathfrak{q}_1}.
\end{array}
$$

This in fact establishes a functor. Let us check, for instance, its compatibility with the componentwise composition. So take four natural transformation

$$
T_1 \xrightarrow{\alpha_1} T_2 \xrightarrow{\alpha_2} T_3, \qquad\qquad F_1 \xrightarrow{\beta_1} F_2 \xrightarrow{\beta_2} F_3,
$$

and assume that $T_i = H(-, \mathfrak{q}_i)$, $i = 1, 2, 3$. By definition we obtain the following diagram

$$\begin{tikzcd}
F_1(\mathfrak{q}_1) \arrow[rr, "\alpha_1 \otimes_{\mathcal{A}} \beta_1"] \arrow[dr, "\beta_{1\,\mathfrak{q}_1}"] \arrow[ddrr, bend right, "(\beta_2 \circ \beta_1)_{\mathfrak{q}_1}"'] & & F_2(\mathfrak{q}_2) \arrow[rr, "\alpha_2 \otimes_{\mathcal{A}} \beta_2"] \arrow[dr, "\beta_{2\,\mathfrak{q}_2}"] & & F_3(\mathfrak{q}_3) \\
& F_2(\mathfrak{q}_1) \arrow[ur, "F_2(\alpha_{1\,\mathfrak{q}_1}(1_{\mathfrak{q}_1}))"] \arrow[dr, "\beta_{2\,\mathfrak{q}_1}"] & & F_3(\mathfrak{q}_2) \arrow[ur, "F_3(\alpha_{2\,\mathfrak{q}_2}(1_{\mathfrak{q}_2}))"] & \\
& & F_3(\mathfrak{q}_1) \arrow[ur, "F_3(\alpha_{1\,\mathfrak{q}_1}(1_{\mathfrak{q}_1}))"] \arrow[uurr, bend right, "F_3((\alpha_2 \circ \alpha_1)_{\mathfrak{q}_1}(1_{\mathfrak{q}_1}))"'] & &
\end{tikzcd}$$

which commutes by the naturality of β_2 (the middle square) and the equality

$$(\alpha_2 \circ \alpha_1)_{\mathfrak{q}_1}(1_{\mathfrak{q}_1}) = \alpha_{2\,\mathfrak{q}_2}(1_{\mathfrak{q}_2}) \circ \alpha_{1\,\mathfrak{q}_1}(1_{\mathfrak{q}_1}),$$

that can be deduced from Eq. (1). The remaining verifications are left to the reader. The bi-functor $- \underset{\mathcal{A}}{\otimes} -$ is referred to as *the tensor product functor* and enjoys the following properties:

(a) For every object $F \in \mathsf{Funct}\big(\mathcal{A}, \mathrm{Mod}_{\Bbbk}\big)$, the functor $- \underset{\mathcal{A}}{\otimes} F : \mathsf{Funct}\big(\mathcal{A}^{op}, \mathrm{Mod}_{\Bbbk}\big) \to \mathrm{Mod}_{\Bbbk}$ is a right exact and direct sums preserving functor.

(b) For every object $T \in \mathsf{Funct}\big(\mathcal{A}^{op}, \mathrm{Mod}_{\Bbbk}\big)$, the functor $T \underset{\mathcal{A}}{\otimes} - : \mathsf{Funct}\big(\mathcal{A}, \mathrm{Mod}_{\Bbbk}\big) \to \mathrm{Mod}_{\Bbbk}$ is a right exact and direct sums preserving functor.

(c) For every object $\mathfrak{q} \in \mathcal{A}$, $F \in \mathsf{Funct}\big(\mathcal{A}, \mathrm{Mod}_{\Bbbk}\big)$ and $T \in \mathsf{Funct}\big(\mathcal{A}^{op}, \mathrm{Mod}_{\Bbbk}\big)$, we have

$$T \underset{\mathcal{A}}{\otimes} \mathrm{Hom}_{\mathcal{A}}\big(\mathfrak{q}, -\big) = T(\mathfrak{q}^o), \;\; \mathrm{Hom}_{\mathcal{A}}\big(-, \mathfrak{q}\big) \underset{\mathcal{A}}{\otimes} F = F(\mathfrak{q}).$$

(d) The following property is the one expressed in [4, Theorem 2.2]. Let $T \in \mathsf{Funct}\big(\mathcal{A}^{op}, \mathrm{Mod}_{\Bbbk}\big)$, $F \in \mathsf{Funct}\big(\mathcal{A}, \mathrm{Mod}_{\Bbbk}\big)$ and $M \in \mathrm{Mod}_{\Bbbk}$. Then an *$\mathcal{A}$-balanced transformation* $\nu_{-} : T(-) \otimes_{\Bbbk} F(-) \to M$ is a family of $\Bbbk$-linear maps

$$\Big\{\nu_{\mathfrak{q}} : T(\mathfrak{q}^o) \otimes_{\Bbbk} F(\mathfrak{q}) \longrightarrow M\Big\}_{\mathfrak{q} \in \mathcal{A}}$$

such that, for every morphism $f : \mathfrak{p} \to \mathfrak{q}$ in $\mathcal{A}$, the following diagram

$$\begin{tikzcd}
T(\mathfrak{q}^o) \otimes_{\Bbbk} F(\mathfrak{p}) \arrow[r, "T(\mathfrak{q}^o)\otimes_{\Bbbk} F(f)"] \arrow[d, "T(f^o)\otimes_{\Bbbk} F(\mathfrak{p})"'] & T(\mathfrak{q}^o) \otimes_{\Bbbk} F(\mathfrak{q}) \arrow[d, "\nu_{\mathfrak{q}}"] \\
T(\mathfrak{p}^o) \otimes_{\Bbbk} F(\mathfrak{p}) \arrow[r, "\nu_{\mathfrak{p}}"'] & M
\end{tikzcd}$$

commutes. There is a universal $\mathscr{A}$-balanced transformation $\varphi : T(-) \otimes_{\Bbbk} F(-) \to T \otimes_{\mathscr{A}} F$ in the sense that for any other $\mathscr{A}$-balanced transformation $\nu_{-} : T(-) \otimes_{\Bbbk} F(-) \to M$, there exists a $\Bbbk$-linear map $\overline{\nu} : T \otimes_{\mathscr{A}} F \to M$ such that $\overline{\nu} \circ \varphi_{\mathfrak{q}} = \nu_{\mathfrak{q}}$, for every object $\mathfrak{q} \in \mathscr{A}$. Furthermore, if $\xi : T_1 \to T_2$ and $\chi : F_1 \to F_2$ are two natural transformations, then $(\xi \otimes_{\mathscr{A}} \chi) \circ \varphi^1_{\mathfrak{q}} = \varphi^2_{\mathfrak{q}} \circ (\xi_{\mathfrak{q}} \otimes_{\Bbbk} \chi_{\mathfrak{q}})$, for every object $\mathfrak{q} \in \mathscr{A}$, where φ^i_{-} is the universal $\mathscr{A}$-balanced transformation associated with the pair (T_i, F_i).

Remark 1 Properties (a)–(d) listed above are indeed consequences of the universal property of *coend* (see [16, pp. 222] for the pertinent definition of this object). Namely, the tensor product $T \underset{\mathscr{A}}{\otimes} F$ of two functors $T \in \mathsf{Funct}\big(\mathscr{A}^{op}, \mathrm{Mod}_{\Bbbk}\big)$ and $F \in \mathsf{Funct}\big(\mathscr{A}^{op}, \mathrm{Mod}_{\Bbbk}\big)$, can be realized as the following coend:

$$T \underset{\mathscr{A}}{\otimes} F \;\cong\; \frac{\bigoplus_{(\mathfrak{p}^o,\, \mathfrak{p}) \in \mathscr{A}^{op} \times \mathscr{A}} T(\mathfrak{q}^o) \otimes_{\Bbbk} F(\mathfrak{p})}{\mathscr{J}_{T,F}},$$

where $\mathscr{J}_{T,F}$ is the sub $\Bbbk$-module of the numerator generated by the set

$$\Big\{ T(f^o)(u) \otimes_{\Bbbk} v - u \otimes_{\Bbbk} F(f)(v) |\; u \in T(\mathfrak{q}^o),\; v \in F(\mathfrak{p}),\; f \in \mathrm{Hom}_{\mathscr{A}}\big(\mathfrak{p},\, \mathfrak{q}\big) \Big\}_{(\mathfrak{p}^o,\, \mathfrak{p}) \in \mathscr{A}^{op} \times \mathscr{A}}.$$

The interested reader can consult [19, pp. 26, 27], for more details.

From now on, we denote by A the ring with enough orthogonal idempotents attached to the small $\Bbbk$-linear category $\mathscr{A}$. As we have mentioned above, its underlying $\Bbbk$-module is the direct sum

$$A = \bigoplus_{(\mathfrak{p},\, \mathfrak{q}) \in \mathscr{A} \times \mathscr{A}} \mathrm{Hom}_{\mathscr{A}}\big(\mathfrak{p},\, \mathfrak{q}\big)$$

with an associative multiplication induced by the composition law. The categories of right and left unital A-modules are, respectively, denoted by Mod_A and ${}_A\mathrm{Mod}$. The tensor product over this ring will be denoted by $- \otimes_A -$. Notice that any (left) unital A-module M decomposes as a direct sum of $\Bbbk$-modules of the form:

$$M = \oplus_{\mathfrak{p} \in \mathscr{A}} 1_{\mathfrak{p}} M, \quad \text{where } 1_{\mathfrak{p}} M := \Big\{ y \in M |\; y = 1_{\mathfrak{p}} z, \;\; \text{for some} \;\; z \in M \Big\}, \quad \mathfrak{p} \in \mathscr{A}. \quad (3)$$

Given an homogeneous element $a \in A$, which belongs to $\mathrm{Hom}_{\mathcal{A}}(\mathfrak{p}, \mathfrak{q})$, the left multiplication by this a leads to a $\Bbbk$-linear map $\ell_a : 1_{\mathfrak{p}}M \to 1_{\mathfrak{q}}M$, $1_{\mathfrak{p}}y \mapsto 1_{\mathfrak{q}}(ay)$.

It is well know from Gabriel's classical result [11, Proposition 2, pp. 347] that the following functors establish an equivalence of categories

$$
\begin{array}{ccc@{\qquad}ccc}
\mathsf{Funct}\big(\mathcal{A}, \mathrm{Mod}_{\Bbbk}\big) & \xrightarrow{\ \mathscr{L}\ } & {}_{A}\mathrm{Mod}, & \mathsf{Funct}\big(\mathcal{A}^{op}, \mathrm{Mod}_{\Bbbk}\big) & \xrightarrow{\ \mathscr{R}\ } & \mathrm{Mod}_{A} \\
F & \longrightarrow & \bigoplus_{\mathfrak{p} \in \mathcal{A}} F(\mathfrak{p}) & T & \longrightarrow & \bigoplus_{\mathfrak{p} \in \mathcal{A}} T(\mathfrak{p}^{o}) \\
\theta & \longrightarrow & \bigoplus_{\mathfrak{p} \in \mathcal{A}} \theta_{\mathfrak{p}}, & \vartheta & \longrightarrow & \bigoplus_{\mathfrak{p} \in \mathcal{A}} \vartheta_{\mathfrak{p}^{o}}
\end{array}
\tag{4}
$$

These equivalences of categories can also be checked directly by hand. For instance, using the previous notations, one can easily check that the following functor

$$
\begin{array}{ccc}
{}_{A}\mathrm{Mod} & \xrightarrow{\ \mathscr{P}\ } & \mathsf{Funct}\big(\mathcal{A}, \mathrm{Mod}_{\Bbbk}\big) \\
M & \longrightarrow & \Big[\mathfrak{p} \longrightarrow 1_{\mathfrak{p}}M, \quad a \longrightarrow \ell_a\Big],
\end{array}
$$

whose action on morphism is the obvious one, is an inverse of $\mathscr{L}$. Using the functors displayed in Eq. (4), we establish the subsequent useful lemma:

Lemma 2 *Let $\mathcal{A}$ be a small $\Bbbk$-linear category and A it associated ring with enough orthogonal idempotents. Then the following diagram is commutative up to natural isomorphism*

$$
\begin{tikzcd}
\mathsf{Funct}\big(\mathcal{A}^{op}, \mathrm{Mod}_{\Bbbk}\big) \times \mathsf{Funct}\big(\mathcal{A}, \mathrm{Mod}_{\Bbbk}\big) \arrow[d, "\mathscr{R} \times \mathscr{L}"'] \arrow[dr, "- \otimes_{\mathcal{A}} -"] & \\
& \mathrm{Mod}_{\Bbbk} \\
\mathrm{Mod}_{A} \times {}_{A}\mathrm{Mod} \arrow[ur, "- \otimes_{A} -"'] &
\end{tikzcd}
$$

Proof Let $T \in \mathsf{Funct}\big(\mathcal{A}^{op}, \mathrm{Mod}_{\Bbbk}\big)$ and $F \in \mathsf{Funct}\big(\mathcal{A}, \mathrm{Mod}_{\Bbbk}\big)$. By Mitchell's Theorem [18, Theorem 4.5.2], there is no loss of generality in assuming that T is of the form $\mathrm{Hom}_{\mathcal{A}}(-, \mathfrak{q})$, for some object $\mathfrak{q} \in \mathcal{A}$. In this case, we know that $T \otimes_{\mathcal{A}} F = F^{*}(T) = F(\mathfrak{q})$ and $\mathscr{R}(T) = \oplus_{\mathfrak{p} \in \mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(\mathfrak{p}, \mathfrak{q}) = 1_{\mathfrak{q}}A$. Therefore,

$$\begin{aligned} 1_{\mathfrak{q}} A \otimes_A \mathscr{L}(F) &= 1_{\mathfrak{q}} A \otimes_A \Big(\oplus_{\mathfrak{p} \in \mathcal{A}} F(\mathfrak{p}) \Big) \\ &\cong 1_{\mathfrak{q}} \Big(\oplus_{\mathfrak{p} \in \mathcal{A}} F(\mathfrak{p}) \Big) \\ &\cong F(\mathfrak{q}) \;=\; T \otimes_{\mathcal{A}} F, \end{aligned}$$

and this finishes the proof. □

Let $\mathbb{T}$ and $\mathbb{F}$ be two objects of the category of functors $\mathsf{Funct}\Big(\mathcal{A}^{op} \times \mathcal{A},\ \mathrm{Mod}_{\Bbbk}\Big)$. For every object $\mathfrak{q} \in \mathcal{A}$, we then get two functors

$$\begin{aligned} \mathbb{T}(-, \mathfrak{q}) :\ \mathcal{A}^{op} &\longrightarrow \mathrm{Mod}_{\Bbbk} \\ \mathbb{F}(\mathfrak{q}^o, -) :\ \mathcal{A} &\longrightarrow \mathrm{Mod}_{\Bbbk} \end{aligned}$$

In this way, using the the bifunctor $- \underset{\mathcal{A}}{\otimes} -$ defined above, we can define a new object in that category, which is given as follows:

$$\begin{aligned} \mathbb{T} \otimes_{\mathcal{A}} \mathbb{F} : \mathcal{A}^{op} \times \mathcal{A} &\longrightarrow \mathrm{Mod}_{\Bbbk} \\ (\mathfrak{p}^o, \mathfrak{q}) &\longrightarrow \mathbb{T}(-, \mathfrak{q}) \underset{\mathcal{A}}{\otimes} \mathbb{F}(\mathfrak{p}^o, -) \\ (f^o, g) &\longrightarrow \mathbb{T}(-, g) \underset{\mathcal{A}}{\otimes} \mathbb{F}(f^o, -). \end{aligned}$$

The previous discussion serves to announce the following result:

Proposition 1 *Let $\mathcal{A}$ be a small $\Bbbk$-linear category. Then the category $\mathfrak{M} := \mathsf{Funct}\Big(\mathcal{A}^{op} \times \mathcal{A},\ \mathrm{Mod}_{\Bbbk}\Big)$ is a strict monoidal category with tensor product given by the bifunctor $- \otimes_{\mathcal{A}} -$ and with identity object $\mathbb{I}_{\mathfrak{M}} := \mathrm{Hom}_{\mathcal{A}}\big(-,\ -\big)$ the functor which acts on objects by sending $(\mathfrak{p}^o, \mathfrak{q}) \to \mathrm{Hom}_{\mathcal{A}}\big(\mathfrak{p},\ \mathfrak{q}\big)$ and on morphisms by sending $(f^o, g) \to \mathrm{Hom}_{\mathcal{A}}\big(f^o,\ g\big)$.*

Proof We only sketch the main steps of the proof. The left unit constraint is given as follows:

$$\big(\mathbb{I}_{\mathfrak{M}} \otimes_{\mathcal{A}} \mathbb{F}\big)(\mathfrak{p}^o, \mathfrak{q}) \;=\; \mathbb{I}_{\mathfrak{M}}(-, \mathfrak{q}) \underset{\mathcal{A}}{\otimes} \mathbb{F}(\mathfrak{p}^o, -) \;=\; H(-, \mathfrak{q}) \underset{\mathcal{A}}{\otimes} \mathbb{F}(\mathfrak{p}^o, -) \;=\; \mathbb{F}(\mathfrak{p}^o, \mathfrak{q}),$$

for every pair of objects $(\mathfrak{p}^o, \mathfrak{q}) \in \mathcal{A}^{op} \times \mathcal{A}$. As for the right unit constraint, given an object $\mathbb{T} \in \mathfrak{M}$, we know that

$$\big(\mathbb{T} \otimes_{\mathcal{A}} \mathbb{I}_{\mathfrak{M}}\big)(\mathfrak{p}^o, \mathfrak{q}) \;=\; \mathbb{T}(-, \mathfrak{q}) \underset{\mathcal{A}}{\otimes} \mathbb{I}_{\mathfrak{M}}(\mathfrak{p}^o, -),$$

for every pair of objects $(\mathfrak{p}^o, \mathfrak{q})$. Now, without loss of generality we can assume that $\mathbb{T}(-, \mathfrak{q}) = H(-, \mathfrak{p}')$ for some $\mathfrak{p}' \in \mathcal{A}$, then we get that

$$\begin{aligned} \mathbb{T}(-, \mathfrak{q}) \underset{\mathcal{A}}{\otimes} \mathbb{I}_{\mathfrak{M}}(\mathfrak{p}^o, -) &= \mathbb{I}_{\mathfrak{M}}(\mathfrak{p}^o, \mathfrak{p}') \\ &= \mathrm{Hom}_{\mathcal{A}}\big(\mathfrak{p},\ \mathfrak{p}'\big) \\ &= \mathbb{T}(\mathfrak{p}^o, \mathfrak{q}). \end{aligned}$$

This gives the desired unit constraints.

Now let us sketch the associativity constraint. Given $\mathbb{G}$ another object of $\mathfrak{M}$ and fixing a pair of objects $(\mathfrak{p}^o, \mathfrak{q}) \in \mathcal{A}^{op} \times \mathcal{A}$, we have the following equalities:

$$\begin{aligned} \Big(\Big(\mathbb{T} \otimes_{\mathcal{A}} \mathbb{F}\Big) \otimes_{\mathcal{A}} \mathbb{G}\Big)(\mathfrak{p}^o, \mathfrak{q}) &= \Big(\mathbb{T} \otimes_{\mathcal{A}} \mathbb{F}\Big)(-, \mathfrak{q}) \underset{\mathcal{A}}{\otimes} \mathbb{G}(\mathfrak{p}^o, -) \\ \Big(\mathbb{T} \otimes_{\mathcal{A}} \Big(\mathbb{F} \otimes_{\mathcal{A}} \mathbb{G}\Big)\Big)(\mathfrak{p}^o, \mathfrak{q}) &= \mathbb{T}(-, \mathfrak{q}) \underset{\mathcal{A}}{\otimes} \Big((\mathbb{F} \otimes_{\mathcal{A}} \mathbb{G})(\mathfrak{p}^o, -)\Big). \end{aligned}$$

Without loss of generality, we may assume that $\mathbb{T}(-, \mathfrak{q}) = \mathrm{Hom}_{\mathcal{A}}\big(-,\ \mathfrak{p}'\big)$, for some $\mathfrak{p}' \in \mathcal{A}$. This implies, on the one hand, that

$$\mathbb{T}(-, \mathfrak{q}) \underset{\mathcal{A}}{\otimes} \Big((\mathbb{F} \otimes_{\mathcal{A}} \mathbb{G})(\mathfrak{p}^o, -)\Big) \;=\; (\mathbb{F} \otimes_{\mathcal{A}} \mathbb{G})(\mathfrak{p}^o, \mathfrak{p}') \;=\; \mathbb{F}(-, \mathfrak{p}') \underset{\mathcal{A}}{\otimes} \mathbb{G}(\mathfrak{p}^o, -). \tag{5}$$

On the other hand, we also have

$$(\mathbb{T} \otimes_{\mathcal{A}} \mathbb{F})(\mathfrak{r}^o, \mathfrak{q}) \;=\; \mathbb{T}(-, \mathfrak{q}) \underset{\mathcal{A}}{\otimes} \mathbb{F}(\mathfrak{r}^o, -) \;=\; \mathbb{F}(\mathfrak{r}^o, \mathfrak{p}'),$$

for every object $\mathfrak{r} \in \mathcal{A}$, which implies that

$$\Big(\mathbb{T} \otimes_{\mathcal{A}} \mathbb{F}\Big)(-, \mathfrak{q}) \underset{\mathcal{A}}{\otimes} \mathbb{G}(\mathfrak{p}^o, -) \;=\; \mathbb{F}(-, \mathfrak{p}') \underset{\mathcal{A}}{\otimes} \mathbb{G}(\mathfrak{p}^o, -). \tag{6}$$

Comparing Eqs. (5) and (6), we arrive to the equality

$$\Big(\Big(\mathbb{T} \otimes_{\mathcal{A}} \mathbb{F}\Big) \otimes_{\mathcal{A}} \mathbb{G}\Big)(\mathfrak{p}^o, \mathfrak{q}) \;=\; \Big(\mathbb{T} \otimes_{\mathcal{A}} \Big(\mathbb{F} \otimes_{\mathcal{A}} \mathbb{G}\Big)\Big)(\mathfrak{p}^o, \mathfrak{q}),$$

for every pair of objects $(\mathfrak{p}^o, \mathfrak{q}) \in \mathcal{A}^{op} \times \mathcal{A}$. Therefore, we have that $\Big(\Big(\mathbb{T} \otimes_{\mathcal{A}} \mathbb{F}\Big) \otimes_{\mathcal{A}} \mathbb{G}\Big) = \Big(\mathbb{T} \otimes_{\mathcal{A}} \Big(\mathbb{F} \otimes_{\mathcal{A}} \mathbb{G}\Big)\Big)$, and this shows the associativity constraint. $\square$

The objects of the monoidal category $\mathfrak{M}$ are referred to, in the literature, as $\mathcal{A}$-*bimodules*. Here we also adopt this terminology.

3 BOCSES are Equivalent to Corings

The notation is that of the previous section. Thus $\mathcal{A}$ still denotes a small $\Bbbk$-linear category, A is its associated ring with enough orthogonal idempotents, and $\mathfrak{M}$ denotes the category of bi-functors $\mathfrak{M} := \mathsf{Funct}\Big(\mathcal{A}^{op} \times \mathcal{A},\ \mathrm{Mod}_{\Bbbk}\Big)$. We denote by ${}_A\mathrm{Mod}_A$ the category of all unital A-bimodules (i.e., bimodule which are left and right unital).

Definition 1 (*A. V. Roiter*) A *BOCS* over the category $\mathcal{A}$ is a comonoid in the monoidal category of $\mathcal{A}$-bimodules $\mathfrak{M} = \mathsf{Funct}\Big(\mathcal{A}^{op} \times \mathcal{A},\ \mathrm{Mod}_{\Bbbk}\Big)$ of Proposition 1.

Thus a BOCS is three-tuple $(\mathbb{C}, \delta, \xi)$ consisting of an $\mathcal{A}$-bimodule $\mathbb{C}$ (i.e., $\mathbb{C} \in \mathfrak{M}$) and two natural transformations

$$\delta : \mathbb{C} \underset{\text{comultiplication}}{\longrightarrow} \mathbb{C} \otimes_{\mathcal{A}} \mathbb{C}, \qquad \xi : \mathbb{C} \underset{\text{counit}}{\longrightarrow} \mathbb{I}_{\mathfrak{M}},$$

satisfying the usual counitary and coassociativity properties.

A *morphism of BOCSES* is a natural transformation $\phi : \mathbb{C} \to \mathbb{C}'$ such that

$$\delta' \circ \phi \;=\; (\phi \otimes_{\mathcal{A}} \phi) \circ \delta, \qquad \xi' \circ \phi \;=\; \xi.$$

BOCSES over $\mathcal{A}$ and their morphisms form a category which we denote by $\mathcal{A}$-Bocses.

Our next goal is to show that the category $\mathcal{A}$-Bocses is equivalent to the category of corings over the ring with enough orthogonal idempotents A, which we denote by A-$\mathsf{corings}$ (i.e., the category of comonoids in the monoidal category of bimodules ${}_A\mathrm{Mod}_A$), see [8] for additional facts on corings over rings with local units. We start with the following lemma, which can be seen as a variation of the equivalences of categories given explicitly in Eq. (4).

Lemma 3 *Let $\mathcal{A}$ be a small $\Bbbk$-linear category and A its associated ring with enough orthogonal idempotents. Then, the following functor establishes an equivalence of categories*

$$\begin{array}{rcl} \mathfrak{M} := \mathsf{Funct}\Big(\mathcal{A}^{op} \times \mathcal{A},\ \mathrm{Mod}_{\Bbbk}\Big) & \xrightarrow{\qquad\quad \mathscr{F} \qquad\quad} & {}_A\mathrm{Mod}_A \\ \mathbb{F} & \longrightarrow & \bigoplus_{(\mathfrak{p}^o,\, \mathfrak{q}) \in \mathcal{A}^{op} \times \mathcal{A}} \mathbb{F}(\mathfrak{p}^o, \mathfrak{q}) \\ \theta & \longrightarrow & \bigoplus_{(\mathfrak{p}^o,\, \mathfrak{q}) \in \mathcal{A}^{op} \times \mathcal{A}} \theta_{(\mathfrak{p}^o, \mathfrak{q})} \end{array}$$

Proof Using the functors $\mathscr{R}$ and $\mathscr{L}$ of Eq. (4), one get the A-bimodule structure on $\mathscr{F}(\mathbb{F})$, for a given $\mathbb{F} \in \mathfrak{M}$. A direct verification shows that $\mathscr{F}$ is a well defined functor. The inverse functor is defined as follows: To every unital A-bimodule X, one corresponds the bi-functor $\mathscr{G}(X) : \mathcal{A}^{op} \times \mathcal{A} \to \mathrm{Mod}_{\Bbbk}$ which acts over objects by $(\mathfrak{p}^o, \mathfrak{q}) \to 1_{\mathfrak{q}} X 1_{\mathfrak{p}}$ (the $\Bbbk$-submodule of X generated by the set of elements $\{1_{\mathfrak{q}} x 1_{\mathfrak{p}} \,|\, x \in X\}$), and acts over morphisms as follows:

$$\Big[(f^o, g) : (\mathfrak{p}_1^o, \mathfrak{q}_1) \to (\mathfrak{p}_2^o, \mathfrak{q}_2)\Big] \longrightarrow \Big[\mathscr{G}(f^o, g) : 1_{\mathfrak{q}_1} X 1_{\mathfrak{p}_1} \to 1_{\mathfrak{q}_2} X 1_{\mathfrak{p}_2}, \\ \Big(1_{\mathfrak{q}_1} x 1_{\mathfrak{p}_1} \mapsto 1_{\mathfrak{q}_2} g x f 1_{\mathfrak{p}_2}\Big)\Big],$$

where the notation gxf stands for the bi-action of A on X, after identifying the elements g, f with their images in A. The rest of the proof is left to the reader. $\square$

Proposition 2 *Let $\mathcal{A}$ be a small $\Bbbk$-linear category and A its associated ring with enough orthogonal idempotents. The functor $\mathscr{F}$ defined in Lemma 3 establishes a monoidal equivalence between the monoidal categories $\mathsf{Funct}\Big(\mathcal{A}^{op} \times \mathcal{A}, \mathrm{Mod}_{\Bbbk}\Big)$ and ${}_A\mathrm{Mod}_A$.*

Proof Fix $\mathbb{T}$ and $\mathbb{F}$ two objects of the category $\mathfrak{M}$. For every pair of objects $(\mathfrak{p}^o, \mathfrak{q}) \in \mathcal{A}^{op} \times \mathcal{A}$, we have by Lemma 2 an isomorphism of $\Bbbk$-modules

$$\begin{aligned} \mathbb{T}(-, \mathfrak{q}) \otimes_{\mathcal{A}} \mathbb{F}(\mathfrak{p}^o, -) &\cong \mathscr{R}\Big(\mathbb{T}(-, \mathfrak{q})\Big) \otimes_A \mathscr{L}\Big(\mathbb{F}(\mathfrak{p}^o, -)\Big) \\ &\cong \Big(\bigoplus_{\mathfrak{r} \in \mathcal{A}} \mathbb{T}(\mathfrak{r}^o, \mathfrak{q})\Big) \otimes_A \Big(\bigoplus_{\mathfrak{t} \in \mathcal{A}} \mathbb{F}(\mathfrak{p}^o, \mathfrak{t})\Big). \end{aligned}$$

Applying the functor $\bigoplus_{(\mathfrak{p}^o, \mathfrak{q}) \in \mathcal{A}^{op} \times \mathcal{A}}(-)$ (i.e., the direct sum functor) to the latter isomorphism, we obtain an isomorphism of an A–bimodules as follows:

$$\begin{aligned} \bigoplus_{(\mathfrak{p}^o, \mathfrak{q}) \in \mathcal{A}^{op} \times \mathcal{A}} \Big(\mathbb{T}(-, \mathfrak{q}) \otimes_{\mathcal{A}} \mathbb{F}(\mathfrak{p}^o, -)\Big) &\cong \bigoplus_{(\mathfrak{p}^o, \mathfrak{q}) \in \mathcal{A}^{op} \times \mathcal{A}} \Big(\Big(\bigoplus_{\mathfrak{r} \in \mathcal{A}} \mathbb{T}(\mathfrak{r}, \mathfrak{q})\Big) \otimes_A \Big(\bigoplus_{\mathfrak{t} \in \mathcal{A}} \mathbb{F}(\mathfrak{p}^o, \mathfrak{t})\Big)\Big) \\ &\cong \Big(\bigoplus_{(\mathfrak{r}^o, \mathfrak{q}) \in \mathcal{A}^{op} \times \mathcal{A}} \mathbb{T}(\mathfrak{r}^o, \mathfrak{q})\Big) \otimes_A \Big(\bigoplus_{(\mathfrak{p}^o, \mathfrak{t}) \in \mathcal{A}^{op} \times \mathcal{A}} \mathbb{F}(\mathfrak{p}^o, \mathfrak{t})\Big). \end{aligned}$$

Therefore, we have an isomorphism $\mathscr{F}(\mathbb{T} \otimes_{\mathcal{A}} \mathbb{F}) \cong \mathscr{F}(\mathbb{T}) \otimes_A \mathscr{F}(\mathbb{F})$ of an A-bimodules, for every pair of objects $\mathbb{T}, \mathbb{F} \in \mathfrak{M}$. It is not difficult to check that this isomorphism is in fact a natural isomorphism. Lastly, the image of the identity object $\mathbb{I}_{\mathfrak{M}}$, is

$$\mathscr{F}(\mathbb{I}_{\mathfrak{M}}) \;=\; \bigoplus_{(\mathfrak{p}^o, \mathfrak{q}) \in \mathcal{A}^{op} \times \mathcal{A}} \mathbb{I}_{\mathfrak{M}}(\mathfrak{p}^o, \mathfrak{q}) \;=\; \bigoplus_{(\mathfrak{p}^o, \mathfrak{q}) \in \mathcal{A}^{op} \times \mathcal{A}} \mathrm{Hom}_{\mathcal{A}}\big(\mathfrak{p}, \mathfrak{q}\big) \;=\; A$$

is the identity object of the monoidal category ${}_A\mathrm{Mod}_A$. The rest of constraints that $\mathscr{F}$ should satisfy are immediate and this finishes the proof. $\square$

Now, we state the promised equivalence of categories:

Corollary 1 *Let $\mathcal{A}$ be a small $\mathbb{k}$-linear category and A its associated ring with enough orthogonal idempotents. The functor $\mathscr{F}$ of Lemma 3 induces an equivalence of categories between A-corings and $\mathcal{A}$-Bocses such that the following diagram*

$$\begin{tikzcd} \mathcal{A}\text{-Bocses} \arrow[r, "\mathscr{F}", "\cong"'] \arrow[d] & A\text{-corings} \arrow[d] \\ \mathsf{Funct}\left(\mathcal{A}^{op} \times \mathcal{A},\ \mathrm{Mod}_{\mathbb{k}}\right) \arrow[r, "\mathscr{F}", "\cong"'] & {}_A\mathrm{Mod}_A \end{tikzcd}$$

commutes, where the vertical arrows are the canonical forgetful functors.

Proof It is a consequence of Lemma 3 and Proposition 2. □

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References

1. Abrams, G.D.: Morita equivalence for rings with local units. Comm. Algebra **11**(8), 801–837 (1983)
2. Ánh, P.N., Márki, L.: Rees matrix rings. J. Algebra **81**, 340–369 (1983)
3. Ánh, P.N., Márki, L.: Morita equivalence for rings without identity. Tsukuba J. Math. **11**(1), 1–16 (1987)
4. Bautista, R., Colavita, L., Salmerón, L.: On adjoint functors in representation theory. Representation Theory. Lecture Notes in Mathematics, vol. 903, pp. 9–25. Springer, Berlin (1981)
5. Beck, J.: Distributive laws. Seminar on Triples and Categorical Homology Theory. Lecture Notes in Mathematics, vol. 80, pp. 119–140. Springer, Berlin (1969)
6. Brzeziński, T.: The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties. Algebra Represent. Theory **5**, 389–410 (2002)
7. Eilenberg, S., Moore, J.C.: Adjoint functors and triples. Illinois J. Math. **9**, 381–398 (1965)
8. El Kaoutit, L.: Corings over rings with local units. Math. Nachr. **282**(5), 726–747 (2009)
9. Freyd, P.: Abelian Categories. Introduction An to the Theory of Functors. Harper's Series in Modern Mathematics. Harper & Row Publishers, New York (1964) ; Reprints in Theory and Applications of Categories. No. **3**, 23–164 (2003)
10. Fuller, K.R.: On rings whose left modules are direct sums of finitely generated modules. Proc. Amer. Math. Soc. **54**, 39–44 (1976)
11. Gabriel, P.: Des catégories abéliennes. Bulletin de la S. M. F., tome **90**, 323–448 (1962)
12. Guzman, F.: Cointegration, relative cohomology for comdules, and coseparable coring. J. Algebra **126**, 211–224 (1989)
13. Jonah, D.W.: Cohomology of coalgebras. Mem. Amer. Math. Soc. **82** (1968)
14. Kleiner, M.: The dual ring to a coring with a grouplike. Proc. Amer. Math. Soc. **91**, 540–542 (1984)
15. Kleisli, H.: Every standard construction is induced by a pair of adjoint functors. Proc. Amer. Math. Soc. **16**, 544–546 (1965)
16. Mac Lane, S.: Categories for the working mathematician. Graduate Texts in Mathematics, vol. 5. Springer, New York (1998)

17. Masuoka, A.: Corings and invertible bimodules. Tsukuba. J. Math. **13**(2), 353–362 (1989)
18. Mitchell, B.: Theory of Categories. New York and London ed., Academic Press (1965)
19. Mitchell, B.: Rings with several objects. Adv. Math. **8**, 1–161 (1972)
20. Roiter, A.V.: Matrix problems and representations of BOCS's. Representation Theory I. Lecture Notes in Mathematics, vol. 831, pp. 288–324. Springer, Berlin (1980)
21. Sweedler, M.: The predual theorem to the Jacobson-Bourbaki theorem. Trans. Amer. Math. Soc. **213**, 391–406 (1975)