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We recognize Harada's generalized categories of diagrams as a particular case of modules over a monad defined on a finite direct product of additive categories. We work in the dual (albeit formally equivalent) situation, that is, with comodules over comonads. With this conceptual tool at hand, we obtain several of the Harada results with simpler proofs, some of them under more general hypothesis, besides with a characterization of the normal triangular matrix comonads that are hereditary, that is, of homological dimension less than or equal to 1. Our methods rest on a matrix representation of additive functors and natural transformations, which allows us to adapt typical algebraic manipulations from Linear Algebra to the additive categorical setting.

1. Introduction

Every complete set of pairwise orthogonal idempotent elements $e_1, \ldots, e_n$ of a unital ring R allows to express the ring as a generalized $n \times n$–matrix ring with entries in the bimodules $e_i R e_j$ for $1 \leq i, j \leq n$. The origin of this decomposition can be traced back to the seminal work of Peirce [1]. Beyond the role of matrix rings in the classification of semi-simple artinian rings (i.e. Wedderburn–Artin's Theorem), these generalized matrix rings were used to investigate rings of low homological dimension in the framework of a systematic program of studying non commutative rings of finite homological dimension promoted by Eilenberg, Ikeda, Jans, Kaplansky, Nagao, Nakayama, Rosenberg and Zelinsky among others, see [2–6]. In particular, Chase [7] and Harada [8] used generalized triangular matrix rings to investigate the structure of the semi-primary hereditary rings (i.e. semi-primary rings with homological dimension 1).

Inspired by the study of homological properties of abelian categories of diagrams given in [9, Section IX], Harada formulated in [10] versions of some of his results on hereditary triangular matrix rings from [8] in the framework of the so-called abelian categories of

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generalized commutative diagrams. In this paper, we recognize these categories as the categories of modules (or algebras) over suitable monads. This allows, apart from giving a more conceptual treatment of these categories, to obtain most of the main results from [10] with sharper (and simpler, we think) proofs. Our methods, based on a sort of 'Linear Algebra' for additive functors and natural transformations, also provide some new results, including a characterization of hereditary categories of generalized commutative diagrams.

We work in the dual, although formally equivalent, situation than in [10]. Thus, we consider a comonad defined on a finite direct product of additive categories, and we investigate how to express in matrix form both its comultiplication and its counit. In this way, its category of comodules (or coalgebras) is expressed in such a way that Harada's categories of generalized diagrams become a particular case (namely, the categories of modules over a normal triangular monad). This is done in Section 2.

In Section 3, we generalize [10, Theorem 2.3] in several directions. On one hand, we do not assume that the base categories are abelian and, on the other, our categories of comodules are more general than the dual of the categories considered in [10] (see Theorem 3.4). We derive our Theorem 3.4 from a simpler result, namely Theorem 3.1, in conjunction with the kind of linear algebra for functors developed in Section 2.

In Section 4, we give full proofs of the dual form of some results from [10]. These results deal with the structure of the injective comodules over a hereditary normal triangular matrix comonad. In particular, they contain part of the dual of [10, Theorem 3.6]. We also prove a characterization of hereditary normal triangular matrix comonads (see Theorem 4.6).

Section 5 illustrates our general results by giving a characterization of the bipartite coalgebras (see [11,12]) that are right hereditary. Our methods allow us to work over a general commutative ring.

2. Matrix comonads and their categories of comodules

In this section, we introduce the notion of a matrix comonad and we describe its category of comodules. We develop a kind of linear algebra for matrix functors and their matrix natural transformations.

2.1. *Basic notions and notation*

All the categories in this paper are assumed to be additive, and all functors between them are additive. If A is an object of a category, then its identity morphism is also denoted by A. We shall use the standard notation for composition of functors and/or natural transformations, see [13].

If A, A' are objects of an additive category $\mathcal{A}$, then $A \oplus A'$ denotes its direct sum, and similarly for any finite collection of objects. Recall that the direct sum of finitely many objects is both the product and the coproduct of the family in the category. The symbol $\oplus$ will be used also for the direct sum of morphisms.

If $F, G : \mathcal{A} \to \mathcal{B}$ are functors, then its direct sum functor $F \oplus G : \mathcal{A} \to \mathcal{B}$ is given by the composition

$$F \oplus G : \mathcal{A} \xrightarrow{\Delta} \mathcal{A} \times \mathcal{A} \xrightarrow{F \times G} \mathcal{B} \times \mathcal{B} \xrightarrow{\oplus} \mathcal{B} ,$$

where Δ denotes the diagonal functor, and $\mathcal{C} \times \mathcal{D}$ denotes the product category of two categories $\mathcal{C}, \mathcal{D}$. Thus, $(F \oplus G)(f) = F(f) \oplus G(f)$ for any morphism f of $\mathcal{A}$. The direct sum of finitely many functors is defined analogously.

Let $\eta : F \to G \oplus H$ be a natural transformation, where $F, G, H : \mathcal{A} \to \mathcal{B}$ are functors. By $\pi_G : G \oplus H \to G$ (resp. $\pi_H : G \oplus H \to H$) we denote the natural transformation defined by the canonical projection. Therefore, η is uniquely determined by the natural transformations $\mu = \pi_G \eta : F \to G$ and $\nu = \pi_H \eta : F \to H$. We will then use the notation $\eta = \mu \dotplus \nu$.

We will consider comonads (or cotriples) on $\mathcal{A}$. We refer to [13] for details on (co)monads and their categories of (co)modules (or (co)algebras).

2.2. *Matrix notation*

Let $\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B}_1, \ldots, \mathcal{B}_m$ be additive categories and consider the product categories

$$\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n, \quad \mathcal{B} = \mathcal{B}_1 \times \cdots \times \mathcal{B}_m. \tag{1}$$

For every $j = 1, \ldots, n$, let $\pi_j : \mathcal{A} \to \mathcal{A}_j$ (resp. $\iota_j : \mathcal{A}_j \to \mathcal{A}$) denote the canonical projection (resp. injection) functor, and similarly for $\mathcal{B}$. An additive functor $F : \mathcal{A} \to \mathcal{B}$ is determined by $m \times n$ functors $F_{ji} = \pi_i F \iota_j : \mathcal{A}_j \to \mathcal{B}_i, j = 1, \ldots, n, i = 1, \ldots, m$, since, from the equalities

$$1_{\mathcal{A}} = \iota_1 \pi_1 \oplus \cdots \oplus \iota_n \pi_n, \quad 1_{\mathcal{B}} = \iota_1 \pi_1 \oplus \cdots \oplus \iota_m \pi_m,$$

we obtain

$$F = \bigoplus_{i,j} \iota_i \pi_i F \iota_j \pi_j = \bigoplus_{i,j} \iota_i F_{ji} \pi_j.$$

This means that, given a morphism $f = (f_1, \ldots, f_n)$ in $\mathcal{A}$, we have

$$Ff = (\oplus_j F_{j1} f_j, \ldots, \oplus_j F_{jm} f_j).$$

This expression can be represented in matrix form as

$$Ff = \begin{pmatrix} F_{11} & F_{21} & \cdots & F_{n1} \\ F_{12} & F_{22} & \cdots & F_{n2} \\ \vdots & \vdots & & \vdots \\ F_{1m} & F_{2m} & \cdots & F_{nm} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

Now, if $\eta : F \to G$ is a natural transformation, where $G : \mathcal{A} \to \mathcal{B}$ is a second functor, then we have the natural transformations

$$\eta^{ij} = \pi_j \eta \iota_i : F_{ij} \to G_{ij}, \qquad i = 1, \ldots, n, \ j = 1, \ldots, m, \tag{2}$$

which completely determine η as follows. For each $i = 1, \ldots, n$, and each $A \in \mathcal{A}$, we consider the canonical morphism $\xi_A^i : \iota_i \pi_i A \to A$ given by the decomposition $A = \iota_1 \pi_1 A \oplus \cdots \oplus \iota_n \pi_n A$. From the naturality of η, we get the following commutative diagrams

$$FA = F\iota_1\pi_1 A \oplus \cdots \oplus F\iota_n\pi_n A \xrightarrow{\eta_A} GA = G\iota_1\pi_1 A \oplus \cdots \oplus G\iota_n\pi_n A$$

$$F\xi_A^i \Big\uparrow \qquad\qquad\qquad\qquad G\xi_A^i \Big\uparrow$$

$$F\iota_i\pi_i A \xrightarrow{\quad\eta_{\iota_i\pi_i A}\quad} G\iota_i\pi_i A$$

which show that $\eta_A = \oplus_i \eta_{\iota_i \pi_i A}$ and, thus, for each $j = 1, \ldots, n$, we have

$$\pi_j \eta_A = \oplus_i \pi_j \eta_{\iota_i \pi_i A} = \oplus_i \eta_{\pi_i A}^{ij}.$$

Since η_A is defined by the morphisms $\pi_j \eta_A$, $j = 1, \ldots, m$, we get that η is entirely determined by η^{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, m$. So we can represent our natural transformation in a matrix form

$$\eta = \begin{pmatrix} \eta^{11} & \eta^{21} & \cdots & \eta^{n1} \\ \eta^{12} & \eta^{22} & \cdots & \eta^{n2} \\ \vdots & \vdots & & \vdots \\ \eta^{1m} & \eta^{2m} & \cdots & \eta^{nm} \end{pmatrix} : \begin{pmatrix} F_{11} & F_{21} & \cdots & F_{n1} \\ F_{12} & F_{22} & \cdots & F_{n2} \\ \vdots & \vdots & & \vdots \\ F_{1m} & F_{2m} & \cdots & F_{nm} \end{pmatrix} \longrightarrow \begin{pmatrix} G_{11} & G_{21} & \cdots & G_{n1} \\ G_{12} & G_{22} & \cdots & G_{n2} \\ \vdots & \vdots & & \vdots \\ G_{1m} & G_{2m} & \cdots & G_{nm} \end{pmatrix}.$$

2.3. *Matrix operations*

Let

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

be functors, where $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$, and $\mathcal{A}, \mathcal{B}$ are finite products as in (1). Then, the composite functor $H = GF$ is represented in matrix form as the 'usual matrix product' of matrices representing G and F. This means that

$$H = \begin{pmatrix} H_{11} & H_{21} & \cdots & H_{n1} \\ H_{12} & H_{22} & \cdots & H_{n2} \\ \vdots & \vdots & & \vdots \\ H_{1l} & H_{2l} & \cdots & H_{nl} \end{pmatrix},$$

where

$$H_{ji} = \bigoplus_{k=1}^{m} G_{ki} F_{jk}, \qquad i = 1, \ldots, l, \; j = 1, \ldots, n.$$

With respect to the vertical composition of natural transformations, let $\eta : F \to F'$, $\mu : F' \to F''$ be natural transformations, where $F, F', F'' : \mathcal{A} \to \mathcal{B}$ are functors. A straightforward argument shows that the matrix representing $\mu\eta$ is the 'Hadamard product' of the matrices that represent μ and η, namely

$$\mu\eta = \begin{pmatrix} \mu^{11} & \mu^{21} & \cdots & \mu^{n1} \\ \mu^{12} & \mu^{22} & \cdots & \mu^{n2} \\ \vdots & \vdots & & \vdots \\ \mu^{1m} & \mu^{2m} & \cdots & \mu^{nm} \end{pmatrix} \cdot \begin{pmatrix} \eta^{11} & \eta^{21} & \cdots & \eta^{n1} \\ \eta^{12} & \eta^{22} & \cdots & \eta^{n2} \\ \vdots & \vdots & & \vdots \\ \eta^{1m} & \eta^{2m} & \cdots & \eta^{nm} \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} \mu^{11}\eta^{11} & \mu^{21}\eta^{21} & \cdots & \mu^{n1}\eta^{n1} \\ \mu^{12}\eta^{12} & \mu^{22}\eta^{22} & \cdots & \mu^{n2}\eta^{n2} \\ \vdots & \vdots & & \vdots \\ \mu^{1m}\eta^{1m} & \mu^{2m}\eta^{2m} & \cdots & \mu^{nm}\eta^{nm} \end{pmatrix}$$

Finally, let us consider the horizontal composition of natural transformations. So let $\eta : F \to F'$ with $F, F' : \mathcal{A} \to \mathcal{B}$, and $G : \mathcal{B} \to \mathcal{C}$. The natural transformation $G\eta : GF \to GF'$ is easily shown to have a matrix representation

$$G\eta = \begin{pmatrix} (G\eta)^{11} & (G\eta)^{21} & \cdots & (G\eta)^{n1} \\ (G\eta)^{12} & (G\eta)^{22} & \cdots & (G\eta)^{n2} \\ \vdots & \vdots & & \vdots \\ (G\eta)^{1l} & (G\eta)^{2l} & \cdots & (G\eta)^{nl} \end{pmatrix},$$

where

$$(G\eta)^{ji} = \bigoplus_{k=1}^{m} G_{ki}\eta^{jk}, \qquad i = 1, \ldots, l, \ j = 1, \ldots, n.$$

That is, the composition $G\eta$ leads to the 'usual matrix product'. This also holds for the composition in the opposite order, that is, given a natural transformation $\gamma : G \to G'$, for functors $G, G' : \mathcal{B} \to \mathcal{C}$, and $F : \mathcal{A} \to \mathcal{B}$, the matrix representation of the natural transformation $\gamma F : GF \to G'F$ is given by

$$\gamma F = \begin{pmatrix} (\gamma F)^{11} & (\gamma F)^{21} & \cdots & (\gamma F)^{n1} \\ (\gamma F)^{12} & (\gamma F)^{22} & \cdots & (\gamma F)^{n2} \\ \vdots & \vdots & & \vdots \\ (\gamma F)^{1l} & (\gamma F)^{2l} & \cdots & (\gamma F)^{nl} \end{pmatrix},$$

where

$$(\gamma F)_{ji} = \bigoplus_{k=1}^{m} \gamma^{ki} F_{jk}, \qquad i = 1, \ldots, l, \ j = 1, \ldots, n.$$

2.4. *Matrix comonads*

Fix a category $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. Let $F : \mathcal{A} \to \mathcal{A}$ be an endofunctor, and $\delta : F \to F^2$ a natural transformation. In matrix form,

$$\delta = (\delta^{ij}) : (F_{ij}) \to (F_{ij})(F_{ij}),$$

where $\delta^{ij} : F_{ij} \to \oplus_k F_{kj} F_{ik}$ is a natural transformation for every $i, j = 1, \ldots, n$, which is uniquely expressed as

$$\delta^{ij} = \delta^{i1j} \dotplus \delta^{i2j} \dotplus \cdots \dotplus \delta^{inj}$$

for some natural transformations $\delta^{ikj} : F_{ij} \to F_{kj} F_{ik}$ with $k = 1, \ldots, n$. The coassociativity of F is given by the equation $(\delta F)\delta = (F\delta)\delta$ or, equivalently,

$$(\delta F)^{ij}\delta^{ij} = (F\delta)^{ij}\delta^{ij}, \quad 1 \le i, j \le n. \tag{4}$$

We know that

$$(\delta F)^{ij} = \bigoplus_{k} \delta^{kj} F_{ik} \quad \text{and} \quad (F\delta)^{ij} = \bigoplus_{l} F_{lj}\delta^{il}.$$

Observe that for each $i, j, k = 1, \ldots, n$, the natural transformation $\delta^{kj} F_{ik} : F_{kj} F_{ik} \to \bigoplus_l F_{lj} F_{kl} F_{ik}$ is given by

$$\delta^{kj} F_{ik} = \delta^{k1j} F_{ik} \dotplus \delta^{k2j} F_{ik} \dotplus \cdots \dotplus \delta^{knj} F_{ik}$$

while, for each $i, j, l = 1, \ldots, n$, the natural transformation $F_{lj}\delta^{il} : F_{lj}F_{il} \to \bigoplus_k F_{lj}F_{kl}F_{ik}$ is defined by

$$F_{lj}\delta^{il} = F_{lj}\delta^{i1l} \dotplus F_{lj}\delta^{i2l} \dotplus \cdots \dotplus F_{lj}\delta^{inl}.$$

We then obtain that

$$(\delta F)^{ij}\delta^{ij} = \left(\bigoplus_k \delta^{kj}F_{ik}\right)\left(\dotplus_k \delta^{ikj}\right) = \dotplus_{k,l}(\delta^{klj}F_{ik})\delta^{ikj}$$

and

$$(F\delta)^{ij}\delta^{ij} = \left(\bigoplus_l F_{lj}\delta^{il}\right)\left(\dotplus_l \delta^{ilj}\right) = \dotplus_{k,l}(F_{lj}\delta^{ikl})\delta^{ilj}.$$

Both vectors should be equal, which is equivalent to

$$(\delta^{klj}F_{ik})\delta^{ikj} = (F_{lj}\delta^{ikl})\delta^{ilj}, \quad (1 \le i, j, k, l \le n).$$

That is, all the diagrams

$$
\begin{array}{ccc}
F_{ij} & \xrightarrow{\ \delta^{ilj}\ } & F_{lj}F_{il} \\
{\scriptstyle \delta^{ikj}}\big\downarrow & & \big\downarrow{\scriptstyle F_{lj}\delta^{ikl}} \\
F_{kj}F_{ik} & \xrightarrow[\delta^{klj}F_{ik}]{} & F_{lj}F_{kl}F_{ik}
\end{array}
$$

are commutative.

Next, we will discuss when a comultiplication $\delta : F \to F^2$ is counital, for a given counity $\varepsilon : F \to 1_{\mathcal{A}}$. The general matrix form of ε is

$$\varepsilon = \begin{pmatrix} \varepsilon^1 & 0 & \cdots & 0 \\ 0 & \varepsilon^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \varepsilon^n \end{pmatrix} : \begin{pmatrix} F_{11} & F_{21} & \cdots & F_{n1} \\ F_{12} & F_{22} & \cdots & F_{n2} \\ \vdots & \vdots & & \vdots \\ F_{1n} & F_{2n} & \cdots & F_{nn} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{5}$$

for some natural transformations $\varepsilon^i : F_{ii} \to 1_{\mathcal{A}_i}$ for $i = 1, \ldots, n$. The counitality conditions $(\varepsilon F)\delta = 1_F = (F\varepsilon)\delta$ lead to matrix equalities equivalent to

$$(\varepsilon^j F_{ij})\delta^{ijj} = 1_{F_{ij}} = (F_{ij}\varepsilon^i)\delta^{iij}, \qquad (i, j = 1 \ldots, n),$$

that is, all the diagrams

$$
\begin{array}{ccc}
F_{ij} & \xrightarrow{\ \delta^{ijj}\ } & F_{jj}F_{ij} \\
{\scriptstyle \delta^{iij}}\big\downarrow & \| & \big\downarrow{\scriptstyle \varepsilon^j F_{ij}} \\
F_{ij}F_{ii} & \xrightarrow[F_{ij}\varepsilon^i]{} & F_{ij}
\end{array}
$$

commute.

We have so far proved the following proposition.

PROPOSITION 2.1   *Let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ be the product category of finitely many categories $\mathcal{A}_1, \ldots, \mathcal{A}_n$, and $F : \mathcal{A} \to \mathcal{A}$ any functor. There is a bijective correspondence between*

(1) *Comonads (F, δ, ε).*
(2) *Sets of natural transformations $\{\delta^{ikj} : F_{ij} \to F_{kj}F_{ik} : i, j, k = 1, \ldots, n\}$, and $\{\varepsilon^i : F_{ii} \to 1_{\mathcal{A}_i} : i = 1, \ldots, n\}$ such that*

    (a) *For all $i, j, k, l = 1, \ldots, n$ the diagram*

$$
\begin{array}{ccc}
F_{ij} & \xrightarrow{\ \delta^{ilj}\ } & F_{lj}F_{il} \\
{\scriptstyle \delta^{ikj}} \downarrow & & \downarrow {\scriptstyle F_{lj}\delta^{ikl}} \\
F_{kj}F_{ik} & \xrightarrow[\ \delta^{klj}F_{ik}\ ]{} & F_{lj}F_{kl}F_{ik}
\end{array}
\tag{6}
$$

        conmutes, and
    (b) *for all $i, j = 1, \ldots, n$, the diagram*

$$
\begin{array}{ccc}
F_{ij} & \xrightarrow{\ \delta^{ijj}\ } & F_{jj}F_{ij} \\
{\scriptstyle \delta^{iij}} \downarrow & & \downarrow {\scriptstyle \varepsilon^j F_{ij}} \\
F_{ij}F_{ii} & \xrightarrow[\ F_{ij}\varepsilon^i\ ]{} & F_{ij}
\end{array}
\tag{7}
$$

        commutes.

Remark 2.2   Take $i = j = k$ in diagrams (6) and (7), then $(F_{ii}, \delta^{iii}, \varepsilon^i)$ is a comonad over $\mathcal{A}_i$. Also diagrams (6) and (7) show that each functor F_{ij} is in fact an $F_{jj} - F_{ii}$-bicomodule functor, in the sense of [14, Definition 4.7], see also [15]. Furthermore, if $l = k$ in diagram (6), we get that each of the F_{ik}'s is a balanced bicomodule in a dual sense of [14, Section 3.2]. In this way, if the 'cotensor product' functor $F_{kj} \square_{F_{kk}} F_{ik}$ does exist, then the natural transformation δ^{ikj} factors through $F_{kj} \square_{F_{kk}} F_{ik}$.

Definition 2.3   A comonad (F, δ, ε) on $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ is called *normal* if $F_{ii} = 1_{\mathcal{A}_i}$ and $\varepsilon^i = 1$ for all $i = 1, \ldots, n$. Thus, for a normal comonad, we have necessarily that $\delta^{ijj} = \delta^{iij} = 1_{F_{ij}}$ for all $i, j = 1, \ldots, n$.

Remark 2.4   In the case $n = 2$, we deduce from Proposition 2.1 that a normal comonad (F, δ, ε) is given by natural transformations $\delta^{121} : 1_{\mathcal{A}_1} \to F_{21}F_{12}$ and $\delta^{212} : 1_{\mathcal{A}_2} \to F_{12}F_{21}$ such that $F_{12}\delta^{121} = \delta^{212}F_{12}$ and $F_{21}\delta^{212} = \delta^{121}F_{21}$. Therefore, the normal comonads over $\mathcal{A}_1 \times \mathcal{A}_2$ are in bijection with the wide (right) Morita contexts between $\mathcal{A}_1$ and $\mathcal{A}_2$ as defined in [16,17].

Remark 2.5   It is possible to formulate Proposition 2.1 in dual form, and thus give the structure of the monads on $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$.

2.5. *Comodules over matrix comonads*

Consider a comonad (F, δ, ε) over $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ as in Proposition 2.1. Next, we want to describe its Eilenberg–Moore category $\mathcal{A}_F$ of comodules (or coalgebras). Recall that an object of $\mathcal{A}_F$ is a morphism $\mathbf{d}_A : A \to FA$ in $\mathcal{A}$ such that the following diagrams are commutative:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \mathbf{d}_A\ } & FA \\
\mathbf{d}_A \downarrow & & \downarrow \delta_A \\
FA & \xrightarrow[F\mathbf{d}_A]{} & F^2 A
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\ \mathbf{d}_A\ } & FA \\
& {}_{1_A}\searrow & \downarrow \varepsilon_A \\
& & A
\end{array}
\tag{8}
$$

We want to describe these objects in terms of the categories $\mathcal{A}_1, \ldots, \mathcal{A}_n$. Each structure morphism $\mathbf{d}_A$ is determined by the projections

$$
\pi_j(\mathbf{d}_A) : A_j \to \bigoplus_k F_{kj} A_k, \qquad j = 1, \ldots, n,
$$

and each of them is given by

$$
\pi_j(\mathbf{d}_A) = d_A^{1j} \dotplus d_A^{2j} \dotplus \cdots \dotplus d_A^{nj} \quad \text{with} \quad d_A^{kj} : A_j \to F_{kj} A_k.
$$

Let us see what restrictions impose the commutativity of the diagrams (8) on the morphism d_A^{kj}. We start with the equation $\delta_A \mathbf{d}_A = F(\mathbf{d}_A) \mathbf{d}_A$, which leads to the following system of equations:

$$
\pi_j(\delta_A)\pi_j(\mathbf{d}_A) = \pi_j(F(\mathbf{d}_A))\pi_j(\mathbf{d}_A), \qquad j = 1, \ldots, n.
\tag{9}
$$

On the other hand,

$$
\pi_j \delta_A = \bigoplus_l \delta_{A_l}^{lj}, \quad \text{with} \quad \delta_{A_l}^{lj} : F_{lj} A_l \to \bigoplus_k F_{kj} F_{lk} A_l.
$$

Therefore,

$$
\pi_j(\delta_A)\pi_j(\mathbf{d}_A) = \left(\bigoplus_l \delta_{A_l}^{lj} \right) \left(\dotplus_l d_A^{lj} \right) = \dotplus_l \delta_{A_l}^{lj} d_A^{lj} = \dotplus_l \left(\dotplus_k \delta_{A_l}^{lkj} \right) d_A^{lj} = \dotplus_{k,l} \delta_{A_l}^{lkj} d_A^{lj}.
\tag{10}
$$

On the other hand,

$$
\pi_j(F\mathbf{d}_A)\pi_j(\mathbf{d}_A) = \left(\bigoplus_k F_{kj}\pi_k(\mathbf{d}_A) \right) \left(\dotplus_k d_A^{kj} \right) = \dotplus_k F_{kj}\pi_k(\mathbf{d}_A) d_A^{kj}
\tag{11}
$$

$$
= \dotplus_k \left(\dotplus_l F_{kj}(d_A^{lk}) \right) d_A^{kj} = \dotplus_{k,l} F_{kj}(d_A^{lk}) d_A^{kj}.
$$

We get from (9)–(11) that the commutativity of the first diagram in (8) is equivalent to the commutativity of all the diagrams

$$
\begin{array}{ccc}
A_j & \xrightarrow{\ d_A^{kj}\ } & F_{kj} A_k \\
d_A^{lj} \downarrow & & \downarrow F_{kj} d_A^{lk} \\
F_{lj} A_l & \xrightarrow[\delta_{A_l}^{lkj}]{} & F_{kj} F_{lk} A_l
\end{array}
\tag{12}
$$

for $j, k, l = 1, \ldots, n$.

The second diagram in (8) leads to the equalities $\pi_j(\varepsilon_A \mathbf{d}_A) = 1_{A_j}$ for each $j = 1, \ldots, n$. Therefore,

$$1_{A_j} = \pi_j(\varepsilon_A \mathbf{d}_A) = \pi_j(\varepsilon_A)\pi_j(\mathbf{d}_A) = (0 \oplus \cdots \oplus 0 \oplus \varepsilon^j_{A_j} \oplus 0 \oplus \cdots \oplus 0)\left(\overset{\cdot}{\underset{k}{+}} d^{kj}_A\right) = \varepsilon^j_{A_j} d^{jj}_A,$$

for all $j = 1, \ldots, n$.

The previous discussion gives the following description of the category $\mathcal{A}_F$ of comodules over F.

PROPOSITION 2.6 *Let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ be the product category of finitely many categories $\mathcal{A}_1, \ldots, \mathcal{A}_n$. If (F, δ, ε) is a comonad over $\mathcal{A}$, then the category of F–comodules is described as follows:*

(1) *Objects: They are pairs $(A, \mathbf{d}_A)$ where $A = (A_1, \ldots, A_n) \in \mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ and $\mathbf{d}_A$ is a set of morphisms $\mathbf{d}_A = \{d^{kj}_A : A_j \to F_{kj}A_k : 1 \leq j, k \leq n\}$ such that the diagrams*

$$
\begin{array}{ccc}
A_j & \xrightarrow{d^{kj}_A} & F_{kj}A_k \\
{\scriptstyle d^{lj}_A}\downarrow & & \downarrow{\scriptstyle F_{kj}d^{lk}_A} \\
F_{lj}A_l & \xrightarrow{\delta^{lkj}_{A_l}} & F_{kj}F_{lk}A_l
\end{array}
\qquad
\begin{array}{ccc}
A_j & \xrightarrow{d^{jj}_A} & F_{jj}A_j \\
\| & & \downarrow{\scriptstyle \varepsilon^j_{A_j}} \\
& & A_j
\end{array}
$$
(13)

are commutative for all $j, k, l = 1, \ldots, n$.

(2) *Morphisms: A morphism $f : (A, \mathbf{d}_A) \to (B, \mathbf{d}_B)$ is a set of morphisms $f = \{f_j : A_j \to B_j : j = 1, \ldots, n\}$ such that the diagrams*

$$
\begin{array}{ccc}
A_j & \xrightarrow{d^{kj}_A} & F_{kj}A_k \\
{\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle F_{kj}f_k} \\
B_j & \xrightarrow{d^{kj}_B} & F_{kj}B_k
\end{array}
$$
(14)

commute, for all $j, k = 1, \ldots, n$.

Remark 2.7 Obviously, it is possible to give the dual statements of the above results in the case of monads. So, a monad T over $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ is given by a set of functors $\{T_{ij} : \mathcal{A}_i \to \mathcal{A}_j, i, j = 1, \ldots, n\}$ with two sets of natural transformations $\{\mu^{ikj} : T_{kj}T_{ik} \to T_{ij} : i, j, k = 1 \ldots, n\}$, $\{\eta^i : 1_{\mathcal{A}_i} \to T_{ii}, i = 1, \ldots, n\}$ such that the diagrams

$$
\begin{array}{ccc}
T_{lj}T_{kl}T_{ik} & \xrightarrow{\mu^{klj}T_{ik}} & T_{kj}T_{ik} \\
{\scriptstyle T_{lj}\mu^{ikl}}\downarrow & & \downarrow{\scriptstyle \mu^{ikj}} \\
T_{lj}T_{il} & \xrightarrow{\mu^{ilj}} & T_{ij}
\end{array}
\qquad
\begin{array}{ccc}
T_{ij} & \xrightarrow{\eta^j T_{ij}} & T_{jj}T_{ij} \\
{\scriptstyle T_{ij}\eta^i}\downarrow & & \downarrow{\scriptstyle \mu^{ijj}} \\
T_{ij}T_{ii} & \xrightarrow{\mu^{iij}} & T_{ij}
\end{array}
$$

commute for all i, j, $k \in \{1, \dots, n\}$. The corresponding category of modules (or algebras) is described by dualizing the statements of Proposition 2.6. When T is a normal matrix monad (i.e. $T_{ii} = 1_{\mathcal{A}_i}$ and $\eta^i = 1$ for all $i = 1, \dots, n$), $\mathcal{A}$ is abelian, and $T_{ij} = 0$ for $i > j$, these are the kind of categories which were studied by Harada in [10], and referred to as *categories of generalized diagrams* in abelian categories (see also [9]).

3. Triangular matrix comonads

We have seen in Remark 2.7 that the abelian categories investigated by Harada in [10] are categories of modules over triangular normal matrix monads. In this section, we will take advantage of the fact that the existence of a (co)monad representing these categories to give a more systematic approach to their study. In fact, our main theorem in this section is more general in several directions than [10, Theorem 2.3]. At the same time, we find the proof presented here sharper in some aspects than Harada's one.

We will say (see Definition 3.3) that a matrix comonad $F = (F_{ij})$ is *triangular* if $F_{ij} = 0$ for all $1 \le j < i \le n$. The understanding of the case $n = 2$ is the key to the study of the general case.

3.1. *Triangular matrix comonads of order 2*
Let

$$\mathcal{C} \xrightarrow{R} \mathcal{C} \xrightarrow{V} \mathcal{D} \xrightarrow{S} \mathcal{D} \tag{15}$$

be functors. By Proposition 2.1, every comonad structure over the endofunctor

$$F = \begin{pmatrix} R & 0 \\ V & S \end{pmatrix} \tag{16}$$

of $\mathcal{C} \times \mathcal{D}$ is given by a set of natural transformations

$$\left\{ R \xrightarrow{\delta^R} R^2, \ S \xrightarrow{\delta^S} S^2, \ V \xrightarrow{\rho^V} VR, \ V \xrightarrow{\lambda^V} SV, \ R \xrightarrow{\varepsilon^R} 1_{\mathcal{C}}, \ S \xrightarrow{\varepsilon^S} 1_{\mathcal{D}} \right\} \tag{17}$$

such that the following eight diagrams commute:

$$\tag{18}$$

$$
\begin{array}{ccc}
V \xrightarrow{\rho^V} VR & \qquad & V \xrightarrow{\lambda^V} SV \\
\rho^V \downarrow \quad\quad \downarrow V\delta^R & & \lambda^V \downarrow \quad\quad \downarrow \delta^S V \\
VR \xrightarrow{\rho^V R} VR^2 & & SV \xrightarrow{S\lambda^V} S^2 V
\end{array}
$$

$$
\begin{array}{ccc}
V \xrightarrow{\lambda^V} SV & \qquad & V \xrightarrow{\lambda^V} SV \\
\rho^V \downarrow \quad\quad \downarrow \varepsilon^S V & & \rho^V \downarrow \quad\quad \downarrow S\rho^V \\
VR \xrightarrow{V\varepsilon^R} V & & VR \xrightarrow{\lambda^V R} SVR
\end{array}
\tag{19}
$$

By Remark 2.2, the commutativity of the diagrams (18) just says that $(R, \delta^R, \varepsilon^R)$ and $(S, \delta^S, \varepsilon^S)$ are comonads, while the commutative diagrams (19) say that (V, λ^V, ρ^V) is an $S - R$–bimodule functor.

THEOREM 3.1 *Let*

$$
\begin{pmatrix} R & 0 \\ V & S \end{pmatrix} : \mathcal{C} \times \mathcal{D} \xrightarrow{\hspace{3cm}} \mathcal{C} \times \mathcal{D}
$$

be a triangular matrix comonad of order 2, with the comonad structure given by a sextuple of natural transformations as in (17) *satisfying the conditions* (18) *and* (19). *If the equalizer of every pair of arrows of the form* $VC \mathrel{\substack{\rho_C^V \\ \longrightarrow \\ \longrightarrow \\ Vd_C}} VRC$, *where* (C, d_C) *is any R–comodule, do exist in* $\mathcal{D}$, *and S preserves all these equalizers, then there exists a functor* $T : \mathcal{C}_R \to \mathcal{D}_S$ *such that the diagram*

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ V\ } & \mathcal{D} \\
L \downarrow & & \uparrow U \\
\mathcal{C}_R & \xrightarrow{\ T\ } & \mathcal{D}_S
\end{array}
$$

commutes, where $L : \mathcal{C} \to \mathcal{C}_R$ *is the free functor and* $U : \mathcal{D}_S \to \mathcal{D}$ *is the forgetful functor. Moreover, there exists an equivalence of categories*

$$
(\mathcal{C} \times \mathcal{D})_{\begin{pmatrix} R & 0 \\ V & S \end{pmatrix}} \cong (\mathcal{C}_R \times \mathcal{D}_S)_{\begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}}
$$

Proof The existence of T is proved in [14, Proposition 4.29] in a different context. We give a direct construction for the convenience of the reader. Given an R–comodule (C, d_C), define an object TC of $\mathcal{D}$ as the equalizer

$$
TC \xrightarrow{\ \iota_C\ } VC \mathrel{\substack{\rho_C^V \\ \longrightarrow \\ \longrightarrow \\ Vd_C}} VRC.
\tag{20}
$$

The S–coaction $\lambda_C^T : TC \to STC$, making TC an S–comodule, is given by the universal property of the equalizer at the bottom of the following diagram.

$$TC \xrightarrow{\iota_C} VC \underset{Vd_C}{\overset{\rho_C^V}{\rightrightarrows}} VRC$$

with vertical maps λ_C^T, λ_C^V, λ_{RC}^V and bottom row $STC \xrightarrow{S\iota_C} SVC \underset{SVd_C}{\overset{S\rho_C^V}{\rightrightarrows}} SVRC.$

Some straightforward computations show that this gives the object part of a functor $T : \mathcal{C}_R \to \mathcal{D}_S$. We only make explicit here its definition on morphisms. Given a morphism of R–comodules $f : (C, d_C) \to (C', d_{C'})$, the morphism $Tf : TC \to TC'$ is uniquely determined by the universal property of the equalizer in the bottom row of the serially commutative diagram

$$TC \xrightarrow{\iota_C} VC \underset{Vd_C}{\overset{\rho_C^V}{\rightrightarrows}} VRC$$
$$TC' \xrightarrow{\iota_{C'}} VC' \underset{Vd_{C'}}{\overset{\rho_{C'}^V}{\rightrightarrows}} VRC'.$$

with vertical maps Tf, Vf, VRf.

In order to construct an equivalence $E : (\mathcal{C} \times \mathcal{D})_{\begin{pmatrix} R & 0 \\ V & S \end{pmatrix}} \to (\mathcal{C}_R \times \mathcal{D}_S)_{\begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}}$, let

us describe the objects of these categories of comodules. By Proposition 2.6, a comodule over $\begin{pmatrix} R & 0 \\ V & S \end{pmatrix}$ consists of an object $A = (C, D) \in \mathcal{C} \times \mathcal{D}$ and a tern of morphisms

$$\mathbf{d}_A = \{\, C \xrightarrow{\rho_C} RC \,,\; D \xrightarrow{d} VC \,,\; D \xrightarrow{\rho_D} SD \,\}$$

such that the following diagrams commute:

$$\begin{array}{ccc} C \xrightarrow{\rho_C} RC & & D \xrightarrow{\rho_D} SD \\ \rho_C \downarrow \quad \downarrow R\rho_C & & \rho_D \downarrow \quad \downarrow S\rho_D \\ RC \xrightarrow{\delta_C^R} R^2C & & SD \xrightarrow{\delta_D^S} S^2C \end{array}$$

$$\begin{array}{ccc} D \xrightarrow{d} VC & & D \xrightarrow{\rho_D} SD \\ d \downarrow \quad \downarrow V\rho_C & & d \downarrow \quad \downarrow Sd \\ VC \xrightarrow{\rho_C^V} VRC & & VC \xrightarrow{\lambda_C^V} SVC \end{array} \tag{21}$$

$$\begin{array}{ccc} C \xrightarrow{\rho_C} RC & D \xrightarrow{\rho_D} SD \\ \quad \searrow \downarrow \varepsilon_C^R & \quad \searrow \downarrow \varepsilon_D^S \\ C & D. \end{array} \tag{22}$$

On the other hand, a comodule over $\begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$ is just a pair of comodules $((C, \rho_C),$ $(D, \rho_D)) \in \mathcal{C}_R \times \mathcal{D}_S$ connected with a morphism of S–comodules $d' : D \to TC$. This last condition is just the commutativity of the diagram

$$\begin{array}{ccc} D & \xrightarrow{\ d'\ } & TC \\ {\scriptstyle \rho_D}\downarrow & & \downarrow{\scriptstyle \lambda_C^T} \\ SD & \xrightarrow{\ Sd'\ } & STC. \end{array}$$

(23)

The functor E will send a tern $\mathbf{d}_A = (\rho_C, d, \rho_D)$ satisfying the conditions (21) and (22) to the pair of comodules $(C, \rho_C), (D, \rho_D) \in \mathcal{C}_R \times \mathcal{D}_S$ with the morphism $d' : D \to TC$ given by the universal property of the equalizer in the following diagram

$$TC \xrightarrow{\ \iota_C\ } VC \mathrel{\mathop{\rightrightarrows}^{\rho_C^V}_{Vd_C}} VRC.$$

$$\begin{array}{c} {\scriptstyle d'}\uparrow \nearrow {\scriptstyle d} \\ D \end{array}$$

The existence and uniqueness of d' is guaranteed by the commutativity of the third diagram of (21), while the fact that d' becomes a morphism of S–comodules (namely, the commutativity of (23)) is given by the fourth diagram in (21). This gives the object part of the functor E. In the other direction, there is a functor E' defined on objects by sending a morphism of comodules $d' : D \to TC$, for $(C, \rho_C) \in \mathcal{C}_R$, $(D, \rho_D) \in \mathcal{D}_S$, to the morphism $d = \iota_C d'$. The computation

$$\rho_C^V d = \rho_C^V \iota_C d' = V\rho_C \iota_C d' = V\rho_C d$$

shows that d makes commute the third diagram in (21), while the computation

$$\lambda_C^V d = \lambda_C^V \iota_C d' = S\iota_C \lambda_C^T d' = S\iota_C Sd' \rho_D = (Sd)\rho_D$$

is just the commutativity of the last diagram of (21).

It is not hard to see that E and E' are mutually inverse. $\qquad\square$

Remark 3.2   A standard argument shows that, if $\mathcal{C}$ and $\mathcal{D}$ are abelian categories, and V, R and S are left exact functors, then T is a left exact functor between abelian categories.

3.2. *Triangular matrix comonads*

Definition 3.3   A functor $F : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ which is endowed with a comonad structure (F, δ, ε) as in Proposition 2.1 is called a *triangular matrix comonad of order* n over $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ if $F_{ji} = 0$ for all $j > i$, that is,

$$F = \begin{pmatrix} F_{11} & 0 & 0 & \cdots & 0 & 0 \\ F_{12} & F_{22} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_{1n} & F_{2n} & F_{3n} & \cdots & F_{n-1\,n} & F_{nn} \end{pmatrix}.$$

(24)

Given $1 \le m < n$, consider the categories

$$\mathcal{A}^{\le m} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m, \quad \mathcal{A}^{>m} = \mathcal{A}_{m+1} \times \cdots \times \mathcal{A}_n.$$

All these categories can be considered, in an obvious way, as full subcategories of $\mathcal{A}$. For instance, an object $A = (A_1, \ldots, A_m)$ of $\mathcal{A}^{\le m}$ is identified with the object $(A_1, \ldots, A_m,$

$0, \ldots, 0$) of $\mathcal{A}$. It is clear that there are canonical functors (the projection functors)

$$\pi_{\leq m} : \mathcal{A} \longrightarrow \mathcal{A}^{\leq m}, \qquad \pi_{>m} : \mathcal{A} \longrightarrow \mathcal{A}^{>m}.$$

By Proposition 2.1, the triangular matrix comonad (F, δ, ε) over $\mathcal{A}$ gives rise to the triangular matrix comonads $(F^{\leq m}, \delta_{\leq m}, \varepsilon_{\leq m})$ and $(F^{>m}, \delta_{>m}, \varepsilon_{>m})$ over $\mathcal{A}^{\leq m}$ and $\mathcal{A}^{>m}$ defined, respectively, by

$$F^{\leq m} = \begin{pmatrix} F_{11} & 0 & 0 & \cdots & 0 \\ F_{12} & F_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ F_{1m} & F_{2m} & F_{3m} & \cdots & F_{mm} \end{pmatrix},$$

$$\delta_{\leq m} = \begin{pmatrix} \delta^{11} & 0 & 0 & \cdots & 0 \\ \delta^{12} & \delta^{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \delta^{1m} & \delta^{2m} & \delta^{3m} & \cdots & \delta^{mm} \end{pmatrix}, \qquad \varepsilon_{\leq m} = \begin{pmatrix} \varepsilon^1 & 0 & \cdots & 0 \\ 0 & \varepsilon^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \varepsilon^m \end{pmatrix}, \qquad (25)$$

and

$$F^{>m} = \begin{pmatrix} F_{m+1\,m+1} & 0 & 0 & \cdots & 0 \\ F_{m+1\,m+2} & F_{m+2\,m+2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ F_{m+1\,n} & F_{m+2\,n} & F_{m+3\,n} & \cdots & F_{nn} \end{pmatrix},$$

$$\delta_{>m} = \begin{pmatrix} \delta^{m+1\,m+1} & 0 & 0 & \cdots & 0 \\ \delta^{m+1\,m+2} & \delta^{m+2\,m+2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \delta^{m+1\,n} & \delta^{m+2\,n} & \delta^{m+3\,n} & \cdots & \delta^{nn} \end{pmatrix}, \varepsilon_{>m} = \begin{pmatrix} \varepsilon^{m+1} & 0 & \cdots & 0 \\ 0 & \varepsilon^{m+2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \varepsilon^n \end{pmatrix}. \quad (26)$$

The functor $F^m : \mathcal{A}^{\leq m} \to \mathcal{A}^{>m}$ given by

$$F^m = \begin{pmatrix} F_{1\,m+1} & F_{2\,m+1} & \cdots & F_{m\,m+1} \\ F_{1\,m+2} & F_{2\,m+2} & \cdots & F_{m\,m+2} \\ \vdots & \vdots & & \vdots \\ F_{1\,n} & F_{2\,n} & \cdots & F_{mn} \end{pmatrix} \qquad (27)$$

is an $F^{>m} - F^{\leq m}$–bimodule functor with the structure natural transformations $\lambda_m : F^m \to F^m F^{\leq m}$ and $\rho_m : F^m \to F^{>m} F^m$ defined by

$$\lambda_m^{ij} : F_{ij} \to \bigoplus_{k=1}^m F_{kj} F_{ik}, \quad \lambda_m^{ij} = \delta^{i1j} \dotplus \cdots \dotplus \delta^{imj}$$

and

$$\rho_m^{ij} : F_{ij} \to \bigoplus_{k=m+1}^n F_{kj} F_{ik}, \quad \lambda_m^{ij} = \delta^{im+1j} \dotplus \cdots \dotplus \delta^{inj},$$

for $1 \leq i \leq m, m+1 \leq j \leq n$. Therefore,

$$F = \begin{pmatrix} F^{\leq m} & 0 \\ F^m & F^{>m} \end{pmatrix},$$

as comonads.

The following consequence of Theorem 3.1 is the basic tool for the study of triangular hereditary comonads in the next section. The particular case where $F_{ii} = 1_{\mathcal{A}_i}$ for all $i = 1, \ldots, n$ was already stated in its dual form in [10, Theorem 2.3] under the additional hypothesis that the categories $\mathcal{A}_i$ are abelian.

THEOREM 3.4   *Let F be a triangular matrix comonad on $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ as in Definition 3.3. Given an integer m with $1 \leq m < n$, assume that all equalizers do exist in $\mathcal{A}_i$ for $i = 1, \ldots, m$, and that the functors F_{ij} preserve equalizers for $1 \leq i \leq j \leq m$. Then, there exists a functor*

$$T^m : \mathcal{A}^{\leq m}_{F^{\leq m}} \xrightarrow{\hspace{2cm}} \mathcal{A}^{>m}_{F^{>m}} \tag{28}$$

such that the category of F-comodules is isomorphic to the category of G^m-comodules, where G^m is the normal triangular matrix comonad defined by T^m. That is,

$$\mathcal{A}_F \cong (\mathcal{A}^{\leq m}_{F^{\leq m}} \times \mathcal{A}^{>m}_{F^{>m}})_{G^m}, \quad \text{where } G^m = \begin{pmatrix} 1 & 0 \\ T^m & 1 \end{pmatrix}.$$

Proof   Apply Theorem 3.1 with $R = F^{\leq m}$, $S = F^{>m}$, and $V = F^m$. $\square$

Remark 3.5   If the categories $\mathcal{A}_i$ are abelian for $i = 1, \ldots, n$, and all the functors F_{ij}, for $1 \leq i < j \leq n$ are left exact, then the functor F is also left exact, as well as the functors $F^{>m}$, F^m, and $F^{\leq m}$, for every $2 \leq m \leq n - 1$. On the other hand, the functor $T^m : \mathcal{A}^{\leq m}_{F^{\leq m}} \to \mathcal{A}^{>m}_{F^{>m}}$, constructed as an equalizer (see the proof of Theorem 3.1) becomes left exact, see Remark 3.2.

4. Hereditary categories of comodules

An abelian category with enough injectives is said to be *hereditary* if its global homological dimension is 0 or 1, that is, for every epimorphism $E_0 \to E_1$, if E_0 is injective, then E_1 is injective. Our aim is to characterize when the category of comodules $\mathcal{A}_F$ of a normal triangular matrix comonad $F = (F, \delta, \varepsilon)$ over $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ is hereditary. We are denoting by $\delta : F \to F^2$ the comultiplication of F, and by $\varepsilon : F \to \mathcal{A}$ its counit. The shape of the matrix of functors representing F is

$$F = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ F_{12} & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_{1n} & F_{2n} & F_{3n} & \cdots & F_{n-1\,n} & 1 \end{pmatrix}. \tag{29}$$

We assume that the categories $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are abelian with enough injectives, and so is $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. If $F : \mathcal{A} \to \mathcal{A}$ is a left exact functor, then $\mathcal{A}_F$ is also abelian and the forgetful functor $\mathcal{A}_F \to \mathcal{A}$ is exact, see [18]. On the other hand, F is exact if and only if F_{ij} is exact for all $i, j = 1, \ldots, n$.

First, we analyse the case $n = 2$.

4.1. *The case $n = 2$*

Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, for $\mathcal{A}_1, \mathcal{A}_2$ abelian categories, and let

$$F = \begin{pmatrix} 1 & 0 \\ F_{12} & 1 \end{pmatrix} : \mathcal{A}_1 \times \mathcal{A}_2 \longrightarrow \mathcal{A}_1 \times \mathcal{A}_2$$

be a functor with $F_{12} : \mathcal{A}_1 \to \mathcal{A}_2$ a left exact functor. Consider the unique structure of normal triangular matrix comonad on the left exact functor F. According to Proposition 2.6, any F–comodule can be identified with a pair (X, d_X^{12}), where $X = (X_1, X_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ and $d_X^{12} : X_2 \to F_{12}X_1$. By convenience, the free functor will be denoted by $F : \mathcal{A} \to \mathcal{A}_F$. By Inj.dim$(X)$, we denote the injective dimension of an object X in some abelian category with enough injectives.

PROPOSITION 4.1 *Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ have enough injectives and that F_{12} is a left exact functor that preserves injectives. Then,*

(1) *$\mathcal{A}_F$ has enough injectives.*
(2) *Each injective comodule in $\mathcal{A}_F$ is, up to isomorphisms, of the form $F(E)$, for some injective object E in $\mathcal{A}$. In particular, the arrow d_X^{12} is a split epimorphism of $\mathcal{A}_2$ for any injective F-comodule (X, d_X^{12}).*
(3) *Given an F-comodule (X, d_X^{12}), we have*

$$\text{Inj.dim}\left(\left(X, d_X^{12} \right) \right) \leq \max\{\text{Inj.dim}(X_1), \text{Inj.dim}(X_2)\} + 1$$

Proof

(1) Observe that $F : \mathcal{A} \to \mathcal{A}_F$ preserves injectives, since it is right adjoint to the forgetful functor $\mathcal{A}_F \to \mathcal{A}$, which is exact [18, Proposition 5.3]. So, given an injective object $E = (E_1, E_2)$ in $\mathcal{A}$, we have an injective F-comodule $F(E) = (E_1, F_{12}E_1 \oplus E_2, \pi)$, where $\pi : F_{12}E_1 \oplus E_2 \to F_{12}E_1$ is the canonical projection (this is given by the comonad structure of F). Now, for every F-comodule (X, d_X^{12}), we can consider monomorphisms $\iota_i : X_i \to E_i$ in $\mathcal{A}_i$, with E_i injective for $i = 1, 2$. So we have a monomorphism of F-comodules

$$\left(\iota_1, F_{12}\iota_1 \circ d_X^{12} \dotplus \iota_2 \right) : \left(X_1, X_2, d_X^{12} \right) \to (E_1, F_{12}E_1 \oplus E_2, \pi) \qquad (30)$$

which shows that $\mathcal{A}_F$ has enough injectives.
(2) If we assume that the F-comodule (X, d_X^{12}) is injective, then the monomorphism (30) splits, so that there exists a morphism of comodules $(\alpha, \beta) : (E_1, F_{12}E_1 \oplus E_2) \to (X_1, X_2, d_X^{12})$ which splits $(\iota_1, F_{12}\iota_1 \circ d_X^{12} \dotplus \iota_2)$. Thus, X_1 is isomorphic to a direct summand of the injective object E_1, so it is injective. Therefore, without loss of generality, we can suppose that X_1 is injective and that $E_1 = X_1$. In this way, we get that $\beta \circ (d_X^{12} \dotplus \iota_2) = 1_{X_2}$, which shows that X_2 is isomorphic to a direct summand of the injective object $F_{12}E_1 \oplus E_2$ and so it is injective too. On the other hand, since (α, β) is a comodule map, we obtain that $\pi = d_X^{12} \circ \beta$. Since π is a split epimorphism, we deduce that d_X^{12} is a split epimorphism. This implies that there exists an isomorphism $\omega : X_2 \to F_{12}X_1 \oplus E_2'$ in $\mathcal{A}_2$ such that $\pi' \circ \omega = d_X^{12}$,

where π' is the obvious canonical projection. Thus,

$$(1_{X_1}, \omega) : \left(X_1, X_2, d_X^{12}\right) \to F\left(X_1, E_2'\right) = \left(X_1, F_{12}X_1 \oplus E_2', \pi'\right)$$

is an isomorphism of F-comodules.

(3) Put $m = \max\{\text{Inj.dim}(\pi_1(X)), \text{Inj.dim}(\pi_2(X))\}$, and take, for an F-comodule $\left(X, d_X^{12}\right)$, a resolution in $\mathcal{A}_F$

$$0 \longrightarrow \left(X, d_X^{12}\right) \xrightarrow{f^0} \left(E^0, d_{E^0}^{12}\right) \longrightarrow \ldots \longrightarrow \left(E^m, d_{E^m}^{12}\right) \xrightarrow{f^m} \left(C, d_C^{12}\right) \longrightarrow 0$$

with $(E^i, d_{E^i}^{12})$ injective for all $0 \leq i \leq m$. We need to show that (C, d_C^{12}) is injective too. If we apply the projection functor π_i, for $i = 1, 2$, to this resolution, by part (2), we get a resolution in $\mathcal{A}_i$ for $\pi_i(X)$ with $\pi_i(E^k)$ injective and $d_{E^k}^{12}$ split epimorphism for $k = 0, \ldots, m$. Since $\text{Inj.dim}(\pi_i(X)) \leq m$, we deduce that $\pi_i(f^m)$ is a split epimorphism and $\pi_i(C)$ is injective for $i = 1, 2$. We know that $d_C^{12} \circ \pi_2(f^m) = F_{12}(\pi_1(f^m)) \circ d_{E^m}^{12}$. Hence, d_C^{12} is a split epimorphism and (C, d_C^{12}) is injective. $\qquad\square$

By $\text{gl.dim}(\mathcal{A})$, we denote the global homological dimension of an abelian category with enough injectives $\mathcal{A}$.

COROLLARY 4.2 *The assumptions are that of Proposition* 4.1. *Then, we have*

$$\text{gl.dim}(\mathcal{A}_F) \leq \max\{\text{gl.dim}(\mathcal{A}_i) : i = 1, 2\} + 1 \leq \text{gl.dim}(\mathcal{A}_F) + 1$$

Next, we give the desired characterization for hereditary categories of comodules over a triangular normal (2×2)-matrix comonad.

THEOREM 4.3 *Let* $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ *for* $\mathcal{A}_1, \mathcal{A}_2$ *abelian categories with enough injectives, and* $F_{12} : \mathcal{A}_1 \to \mathcal{A}_2$ *a left exact functor. Consider the unique normal comonad structure on the endofunctor*

$$F = \begin{pmatrix} 1 & 0 \\ F_{12} & 1 \end{pmatrix} : \mathcal{A} \longrightarrow \mathcal{A}.$$

The category of comodules $\mathcal{A}_F$ *is hereditary if, and only if, the following conditions are satisfied:*

(a) F_{12} *preserves injectives.*
(b) $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are hereditary.*
(c) $F_{12}p$ *is a split epimorphism for each epimorphism* $p : E_1 \to E_1'$ *between injective objects in* $\mathcal{A}_1$.

Proof Suppose that $\mathcal{A}_F$ is hereditary. Given an injective object E_1 in $\mathcal{A}_1$, we know that $F(E_1, 0) = (E_1, F_{12}E_1, 1_{F_{12}E_1})$ is an injective F-comodule. Now, since $(0, 1_{F_{12}E_1}) : (E_1, F_{12}E_1) \to (0, F_{12}E_1)$ is an epimorphism in the hereditary category $\mathcal{A}_F$, we have that $(0, F_{12}E_1)$ is an injective object in $\mathcal{A}_F$. Henceforth, it is clear that $F_{12}E_1$ is injective in $\mathcal{A}_1$, from which (a) is derived. The statement (b) follows from Corollary 4.2. For the proof of (c), given an epimorphism $p : E_1 \to E_1'$ in $\mathcal{A}_1$, we get an epimorphism $(p, 1) :$

$F(E_1, 0) = (E_1, F_{12}E_1, 1) \rightarrow (E_1', F_{12}E_1, F_{12}p)$ in $\mathcal{A}_F$. Since $F(E_1, 0)$ is injective, then so is $(E_1', F_{12}E_1, F_{12}p)$. Therefore, $F_{12}p$ is a split epimorphism, by the characterization of the injectives in Proposition 4.1.

Conversely, consider an F–comodule $\left(X, d_X^{12}\right)$ and a resolution in $\mathcal{A}_F$

$$0 \longrightarrow \left(X, d_X^{12}\right) \longrightarrow F(E_1, E_2) \xrightarrow{(f_1, f_2)} (C, d_C^{12}) \longrightarrow 0,$$

with $(E_1, E_2) \in \mathcal{A}$ injective. Since F_{12} preserves injectives, we have that E_1 and $F_{12}E_1 \oplus E_2$ are injective. As f_1 and f_2 are epimorphism, and $\mathcal{A}_i$ is hereditary for $i = 1, 2$, we deduce that C_1 and C_2 are injective. Using the fact that $F_{12}(f_1)$ is a split epimorphism, we obtain from the commutative diagram

$$
\begin{array}{ccc}
F_{12}E_1 \oplus E_2 & \xrightarrow{\ f_2\ } & C_2 \\
{\scriptstyle \pi} \downarrow & & \downarrow {\scriptstyle d_C^{12}} \\
F_{12}E_1 & \xrightarrow[F_{12}f_1]{} & F_{12}C_1
\end{array}
$$

that d_C^{12} is a split epimorphism and so (C, d_C^{12}) is injective. This shows that the injective dimension of $\left(X, d_X^{12}\right)$ is less or equal than 1, which means that $\mathcal{A}_F$ is a hereditary category. $\qquad\square$

4.2. *The case $n \geq 3$*

Let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ denote a product of categories and $F : \mathcal{A} \rightarrow \mathcal{A}$ a normal triangular matrix comonad.

Propositions 4.4 and 4.5 below, which will be used to deduce our main result in this section (Theorem 4.6), contain part of the dual form of [10, Theorem 3.6].

PROPOSITION 4.4 *Assume that $\mathcal{A}_i$ is an abelian category with enough injectives for every $i \in \{1, \ldots, n\}$, and that each of the functors $F_{ij} : \mathcal{A}_i \rightarrow \mathcal{A}_j$, for $1 \leq i < j \leq n$, is left exact. If the category of comodules $\mathcal{A}_F$ is hereditary, then we have:*

(1) *Each injective object $(X, \mathbf{d}_X)$ of $\mathcal{A}_F$ is, up to isomorphisms, of the form $X = F(E_1, \ldots, E_n)$ for some injective object $(E_1, \ldots, E_n) \in \mathcal{A}$. In particular, each arrow $d_X^{ij} : X_j \rightarrow F_{ij}(X_i)$ is a split epimorphism.*

(2) *If $i < j < k$ in $\{1, \ldots, n\}$, then, for every injective object E_i in $\mathcal{A}_i$, we have that $\delta_{E_i}^{ijk} : F_{ik}E_i \rightarrow F_{kj}F_{ik}E_i$ is a split epimorphism.*

(3) *For each $m \in \{1, \ldots, n-1\}$, $T^m p$ is a split epimorphism, for any epimorphism $p : (E, \mathbf{d}_E) \rightarrow (E', \mathbf{d}_{E'})$ between injective $F^{\leq m}$-comodules.*

Proof

(1) Since F is left exact, then the forgetful functor $\mathcal{A}_F \rightarrow \mathcal{A}$ is exact. Therefore, the free functor $F : \mathcal{A} \rightarrow \mathcal{A}_F$ preserves injectives, with implies that $F(E_1, \ldots, E_n)$ is an injective F–comodule for every injective $(E_1, \ldots, E_n) \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. Let us prove that every injective F–comodule is of this form by induction on n. For $n = 1$, there is nothing to prove, so let $n > 1$. By Theorem 3.4, we can identify

$\mathcal{A}_F$ with the category of comodules $(\mathcal{A}_1 \times \mathcal{A}_{F>1}^{>1})_{G^1}$. By Theorem 4.3, we have that $\mathcal{A}_{F>1}^{>1}$ is hereditary and that T^1 preserves injectives. In this case, $T^1 = F^1$, see Equation (27). Therefore, using Proposition 4.1, we have that any injective object of $\mathcal{A}_F$ is of the form $(X, \mathbf{d}_X) = (E_1, F^1(E_1) \oplus (V, \mathbf{d}_V))$ for some injective objects $E_1 \in \mathcal{A}_1$ and $(V, \mathbf{d}_V) \in \mathcal{A}_{F>1}^{>1}$. By induction hypothesis, we get that $(V, \mathbf{d}_V) = F^{>1}(E_2, \ldots, E_n)$ for some injectives E_j in $\mathcal{A}_j$, $j = 2, \ldots, n$. Hence $(X, \mathbf{d}_X) = F(E_1, E_2, \ldots, E_n)$ for some injective objects E_i in $\mathcal{A}_i$, $i = 1, \ldots, n$. By Theorem 4.3,

$$\left(d_X^{12}, \ldots, d_X^{1n} \right) : F^1 E_1 = (F_{12} E_1, \ldots, F_{1n} E_1) \to (E_2, \ldots, E_n)$$

is a split epimorphism, which implies that d_X^{1j} is a split epimorphism for $j = 2, \ldots, n$. Since by induction we know that each of the d_V^{ij}, $2 \leq i < j \leq n$ is a split epimorphism, we deduce that d_X^{ij}, for every $i < j$ in $\{1, \ldots, n\}$, is a split epimorphism.

(2) It follows from (1), since any object of the form $F(0, \ldots, 0, E_i, 0 \ldots, 0)$, for $i = 1, \ldots, n-2$ is injective in $\mathcal{A}_F$ (recall that the structure morphisms are exactly

$$d_{F(0, \ldots, 0, E_i, 0 \ldots, 0)}^{jk} = \delta_{E_i}^{ijk}, \quad i < j < k).$$

(3) By Theorem 3.4, we know that $(\mathcal{A}_{F \leq m}^{\leq m} \times \mathcal{A}_{F > m}^{> m})_{G^m}$ is hereditary, where G^m is the triangular (2×2)-matrix comonad constructed using the functor T^m from (28). Now, we conclude using Remark 3.5 and Theorem 4.3. $\qquad\square$

PROPOSITION 4.5   *Let $n \geq 2$ be a positive integer. Assume that $\mathcal{A}_i$ is an abelian category with enough injectives for all $i \in \{1, \ldots, n\}$, and that $F_{ij} : \mathcal{A}_i \to \mathcal{A}_j$ is left exact for every $1 \leq i < j \leq n$. If the category of comodules $\mathcal{A}_F$ is a hereditary category, then the following statements hold:*

(1)   Each of the functors F_{ij} preserves injectives.
(2) $F_{ij} p$ is a split epimorphism, for any epimorphism $p : E_i \to E_i'$ between injective objects in $\mathcal{A}_i$, for every $1 \leq i < j \leq n$.

Proof

(1) For $n = 2$, the claim follows from Theorem 4.3. Assume $n \geq 3$, we will proceed by induction on n. By Theorem 3.4, the hereditary category $\mathcal{A}_F$ is isomorphic $(\mathcal{A}_1 \times \mathcal{A}_{F>1}^{>1})_{F^1}$ (here, $T^1 = F^1$). From Theorem 4.3, we get that $\mathcal{A}_{F>1}^{>1}$ is hereditary. By induction hypothesis, all the functors F_{ij} with $2 \leq i < j$ preserve injectives. Given an injective object E_1 in $\mathcal{A}_1$, we know from the proof of Theorem 3.4 that $F^1 E_1 = ((F_{12} E_1, \ldots, F_{1n} E_1), \mathbf{d}_{F^1 E_1})$ is injective in $\mathcal{A}_{F>1}^{>1}$. On the other hand, by Proposition 4.4.(1), there exists an injective object $(E_2, \ldots, E_n) \in \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$ such that $F^1 E_1 = F^{>1}(E_2, \ldots, E_n)$. Therefore, since each F_{2j} for $j = 3, \ldots n$ preserves injectives, we obtain that $F_{1k}(E_1)$ is injective for every $k = 2, \ldots, n$. This completes the induction.

(2) Use induction on n and Theorems 3.4 and 4.3. $\qquad\square$

The following is our main result in this section.

THEOREM 4.6 *Assume that $\mathcal{A}_i$ is an abelian category with enough injectives for all $i \in \{1, \ldots, n\}$, and consider a normal triangular $n \times n$-matrix comonad $F = (F_{ij}) : \mathcal{A} \to \mathcal{A}$ such that $F_{ij} : \mathcal{A}_i \to \mathcal{A}_j$ is left exact for every $1 \leq i < j \leq n$. Then the category of comodules $\mathcal{A}_F$ is hereditary if, and only if, the following conditions are fulfilled:*

(a) *T^{n-1} and every F_{ij} preserves injectives for $1 \leq i < j \leq n$.*
(b) *For each $m \in \{1, \ldots, n-1\}$, $T^m p$ is a split epimorphism, for every epimorphism $p : (E, \mathbf{d}_E) \to (E', \mathbf{d}_{E'})$ between injective $F^{\leq m}$-comodules.*
(c) *For every injective object E_i in $\mathcal{A}_i$, its image $\delta_{E_i}^{ijk}$ for $i < j < k$ is a split epimorphism.*
(d) *Each of the categories $\mathcal{A}_i$ is hereditary.*

Proof

($\Leftarrow$) We use induction on n. For $n = 2$, this implication is given by Theorem 4.3, since in this case we have $T^1 = F^1 = F_{12}$.

Let $n \geq 3$, and suppose that the implication is true for any category of comodules over a normal triangular matrix comonad constructed using a linearly ordered set of length $n - 1$. Without loss of generality we can suppose by Theorem 3.4 that $\mathcal{A}_F = (\mathcal{A}_{F^{\leq n-1}}^{\leq n-1} \times \mathcal{A}_n)_{G^{n-1}}$, where G^{n-1} is the triangular 2×2-matrix comonad associated to the functor $T^{n-1} : \mathcal{A}_{F^{\leq n-1}}^{\leq n-1} \to \mathcal{A}_n$ given in (28). Since the axioms (a)-(d) are satisfied for the triangular $(n-1) \times (n-1)$-matrix comonad $F^{\leq n-1}$, by induction hypothesis we know that its category of comodules $\mathcal{A}_{F^{\leq n-1}}^{\leq n-1}$ is hereditary. Henceforth, by Remark 3.5, Theorem 4.3 can be applied, as $\mathcal{A}_n$ is already assumed to be hereditary. Therefore, $\mathcal{A}_F$ is hereditary since T^{n-1} preserves injectives.

($\Rightarrow$) Conditions (b) and (c) follow from Proposition 4.4. By Proposition 4.5.(1), we know that F_{ij} preserves injectives for every $1 \leq i < j \leq n$. Since, by Theorems 3.4 and 4.3, we get that T^{n-1} preserves injectives, we conclude (a). We use induction to prove (d). For $n = 2$, this implication is clear from Theorem 4.3. Suppose that (d) holds for any hereditary category of comodules over a triangular matrix comonad which was constructed by using a linearly ordered set of length $n - 1$. By Remark 3.5, we know that T^m is a left exact functor for any $m \in \{1, \ldots, n-1\}$. Thus, by Theorems 4.3 and 3.4, we know that $\mathcal{A}_{F^{\leq n-1}}^{\leq n-1}$ and $\mathcal{A}_n$ are hereditary. Hence $\mathcal{A}_i$, $i = 1, \ldots, n$ are hereditary. This gives to us condition (d). $\square$

5. Hereditary triangular matrix coalgebras

We illustrate our results by applying some of them to categories of comodules over coalgebras. We refer to [19] for basic information on coalgebras over commutative rings and their categories of comodules.

5.1. *Triangular matrix coalgebras*

Let C and D be coalgebras over a commutative ring K, and M be a $C - D$–bicomodule. Then, the functors $-\otimes_K C$, $-\otimes_K D$ give comonads over the category Mod_K of K–modules, and $-\otimes_K M : \mathsf{Mod}_K \to \mathsf{Mod}_K$ becomes a $(-\otimes_K D)-(-\otimes_K C)$–bicomodule functor (see Subsection 3.1). These functors define a triangular matrix comonad on $\mathsf{Mod}_K \times \mathsf{Mod}_K = \mathsf{Mod}_{K \times K}$ represented by the $K \times K$–coalgebra

$$E = \begin{pmatrix} C & 0 \\ M & D \end{pmatrix}.$$

The comultiplication of this coalgebra over $R = K \times K$ is given by

$$\Delta \begin{pmatrix} c & 0 \\ m & d \end{pmatrix} = \sum_{(c)} \begin{pmatrix} c_{(1)} & 0 \\ 0 & 0 \end{pmatrix} \otimes_R \begin{pmatrix} c_{(2)} & 0 \\ 0 & 0 \end{pmatrix} + \sum_{(d)} \begin{pmatrix} 0 & 0 \\ 0 & d_{(1)} \end{pmatrix} \otimes_R \begin{pmatrix} 0 & 0 \\ 0 & d_{(2)} \end{pmatrix}$$

$$+ \sum_{(m)} \begin{pmatrix} m_{(-1)} & 0 \\ 0 & 0 \end{pmatrix} \otimes_R \begin{pmatrix} 0 & 0 \\ m_{(0)} & 0 \end{pmatrix} + \sum_{(m)} \begin{pmatrix} 0 & 0 \\ m_{(0)} & 0 \end{pmatrix} \otimes_R \begin{pmatrix} 0 & 0 \\ 0 & m_{(1)} \end{pmatrix},$$

$$(31)$$

and its couinty is defined by

$$\varepsilon \begin{pmatrix} c & 0 \\ m & d \end{pmatrix} = \begin{pmatrix} \varepsilon_C(c) & 0 \\ 0 & \varepsilon_D(d) \end{pmatrix},$$

where we are using Heyneman–Sweedler's notation.

Assume that C and M are flat as K–modules. The category of right C–comodules is denoted by Comod_C, and similarly for any other coalgebra over a commutative ring. By Theorem 3.1, we have the normal triangular matrix comonad $G = \begin{pmatrix} 1 & 0 \\ -\square_C M & 1 \end{pmatrix}$ on $\mathsf{Comod}_C \times \mathsf{Comod}_D$, and the equivalence of categories

$$(\mathsf{Comod}_C \times \mathsf{Comod}_D)_G \cong (\mathsf{Mod}_{K \times K})_{-\otimes_R E} = \mathsf{Comod}_E. \qquad (32)$$

5.2. *Base change ring by a Frobenius algebra*

Let R be a commutative Frobenius algebra over a commutative ring K (see [20]), that is, the functor $- \otimes_K R : \mathsf{Mod}_K \to \mathsf{Mod}_R$ is right adjoint to the forgetful functor from Mod_R to Mod_K (see [21] and [22, Remark 2.3.1]). The counit of this adjunction evaluated at K gives the Frobenius functional $\psi : R \to K$. If η denotes the unit, then $\eta_R(1) = \sum_i e_i \otimes f_i \in R \otimes_K R$ is the Casimir element. Given any R–coalgebra E, we get the adjoint pairs of functors

$$\mathsf{Comod}_E \underset{-\otimes_R E}{\overset{\longrightarrow}{\longleftarrow}} \mathsf{Mod}_R \underset{-\otimes_K R}{\overset{\longrightarrow}{\longleftarrow}} \mathsf{Mod}_K,$$

where the unlabelled arrows denote the forgetful functors that are left adjoints to $- \otimes_R E$ and $- \otimes_K R$. By composing these adjoint pairs we obtain the adjoint pair

$$\mathsf{Comod}_E \underset{-\otimes_K E}{\overset{\longrightarrow}{\longleftarrow}} \mathsf{Mod}_K , \qquad (33)$$

where we are using the isomorphism $E \cong R \otimes_R E$. In this way, we obtain a comonad $- \otimes_K E : \mathsf{Mod}_K \to \mathsf{Mod}_K$, which is determined by the structure of K–coalgebra on E

with comultiplication

$$E \xrightarrow{\tilde{\Delta}} E \otimes_K E, \qquad x \longmapsto \sum_{(x),i} x_{(1)} e_i \otimes f_i x_{(2)},$$

and counit $\tilde{\varepsilon} = \psi \circ \varepsilon$, where $\varepsilon : E \to R$ is the counit of the R–coalgebra E.

The adjoint pair (33) gives rise to the comparison functor $V : \mathsf{Comod}_E \to (\mathsf{Mod}_K)_{-\otimes E}$ (see [13, Section 3.2]). If E is flat as an R–module, then Comod_E is an abelian category and the faithful forgetful functor $\mathsf{Comod}_E \to \mathsf{Mod}_R$ is exact [18]. This easily implies that the forgetful functor $\mathsf{Comod}_E \to \mathsf{Mod}_K$ satisfies the hypotheses of Beck's theorem (precisely, the dual of [13, Theorem 3.3.10]) and, hence, V is an equivalence of categories. We get thus that the categories of right comodules over the R–coalgebra (E, Δ, ε) and of right comodules over the K–coalgebra $(E, \tilde{\Delta}, \tilde{\varepsilon})$ are equivalent.

5.3. *Bipartite coalgebras*

Let C, D be K–coalgebras, and M be a $C - D$–bicomodule. Assume that C, D, M are flat K–modules. In the case of the coalgebra $E = \begin{pmatrix} C & 0 \\ M & D \end{pmatrix}$, the base ring $R = K \times K$ is a Frobenius K–algebra with Frobenius functional $\psi : R \to K$ given by $\psi(a, b) = a + b$ for all $(a, b) \in K \times K$, and Casimir element $e = u \otimes u + v \otimes v \in R \otimes R$, where $u = (1, 0)$, $v = (0, 1)$. By 5.2 , E is a K–coalgebra. Explicitly, the comultiplication and counit are

$$\tilde{\Delta} \begin{pmatrix} c & 0 \\ m & d \end{pmatrix} = \sum_{(c)} \begin{pmatrix} c_{(1)} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c_{(2)} & 0 \\ 0 & 0 \end{pmatrix} + \sum_{(d)} \begin{pmatrix} 0 & 0 \\ 0 & d_{(1)} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & d_{(2)} \end{pmatrix} \quad (34)$$

$$+ \sum_{(m)} \begin{pmatrix} m_{(-1)} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ m_{(0)} & 0 \end{pmatrix} + \sum_{(c)} \begin{pmatrix} 0 & 0 \\ m_{(0)} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & m_{(1)} \end{pmatrix},$$

and

$$\tilde{\varepsilon} \begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \varepsilon_C(c) + \varepsilon_D(d)$$

We recover thus the construction of a bipartite K–coalgebra from [11, p.91] (called triangular matrix coalgebra in [12]).

In the following theorem, we say that a K–coalgebra C is *right hereditary* if the category Comod_C is hereditary.

THEOREM 5.1 *Let C, D be K–coalgebras, and M be a C–D–bicomodule. Assume that C, D and M are flat as K–modules. The bipartite K–coalgebra $\begin{pmatrix} C & 0 \\ M & D \end{pmatrix}$ is right hereditary if and only if the following conditions hold.*

(1) *$U \square_C M$ is an injective right D–comodule for every injective right C–comodule U ;*
(2) *C and D are right hereditary;*
(3) *$p \square_C M$ is a split epimorphism for each epimorphism $p : E_1 \to E_1'$ of injective right C–comodules E_1, E_1'.*

Proof Since we assume M, C and D to be flat over K, it follows that the functor $-\square_C M : \mathsf{Comod}_C \to \mathsf{Comod}_D$ is left exact. Now, the theorem follows from Theorem 4.3, the equivalence of categories (32), and the equivalence of categories given at the end of paragraph 5.2. □

5.4. *Generalized matrix coalgebras*

Let M_{ij}, $i, j = 1, \ldots, n$, be a set of modules over a commutative ring K, and consider the endofunctors $F_{ij} = M_{ij} \otimes_K - : \mathsf{Mod}_K \to \mathsf{Mod}_K$. Then the matrix functor $F : \mathsf{Mod}_K^n \to \mathsf{Mod}_K^n$ has the structure of a comonad if and only if there exists a set of natural transformations δ^{ikj} and ε^i as in Proposition 2.1. These natural transformations are determined by linear maps $\phi_{ikj} = \delta_K^{ikj} : M_{ij} \to M_{kj} \otimes M_{ik}$ and $\epsilon_i = \varepsilon_K^i : M_{ii} \to K$. We thus obtain a K^n-coalgebra

$$F(K) = \begin{pmatrix} M_{11} & M_{21} & \cdots & M_{n1} \\ M_{12} & M_{22} & \cdots & M_{n2} \\ \vdots & \vdots & & \vdots \\ M_{1n} & M_{2n} & \cdots & M_{nn} \end{pmatrix}.$$

By Remark 2.2, each of the entries in the diagonal of this matrix is a K–coalgebra, and M_{ij} is an $M_{jj} - M_{ii}$–bicomodule. Also, every ϕ_{ikj} factors trough the cotensor product $M_{kj}\square_{M_{kk}}M_{ik}$. Using the base change of ring from 5.2 with the Frobenius K–algebra $R = K^n$ we get the 'comatrix' K-coalgebra of [12, Section 2]. We prefer the name matrix coalgebras to avoid confusion with the notion of a comatrix coring (and, in particular, comatrix coalgebra) from [23], which is a different construction. As in 5.2, the comultiplication and counit of this matrix K–coalgebra can be computed explicitly.

5.5. *Triangular matrix corings*

More generally, we may consider corings $\mathfrak{C}$ and $\mathfrak{D}$ over different base rings A and B, respectively (see [19]). Given a $\mathfrak{C} - \mathfrak{D}$–bicomodule M, we get the triangular matrix $A \times B$-coring $\begin{pmatrix} \mathfrak{C} & 0 \\ M & \mathfrak{D} \end{pmatrix}$. Under suitable flatness conditions, a result similar to Theorem 5.1 may be formulated for corings.

Remark 5.2 Let $\mathfrak{C}$ be an R-coring which is flat as left R-module. Assume that R is a (possibly non commutative) Frobenius algebra over a commutative ring K. Since the functor $R \otimes_K - : \mathsf{Mod}_K \to \mathsf{Mod}_R$ is then a right adjoint to the forgetful functor from the category of right R–modules Mod_R to Mod_K, the arguments from 5.2 run here to prove that the category of right $\mathfrak{C}$–comodules is equivalent to the category of right comodules over a certain K-coalgebra built from R and $\mathfrak{C}$. In particular, when $\mathfrak{C} = R$ endowed with the trivial coring structure, we get the main result from [24], namely, that the category of right R-modules is equivalent to the category of right R-comodules.

Disclosure statement

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