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Corings over rings with local units

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We show that the bicategory whose 0-cells are corings over rings with local units is bi-equivalent to the bicategory of comonads in (right) unital modules whose underlying functors are right exact and preserve direct sums. A base ring extension of a coring by an adjunction is introduced as well.

1 Introduction

Corings over rings with identity have been intensively studied in the last years. A detailed discussion can be found in [8]. It is well-known that any coring entails a comonad (i.e., cotriple) in the category of right modules over the base ring. The converse is also true for the case of coalgebras, taking functors which are right exact and preserve direct sums, as was checked in [11]. Thus, the categorical study of corings and their comodules is entirely linked to the study of certain comonads and their universal cogenerators (see [11]) in non necessarily monoidal categories. For instance, many new examples of corings can be built using earlier constructions in comonads theory (e.g. the distributive laws of J. Beck [5]), or just by considering comonads with a special cogenerator.

In this paper, we study these relationships in the context of corings over rings with local units in the sense of [1, 2]. This class of rings arose naturally in the definition of infinite comatrix corings, see [13], [19] and [9].

In what follows an additive covariant functor is said to be *continuous* if it is right exact and preserves direct sums.

The paper is organized as follows. Section 2 is rather technical, and it is devoted to introduce a 2-category with objects (0-cells) comonads in Grothendieck categories, using earlier results from [24]. In Section 3, we extend a result of Watts [31] to the case of rings with local units (Lemma 3.3), and use this to prove that each comonad induces a coring whenever its underlying functor is continuous (Proposition 3.5). In Section 4, we establish a bi-equivalence between the bicategory of unital bimodules and the bicategory whose 0-cells are rings with local units and having the categories of continuous functors over unital right modules as Hom-Categories (Proposition 4.1, compare with [20, Proposition 2.1]). We, then, deduce an equivalence of categories between the category of corings over a fixed ring with local units and the category of comonads in right unital modules whose underlying functors are continuous (Corollary 4.3). Lastly, we derive a bi-equivalence between the bicategory of all corings over rings with local units and the bicategory of comonads in right unital modules with continuous underlying functors (Theorem 4.6). The left version of these results is similarly obtained and will not be considered. In Section 5 we apply results from Sections 2 and 3 to introduce a base ring extension of a coring by an adjunction.

Throughout $\mathbb{K}$ denotes a commutative ring with identity 1.

2 Review on the 2-category of comonads

In this section we observe that Grothendieck categories form a class of objects (0-cells) in a 2-category whose 1-cells are continuous functors. These will be needed to introduce the 2-category of comonads using the formalism

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of [24]. For general definitions of bicategories and their homomorphisms we refer the reader to the fundamental paper [6].

Recall from [11] that a comonad (or cotriple) in a category $\mathcal{A}$ is a three-tuple (F, δ, ξ) consisting of a functor $F : \mathcal{A} \to \mathcal{A}$ and natural transformations $\delta_- : F(-) \to F \circ F(-) = F^2(-)$, $\xi_- : F(-) \to id_{\mathcal{A}}(-)$ such that

$$\begin{array}{ccc} F(-) & \xrightarrow{\delta_-} & F^2(-) \\ {\scriptstyle \delta_-}\downarrow & & \downarrow{\scriptstyle \delta_{F(-)}} \\ F^2(-) & \xrightarrow{F(\delta_-)} & F^3(-) \end{array} \qquad \begin{array}{ccc} F(-) & \xrightarrow{\delta_-} & F^2(-) \\ {\scriptstyle \delta_-}\downarrow & \searrow^{=} & \downarrow{\scriptstyle F(\xi_-)} \\ F^2(-) & \xrightarrow{\xi_{F(-)}} & F(-) \end{array} \tag{2.1}$$

are commutative diagrams. It is well-known from [22] that any adjunction $S : \mathcal{B} \rightleftarrows \mathcal{A} : T$ with S left adjoint to T (notation $S \dashv T$) induces a comonad $(ST, S\eta_T, \zeta)$ in $\mathcal{A}$, where $\eta : id_{\mathcal{B}} \to TS$ and $\zeta : ST \to id_{\mathcal{A}}$ are, respectively, the unit and the counit of this adjunction. Using the terminology of [11, Proposition 2.1], we say that the adjunction $S \dashv T$ cogenerates the comonad $(ST, S\eta_T, \zeta)$. Now, the dual version of [11, Theorem 2.2] asserts that for any comonad (F, δ, ξ) in $\mathcal{A}$, there exists a universal cogenerator, that is, a category $\mathcal{A}^F$ and an adjunction $S^F : \mathcal{A}^F \rightleftarrows \mathcal{A} : T^F$ cogenerating (F, δ, ξ) with the following universal property: If $S : \mathcal{B} \rightleftarrows \mathcal{A} : T$ is another cogenerator of (F, δ, ξ), then there exists a unique functor $L : \mathcal{B} \to \mathcal{A}^F$ such that $S^F \circ L = S$ and $L \circ T = T^F$. The objects of $\mathcal{A}^F$ are referred to as *comodules*, they are pairs (X, d^X) consisting of an object $X \in \mathcal{A}$ and a morphism $d^X : X \to F(X)$ in $\mathcal{A}$ such that

$$\begin{array}{ccc} X & \xrightarrow{d^X} & F(X) \\ {\scriptstyle d^X}\downarrow & & \downarrow{\scriptstyle \delta_X} \\ F(X) & \xrightarrow{F(d^X)} & F^2(X) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{d^X} & F(X) \\ & \searrow^{=} & \downarrow{\scriptstyle \xi_X} \\ & & X \end{array} \tag{2.2}$$

are commutative diagrams. A morphism in $\mathcal{A}^F$ is a morphism $f : X \to X'$ in $\mathcal{A}$, where $\left(X, d^X\right)$, $\left(X', d^{X'}\right) \in \mathcal{A}^F$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle d^X}\downarrow & & \downarrow{\scriptstyle d^{X'}} \\ F(X) & \xrightarrow{F(f)} & F(X') \end{array} \tag{2.3}$$

is a commutative diagram. The functor S^F is then the forgetful functor $S^F\left(X, d^X\right) = X$ for every comodule $\left(X, d^X\right)$, and the functor T^F is defined by $T^F(Y) = (F(Y), \delta_Y)$, $T^F(f) = F(f)$ for any object Y and morphism f of $\mathcal{A}$.

Given (F, δ, ξ) a comonad in $\mathcal{A}$, the Kleisli [23] category $\mathcal{A}_K$, is defined as follows. The objects of $\mathcal{A}_K$ are the same as those of $\mathcal{A}$. For each pair of objects Y, Y', the morphism set is defined by

$$\mathrm{Hom}_{\mathcal{A}_K}(Y, Y') := \mathrm{Hom}_{\mathcal{A}}(F(Y), Y').$$

The composition law and identities are canonically derived using δ and ξ. As was proved in [23], there is an adjunction $S : \mathcal{A}_K \rightleftarrows \mathcal{A} : T$ cogenerating (F, δ, ξ), where $S(Y) = F(Y)$ and $S(f) = F(f) \circ \delta_Y$ for any object Y and morphism f of $\mathcal{A}_K$, and $T(X) = X$, $T(g) = \xi_{X'} \circ F(g)$ for any objects X, X' and morphism $g : X \to X'$ of $\mathcal{A}$. The unique factorizing functor is given as follows. The adjunction $S^F \dashv T^F$ gives a natural isomorphism

$$\mathrm{Hom}_{\mathcal{A}_K}(X, X') \cong \mathrm{Hom}_{\mathcal{A}^F}\left(T^F(X), T^F(X')\right),$$

which defines a functor $L : \mathcal{A}_K \to \mathcal{A}^F$ by $L(X) = T^F(X) = (F(X), \delta_X)$, for any object $X \in \mathcal{A}_K$, and $L(f) = F(f) \circ \delta_X : F(X) \to F(X')$, for any morphism $f \in \mathrm{Hom}_{\mathcal{A}_K}(X, X')$. Clearly $L \circ T = T^F$, $S^F \circ L = S$, and L is a fully faithful functor.

Lemma 2.1 *Consider an adjunction* $S : \mathcal{A} \rightleftarrows \mathcal{B} : T$ *,* $S \dashv T$ *with unit* $\eta : id_{\mathcal{A}} \to TS$ *and counit* $\zeta : ST \to id_{\mathcal{B}}$ *together with* (F, δ, ξ) *a comonad in* $\mathcal{A}$*. Then*

1. $[S \dashv T](F, \delta, \xi) := (SFT, SF\eta_{FT} \circ S\delta_T, \zeta \circ S\xi_T)$ *is a comonad in* $\mathcal{B}$*.*
2. *There is a functor* $\mathscr{S} : \mathcal{A}^F \longrightarrow \mathcal{B}^{SFT}$ *defined by*

$$\left(X, d^X\right) \longrightarrow \left(S(X), d^{S(X)} = SF\eta_X \circ Sd^X\right) \quad (f \to S(f)).$$

Proof. (1) This is [22, Theorem 4.2].
(2) It is clear that we have an adjunction

$$\mathcal{A}^F \underset{T^F \circ T}{\overset{S \circ S^F}{\rightleftarrows}} \mathcal{B}$$

with $S \circ S^F \dashv T^F \circ T$ which cogenerates the comonad $(SFT, SF\eta_{FT} \circ S\delta_T, \zeta \circ S\xi_T)$. Following the proof of the dual version of [11, Theorem 2.2], we obtain that the stated functor is the well-known unique factorization functor. □

Following [4], a morphism between two comonads (F, δ, ξ) and (F', δ', ξ') in a category $\mathcal{A}$ is a natural transformation $\Phi_- : F(-) \to F'(-)$ which turns commutative the following diagrams

$$\begin{array}{ccc} F(-) & \xrightarrow{\Phi_-} & F'(-) \\ {\scriptstyle \delta_-}\downarrow & & \downarrow{\scriptstyle \delta'_-} \\ F^2(-) & \xrightarrow{\varrho(\Phi)_-} & (F')^2(-) \end{array} \qquad \begin{array}{ccc} F(-) & \xrightarrow{\Phi_-} & F'(-) \\ {\scriptstyle \xi_-}\downarrow & & \downarrow{\scriptstyle \xi'_-} \\ id_{\mathcal{A}}(-) & = & id_{\mathcal{A}}(-), \end{array} \tag{2.4}$$

where $\varrho(\Phi)_-$ is the natural transformation defined by

$$\varrho(\Phi)_- := F'\Phi_- \circ \Phi_{F(-)} = \Phi_{F'(-)} \circ F\Phi_-. \tag{2.5}$$

Given a morphism of comonads $\Phi : F \to F'$, we have an induction functor

$$(-)_\Phi : \mathcal{A}^F \longrightarrow \mathcal{A}^{F'}$$

defined by $\left(X, d^X\right)_\Phi = \left(X, \Phi_X \circ d^X\right)$ for any $\left(X, d^X\right) \in \mathcal{A}^F$ and the identity on morphisms.

Example 2.2 Let (F, δ, ξ) be any comonad in a category $\mathcal{A}$. Clearly $\xi : F \to id_{\mathcal{A}}$ is morphism of comonads, where $id_{\mathcal{A}}$ is endowed with a trivial comonad structure. If we consider any adjunction $S : \mathcal{A} \rightleftarrows \mathcal{B} : T$ with $S \dashv T$ and $[S \dashv T](F, \delta, \xi)$ the associated comonad of Lemma 2.1, then we can easily check that $S\xi_T : SFT \to ST$ is also a morphism of comonads in $\mathcal{B}$, where $(ST, S\eta_T, \zeta)$ is the comonad cogenerated by $S \dashv T$.

Next, we are going to look at the case in which certain comonads and their morphisms form a set-category and try to interpret the above constructions by means of functors between those categories. The following definition makes sense after [10, Lemma 5.1], where it was proved that, over Grothendieck category, natural transformations between continuous functors form a set.

Definition 2.3 Let $\mathcal{A}$ be a Grothendieck category. We define the category of comonads in $\mathcal{A}$ and denote it by $\mathcal{A}$-*comonad*, as the category whose objects are comonads (F, δ, ξ) in $\mathcal{A}$ with $F : \mathcal{A} \to \mathcal{A}$ a continuous functor (thus it also preserves inductive limits); and whose morphisms are natural transformations satisfying the commutativity of the diagrams (2.4).

With this definition we can give an elegant restatement of Lemma 2.1(1).

Proposition 2.4 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two Grothendieck categories together with an adjunction* $S : \mathcal{A} \rightleftarrows \mathcal{B} : T$ *,* $S \dashv T$ *whose unit and counit are, respectively,* $\eta : id_{\mathcal{A}} \to TS$ *and* $\zeta : ST \to id_{\mathcal{B}}$*. Assume that* S *and* T *are continuous functors.*

1. *The assignment of Lemma* 2.1 (1)

$$(F,\delta,\xi) \longrightarrow [S \dashv T](F,\delta,\xi), \quad [\Phi : F \to F'] \longrightarrow \Big[[S \dashv T](\Phi) = [S\Phi_T : SFT \to SF'T]\Big]$$

defines a functor $[S \dashv T] : \mathcal{A}$*-comonad* $\to \mathcal{B}$*-comonad.*

2. *Given* $P : \mathcal{B} \rightleftarrows \mathcal{C} : Q$ *another adjunction where* $\mathcal{C}$ *is a Grothendieck category and* P, Q *are continuous functors. Then, we have the following composition*

$$[P \dashv Q] \circ [S \dashv T] = [PS \dashv TQ],$$

where $[P \dashv Q] : \mathcal{B}$*-comonad* $\to \mathcal{C}$*-comonad and* $[PS \dashv TQ] : \mathcal{A}$*-comonad* $\to \mathcal{C}$*-comonad are functors defined as above.*

Proof. (1) We only show that

$$S\Phi_T = \overline{\Phi} : \ \overline{F} = SFT \longrightarrow \overline{F'} = SF'T$$

is a morphism of comonads where (F,δ,ξ) and (F',δ',ξ') are comonads in $\mathcal{A}$. We have

$$\zeta \circ S\xi'_T \circ \overline{\Phi} = \zeta \circ S\xi'_T \circ S\Phi_T = \zeta \circ S\big(\xi' \circ \Phi\big)_T = \zeta \circ S\xi_T$$

which shows that the second diagram in Equation (2.4) commutes. On the other hand, we have

$$\begin{aligned}
SF'\eta_{F'T} \circ S\delta'_T \circ S\Phi_T &= S\big(F'\eta_{F'} \circ \delta' \circ \Phi\big)_T \\
&= S\big(F'\eta_{F'} \circ \varrho(\Phi) \circ \delta\big)_T && \text{(by (2.4))} \\
&= S\big(F'\eta_{F'} \circ F'\Phi \circ \Phi_F \circ \delta\big)_T && \text{(by (2.5))} \\
&= S\big(F'(\eta_{F'} \circ \Phi) \circ \Phi_F \circ \delta\big)_T \\
&= S\big(F'TS\Phi \circ F'\eta_F \circ \Phi_F \circ \delta\big)_T && (\eta_- \text{ is natural}) \\
&= S\big(F'TS\Phi \circ \Phi_{TSF} \circ F\eta_F \circ \delta\big)_T && (\Phi_- \text{ is natural}) \\
&= SF'TS\Phi_T \circ S\Phi_{TSFT} \circ SF\eta_{FT} \circ S\delta_T \\
&= \varrho(S\Phi_T) \circ SF\eta_{FT} \circ S\delta_T.
\end{aligned}$$

Thus the first diagram in Equation (2.4) commutes for $\overline{F}$, $\overline{F'}$ and $\overline{\Phi} = S\Phi_T$.
(2) Straightforward. □

Using Definition 2.3, one can adapt the formalism introduced in [24] (see also [27]) for monads in arbitrary 2-category, to the setting of Grothendieck categories as follows: First, using [10, Lemma 5.1], we obtain a 2-category constructed by the following data:

- *Objects* (0*-cells*): All Grothendieck categories.
- 1*-cells*: An 1-cell from $\mathcal{B}$ to $\mathcal{A}$ is a continuous functor $F : \mathcal{A} \to \mathcal{B}$.
- 2*-cells*: Natural transformations.

Associated to this 2-category, we construct, as in [24], the right Eilenberg-Moore 2-category of comonads:

- *Objects* (0*-cells*): They are pairs $(F,\delta,\xi : \mathcal{A})$ consisting of a Grothendieck category $\mathcal{A}$ and a comonad $\boldsymbol{F} = (F,\delta,\xi)$ in $\mathcal{A}$ such that $F : \mathcal{A} \to \mathcal{A}$ is a continuous functor (i.e., $\boldsymbol{F}$ is an object of the category $\mathcal{A}$-comonad of Definition 2.3).

- 1-*cells*: An 1-cell from $(\boldsymbol{G} : \mathcal{B})$ to $(\boldsymbol{F} : \mathcal{A})$ (here $\boldsymbol{G} = (G, \Omega, \gamma) \in \mathcal{B}$-comonad), is a pair (S, s) consisting of a continuous functor $S : \mathcal{A} \to \mathcal{B}$, and a natural transformation $\mathsf{s} : SF \longrightarrow GS$ satisfying the commutativity of the following two diagrams

$$\begin{array}{ccc} SF & \xrightarrow{\ \mathsf{s}\ } & GS \\ {\scriptstyle S\xi}\downarrow & & \downarrow{\scriptstyle \gamma_S} \\ S & = & S, \end{array} \qquad \begin{array}{ccccc} SF & \xrightarrow{\ \mathsf{s}\ } & GS & \xrightarrow{\ \Omega_S\ } & G^2S \\ {\scriptstyle S\delta}\downarrow & & & \nearrow{\scriptstyle G\mathsf{s}} & \\ SF^2 & \xrightarrow{\ \mathsf{s}_F\ } & GSF & & \end{array} \tag{2.6}$$

The identity 1-cell for a given object $(\boldsymbol{F} : \mathcal{A})$ is provided by $\big(id_{\mathcal{A}}, id_{F(-)}\big)$.

- 2-*cells*: Given (S, s) and (S', s') two 1-cells from $(\boldsymbol{G} : \mathcal{B})$ to $(\boldsymbol{F} : \mathcal{A})$, a 2-cell $(S, \mathsf{s}) \to (S', \mathsf{s}')$ is a natural transformation $\alpha : SF \to S'$ turning commutative the following diagram

$$\begin{array}{ccccc} SF & \xrightarrow{\ S\delta\ } & SF^2 & \xrightarrow{\ \mathsf{s}_F\ } & GSF \\ {\scriptstyle S\delta}\downarrow & & & & \downarrow{\scriptstyle G\alpha} \\ SF^2 & \xrightarrow{\ \alpha_F\ } & S'F & \xrightarrow{\ \mathsf{s}'\ } & GS'. \end{array} \tag{2.7}$$

The category constructed by all 1 and 2-cells from $(\boldsymbol{G} : \mathcal{B})$ to $(\boldsymbol{F} : \mathcal{A})$ will be denoted by ${}_{\boldsymbol{F}}\mathscr{C}_{\boldsymbol{G}}$. The laws composition are given as follows. Let (S, s), (S', s'), and (S'', s'') be three 1-cells from $(\boldsymbol{G} : \mathcal{B})$ to $(\boldsymbol{F} : \mathcal{A})$ with 2-cells $\alpha : (S, \mathsf{s}) \to (S', \mathsf{s}')$ and $\alpha' : (S', \mathsf{s}') \to (S'', \mathsf{s}'')$. Then

$$\alpha' \,\underline{\circ}\, \alpha = \alpha' \circ \alpha_F \circ S\delta, \quad \text{where} \quad \boldsymbol{F} = (F, \delta, \xi). \tag{2.8}$$

Given (P, p) and (P', p') two 1-cells from $(\boldsymbol{H} : \mathcal{C})$ to $(\boldsymbol{G} : \mathcal{B})$, together with 2-cells $\alpha : (S, \mathsf{s}) \to (S', \mathsf{s}')$ and $\beta : (P, \mathsf{p}) \to (P', \mathsf{p}')$, the vertical composition is given by

$$(S, \mathsf{s}) \,.\, (P, \mathsf{p}) = (PS, \mathsf{p}_S \circ P\mathsf{s}), \quad \text{and} \quad (S', \mathsf{s}') \,.\, (P', \mathsf{p}') = (P'S', \mathsf{p}'_{S'} \circ P'\mathsf{s}'). \tag{2.9}$$

The horizontal composition $\alpha \,.\, \beta : (PS, \mathsf{p}_S \circ P\mathsf{s}) \to \big(P'S', \mathsf{p}'_{S'} \circ P'\mathsf{s}'\big)$ is defined by

$$\begin{array}{ccccccc} PSF & \xrightarrow{\ PS\delta\ } & PSF^2 & \xrightarrow{\ P\mathsf{s}_F\ } & PGSF & \xrightarrow{\ PG\alpha\ } & PGS' \\ & \searrow{\scriptstyle \alpha\,.\,\beta} & & & & & \downarrow{\scriptstyle \beta_{S'}} \\ & & & & & & P'S' \end{array} \tag{2.10}$$

Associated to an 1-cell $(S, \mathsf{s}) \in {}_{\boldsymbol{F}}\mathscr{C}_{\boldsymbol{G}}$, there is a functor connecting the universal cogenerators. Namely, there is an additive functor

$$\mathscr{S} : \mathcal{A}^F \longrightarrow \mathcal{B}^G, \tag{2.11}$$

sending

$$\big((X, d^X) \longrightarrow \big(S(X), \mathsf{s}_X \circ Sd^X\big)\big) \quad (f \to S(f)),$$

which clearly turns commutative the following diagram

$$\begin{array}{ccc} \mathcal{A}^F & \xrightarrow{\ \mathscr{S}\ } & \mathcal{B}^G \\ {\scriptstyle S^F}\downarrow & & \downarrow{\scriptstyle S^G} \\ \mathcal{A} & \xrightarrow{\ S\ } & \mathcal{B} \end{array} \tag{2.12}$$

As in the case of an arbitrary 2-category [24], one can substitute the above 2-cells (reduced forms) by the unreduced forms, that is, natural transformations of the form $\alpha : SF \to GS'$ satisfying adequate conditions. The bijection between reduced forms and unreduced forms established in [24] for monads in 2-category, is interpreted in our setting by the forthcoming proposition whose proof is based upon the following. Recall that an object V of an additive category $\mathcal{G}$ with direct sums and cokernels, is said to be a *subgenerator*, if every object of $\mathcal{G}$ is a sub-object of a V-generated one.

Lemma 2.5 *Let $\mathcal{A}$ be a Grothendieck category and (F, δ, ξ) be an object of the category $\mathcal{A}$-comonad. Then $\mathcal{A}^F$ is an additive category with direct sums and cokernels. Furthermore, if U is a generator of $\mathcal{A}$ then $\big(F(U), \delta_U\big)$ is a subgenerator of $\mathcal{A}^F$.*

P r o o f. It is immediate since F preserves direct sums and cokernels. □

Proposition 2.6 *Let $\mathcal{A}$ and $\mathcal{B}$ be two Grothendieck categories, and $\boldsymbol{F} = (F, \delta, \xi) \in \mathcal{A}$-comonad, $\boldsymbol{G} = (G, \Omega, \gamma) \in \mathcal{B}$-comonad. Considering (S, s) and (S', s') two 2-cells from $(\boldsymbol{G} : \mathcal{B})$ to $(\boldsymbol{F} : \mathcal{A})$, with the associated functors $\mathscr{S}, \mathscr{S}' : \mathcal{A}^F \to \mathcal{B}^G$ as in Equation (2.11), then the natural transformations $Nat(\mathscr{S}, \mathscr{S}')$ form a set. Moreover, there is a bijection*

$$\mathrm{Hom}_{\boldsymbol{F}\mathscr{C}_{\boldsymbol{G}}}\big((S, \mathsf{s}), (S', \mathsf{s}') \big) \simeq Nat(\mathscr{S}, \mathscr{S}'),$$

explicitly given by

$$\Big([\alpha : SF \to S'] \longmapsto [\boldsymbol{\alpha} : \mathscr{S} \to \mathscr{S}'] \Big), \quad \Big(S'\xi_{-} \circ \boldsymbol{\beta}_{T^F(-)} \longleftarrow\!\!\mapsfrom \boldsymbol{\beta} \Big),$$

where for every object $\big(X, d^X\big) \in \mathcal{A}^F$, $\boldsymbol{\alpha}_X = \alpha_X \circ S d^X : S(X) \to S'(X)$.

P r o o f. We first prove that $Nat(\mathscr{S}, \mathscr{S}')$ is a set. To do this, we follow the proof of [10, Lemma 5.1]. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} : \mathscr{S} \to \mathscr{S}'$ be two natural transformations and let U be a generator of $\mathcal{A}$. So $(F(U), \delta_U)$ is a sub-generator of $\mathcal{A}^F$, by Lemma 2.5. We claim that if $\boldsymbol{\alpha}_{(F(U),\, \delta_U)} = \boldsymbol{\beta}_{(F(U),\, \delta_U)}$ then $\boldsymbol{\alpha} = \boldsymbol{\beta}$. Considering any object $\big(X, d^X\big)$ of $\mathcal{A}^F$ with an epimorphism $\pi : U^{(I)} \to X \to 0$ in $\mathcal{A}$, for some set I, we obtain a diagram

$$\begin{array}{ccccc} F(U^{(I)}) \cong F(U)^{(I)} & \xrightarrow{\ \pi'\ } & F(X) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle d^X} & & \\ & & X & & \end{array}$$

of morphisms of $\mathcal{A}^F$. Since $\boldsymbol{\alpha}_{\left(F(U)^{(I)},\, d^{F(U)^{(I)}}\right)} = \boldsymbol{\beta}_{\left(F(U)^{(I)},\, d^{F(U)^{(I)}}\right)}$, we have

$$\mathscr{S}'(\pi')\Big(\boldsymbol{\alpha}_{\left(F(U)^{(I)},\, d^{F(U)^{(I)}}\right)} - \boldsymbol{\beta}_{\left(F(U)^{(I)},\, d^{F(U)^{(I)}}\right)}\Big) = \Big(\boldsymbol{\alpha}_{(F(X),\, \delta_X)} - \boldsymbol{\beta}_{(F(X),\, \delta_X)}\Big)\, \mathscr{S}(\pi') = 0.$$

By hypothesis, Lemma 2.5 and diagram (2.12), we know that $\mathscr{S}$ preserves epimorphisms. Therefore, $\boldsymbol{\alpha}_{(F(X),\, \delta_X)} = \boldsymbol{\beta}_{(F(X),\, \delta_X)}$, as π' is an epimorphism. This implies that

$$\mathscr{S}'\big(d^X\big) \circ \big(\boldsymbol{\alpha}_{(X,\, d^X)} - \boldsymbol{\beta}_{(X,\, d^X)}\big) = \big(\boldsymbol{\alpha}_{(F(X),\, \delta_X)} - \boldsymbol{\beta}_{(F(X),\, \delta_X)}\big) \circ \mathscr{S}\big(d^X\big) = 0.$$

Applying the functor S'^G and using the diagram (2.12) for (S', s'), we get

$$S'S^F\big(d^X\big) \circ S'^G\big(\boldsymbol{\alpha}_{(X,\, d^X)} - \boldsymbol{\beta}_{(X,\, d^X)}\big) \;=\; 0.$$

Composing with the map $S'(\xi_X)$, we obtain that $S'^G\big(\boldsymbol{\alpha}_{(X,\, d^X)} - \boldsymbol{\beta}_{(X,\, d^X)}\big) = 0$ in $\mathcal{B}$, that is, $\boldsymbol{\alpha}_{(X,\, d^X)} = \boldsymbol{\beta}_{(X,\, d^X)}$ in $\mathcal{B}^G$, and this proves the claim.

For the stated bijection, we only prove that the mutually inverse maps are well-defined. Starting with a 2-cell $\alpha : SF \to S'$, and taking an arbitrary object $\big(X, d^X\big) \in \mathcal{A}^F$, we need to demonstrate that $\boldsymbol{\alpha}_X = \alpha_X \circ Sd^X$ is a morphism of the category $\mathcal{B}^G$, so

$$
\begin{aligned}
d^{S'(X)} \circ \alpha_X \circ Sd^X &= \mathsf{s}'_X \circ S'd^X \circ \alpha_X \circ Sd^X \\
&= \mathsf{s}'_X \circ \alpha_{F(X)} \circ SFd^X \circ Sd^X \\
&= \big(\mathsf{s}'_X \circ \alpha_{F(X)} \circ S\delta_X\big) \circ Sd^X && \text{(by (2.2))} \\
&= G\alpha_X \circ \mathsf{s}_{F(X)} \circ S\delta_X \circ Sd^X && \text{(by (2.7))} \\
&= G\alpha_X \circ \mathsf{s}_{F(X)} \circ SFd^X \circ Sd^X \\
&= G\alpha_X \circ GSd^X \circ \mathsf{s}_X \circ Sd^X && (\mathsf{s}_- \text{ is natural}) \\
&= G\big(\alpha_X \circ Sd^X\big) \circ d^{S(X)}.
\end{aligned}
$$

Obviously $\boldsymbol{\alpha}_-$ is natural. Conversely, starting with a natural transformation $\boldsymbol{\beta} : \mathscr{S} \to \mathscr{S}'$, its image is the natural transformation $\beta : SF \to S'$ defined in every object $Y \in \mathcal{A}$ by $\beta_Y = S'\xi_Y \circ \boldsymbol{\beta}_{T^F(Y)}$. We need to show that β satisfies the 2-cell condition. In one hand, we have

$$
\begin{aligned}
G\beta_Y \circ \mathsf{s}_{F(Y)} \circ S\delta_Y &= GS'\xi_Y \circ G\boldsymbol{\beta}_{T^F(Y)} \circ \mathsf{s}_{F(Y)} \circ S\delta_Y \\
&= GS'\xi_Y \circ G\boldsymbol{\beta}_{T^F(Y)} \circ d^{SF(Y)} \\
&= GS'\xi_Y \circ d^{S'F(Y)} \circ \boldsymbol{\beta}_{T^F(Y)} && \big(\boldsymbol{\beta}_{T^F(Y)} \text{ satisfies (2.3)}\big) \\
&= GS'\xi_Y \circ \mathsf{s}'_{F(Y)} \circ S'\delta_Y \circ \boldsymbol{\beta}_{T^F(Y)} \\
&= GS'\xi_Y \circ \mathsf{s}'_{F(Y)} \circ \boldsymbol{\beta}_{T^F(F(Y))} \circ S\delta_Y && (\boldsymbol{\beta}_- \text{ is natural}) \\
&= \mathsf{s}'_Y \circ S'F\xi_Y \circ \boldsymbol{\beta}_{T^F(F(Y))} \circ S\delta_Y && (\mathsf{s}'_- \text{ is natural}) \\
&= \mathsf{s}'_Y \circ \boldsymbol{\beta}_{T^F(Y)} \circ SF\xi_Y \circ S\delta_Y && (\boldsymbol{\beta}_- \text{ is natural}) \\
&= \mathsf{s}'_Y \circ \boldsymbol{\beta}_{T^F(Y)}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\mathsf{s}'_Y \circ \beta_{F(Y)} \circ S\delta_Y &= \mathsf{s}'_Y \circ S'\xi_{F(Y)} \circ \boldsymbol{\beta}_{T^F(F(Y))} \circ S\delta_Y \\
&= \mathsf{s}'_Y \circ S'\xi_{F(Y)} \circ S'\delta_Y \circ \boldsymbol{\beta}_{T^F(Y)} \\
&= \mathsf{s}'_Y \circ \boldsymbol{\beta}_{T^F(Y)}.
\end{aligned}
$$

Therefore, $G\beta_Y \circ \mathsf{s}_{F(Y)} \circ S\delta_Y = \mathsf{s}'_Y \circ \beta_{F(Y)} \circ S\delta_Y$, for every object $Y \in \mathcal{A}$, and this gives the needed condition. □

3 Comonads and corings over rings with local units

We will consider rings without identity although we assume that a set of identities is given. Following [1] (see also [2] and [3]) a $\mathbb{K}$-module A is said to be a *ring with local units* if for every $a_1, \dots, a_n$ in A, there exists an idempotent element $e \in \mathrm{Idemp}(A)$ (the set of all idempotent elements) such that

$$
a_i e = e a_i = a_i, \quad i = 1, \dots, n.
$$

We say that e is a *unity* for the set $\{a_1, \dots, a_n\}$. This is equivalent to say that for every $a, a' \in A$, there exists a ring with identity of the form $A_e = eAe$ for some idempotent element $e \in A$ such that $a, a' \in A_e$. For instance, every induced ring from a $\mathbb{K}$-additive small category is a ring with local units, in such case it is a ring with enough orthogonal idempotents, see [17] and [16].

For any right A-module X (i.e., a $\mathbb{K}$-module X with associative $\mathbb{K}$-linear right A-action $\mu_X : X \otimes_{\mathbb{K}} A \to X$), XA denotes the right A-submodule

$$
XA = \left\{ \sum_{1 \le i \le n} x_i a_i \mid x_i \in X,\, a_i \in A, \text{ and } n \in \mathbb{N} \right\}.
$$

A morphism between two rings with local units is a morphism of rings $\psi : B \to A$ (i.e., compatible with multiplications) satisfying the following condition: For every $a \in A$, there exists $f \in \mathrm{Idemp}(B)$ such that $a\psi(f) = \psi(f)a = a$. Observe that this condition is equivalent to say that for every $e \in \mathrm{Idemp}(A)$, there exists $f \in \mathrm{Idemp}(B)$ such that $e\psi(f) = \psi(f)e = e$.

The construction of the usual tensor product over rings with identity can be directly transferred to rings with local units, and the most useful properties of this product are preserved. We use the same symbol $\otimes_A$ to denote the tensor product between A-modules and A-linear morphisms for any ring with local units A.

Let A be a ring with local units, and $e \in \mathrm{Idemp}(A)$. The underlying $\mathbb{K}$-module of the right A-module eA is a direct summand of A with decomposition $A = eA \oplus \langle a - ea \mid a \in A \rangle$. Associated to eA there are two $\mathbb{K}$-linear natural transformations

$$\begin{array}{ll} \gamma_{e,X} : X \longrightarrow eA \otimes_A X, & \tau_{e,X} : eA \otimes_A X \longrightarrow X, \\ x \longmapsto e \otimes_A x, & ea \otimes_A x \longmapsto eax, \end{array} \tag{3.1}$$

for every right A-module X. Moreover, if X is an (A, B)-bimodule (B is another ring with local units), then $\gamma_{e,X}$ and $\tau_{e,X}$ are clearly right B-linear. Taking $e' \in \mathrm{Idemp}(A)$ another idempotent and $f : eA \to e'A$ a right A-linear map, there are two commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\gamma_{e,X}} & eA \otimes_A X \\ {\scriptstyle\lambda_{f(e)}}\downarrow & & \downarrow{\scriptstyle f\otimes_A X} \\ X & \xrightarrow{\gamma_{e',X}} & e'A \otimes_A X, \end{array} \qquad \begin{array}{ccc} eA \otimes_A X & \xrightarrow{\tau_{e,X}} & X \\ {\scriptstyle f\otimes_A X}\downarrow & & \downarrow{\scriptstyle\lambda_{f(e)}} \\ e'A \otimes_A X & \xrightarrow{\tau_{e',X}} & X \end{array} \tag{3.2}$$

where $\lambda_{f(e)} : X \to X$ is the left multiplication by $f(e)$, and X any right A-module. Following [1], there is a partial ordering on $\mathrm{Idemp}(A)$ defined by

$$e \leq e' \Longleftrightarrow e = ee' = e'e$$

for every $e, e' \in \mathrm{Idemp}(A)$. Taking X_A any right A-module, and $e, e' \in \mathrm{Idemp}(A)$ such that $e \leq e'$, we can define a canonical injection $\mu_{ee'} : Xe \to Xe'$, $\mu_e : Xe \to X$. Furthermore, if $e \leq e' \leq e''$, then it is clear that $\mu_{ee''} = \mu_{e'e''} \circ \mu_{ee'}$. Thus $\{(Xe, \mu_e)\}_{e \in \mathrm{Idemp}(A)}$ is a directed system of $\mathbb{K}$-submodule of X. In this way it is obvious that ${}_AA = \varinjlim(Ae)$ and $A_A = \varinjlim(eA)$

A right A-module X is said to be *unital* if $XA = X$ (or $X \cong X \otimes_A A$ as right A-module, where the isomorphism should be given by the right A-action). Equivalently, for every element $x \in X$, there exists $e \in \mathrm{Idemp}(A)$ such that $xe = x$. We denote by $\mathcal{M}_A$ the full subcategory of the category of right A-modules whose objects are all unital right A-modules. An easy argument shows that XA is the largest unital right A-submodule of the right A-module X. On the other hand, a right A-module X is unital if and only if $\varinjlim(Xe) = X$ in the category of $\mathbb{K}$-modules. Given B another ring with local units, an *unital* (B, A)*-bimodule* is a (B, A)-bimodule which is unital as a left B-module and as a right A-module. Over the same ring, an A-bimodule X is unital if and only if for every $x \in X$, there exists $e \in \mathrm{Idemp}(A)$ such that $ex = xe = x$. In this way a morphism of rings with local units $\psi : B \to A$ induces a structure of an unital B-bimodule over A, and preserves the usual adjunction between the categories of unital right modules

$$- \otimes_B A : \mathcal{M}_B \rightleftarrows \mathcal{M}_A : \mathcal{O}.$$

The definition of corings over rings with identity as was introduced in [28] can be directly extended, using unital bimodules, to rings with local units. Let A be a ring with local units, an A*-coring* is a three-tuple $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ consisting of an unital A-bimodule $\mathfrak{C}$ and two A-bilinear maps

$$\mathfrak{C} \xrightarrow{\Delta_{\mathfrak{C}}} \mathfrak{C} \otimes_A \mathfrak{C}\,, \qquad \mathfrak{C} \xrightarrow{\varepsilon_{\mathfrak{C}}} A$$

such that $(\Delta_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ \Delta_{\mathfrak{C}} = (\mathfrak{C} \otimes_A \Delta_{\mathfrak{C}}) \circ \Delta_{\mathfrak{C}}$ and $(\varepsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ \Delta_{\mathfrak{C}} = (\mathfrak{C} \otimes_A \varepsilon_{\mathfrak{C}}) \circ \Delta_{\mathfrak{C}} = \mathfrak{C}$. A morphism of A-corings is an A-bilinear map $\phi : \mathfrak{C} \to \mathfrak{C}'$ which satisfies $\varepsilon_{\mathfrak{C}'} \circ \phi = \varepsilon_{\mathfrak{C}}$ and $\Delta_{\mathfrak{C}'} \circ \phi = (\phi \otimes_A \phi) \circ \Delta_{\mathfrak{C}}$. We denote by A*-coring* the category of all A-corings and their morphisms.

A right $\mathfrak{C}$-comodule is a pair (M, ρ_M) consisting of an unital right A-module M and a right A-linear map $\rho_M : M \to M \otimes_A \mathfrak{C}$, called right $\mathfrak{C}$-coaction, such that $(M \otimes_A \Delta_{\mathfrak{C}}) \circ \rho_M = (\rho_M \otimes_A \mathfrak{C}) \circ \rho_M$ and $(M \otimes_A \varepsilon_{\mathfrak{C}}) \circ \rho_M = M$. A morphism of right $\mathfrak{C}$-comodules is a right A-linear map $f : M \to M'$ satisfying $\rho_{M'} \circ f = (f \otimes_A \mathfrak{C}) \circ \rho_M$. Right $\mathfrak{C}$-comodules and their morphisms form a not necessarily abelian category which we denote by $\mathcal{M}^{\mathfrak{C}}$ (see [14, Section 1]). For every unital right A-module X the pair $(X \otimes_A \mathfrak{C}, X \otimes_A \Delta_{\mathfrak{C}})$ is clearly a right $\mathfrak{C}$-comodule. This establishes in fact a functor $- \otimes_A \mathfrak{C} : \mathcal{M}_A \to \mathcal{M}^{\mathfrak{C}}$ with the forgetful functor $U_{\mathfrak{C}} : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ as a left adjoint (see [21]).

Example 3.1 Of course every ring with local units A is trivially an A-coring with comultiplication the isomorphism $A \cong A \otimes_A A$ and counit the identity map A.

1. ([28]) Let $\psi : B \to A$ be a morphism of rings with local units and consider the unital A-bimodule $A \otimes_B A$ with the following two maps

$$\begin{aligned} \Delta : A \otimes_B A &\longrightarrow (A \otimes_B A) \otimes_A (A \otimes_B A), & \varepsilon : A \otimes_B A &\longrightarrow A, \\ a \otimes_B a' &\longmapsto a \otimes_B e \otimes_A e \otimes_B a', & a \otimes_B a' &\longmapsto aa', \end{aligned}$$

where $e \in \psi(\mathrm{Idemp}(B))$ such that $ea = ae = a$ and $a'e = ea' = a'$; that is, e is a unity for both a and a'. To check that $\Delta(a \otimes_B a')$ is independent of the choice of the unity, let us consider another unity $e' \in \psi(\mathrm{Idemp}(B))$ for both a and a'. By definition there exists $e'' \in \psi(\mathrm{Idemp}(B))$ a unity for e and e'. Of course e'' is also a unity for both a and a'. Now, we have

$$\begin{aligned} a \otimes_B e \otimes_A e \otimes_B a' &= a \otimes_B ee'' \otimes_A e''e \otimes_B a' \qquad (\text{since } e = ee'' = e''e) \\ &= ae \otimes_B e'' \otimes_A e'' \otimes_B ea' \\ &= a \otimes_B e'' \otimes_A e'' \otimes_B a'. \end{aligned}$$

Similarly, we get $a \otimes_B e' \otimes_A e' \otimes_B a' = a \otimes_B e'' \otimes_A e'' \otimes_B a'$, and so $\Delta(a \otimes_B a')$ is independent from the choice of e. An easy verification shows now that $(A \otimes_B A, \Delta, \varepsilon)$ is an A-coring.

2. Let M be an unital A-bimodule over a ring with local units A. Consider the direct sum of an A-bimodules $\mathfrak{C} := A \oplus M$ together with the A-bilinear maps

$$\begin{aligned} \Delta : \mathfrak{C} &\longrightarrow \mathfrak{C} \otimes_A \mathfrak{C}, & \varepsilon : \mathfrak{C} &\longrightarrow A, \\ (a, m) &\longmapsto (a, 0) \otimes_A (e, 0) + (0, m) \otimes_A (e, 0) & (a, m) &\longmapsto a, \\ &\qquad + (e, 0) \otimes_A (0, m), \end{aligned}$$

where $e \in \mathrm{Idemp}(A)$ such that $em = me = m$ and $ea = ae = a$. Let us check that Δ is a well-defined map. First we observe that a common unity for a and m does always exist. If $e' \in \mathrm{Idemp}(A)$ is another unity for a and m, then one can consider $e'' \in \mathrm{Idemp}(A)$ a unity for e' and e. Therefore, we have

$$\begin{aligned} \Delta(a, m) &= (a, 0) \otimes_A (e, 0) + (0, m) \otimes_A (e, 0) + (e, 0) \otimes_A (0, m) \\ &= (a, 0) \otimes_A (ee'', 0) + (0, m) \otimes_A (ee'', 0) + (e''e, 0) \otimes_A (0, m) \\ &= (a, 0)e \otimes_A (e'', 0) + (0, m)e \otimes_A (e'', 0) + (e'', 0) \otimes_A e(0, m) \\ &= (a, 0) \otimes_A (e'', 0) + (0, m) \otimes_A (e'', 0) + (e'', 0) \otimes_A (0, m). \end{aligned}$$

In the same way, we get $(a, 0) \otimes_A (e'', 0) + (0, m) \otimes_A (e'', 0) + (e'', 0) \otimes_A (0, m) = (a, 0) \otimes_A (e', 0) + (0, m) \otimes_A (e', 0) + (e', 0) \otimes_A (0, m)$, and thus $\Delta(a, m)$ is independent from the choice of the unity e. The three-tuple $(\mathfrak{C}, \Delta, \varepsilon)$ is easily proved to be an A-coring.

3. ([12]) Let ${}_B\Sigma_A$ be an unital bimodule over rings with local units A and B such that Σ_A is a finitely generated and projective unital right module with finite right dual basis $\{(u_i, u_i^*)\}_i \subset \Sigma \times \Sigma^*$ where $\Sigma^* = \mathrm{Hom}_A(\Sigma, A)$. That is, $u = \sum_i u_i u_i^*(u)$, for every $u \in \Sigma$. It is well-known that Σ^* is also an unital (A, B)-bimodule, and thus $\Sigma^* \otimes_B \Sigma$ is an unital A-bimodule. Furthermore, there exist two A-bilinear maps

$$\begin{aligned} \Delta : \Sigma^* \otimes_B \Sigma &\longrightarrow \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \otimes_B \Sigma, & \varepsilon : \Sigma^* \otimes_B \Sigma &\longrightarrow A, \\ u^* \otimes_B u &\longmapsto \textstyle\sum_i u^* \otimes_B u_i \otimes_A u_i^* \otimes_B u, & u^* \otimes_B u &\longmapsto u^*(u). \end{aligned}$$

The canonical isomorphism $\Sigma \otimes_A \Sigma^* \cong \mathrm{End}(\Sigma_A)$ implies that Δ is independent from the choice of this right dual basis, and that $(\Sigma^* \otimes_B \Sigma, \Delta, \varepsilon)$ is an A-coring. This coring is known as *the finite comatrix coring* associated to ${}_B\Sigma_A$.

Example 3.2 To take a specific example of finite comatrix corings over rings with local units, we consider the so called *finitely orthogonal Rees matrix rings* extensively investigated in [2]. Following [3, Example 2] (see also [2] for notions occurring here), let R be a ring with identity, and A a Rees matrix ring over R with canonical decompositions $A \cong Ae \otimes_{eAe} eA$, $e \in \mathrm{Idemp}(A)$ and $R \cong eAe$. If A is left-right finitely orthogonal with respect to e [2, Definition 4.2], then one can easily prove that A is a ring with local units. On the other hand, if we take ${}_{eAe}\Sigma_A = eA$, then the associated finite comatrix A-coring is given by the A-bimodule $Ae \otimes_{eAe} eA$, and its counit is just the above isomorphism $Ae \otimes_{eAe} eA \xrightarrow{\cong} A$ sending $ae \otimes_{eAe} ea' \mapsto aea'$. Therefore, we have an isomorphism of categories $\mathcal{M}^{Ae \otimes_{eAe} eA} \cong \mathcal{M}_A$ via this counit (recall that the counit is always a morphism of corings). Since the right $(Ae \otimes_{eAe} eA)$-comodule Σ is clearly a generator of $\mathcal{M}^{Ae \otimes_{eAe} eA}$, we deduce following Gabriel–Popescu's Theorem [18] that ${}_{eAe}\Sigma$ is a faithfully flat module (here eAe coincides with the endomorphism ring of this comodule). Thus, $- \otimes_{eAe} \Sigma_A : \mathcal{M}_{eAe} \to \mathcal{M}^{Ae \otimes_{eAe} eA}$ establishes an equivalence of categories by using the non unital version of the generalized Descent Theorem [12, Theorem 3.10]. In conclusion, A is Morita equivalent to eAe, and thus to R, which gives an alternative proof of [3, Example 2].

From now on, we fix A, B rings with local units. Let $F : \mathcal{M}_A \to \mathcal{M}_B$ be a continuous functor (thus it also preserves inductive limits). As in the case of rings with identity [31], next we will show that F is naturally isomorphic to the tensor product functor. Another approach, concerning functors valued in abelian groups was given in [15].

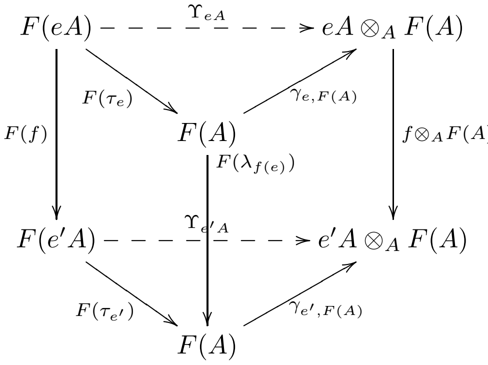
The structure of an (A, B)-bimodule over $F(A)$ comes out by the composition map

$$A \xrightarrow{\ \lambda\ } \mathrm{Hom}_A(A_A, A_A) \xrightarrow{\ F\ } \mathrm{Hom}_B(F(A), F(A)) \tag{3.3}$$

where $\lambda_a : A_A \to A_A$ is the left multiplication by $a \in A$. In the same way, we get an (A, B)-biaction over $F^n(A) = (F \circ \cdots \circ F)(A)$ (n-times). Therefore, one can consider the functor $- \otimes_A F(A) : \mathcal{M}_A \to \mathcal{M}_B$. Now, let's start with an arbitrary idempotent element $e \in \mathrm{Idemp}(A)$, and consider the composed right B-linear map

$$\Upsilon_{eA} = \gamma_{e,F(A)} \circ F(\tau_e) : F(eA) \xrightarrow{F(\tau_e)} F(A) \xrightarrow{\gamma_{e,F(A)}} eA \otimes_A F(A) \tag{3.4}$$

where $\tau_e : eA \to A$ is the canonical injection, and $\gamma_{e,F(A)}$ is the map defined in Equation (3.1). If we take $e' \in \mathrm{Idemp}(A)$ another idempotent element and $f : eA \to e'A$ a right A-linear map, then by Equation (3.2), we get a commutative diagram like this



which shows that Υ_{-} is natural over the set of right A-modules $\{eA\}_{e\in \mathrm{Idemp}(A)}$. Using the projections $\pi_e : A \to eA$ and the maps $\tau_{e,F(A)}$ defined in Equation (3.1), we can also construct a right B-linear map

$$\begin{array}{ccccc} eA \otimes_A F(A) & & \overset{\Theta_{eA}}{-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,\to} & & F(eA) \\ & \searrow^{\tau_{e,F(A)}} & & \nearrow^{F(\pi_e)} & \\ & & F(A) & & \end{array} \tag{3.5}$$

which is natural over $\{eA\}_{e\in \mathrm{Idemp}(A)}$ by Equation (3.2).

Lemma 3.3 *Let $F : \mathcal{M}_A \to \mathcal{M}_B$ be a continuous functor. For every idempotent $e \in \mathrm{Idemp}(A)$, we have*

$$\Upsilon_{eA} \circ \Theta_{eA} = eA \otimes_A F(A),$$
$$\Theta_{eA} \circ \Upsilon_{eA} = F(eA).$$

In particular Υ_{-} extended uniquely to a natural isomorphism

$$\Upsilon^F_{-} : F(-) \longrightarrow - \otimes_A F(A),$$

and $F(A)$ becomes a left unital A-module, and thus an unital (A, B)-bimodule.

Proof. By definition, the left A-action of $F(A)$ is given by the rule

$$a\, x = F(\lambda_a)(x), \ \text{ for every pair } (a, x) \in A \times F(A).$$

Fix an arbitrary idempotent $e \in \mathrm{Idemp}(A)$. So

$$\begin{aligned} \Upsilon_{eA} \circ \Theta_{eA}(ea \otimes_A x) &= \Upsilon_{eA} \circ F(\pi_e)(ea\, x) \\ &= \Upsilon_{eA} \circ F(\pi_e) \circ F(\lambda_{ea})(x) \\ &= \Upsilon_{eA} \circ F(\lambda_{ea})(x) \\ &= \gamma_{e,F(A)} \circ F(\tau_e \circ \lambda_{ea})(x) \\ &= \gamma_{e,F(A)} \circ F(\lambda_{ea})(x) \\ &= \gamma_{e,F(A)}(ea\, x) \\ &= e \otimes_A eax, \\ &= ea \otimes_A x \end{aligned}$$

for every $a \in A$, $x \in F(A)$, and this shows the first equality. To check the second equality, take $y \in F(eA)$ and compute

$$\begin{aligned} \Theta_{eA} \circ \Upsilon_{eA}(y) &= \Theta_{eA} \circ \gamma_{e,F(A)} \circ F(\tau_e)(y) \\ &= F(\pi_e) \circ \tau_{e,F(A)}\big(e \otimes_A F(\tau_e)(y)\big) \\ &= F(\pi_e) \circ \big(e\, F(\tau_e)(y)\big) \\ &= F(\pi_e) \circ \big(F(\lambda_e) \circ F(\tau_e)(y)\big) \\ &= F(\pi_e) \circ \big(F(\lambda_e \circ \tau_e)(y)\big) \\ &= F(\pi_e) \circ F(\tau_e)(y) \\ &= y. \end{aligned}$$

Υ_{-} is clearly extended to unital right A-modules of the form $\oplus_{j\in J}(e_jA)^{(I_j)}$ (J and I_j are sets). Since F preserves direct sums, this extension is also an isomorphism of unital right A-modules. By Mitchell's Theorem [25, Theorem 4.5.2] ([26, Theorem 3.6.5]), Υ_{-} extends uniquely to a natural isomorphism $\Upsilon^F_{-} : F(-) \to - \otimes_A F(A)$ over all unital right A-modules.

We need to show that $F(A)$ is a left unital A-module. We have seen that $eF(A) \cong F(eA)$ for every idempotent $e \in \mathrm{Idemp}(A)$ via Υ^F_{eA}. Since F preserves inductive limits, we get

$$\varinjlim(eF(A)) \cong \varinjlim(F(eA)) \cong F(\varinjlim(eA)) \cong F(A)$$

which implies that ${}_AF(A)$ is left unital and this finishes the proof. □

The following lemma can be deduced from [8, 39.5]. For the sake of completeness, we include a detailed proof.

Lemma 3.4 *Let A, B and C be three rings with local units. We denote by Υ^{χ}_{-} the natural isomorphism of Lemma* 3.3 *associated to the continuous functor χ.*

(a) *Let $F_1,\, F_2 : \mathcal{M}_A \to \mathcal{M}_B$ be two continuous functors. Assuming that there exists a natural transformation $\Xi : F_1 \to F_2$, then*

(1) *The morphism $\Xi_A : F_1(A) \to F_2(A)$ is (A, B)-bilinear.*

(2) *For every unital right A-module X, we have a commutative diagram*

$$\begin{array}{ccc} F_1(X) & \xrightarrow{\ \Upsilon^{F_1}_X\ } & X \otimes_A F_1(A) \\ {\scriptstyle \Xi_X}\downarrow & & \downarrow{\scriptstyle X\otimes_A \Xi_A} \\ F_2(X) & \xrightarrow{\ \Upsilon^{F_2}_X\ } & X \otimes_A F_2(A). \end{array}$$

(b) *Let $F : \mathcal{M}_A \to \mathcal{M}_B$ and $S : \mathcal{M}_B \to \mathcal{M}_C$ be two continuous functors. Then we have $\Upsilon^{F(-)\otimes_B S(B)}_{-} = \Upsilon^F_{-} \otimes_B S(B)$ and the following diagram of natural transformations commutes*

$$\begin{array}{ccc} SF(-) & \xrightarrow{\ \Upsilon^{SF}_{-}\ } & - \otimes_A SF(A) \\ {\scriptstyle \Upsilon^S_{F(-)}}\downarrow & & \downarrow{\scriptstyle -\otimes_A \Upsilon^S_{F(A)}} \\ F(-) \otimes_B S(B) & \xrightarrow{\ \Upsilon^{F(-)\otimes_B S(B)}_{-}\ } & - \otimes_A F(A) \otimes_B S(B) \end{array}$$

P r o o f. (a) (1). By definition, we only need to show that Ξ_A is left A-linear. For every element $a \in A$, we have $\Xi_A \circ F_1(\lambda_a) = F_2(\lambda_a) \circ \Xi_A$ since Ξ_{-} is natural and λ_a is right A-linear. Hence $\Xi_A(ax) = a\Xi_A(x)$, for every pair of elements $(a, x) \in A \times F_1(A)$.

(a) (2). Assuming that X is of the form $X = eA$ for some idempotent element $e \in \mathrm{Idemp}(A)$, we obtain

$$\begin{aligned} \Upsilon^{F_2}_{eA} \circ \Xi_{eA} &= \gamma_{e,F_2(A)} \circ F_2(\tau_e) \circ \Xi_{eA} \\ &= \gamma_{e,F_2(A)} \circ \Xi_A \circ F_1(\tau_e) \qquad (\Xi_{-} \text{ is natural}) \\ &= \big(eA \otimes_A \Xi_A\big) \circ \gamma_{e,F_1(A)} \circ F_1(\tau_e) \quad (\gamma_{e,-} \text{ is natural }) \\ &= \big(eA \otimes_A \Xi_A\big) \circ \Upsilon^{F_1}_{eA}. \end{aligned}$$

For the general case we use a free presentation $\oplus_k(e_kA)^{(I_k)} \to X \to 0$ where $\{e_k\}_k \subseteq \mathrm{Idemp}(A)$, I_k are sets, and the previous case taking into account the hypothesis done over the stated functors.

(b) Straightforward. □

It is clear that any A-coring $(\mathfrak{C}, \Delta, \varepsilon)$ induces, by the three-tuple $((- \otimes_A \mathfrak{C}), - \otimes_A \Delta, - \otimes_A \varepsilon)$, a comonad in $\mathcal{M}_A$. Now, consider (F, δ, ξ) a comonad in $\mathcal{M}_A$ such that $F : \mathcal{M}_A \to \mathcal{M}_A$ is a continuous functor. Our next goal is to prove that $F(A)$ admits a structure of an A-coring. Notice that this has been already observed in [11, p. 398] (without proofs) in the case of commutative rings with identity.

We begin by introducing some convenient notations. Denote by $\Upsilon_{-}^{id_{\mathcal{M}_A}} : id_{\mathcal{M}_A} \to - \otimes_A A$ the canonical natural isomorphism, and by $\Upsilon_{-}^{\chi} : \chi(-) \to - \otimes_A \chi(A)$ the natural isomorphism of Lemma 3.3 associated to the continuous functor $\chi : \mathcal{M}_A \to \mathcal{M}_A$. For every element $a \in A$, $\lambda_a : A_A \to A_A$ stills denoting the left multiplication by a.

Proposition 3.5 *Let A be a ring with local units, and let (F, δ, ξ) be a comonad in $\mathcal{M}_A$ such that F is a continuous functor. Then $\left(F(A), \Upsilon_{F(A)}^{F} \circ \delta_A, \xi_A\right)$ is an A-coring.*

Proof. By Lemma 3.3, $F(A)$ admits a structure of unital A-bimodule. The maps $\Delta = \Upsilon_{F(A)}^{F} \circ \delta_A$ and $\varepsilon = \xi_A$ are A-bilinear by Lemma 3.4(a)(1).

By hypothesis, we have the following diagram

$$\begin{array}{ccccc}
F(A) & \xrightarrow{\ \delta_A\ } & F^2(A) & \xrightarrow{\ \Upsilon^F_{F(A)}\ } & F(A) \otimes_A F(A) \\
\downarrow{\scriptstyle \delta_A} & (r1) & \downarrow{\scriptstyle F(\delta_A)} & (r2) & \downarrow{\scriptstyle \delta_A \otimes_A F(A)} \\
F^2(A) & \xrightarrow{\ \delta_{F(A)}\ } & F^3(A) & \xrightarrow{\ \Upsilon^F_{F^2(A)}\ } & F^2(A) \otimes_A F(A) \\
\downarrow{\scriptstyle \Upsilon^F_{F(A)}} & (r3) & \downarrow{\scriptstyle \Upsilon^{F^2}_{F(A)}} & (r4) & \downarrow{\scriptstyle \Upsilon^F_{F(A)} \otimes_A F(A)} \\
F(A) \otimes_A F(A) & \xrightarrow{\ F(A) \otimes_A \delta_A\ } & F(A) \otimes_A F^2(A) & \xrightarrow{\ F(A) \otimes_A \Upsilon^F_{F(A)}\ } & F(A) \otimes_A F(A) \otimes_A F(A)
\end{array}$$

where the rectangle $(r1)$ is commutative by Equation (2.1), and $(r2)$ by the naturality of Υ_{-}^{F}. Applying Lemma 3.4 to the natural transformation $\delta : F \to F^2$, we get the commutativity of the rectangle $(r3)$. Lastly, Lemma 3.4 applied this time to $\Upsilon_{F(-)}^{F} : F^2(-) \to F(-) \otimes_A F(A)$, gives the commutativity of the rectangle $(r4)$. Therefore, the total diagram is commutative; whence Δ is coassociative. The left counitary property is shown by the following diagram

$$\begin{array}{ccccc}
F(A) & \xrightarrow{\ \delta_A\ } & F^2(A) & \xrightarrow{\ \Upsilon^F_{F(A)}\ } & F(A) \otimes_A F(A) \\
 & \searrow{\scriptstyle =} & \downarrow{\scriptstyle F(\xi_A)} & & \downarrow{\scriptstyle \xi_A \otimes_A F(A)} \\
 & & F(A) & \xrightarrow{\ \Upsilon^F_{A}\ } & A \otimes_A F(A)
\end{array}$$

which is commutative by the naturality of Υ_{-}^{F} and Equation (2.1). Now, this last equation together with Lemma 3.4 applied to $\xi : F \to id_{\mathcal{M}_A}$ give the commutative diagram

$$\begin{array}{ccccc}
F(A) & \xrightarrow{\ \delta_A\ } & F^2(A) & \xrightarrow{\ \Upsilon^F_{F(A)}\ } & F(A) \otimes_A F(A) \\
 & \searrow{\scriptstyle =} & \downarrow{\scriptstyle \xi_{F(A)}} & & \downarrow{\scriptstyle F(A) \otimes_A \xi_A} \\
 & & F(A) & \xrightarrow{\ \Upsilon^{id_{\mathcal{M}_A}}_{F(A)}\ } & F(A) \otimes_A A
\end{array}$$

which leads to the right counitary property, and this finishes the proof. □

Let A be a ring with local units, and let (F, δ, ξ) be a comonad in $\mathcal{M}_A$. Consider the universal cogenerator of this comonad, that is, an adjunction $S^F : \mathcal{M}_A^F \rightleftarrows \mathcal{M}_A : T^F$ such that $F = S^F \circ T^F$, and where $\mathcal{M}_A^F$ is the category of comodules over (F, δ, ξ). The functor S^F is then the forgetful functor: $S^F\left(X, d^X\right) = X$, for every comodule $\left(X, d^X\right)$, and the functor T^F is defined over objects by $T^F(M) = (F(M), \delta_M)$, for every right unital A-module M. Given an A-coring $(\mathfrak{C}, \Delta, \varepsilon)$ it is easily seen that the canonical adjunction

$U_{\mathfrak{C}} : \mathcal{M}^{\mathfrak{C}} \rightleftarrows \mathcal{M}_A : - \otimes_A \mathfrak{C}$ is the universal cogenerator of the associated comonad $\big(- \otimes_A \mathfrak{C}, - \otimes_A \Delta, - \otimes_A \varepsilon\big)$ in $\mathcal{M}_A$. The following proposition compares the cogenerator of the comonad (F, δ, ξ) with the canonical adjunction associated to the coring $\Big(F(A), \Upsilon^A_{F(A)} \circ \delta_A, \xi_A\Big)$ of Proposition 3.5.

Proposition 3.6 *Let A be a ring with local units, and (F, δ, ξ) a comonad in $\mathcal{M}_A$ such that F is a continuous functor with the universal cogenerator $S^F : \mathcal{M}^F_A \rightleftarrows \mathcal{M}_A : T^F$. Considering $F(A)$ as an A-coring with the structure of Proposition* 3.5, *we obtain an isomorphism of categories*

$$\digamma : \mathcal{M}^F_A \underset{}{\overset{\cong}{\rightleftarrows}} \mathcal{M}^{F(A)} : \daleth,$$

such that $S^F = U_{F(A)} \circ \digamma$, and $\digamma \circ T^F \cong - \otimes_A F(A)$ is a natural isomorphism.

P r o o f. Given $\big(X, d^X\big)$ an object of $\mathcal{M}^F_A$, we get a diagram

$$\begin{array}{ccccc}
X & \xrightarrow{\quad d^X \quad} & F(X) & \xrightarrow{\quad \Upsilon^F_X \quad} & X \otimes_A F(A) \\
\downarrow{\scriptstyle d^X} & (r1) & \downarrow{\scriptstyle \delta_X} & (r2) & \downarrow{\scriptstyle X \otimes_A \delta_A} \\
F(X) & \xrightarrow{\quad F(d^X) \quad} & F^2(X) & \xrightarrow{\quad \Upsilon^{F^2}_X \quad} & X \otimes_A F^2(X) \\
\downarrow{\scriptstyle \Upsilon^F_X} & (r3) & \downarrow{\scriptstyle \Upsilon^F_{F(X)}} & (r4) & \downarrow{\scriptstyle X \otimes_A \Upsilon^F_{F(A)}} \\
X \otimes_A F(X) & \xrightarrow{\quad d^X \otimes_A F(A) \quad} & F(X) \otimes_A F(A) & \xrightarrow{\quad \Upsilon^F_X \otimes_A F(A) \quad} & X \otimes_A F(A) \otimes_A F(A).
\end{array} \tag{3.6}$$

The rectangles $(r1)$ and $(r3)$ are by definition commutative. Applying Lemma 3.4 consecutively to the natural transformations $\delta : F \to F^2$ and $\Upsilon^F_{F(-)} : F^2(-) \to F(-) \otimes_A F(A)$, we obtain the commutativity of rectangles $(r2)$ and $(r4)$ $\big($here the natural isomorphism associated to the functor $F(-) \otimes_A F(A)$ is $\Upsilon^{F(-)\otimes_A F(A)}_{-} = \Upsilon^F_{-} \otimes_A F(A)$, see Lemma 3.4(b)$\big)$. Thus, the whole diagram is commutative which shows the coassociativity of the map $\Upsilon^F_X \circ d^X$. We also have another commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\quad d^X \quad} & F(X) & \xrightarrow{\quad \Upsilon^F_X \quad} & X \otimes_A F(A) & & \\
 & & \downarrow{\scriptstyle \xi_X} & & \downarrow{\scriptstyle X \otimes_A \xi_A} & & \\
 & & X & \xrightarrow{\quad \Upsilon^{id_{\mathcal{M}_A}}_X \quad} & X \otimes_A A & \xrightarrow{\quad (\Upsilon^{id_{\mathcal{M}_A}}_X)^{-1} \quad} & X.
\end{array}$$

Therefore, $\big(S^F(X) = X, \rho_X = \Upsilon^F_X \circ d^X\big)$ is a right $F(A)$-comodule. Now any morphism $f : \big(X, d^X\big) \to \big(X', d^{X'}\big)$ in the category $\mathcal{M}^F_A$ entails a commutative diagram

$$\begin{array}{ccccc}
X & \xrightarrow{\quad d^X \quad} & F(X) & \xrightarrow{\quad \Upsilon^F_X \quad} & X \otimes_A F(A) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle f \otimes_A F(A)} \\
X' & \xrightarrow{\quad d^{X'} \quad} & F(X') & \xrightarrow{\quad \Upsilon^F_{X'} \quad} & X' \otimes_A F(A).
\end{array}$$

Hence $\digamma : \mathcal{M}^F_A \to \mathcal{M}^{F(A)}$ defined by

$$\digamma\big(X, d^X\big) = \big(S^F(X), \Upsilon^F_X \circ d^X\big), \quad \text{and} \quad \digamma(f) = S^F(f)$$

is a well-defined functor with inverse $\daleth : \mathcal{M}^{F(A)} \to \mathcal{M}^F_A$ defined by

$$\daleth(Y, \rho_Y) = \Big(Y, d^Y = \big(\Upsilon^F_Y\big)^{-1} \circ \rho_Y\Big), \quad \text{and} \quad \daleth(g) = g.$$

Clearly the underlying right A-module of $\digamma\left(X, d^X\right)$ coincides with that of $\left(X, d^X\right)$. That is $U_{F(A)} \circ \digamma = S^F$. The commutative rectangles $(r2)$ and $(r4)$ in diagram (3.6) assert that $\Upsilon^F_X : \digamma \circ T^F(X) = \digamma(F(X), \delta_X) \to \left(X \otimes_A F(A), X \otimes_A \left(\Upsilon^F_{F(A)} \circ \delta_A\right)\right)$ is an isomorphism of right $F(A)$-comodules, for every right unital module $X \in \mathcal{M}_A$. This leads to the stated natural isomorphism $\Upsilon^F_- : \digamma \circ T^F(-) \cong - \otimes_A F(A)$. $\square$

Remark 3.7 Of course one can work with left unital modules, and prove similar results concerning functors which are right exact and preserve direct sums and the induced corings from their comonads. In this paper we only work with right unital modules. The left-right relationship is omitted.

4 The bi-equivalence of bicategories

In this section we define the bicategory of corings over rings with local units in the same way as in [7], using general methods from [24]. Next, we establish a bi-equivalence between this bicategory and the 2-category of comonads in right unital modules over rings with local units as it was defined in Section 2.

In what follows $\mathscr{B}$ denotes the bicategory of unital bimodules (i.e. 0-cells are all rings with local units, Hom-Categories are categories of unital bimodules). The multiplications are given by the tensor product bi-functors. Let us consider as in Section 2, the 2-category $\widetilde{\mathscr{L}}$ whose 0-cells are all Grothendieck categories of the form $\mathcal{M}_A$ for some ring with local units A, and whose Hom-Categories $\overline{\mathrm{Funct}}(\mathcal{M}_A, \mathcal{M}_B)$ consists of continuous functors. The multiplications are given by the usual compositions of functors and natural transformations. The identity 1-cell of a given 0-cell $\mathcal{M}_A$ is the identity functor $id_{\mathcal{M}_A}$. There is another bicategory which we denote by $\mathscr{L}$ whose class of 0-cells is the class of all rings with local units, and with the same Home-Categories $\overline{\mathrm{Funct}}(\mathcal{M}_A, \mathcal{M}_B)$. Next, we formulate our results using the bicategory $\mathscr{L}$ instead of $\widetilde{\mathscr{L}}$.

Proposition 4.1 *Keep the above notations. There exists a bi-equivalence of bicategories $\mathscr{F} : \mathscr{L} \to \mathscr{B}$ given locally by the functors*

$$\begin{array}{rcl} \mathscr{F}_{A,B} : \overline{\mathrm{Funct}}(\mathcal{M}_A, \mathcal{M}_B) & \longrightarrow & {}_A\mathcal{M}_B \\ F & \longrightarrow & F(A) \\ \eta_- & \longrightarrow & \eta_A, \end{array}$$

for every pair of ring with local units (A, B).

P r o o f. It will be done in three steps.

Step 1. Homomorphism of bicategories. The morphism $\mathscr{F}$ is the identity on the class of objects ($\mathscr{L}$ and $\mathscr{B}$ have the same class of objects). Taking two rings with local units A and B, the stated functor $\mathscr{F}_{A,B}$ is well-defined thanks to Lemmata 3.3 and 3.4 (1). We need to show its compatibility with the horizontal and vertical multiplications. So let C be another ring with local units and consider two morphisms: $\eta_- : F \to F'$ in the category $\overline{\mathrm{Funct}}(\mathcal{M}_A, \mathcal{M}_B)$, and $\zeta : G \to G'$ in the category $\overline{\mathrm{Funct}}(\mathcal{M}_B, \mathcal{M}_C)$. By Lemma 3.4 (1), the morphism

$$\mathscr{F}_{A,B}(F).\mathscr{F}_{B,C}(G) = F(A) \otimes_B G(B) \xrightarrow{\Theta^G_{F(A)}} \mathscr{F}_{A,C}(G \circ F) = GF(A)$$

is $A - C$-bilinear, where $\Theta^G_{F(A)} = \left(\Upsilon^G_{F(A)}\right)^{-1}$ and Υ^-_- is the natural transformation of Lemma 3.3. Moreover, applying Lemma 3.4, we obtain the following commutative diagram

$$\begin{array}{ccc} F(A) \otimes_B G(B) & \xrightarrow{\Theta^G_{F(A)}} & GF(A) \\ \Big\downarrow {\scriptstyle \eta_A \otimes_B \zeta_B = \mathscr{F}(\eta).\mathscr{F}(\zeta)} & & \Big\downarrow {\scriptstyle \mathscr{F}(\eta.\zeta) = \zeta_{F'(A)} \circ G\eta_A} \\ F'(A) \otimes_B G'(B) & \xrightarrow{\Theta^{G'}_{F'(A)}} & G'F'(A) \end{array}$$

This implies that $\mathscr{V}_{F,G} := \Theta^G_{F(A)} : \mathscr{F}(F).\mathscr{F}(G) \cong \mathscr{F}(G \circ F)$ is a natural isomorphism. Clearly $\mathscr{F}(id_{\mathcal{M}_A}) = id_{\mathcal{M}_A}(A) = A$, and the compatibility with the associativity, left and right multiplications by identities 1-cell is fulfilled. Therefore, the pair $(\mathscr{F}, \mathscr{V})$ establishes a homomorphisms of bicategories from $\mathscr{L}$ to $\mathscr{B}$.

Step 2. Local equivalences of Hom-Categories. Given two rings with local units A and B, and consider the stated functor $\mathscr{F}_{A,\,B}$. Define the functor $\mathscr{G}_{A,\,B} : {}_A\mathcal{M}_B \to \overline{\mathrm{Funct}}(\mathcal{M}_B, \mathcal{M}_C)$ acting on objects by $M \to -\otimes_A M$ and on morphisms by $f \to -\otimes_A f$. It is clear that Υ^F_{-} gives a natural isomorphism $\mathscr{G}_{A,\,B} \circ \mathscr{F}_{A,\,B}(F) = -\otimes_A F(A) \cong F$, for any functor $F \in \overline{\mathrm{Funct}}(\mathcal{M}_A, \mathcal{M}_B)$. That is $\mathscr{G}_{A,\,B} \circ \mathscr{F}_{A,\,B} \cong id_{\overline{\mathrm{Funct}}(\mathcal{M}_A, \mathcal{M}_B)}$. Conversely, for any (A,B)-bimodule M, we have a natural isomorphism of bimodules $\mathscr{F}_{A,\,B} \circ \mathscr{G}_{A,\,B}(M) = A \otimes_A M \cong M$. That is $\mathscr{G}_{A,\,B} \circ \mathscr{F}_{A,\,B} \cong id_{{}_A\mathcal{M}_B}$. Therefore, $\mathscr{F}_{-,-}$ are equivalences of Hom-Categories.

Step 3. Surjectivity up to equivalences. It is immediate since $\mathscr{L}$ and $\mathscr{B}$ have the same class of objects which are not altered by $\mathscr{F}$. □

Remark 4.2 As was pointed by the referee there is an alternative proof of Proposition 4.1 which uses results from [20]. There, firm rings (resp. firm modules) (see Remark 5.6 for definitions) were termed regular algebras (resp. regular modules). The regularity of a functor $F : \mathcal{M}_A \to \mathcal{M}_B$ ([20, Definition 1.5]) between categories of firm modules over firm rings A and B, means that $F(A)$ is a firm $A-B$-bimodule. Rings with local units are firm rings. So by Lemma 3.3, right exact and direct sums preserving functors (i.e., continuous) are regular. Hence Proposition 4.1 follows by [20, Proposition 2.1]. Another way to obtain Proposition 4.1 for rings with orthogonal idempotents, is by using the arguments done before [20, Section 2]. For a given ring A with a set of orthogonal idempotents $\{e_i\}_{i\in I}$ $\big($i.e., $A = \oplus_{i\in I} A e_i = \oplus_{i \in I} e_i A\big)$, one can construct a right A-linear map $\varphi : A \to \oplus_{a\in A} A$ such that

$$A \xrightarrow{\ \varphi\ } \oplus_{a\in A} A \xrightarrow{\ \theta\ } A \otimes_{\mathbb{K}} A$$

is a section for the multiplication, where θ is the right A-linear map induced be tensoring with a fixed element of A (see [20, p. 142]). Here φ is defined as follows: If $a \in A$, and $e_{i_1}, \dots, e_{i_n}$ the associated idempotents such that $a = e_{i_1}a + \cdots + e_{i_n}a$, we then take $\varphi(a) = e_{i_1}a \dotplus \cdots \dotplus e_{i_n}a$.

By A-comonad, we denote the category of all comonads in $\mathcal{M}_A$ whose underlying functors are right exact and preserve direct sums, that is, the category of Definition 2.3 associated to the Grothendieck category $\mathcal{M}_A$. The following corollary is a direct consequence of Propositions 4.1 and 3.5.

Corollary 4.3 *Let A be a ring with local units. Then the functor*

$$\mathscr{F} : \ A\text{-}comonad \longrightarrow A\text{-}coring$$

defined by

$$\Big((F,\delta,\xi) \longrightarrow \Big(F(A), \Upsilon^F_{F(A)} \circ \delta_A, \xi_A\Big)\Big), \quad \Big([\Phi : F \to F'] \longrightarrow \Big[\Phi_A : F(A) \to F'(A)\Big]\Big)$$

establishes an equivalence of categories.

The right Eilenberg–Moore bicategory associated to $\mathscr{B}$ is given by following corollary which is the non-unital version of [7, 2.1]

Corollary 4.4 *The following data form a bicategory $\mathscr{R}$:*

- 0*-cells: corings* $(\mathfrak{C} : A)$ (*i.e., A is a ring with local units and $\mathfrak{C}$ is an A-coring*).
- 1*-cells: From* $(\mathfrak{D} : B)$ *to* $(\mathfrak{C} : A)$ *are pairs* $(M, \mathfrak{m})$ *consisting of an unital* (A,B)*-bimodule M and an $A-B$-bilinear map $\mathfrak{m} : \mathfrak{C} \otimes_A M \to M \otimes_B \mathfrak{D}$ compatible with comultiplications and counits, that is,* $\mathfrak{m}$ *satisfies*

$$\begin{aligned} &(M \otimes_B \varepsilon_{\mathfrak{D}}) \circ \mathfrak{m} = \varepsilon_{\mathfrak{C}} \otimes_A M, \\ &(\mathfrak{m} \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \mathfrak{m}) \circ (\Delta_{\mathfrak{C}} \otimes_A M) = (M \otimes_B \Delta_{\mathfrak{D}}) \circ \mathfrak{m}, \end{aligned} \tag{4.1}$$

where the first equality is up to the isomorphism $A \otimes_A M \cong M \otimes_B B$. The identity 1*-cells for a given coring $(\mathfrak{C} : A)$ is given by the pair $(A, \mathfrak{C} \otimes_A A \cong A \otimes_A \mathfrak{C})$.*

- 2-*cells: From* $(M, \mathfrak{m})$ *to* $(M', \mathfrak{m}')$ *are* $A - B$*-bilinear maps* $\mathfrak{a} : \mathfrak{C} \otimes_A M \to M'$ *satisfying*

$$(\mathfrak{a} \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \mathfrak{m}) \circ (\Delta_{\mathfrak{C}} \otimes_A M) \;=\; \mathfrak{m}' \circ (\mathfrak{C} \otimes_A \mathfrak{a}) \circ (\Delta_{\mathfrak{C}} \otimes_A M). \tag{4.2}$$

Laws composition are defined as in [7, 2.1], and given by

$$(M, \mathfrak{m}) \otimes (N, \mathfrak{n}) = \big(M \otimes_B N,\, (M \otimes_B \mathfrak{n}) \circ (\mathfrak{m} \otimes_B N)\big) \tag{4.3}$$

If $\mathfrak{a} : \mathfrak{C} \otimes_A M \to M'$ and $\mathfrak{b} : \mathfrak{D} \otimes_B N \to N'$ are 2-cells, then

$$\mathfrak{a} \otimes \mathfrak{b} \;=\; (M' \otimes_B \mathfrak{b}) \circ \mathfrak{m}' \circ (\mathfrak{C} \otimes_A \mathfrak{a} \otimes_B N) \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B N). \tag{4.4}$$

The resulting category of all 1- and 2-cells from $(\mathfrak{D} : B)$ to $(\mathfrak{C} : A)$ is denoted by ${}_{(\mathfrak{C}:\,A)}\mathscr{R}_{(\mathfrak{D}:\,B)}$.

Let's keep now the notations of Section 2. Then the right Eilenberg–Moore 2-category associated to $\mathscr{L}$ is defined as follows

Corollary 4.5 *The following data form a* 2*-category* $\mathscr{C}$*:*

- 0-cells*: They are pairs* $(\boldsymbol{F} : A)$*, that is,* $\boldsymbol{F} = (F, \delta, \xi) \in A$*-comonad where* A *is a ring with local units* (*i.e.,* F *is a continuous functor*)*.*
- 1-cells*: From* $(\boldsymbol{G} : B)$ *to* $(\boldsymbol{F} : A)$ *is a pair* (S, s) *consisting of a continuous functor* $S : \mathcal{M}_A \to \mathcal{M}_B$ *and a natural transformation* $\mathsf{s} : SF \to GS$ *satisfying the commutativity of diagrams in Equation* (2.6)*.*
- 2-cells*: Given* (S, s) *and* (S', s') *two* 1*-cells from* $(\boldsymbol{G} : B)$ *to* $(\boldsymbol{F} : A)$*, a* 2*-cell* $\alpha : (S, \mathsf{s}) \to (S', \mathsf{s}')$ *is a natural transformation* $\alpha : SF \to S'$ *satisfying the commutativity of diagram in Equation* (2.7)*.*

The category obtained by all 1- and 2-cells from $(\boldsymbol{G} : B)$ to $(\boldsymbol{F} : A)$ is denoted by ${}_{(\boldsymbol{F}:\,A)}\mathscr{C}_{(\boldsymbol{G}:\,B)}$.

The following is our main result of this section.

Theorem 4.6 *There is a bi-equivalence between the bicategory* $\mathscr{R}$ *of corings over rings with locals units* (*Corollary* 4.4)*, and the bicategory* $\mathscr{C}$ *whose objects are comonads with continuous underlying functors over right unital modules* (*Corollary* 4.5)*. This bi-equivalence is locally induced by the functors* $\mathscr{F}_{(-,-)}$ *defined by*

$$\mathscr{F}_{(\boldsymbol{F},\boldsymbol{G})} :\ {}_{(\boldsymbol{F}:\,A)}\mathscr{C}_{(\boldsymbol{G}:\,B)} \longrightarrow {}_{(F(A):\,A)}\mathscr{R}_{(G(B):\,B)}$$

$$(S, \mathsf{s}) \longrightarrow \left(S(A), \Upsilon^{G}_{S(A)} \circ \mathsf{s}_A \circ \left(\Upsilon^{S}_{F(A)}\right)^{-1} \right),$$

$$[\alpha : SF \to S'] \longrightarrow \left[\alpha_A \circ \left(\Upsilon^{S}_{F(A)}\right)^{-1} : F(A) \otimes_A S(A) \to S'(A) \right],$$

where Υ^{-}_{-} *are the natural isomorphisms of Lemma* 3.3*, and where* $(\boldsymbol{F} : A)$ (*resp.* $(\boldsymbol{G} : B)$) *is sent to* $(F(A) : A)$ (*resp. to* $(G(B) : B)$) *is the coring constructed in Proposition* 3.5*.*

Proof. It is a consequence of Proposition 4.1 and [24, Remark 1.1]. □

Remark 4.7 If we want to study any aspect in the right Eilenberg–Moore bicategory $\mathscr{R}$, then it is convenient, using the local equivalences $\mathscr{F}_{-,-}$ stated in Theorem 4.6, to transfer this study to the 2-category $\mathscr{C}$. The local equivalences in the other direction are given by the functors

$$\mathscr{G}_{(\mathfrak{C},\mathfrak{D})} :\ {}_{(\mathfrak{C}:A)}\mathscr{R}_{(\mathfrak{D}:B)} \longrightarrow {}_{(\boldsymbol{F}:\,A)}\mathscr{C}_{(\boldsymbol{G}:\,B)}$$

defined by

$$\Big((M, \mathfrak{m}) \longrightarrow (- \otimes_A M, - \otimes_A \mathfrak{m})\Big),$$

$$\Big([\mathfrak{a} : \mathfrak{C} \otimes_A M \to M'] \longrightarrow \Big[\ - \otimes_A \mathfrak{a} : - \otimes_A \mathfrak{C} \otimes_A M \to - \otimes_A M' \Big]\Big),$$

where $\mathbf{F}$ (resp. $\mathbf{G}$) is the comonad induced by the coring $(\mathfrak{C} : A)$ (resp. by $(\mathfrak{D} : B)$). These local equivalences are not in fact given by [24, Remark 1.1]. Their construction was given separately using direct computations.

5 Base ring extension of a coring by an adjunction

In this section we apply results from Sections 2 and 3, to extend the notion of base ring extension of a coring by a (finitely generated and projective) module, introduced in [7], to the case of rings with local units. This will give a new class of corings over rings with local units which includes some infinite comatrix corings [13].

The following proposition characterizes an adjunction between right unital modules with continuous functors (i.e. right exact and direct sums preserving functors).

Proposition 5.1 *Let A and B be two rings with local units. The following statements are equivalent.*

(i) *There is an adjunction $S : \mathcal{M}_B \rightleftarrows \mathcal{M}_A : T$ with S left adjoint to T, and such that S, T are continuous functors.*

(ii) *There is an unital (B, A)-bimodule Σ such that $h\Sigma$ is finitely generated and projective unital right A-module, for every $h \in \mathrm{Idemp}(B)$.*

P r o o f. (ii) ⇒ (i). We denote by $\Sigma^\dagger = A\mathrm{Hom}_A(\Sigma, A)\, B$ the unital right dual of Σ. The bi-actions are defined as follows: For $a \in A$, $b \in B$, and $\chi \in \Sigma^\dagger$, we have

$$(a.\chi)(x) = a\chi(x), \quad (\chi.b)(x) = \chi(bx), \quad \text{for all} \quad x \in \Sigma.$$

This is clearly the unital part of the (A, B)-bimodule $\mathrm{Hom}_A(\Sigma, A)$. The unit of the adjunction is given by

$$\begin{aligned} \eta_{Y_B} : Y &\longrightarrow Y \otimes_B \Sigma \otimes_A \Sigma^\dagger, \\ y &\longmapsto \textstyle\sum_{i=1}^{n_h} y \otimes_B u_i \otimes_A (u_i^* \circ \pi_h), \end{aligned} \tag{5.1}$$

where $h \in \mathrm{Idemp}(B)$ such that $yh = y$, and $\{(u_i, u_i^*)\}_{1 \le i \le n_h} \subset h\Sigma \times \mathrm{Hom}_A(h\Sigma, A)$ is the finite right dual basis for $h\Sigma$, and $\pi_h : \Sigma \to h\Sigma$ is the canonical projection. We denote by v_i^* the composition

$$v_i^* : \Sigma \xrightarrow{\ \pi_h\ } h\Sigma \xrightarrow{\ u_i^*\ } A \quad \in \Sigma^\dagger.$$

We claim that $\eta_{Y_B}(yb) = \eta_{Y_B}(y)b$, for every $b \in B$ such that $b = hb = bh$. We have

$$\begin{aligned} \sum_{i=1}^{n_h} yb \otimes_B u_i \otimes_A (u_i^* \circ \pi_h) &= \sum_{i=1}^{n_h} y \otimes_B bu_i \otimes_A (u_i^* \circ \pi_h) \\ &= \sum_{i=1}^{n_h} \sum_{i'=1}^{n_h} y \otimes_B u_{i'} \otimes_A u_{i'}^*(bu_i)(u_i^* \circ \pi_h) \\ &= \sum_{i'=1}^{n_h} y \otimes_B u_{i'} \otimes_A \left(\sum_{i=1}^{n_h} u_{i'}^*(bu_i)(u_i^* \circ \pi_h) \right) \\ &= \sum_{i'=1}^{n_h} y \otimes_B u_{i'} \otimes_A \left(\left(\sum_{i=1}^{n_h} u_{i'}^*(bu_i)u_i^* \right) \circ \pi_h \right) \\ &= \sum_{i'=1}^{n_h} y \otimes_B u_{i'} \otimes_A ((u_{i'}^* b) \circ \pi_h) \\ &= \sum_{i'=1}^{n_h} y \otimes_B u_{i'} \otimes_A (u_{i'}^* \circ \pi_h)b, \quad \text{since } b = bh = hb, \end{aligned}$$

and this proves the claim. Next, we prove that η_{Y_B} is independent from the choice of h. So, let's fix an arbitrary element $y \in Y$ and let $h' \in \mathrm{Idemp}(B)$ be another unity for y (i.e., $yh' = y = yh$). We consider as before $\{(x_j, w_j^*)\}_{1 \le j \le n_{h'}} \subset h'\Sigma \times \Sigma^\dagger$ the induced set by the dual basis $\{(x_j, x_j^*)\}_{1 \le j \le n_{h'}}$ of $h'\Sigma$, where $w_j^* = x_j^* \circ \pi_{h'}$. Henceforth, its remains to prove that

$$\sum_{i=1}^{n_h} y \otimes_B u_i \otimes_A v_i^* = \sum_{j=1}^{n_{h'}} y \otimes_B x_j \otimes_A w_j^*.$$

So, let $h'' \in \mathrm{Idemp}(B)$ such that $h = hh'' = h''h$ and $h' = h'h'' = h''h'$, and consider again its corresponding set $\{(z_k, t_k^*)\}_{1 \leq k \leq n_{h''}} \subset h''\Sigma \times \Sigma^\dagger$, where $t_k^* = z_k^* \circ \pi_{h''}$. Using elementary arguments, one can directly check that

$$v_i^* = (v_i^* \circ \tau_{h''}).h'' \quad \text{and} \quad w_j^* = (w_j^* \circ \tau_{h''}).h'', \quad \text{for every pair} \quad (i,j) \in \{1,\ldots,n_h\} \times \{1,\ldots,n_{h'}\},$$

where $\tau_{h''} : h''\Sigma \to \Sigma$ is the canonical injection. On the other hand, we have $\sum_{1 \leq k \leq n_{h''}} (v_i^* \circ \tau_{h''})(z_k)t_i^* = (v_i^* \circ \tau_{h''}).h''$. Taking all these equalities into account, we compute

$$\begin{aligned}
\sum_i^{n_h} y \otimes_B u_i \otimes_A v_i^* &= \sum_i^{n_h} y \otimes_B u_i \otimes_A (v_i^* \circ \tau_{h''})h'' \\
&= \sum_{i,k}^{n_h, n_{h''}} y \otimes_B u_i \otimes_A (v_i^* \circ \tau_{h''})(z_k)t_k^* \\
&= \sum_k^{n_{h''}} y \otimes_B \left(\sum_i^{n_h} u_i (v_i^* \circ \tau_{h''})(z_k) \right) \otimes_A t_k^* \\
&= \sum_k^{n_{h''}} y \otimes_B \left(\sum_i^{n_h} u_i u_i^*(hz_k) \right) \otimes_A t_k^*, \quad z_k \in h''\Sigma \\
&= \sum_k^{n_{h''}} y \otimes_B hz_k \otimes_A t_k^* \\
&= \sum_k^{n_{h''}} y \otimes_B z_k \otimes_A t_k^*.
\end{aligned}$$

Similar computation entails the equality

$$\sum_j^{n_{h'}} y \otimes_B x_j \otimes_A w_j^* = \sum_k^{n_{h''}} y \otimes_B z_k \otimes_A t_k^*,$$

and this proves the desired independence. Therefore, η_{Y_B} is a well-defined right B-linear map, for every right unital B-module Y. Clearly, $\eta_- : id_{\mathcal{M}_B}(-) \to - \otimes_B \Sigma \otimes_A \Sigma^\dagger$ is a natural transformation. The counit is given by

$$\begin{aligned}
\zeta_{X_A} : X \otimes_A \Sigma^\dagger \otimes_B \Sigma &\longrightarrow X, \\
x \otimes_A \varphi \otimes_B u &\longmapsto x\varphi(u).
\end{aligned} \tag{5.2}$$

Lastly, one can easily show that

$$(\zeta_X \otimes_A \Sigma^\dagger) \circ \eta_{X \otimes_A \Sigma^\dagger} = X \otimes_A \Sigma^\dagger, \quad \text{and} \quad \zeta_{Y \otimes_B \Sigma} \circ (\eta_Y \otimes_B \Sigma) = Y \otimes_B \Sigma,$$

for every pair of unital modules (Y_B, X_A) which implies the desired adjunction taking $S(-) = - \otimes_B \Sigma$ and $T(-) = - \otimes_A \Sigma^\dagger$.

(i) ⇒ (ii). By Lemma 3.3 we know that ${}_BS(B)_A$ and ${}_AT(A)_B$ are unital bimodules, and that $S(-) \cong - \otimes_B S(B)$, $T(-) \cong - \otimes_A T(A)$ are natural isomorphisms. Taking ${}_B\Sigma_A = S(B)$, and $h \in \mathrm{Idemp}(B)$, we deduce that $h\Sigma = hS(B) \cong hB \otimes_B S(B) \cong S(hB)$ is a right A-linear isomorphism. Henceforth, it remains to show that $S(hB)$ is a finitely generated and projective module, for every $h \in \mathrm{Idemp}(B)$. So, the natural isomorphism of the stated adjunction gives us the following chain of natural isomorphisms

$$\mathrm{Hom}_A(S(hB), -) \cong \mathrm{Hom}_B(hB, T(-)) \cong T(-)h \cong - \otimes_A T(A)h,$$

for every $h \in \mathrm{Idemp}(B)$. That is, the functor $\mathrm{Hom}_A(S(hB), -)$ preserves inductive limits, and so $S(hB)$ is a finitely generated and projective A-module for every $h \in \mathrm{Idemp}(B)$. □

Remark 5.2 Considering Σ an unital (B, A)-bimodule, we can easily check, using the partial ordering on idempotent elements, that $\varinjlim_h(h\Sigma) \cong \Sigma$ as right unital A-modules. In fact $\{h\Sigma\}_{h\in \mathrm{Idemp}(B)}$ is a split direct system of right unital A-module (see [29, Section 1]). If we assume that Σ satisfies condition (ii) of Proposition 5.1, then Σ_A is locally projective in the sense of [3], equivalently, it is strongly locally projective in the sense of [29, Theorem 2.17].

Remark 5.3 Let A, B, and C be rings with local units. Consider Σ (respectively W) an unital (B, A)-bimodule (respectively (C, B)-bimodule) such that $h\Sigma$ (respectively gW) is finitely generated and projective unital right A-module (respectively B-module), for every $h \in \mathrm{Idemp}(B)$ (respectively $g \in \mathrm{Idemp}(C)$). Then $W \otimes_B \Sigma$ is an unital (C, A)-bimodule such that $g(W \otimes_B \Sigma)$ is finitely generated and projective unital right A-module, for every $g \in \mathrm{Idemp}(C)$. Furthermore, if we put $\Sigma^\dagger = A\mathrm{Hom}_A(\Sigma, A)\, B$ (respectively $W^\dagger = B\mathrm{Hom}_B(W, B)\, C$), then

$$(W \otimes_B \Sigma)^\dagger = A\mathrm{Hom}_A(W \otimes_B \Sigma, A)\, C \;\cong\; \Sigma^\dagger \otimes_B W^\dagger \tag{5.3}$$

is an isomorphism of unital (A, C)-bimodules. Effectively, let $g \in \mathrm{Idemp}(C)$ any idempotent element, so there exists an unital right B-module N such that

$$gW \,\oplus\, N = \bigoplus_{i=1}^{n} h_i B$$

where each $h_i \in \mathrm{Idemp}(B)$. Applying the tensor product $- \otimes_B \Sigma$, we obtain

$$(gW \otimes_B \Sigma) \,\oplus\, (N \otimes_B \Sigma) \cong \bigoplus_{i=1}^{n} (h_i\Sigma)$$

an isomorphism of unital right A-modules. Since the right-hand module is a finitely generated and projective A-module, we get that $gW \otimes_B \Sigma$ is also finitely generated and projective as an A-module, and this proves the first claim. Now, using the adjunctions arising from the proof of Proposition 5.1 and the usual Hom–Tensor adjunction, we get the isomorphism of Equation (5.3).

It is convenient to adopt the notations of the proof of Proposition 5.1. Thus, if Σ is any (B, A)-bimodule we denote by $\Sigma^\dagger = A\mathrm{Hom}_A(\Sigma, A)\, B$. When $h\Sigma$ is a finitely generated and projective right A-module, for some $h \in \mathrm{Idemp}(B)$, we consider the set $\{(u_i, v_i^*)\}_{1\le i\le n_h} \subset h\Sigma \times \Sigma^\dagger$ where $\{(u_i, u_i^*)\}_{1\le i\le n_h} \subset h\Sigma \times \mathrm{Hom}_A(h\Sigma, A)$ is the finite dual basis for $h\Sigma$, where $v_i^* = u_i^* \circ \pi_h$ and $\pi_h : \Sigma \to h\Sigma$ is the canonical projection.

Corollary 5.4 *Let A and B be two rings with local units together with an unital (B, A)-bimodule Σ and a B-coring $(\mathfrak{D}, \Delta_{\mathfrak{D}}, \varepsilon_{\mathfrak{D}})$. Assume that $h\Sigma$ is finitely generated and projective module for every $h \in \mathrm{Idemp}(B)$. Then the unital A-bimodule $\Sigma^\dagger \otimes_B \mathfrak{D} \otimes_B \Sigma$ admits a structure of an A-coring with comultiplication defined by*

$$\Delta : \Sigma^\dagger \otimes_B \mathfrak{D} \otimes_B \Sigma \longrightarrow \Sigma^\dagger \otimes_B \mathfrak{D} \otimes_B \Sigma \otimes_A \Sigma^\dagger \otimes_B \mathfrak{D} \otimes_B \Sigma,$$
$$\varphi \otimes_B d \otimes_B u \longmapsto \sum_{i,(d)} \varphi \otimes_B d_{(1)} \otimes_B u_i \otimes_A v_i^* \otimes_B d_{(2)} \otimes_B u,$$

where $\{(u_i, v_i^*)\}_i \subset h\Sigma \times \Sigma^\dagger$ is the finite set induced by the dual basis of $h\Sigma$, where $h \in \mathrm{Idemp}(B)$ such that $\varphi h = \varphi$, $hu = u$, $d = hd = dh$, and where $\Delta_{\mathfrak{D}}(d) = \sum_{(d)} d_{(1)} \otimes_B d_{(2)}$. The counit is defined by

$$\varepsilon : \Sigma^\dagger \otimes_B \mathfrak{D} \otimes_B \Sigma \longrightarrow A \quad (\varphi \otimes_B d \otimes_B u \longmapsto \varphi(\varepsilon_{\mathfrak{D}}(d)u)).$$

P r o o f. By Proposition 5.1, we know that $- \otimes_B \Sigma : \mathcal{M}_B \to \mathcal{M}_A$ is left adjoint to $- \otimes_A \Sigma^\dagger : \mathcal{M}_A \to \mathcal{M}_B$ with unit $\eta_- : id_{\mathcal{M}_B}(-) \to - \otimes_B \Sigma \otimes_A \Sigma^\dagger$ and counit $\zeta_- : - \otimes_A \Sigma^\dagger \otimes_B \Sigma \to id_{\mathcal{M}_A}(-)$ given explicitly by Equations (5.1) and (5.2). Applying Lemma 2.1 (1) to the comonad $(- \otimes_B \mathfrak{D}, - \otimes_B \Delta_{\mathfrak{D}}, - \otimes_B \varepsilon_{\mathfrak{D}})$ in $\mathcal{M}_B$, we obtain a new comonad (G, Ω, γ) in $\mathcal{M}_A$ with

$$G \;=\; - \otimes_A \Sigma^\dagger \otimes_B \mathfrak{D} \otimes_B \Sigma,$$

$$\Omega_- = \big(\eta_{-\otimes_A \Sigma^\dagger \otimes_B \mathfrak{D}} \otimes_B \mathfrak{D} \otimes_B \Sigma\big) \circ \big(-\otimes_A \Sigma^\dagger \otimes_B \Delta_{\mathfrak{D}} \otimes_B \Sigma\big),$$
$$\gamma_- = \zeta_- \circ \big(-\otimes_A \Sigma^\dagger \otimes_B \varepsilon_{\mathfrak{D}} \otimes_B \Sigma\big).$$

Now, Proposition 3.5 implies that $G(A)$ admits a structure of an A-coring. Since $G(A) \cong \Sigma^\dagger \otimes_B \mathfrak{D} \otimes_B \Sigma$ is obviously an isomorphism of unital A-bimodules, this structure can be transferred to $\Sigma^\dagger \otimes_B \mathfrak{D} \otimes_B \Sigma$ with comultiplication and counit computed explicitly from the maps Ω_A and γ_A. This, in fact, leads exactly to the stated structure. □

Recently, in [19] and [9] new generalizations of infinite comatrix corings, earlier introduced in [13], were given in the context of non-unital (firm) rings. The following example gives another way to construct infinite comatrix corings by using Corollary 5.4.

Example 5.5 Assume that A is a ring with identity 1_A, and denote by $add(A_A)$ the full subcategory of all finitely generated and projective unital right A-modules. Consider a $\mathbb{K}$-additive small category $\mathcal{A}$ and its induced ring with enough orthogonal idempotents $B = \oplus_{(\mathfrak{p},\mathfrak{q})\in\mathcal{A}^2}\mathrm{Hom}_{\mathcal{A}}(\mathfrak{p},\mathfrak{q})$: These are $\{1_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{A}} \subset B$, where each of the $1_{\mathfrak{p}}$'s is the image of the identity $1_{\mathrm{End}_{\mathcal{A}}(\mathfrak{p})}$. Given an additive faithful functor $\omega : \mathcal{A} \to add(A_A)$, we get an unital (B, A)-bimodule $\Sigma = \oplus_{\mathfrak{p}\in\mathcal{A}}\omega(\mathfrak{p})$. It is clear that $1_{\mathfrak{p}}\Sigma = \omega(\mathfrak{p})$, for every object $\mathfrak{p} \in \mathcal{A}$. Therefore, $h\Sigma$ is a finitely generated and projective right A-module, for every $h \in \mathrm{Idemp}(B)$. Finally, considering B as a trivial B-coring, we obtain by Corollary 5.4 an A-coring $\Sigma^\dagger \otimes_B B \otimes_B \Sigma \cong \Sigma^\dagger \otimes_B \Sigma$, where $\Sigma^\dagger = \oplus_{\mathfrak{p}\in\mathcal{A}}\mathrm{Hom}_A(\omega(\mathfrak{p}), A)$.

Remark 5.6 Rings with local units are in fact a sub-class of firm rings. Recall that an associative ring R is firm if the multiplication $R \otimes_R R \to R$ is an isomorphism. Unital modules are extended to firm modules, i.e., a right R-module M with action $M \otimes_R R \to M$ an isomorphism of right R-modules. The results of this paper can be extended to this class of rings by using the categorical version of Lemmata 3.4 and 3.3 stated in [8, 39.3, 39.5] with firm base rings. The fact that right exact and direct sums preserving functors between firm modules are naturally isomorphism to the tensor product functors, has been recently proved by J. Vercruysse in [30, Theorem 3.1] (see also [20, Proposition 1.6]). A characterization of an adjunction whose both functors are right exact and preserve direct sums (as in Proposition 5.1) was extended to the case of firm modules in [30, Theorem 2.4].

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