The bicategories of corings

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Abstract

To a B-coring and a (B, A)-bimodule that is finitely generated and projective as a right A-module an A-coring is associated. This new coring is termed a *base ring extension of a coring by a module*. We study how the properties of a bimodule such as separability and the Frobenius properties are reflected in the induced base ring extension coring. Any bimodule that is finitely generated and projective on one side, together with a map of corings over the same base ring, lead to the notion of a *module-morphism*, which extends the notion of a morphism of corings (over different base rings). A module-morphism of corings induces functors between the categories of comodules. These functors are termed *pull-back* and *push-out* functors, respectively, and thus relate categories of comodules of different corings. We study when the pull-back functor is fully faithful and when it is an equivalence. A generalised descent associated to a morphism of corings is introduced. We define a *category of module-morphisms*, and show that push-out functors are naturally isomorphic to each other if and only if the corresponding module-morphisms are mutually isomorphic. All these topics are studied within a unifying language of *bicategories* and the extensive use is made of interpretation of corings as comonads in the bicategory Bim of bimodules and module-morphisms as 1-cells in the associated bicategories of comonads in Bim.
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1. Introduction

1.1. Motivation and overview

The aim of this paper is to study properties of corings and functors between comodule categories from a *bicategorical* point of view, and thus argue that bicategories provide a natural and unifying point of view on corings.

In context of corings, bicategories arise in a very natural way. The categorical information about rings is contained in a bicategory of bimodules Bim in which objects (0-cells) are rings, 1-cells are bimodules (with composition given by the tensor product) and 2-cells are bilinear maps. This is the most fundamental example of a bicategory, and provides the ideal set-up for studying problems such as Morita theory. Building upon a pioneering work of Street [28], Lack and Street [21] have considered the bicategory EM(B) obtained as the free completion under Eilenberg–Moore objects of a given bicategory B. When B is taken to be Bim, the suitable dual of EM(B) is a bicategory in which objects are corings. The resulting bicategory (and its sub-bicategories) is the main object of studies in the present paper.

Our motivation for studying the bicategory of corings is twofold, deeply rooted in non-commutative geometry. First, it appears that there is a growing appreciation for the language of bicategories in non-commutative geometry. For example, in a recent paper [23] Manin argues that the classification of vector bundles over the non-commutative torus (or the K-theory of this torus) is best explained in terms of a Morita bicategory associated with this torus. This particular example of the role that bicategories play in non-commutative geometry can be seen in a much wider context, as bicategories appear very naturally in quantisation of Poisson manifolds in terms of C^*-algebras or in the theory of von Neumann algebras [22].

Second motivation originates from the appearance of corings in non-commutative algebraic geometry. Here corings feature in two different ways. On one hand, if an A-coring $\mathfrak{C}$ is flat as a left A-module, then the category of its right comodules is a Grothendieck category, hence a non-commutative space or a non-commutative quasi-scheme in the sense of Van den Bergh [30] and Rosenberg [25,26]. Natural isomorphism classes of functors between Grothendieck categories play the role of maps between non-commutative spaces (cf. [27]). In relation to corings, one needs to consider bimodules between corings that induce functors between corresponding comodule categories. Natural maps between these functors arise from morphisms between corresponding bimodules. It turns out that to study all these structures in a unified way one is led to considering a suitable bicategory. In another approach to non-commutative algebraic geometry, certain classes of corings appear as *covers* of non-commutative spaces [20]. Bimodules between corings can then be understood as a change of cover of the underlying space. The change of cover affects the corresponding quasi-scheme, i.e. the category of comodules. Again to study these effects in a uniform way, one should study a bicategory in which corings are 0-cells.

Our presentation and the choice of topics for the present paper are motivated by the above geometric interpretation. We begin by applying (the comonadic version of) the Lack and Street construction to the bicategory of bimodules and describe explicitly the resulting

bicategory of corings REM(Bim). With an eye on the interpretation of corings as covers of non-commutative spaces we introduce the bicategory fREM(Bim), by restricting 1-cells in REM(Bim) to those that arise from adjoint pairs in Bim (finitely generated and projective modules). The Lack and Street construction has an obvious 'left-sided' version resulting in bicategories LEM(Bim) and fLEM(Bim). We show that there is a duality between the hom-categories of fLEM(Bim) and fREM(Bim).

Next we introduce the notion of a base extension of a coring by a bimodule (a 'change of cover' of a non-commutative space). We show that any such base extension gives rise to 1-cells in fLEM(Bim) and fREM(Bim). We study in what way properties of a bimodule such as separability and the Frobenius property are reflected by the resulting base extension coring.

The appearance of 1-cells in fLEM(Bim) and fREM(Bim) associated to a base extension of a coring by a bimodule leads to the notion of a *module-morphism* of corings as a pair consisting of a bimodule that is finitely generated and projective as a right module, and of a coring map. We introduce two functors between categories of comodules induced by a module-morphism. Following the geometric interpretation these are called a *push-out* and a *pull-back* functors. In Section 5 we determine when these functors are inverse equivalences, and also we define and give basic properties of a generalised descent associated to a morphism of corings.

1.2. Notation and preliminaries

We work over a commutative ring k with a unit. All algebras are over k, associative and with a unit. The identity morphism for an object V is also denoted by V. For a ring (algebra) R, the category of right R-modules and right R-linear maps is denoted by $\mathscr{M}_R$. Symmetric notation is used for left modules. As is customary, we often write M_R to indicate that M is a right R-module, etc. The dual module of M_R, consisting of all R-linear maps from M_R to R_R, is denoted by M^*, while the dual of $_RN$ is denoted by $^*N$. The multiplication in the endomorphism ring of a right module (comodule) is given by composition of maps, while the multiplication in the endomorphism ring of a left module (comodule) is given by opposite composition (we always write argument to the right of a function). The symbol $-\otimes_R-$ between maps and modules denotes the tensor product bifunctor over the algebra R.

Let A be an algebra. The comultiplication of an A-coring $\mathfrak{C}$ is denoted by $\varDelta_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}\otimes_A\mathfrak{C}$, and its counit by $\varepsilon_{\mathfrak{C}} : \mathfrak{C} \to A$. To indicate the action of $\varDelta_{\mathfrak{C}}$ on elements we use the Sweedler sigma notation, i.e. for all $c \in \mathfrak{C}$,

$$\varDelta_{\mathfrak{C}}(c) = \sum c_{(1)}\otimes_A c_{(2)},$$
$$(\varDelta_{\mathfrak{C}}\otimes_A\mathfrak{C}) \circ \varDelta_{\mathfrak{C}}(c) = (\mathfrak{C}\otimes_A\varDelta_{\mathfrak{C}}) \circ \varDelta_{\mathfrak{C}}(c) = \sum c_{(1)}\otimes_A c_{(2)}\otimes_A c_{(3)},$$

etc. Gothic capital letters always denote corings. To indicate that $\mathfrak{C}$ is an A-coring we often write $(\mathfrak{C} : A)$. The category of right $\mathfrak{C}$-comodules and right $\mathfrak{C}$-colinear maps is denoted by $\mathscr{M}^{\mathfrak{C}}$. Recall that $\mathscr{M}^{\mathfrak{C}}$ is built upon the category of right A-modules, in the sense that

there is a forgetful functor $\mathscr{M}^{\mathfrak{C}} \to \mathscr{M}_A$. In particular, any right $\mathfrak{C}$-comodule is also a right A-module, and any right $\mathfrak{C}$-comodule map is right A-linear. For a right $\mathfrak{C}$-comodule M, $\varrho^M : M \to M \otimes_A \mathfrak{C}$ denotes the coaction, and $\operatorname{Hom}^{-\mathfrak{C}}(M, N)$ is the k-module of $\mathfrak{C}$-colinear maps $M \to N$. On elements, the action of ϱ^M is expressed by Sweedler's sigma-notation $\varrho^M(m) = \sum m_{(0)} \otimes m_{(1)}$. Symmetric notation is used for left $\mathfrak{C}$-comodules. In particular, the coaction of a left $\mathfrak{C}$-comodule N is denoted by $^N\varrho$, and, on elements, by $^N\varrho(n) = \sum n_{(-1)} \otimes n_{(0)} \in \mathfrak{C} \otimes_A N$.

For any A-coring $\mathfrak{C}$, the dual module $\mathfrak{C}^* = \operatorname{Hom}_{-A}(\mathfrak{C}, A)$ is a k-algebra with the product $f * g(c) = \sum f(g(c_{(1)})c_{(2)})$ and unit $\varepsilon_{\mathfrak{C}}$. This is known as a *right dual ring of* $\mathfrak{C}$. Similarly, the dual module $^*\mathfrak{C} = \operatorname{Hom}_{A-}(\mathfrak{C}, A)$ is a k-algebra with the product $f * g(c) = \sum f(c_{(1)}g(c_{(2)}))$ and unit $\varepsilon_{\mathfrak{C}}$. This is known as a *left dual ring of* $\mathfrak{C}$. The k-linear map $\iota_A : A \to {}^*\mathfrak{C}, a \mapsto [c \mapsto \varepsilon_{\mathfrak{C}}(ca)]$ is an anti-algebra map.

Given A-corings $\mathfrak{C}, \mathfrak{D}$, a *morphism of A-corings* $\mathfrak{D} \to \mathfrak{C}$ is an A-bimodule map $\gamma : \mathfrak{D} \to \mathfrak{C}$ such that $\varDelta_{\mathfrak{C}} \circ \gamma = (\gamma \otimes_A \gamma) \circ \varDelta_{\mathfrak{D}}$ and $\varepsilon_{\mathfrak{C}} \circ \gamma = \varepsilon_{\mathfrak{D}}$. The category of A-corings is denoted by A-**Crg**

If $\mathfrak{D}$ is a B-coring and $\alpha : B \to A$ is an algebra map, then one views A as a B-bimodule via α and defines an A-coring structure on the A-bimodule $A_\alpha[\mathfrak{D}] := A \otimes_B \mathfrak{D} \otimes_B A$ by

$$\varDelta_{A_\alpha[\mathfrak{D}]} : A_\alpha[\mathfrak{D}] \to A_\alpha[\mathfrak{D}] \otimes_A A_\alpha[\mathfrak{D}] \simeq A \otimes_B \mathfrak{D} \otimes_B A \otimes_B \mathfrak{D} \otimes_B A,$$
$$a \otimes_B d \otimes_B a' \mapsto \sum a \otimes_B d_{(1)} \otimes_B 1 \otimes_B d_{(2)} \otimes_B a',$$

and $\varepsilon_{A_\alpha[\mathfrak{D}]} : a \otimes d \otimes a' \mapsto a\varepsilon_{\mathfrak{D}}(d)a'$. $A_\alpha[\mathfrak{D}]$ is known as a *base ring extension of* $\mathfrak{D}$. The construction of a base ring extension allows one to consider morphisms of corings over different rings. Given corings $(\mathfrak{C} : A)$ and $(\mathfrak{D} : B)$ a *morphism of corings* $(\mathfrak{D} : B) \to (\mathfrak{C} : A)$ is a pair (γ, α), where $\alpha : B \to A$ is an algebra map and $\gamma : \mathfrak{D} \to \mathfrak{C}$ is a B-bimodule map such that the induced map $\tilde{\gamma} : A_\alpha[\mathfrak{D}] \to \mathfrak{C}, a \otimes_B d \otimes_B a' \mapsto a\gamma(d)a'$ is a morphism of A-corings.

Recall that, given a right $\mathfrak{C}$-comodule M and a left $\mathfrak{C}$-comodule N one defines a *cotensor product* $M\square_{\mathfrak{C}}N$ by the following exact sequence of k-modules:

$$0 \longrightarrow M\square_{\mathfrak{C}}N \longrightarrow M \otimes_A N \xrightarrow{\omega_{M,N}} M \otimes_A \mathfrak{C} \otimes_A N,$$

where $\omega_{M,N} = \varrho^M \otimes_A N - M \otimes_A^N \varrho$, and ϱ^M and $^N\varrho$ are coactions. Suppose that $\mathfrak{C}$ is flat as a left A-module. A left $\mathfrak{C}$-comodule N is said to be *coflat* (resp. *faithfully coflat*), if the cotensor functor $-\square_{\mathfrak{C}}N : \mathscr{M}^{\mathfrak{C}} \to \mathscr{M}_k$ preserves (resp. preserves and reflects) exact sequences in $\mathscr{M}^{\mathfrak{C}}$. A detailed account of the theory of corings and comodules can be found in [6].

Henceforth, $\{e_i, e_i^*\}$, with $e_i \in \Sigma, e_i^* \in \Sigma^*$ always denotes a finite dual basis for a finitely generated and projective module Σ_A.

2. The bicategories of corings

For general definitions of bicategories and their morphisms we refer the reader to the fundamental paper [2]. Following [2, (2.5)] and the conventions adopted there, the bicategory of bimodules Bim is defined as follows. Objects (i.e. 0-cells) are algebras $A, B, \ldots,$

1-cells from A to B are objects of the category ${}_B\mathscr{M}_A$ of (B, A)-bimodules, and 2-cells are bilinear maps. The composition of 1-cells is given by the tensor product of bimodules, and the identity 1-cell of A is the regular bimodule ${}_AA_A$. Given a bicategory B, the transpose bicategory of B is the bicategory B^{op} obtained from B by reversing 1-cells, while the conjugate bicategory B^{co} is obtained by reversing 2-cells, and the bicategory B^{coop} is obtained by reversing both (cf. [2, Section 3]).

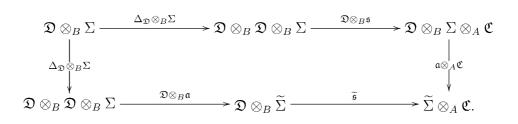## 2.1. The right bicategory of corings

Following [21, p. 249] (cf. [28]), the bicategory $\mathsf{REM(Bim)} := \mathsf{EM(Bim}^{coop})^{coop}$ consists of the following data:

- *Objects*: Corings $(\mathfrak{C} : A)$ (i.e. $\mathfrak{C}$ is an A-coring).
- *1-cells*: A 1-cell from $(\mathfrak{C} : A)$ to $(\mathfrak{D} : B)$ is a pair $(\Sigma, \mathfrak{s})$ consisting of a (B, A)-bimodule Σ and a (B, A)-bilinear map $\mathfrak{s} : \mathfrak{D}\otimes_B\Sigma \to \Sigma\otimes_A\mathfrak{C}$ rendering commutative the following diagrams

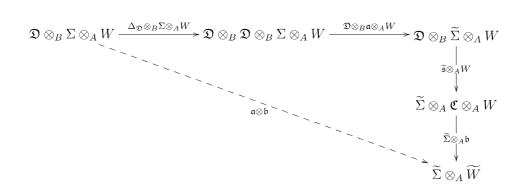The identity 1-cell of an object $(\mathfrak{C} : A)$ is given by $(A, \mathfrak{C})$.
- *2-cells*: Given 1-cells $(\Sigma, \mathfrak{s})$ and $(\widetilde{\Sigma}, \widetilde{\mathfrak{s}})$ from $(\mathfrak{C} : A)$ to $(\mathfrak{D} : B)$, 2-cells are defined as (B, A)-bilinear maps $\mathfrak{a} : \mathfrak{D}\otimes_B\Sigma \to \widetilde{\Sigma}$ rendering commutative the following diagram

The category consisting of all 1 and 2-cells from $(\mathfrak{C} : A)$ to $(\mathfrak{D} : B)$ is denoted by ${}_{(\mathfrak{D}:B)}\mathscr{R}_{(\mathfrak{C}:A)}$. The composition of 2-cells is defined as follows: Let $(\Sigma, \mathfrak{s})$, $(\widetilde{\Sigma}, \widetilde{\mathfrak{s}})$ be 1-cells from $(\mathfrak{C} : A)$ to $(\mathfrak{D} : B)$, and $(W, \mathfrak{w})$, $(\widetilde{W}, \widetilde{\mathfrak{w}})$ be 1-cells from $(\mathfrak{E} : C)$ to $(\mathfrak{C} : A)$. The composition of 1-cells leads to the following 1-cells from $(\mathfrak{E} : C)$ to $(\mathfrak{D} : B)$:

$$(\Sigma\otimes_A W, (\Sigma\otimes_A\mathfrak{w}) \circ (\mathfrak{s}\otimes_A W)) \quad \text{and} \quad (\widetilde{\Sigma}\otimes_A\widetilde{W}, (\widetilde{\Sigma}\otimes_A\widetilde{\mathfrak{w}}) \circ (\widetilde{\mathfrak{s}}\otimes_A\widetilde{W})).$$

If $\mathfrak{a} : \mathfrak{D}\otimes_B \Sigma \to \widetilde{\Sigma}$ and $\mathfrak{b} : \mathfrak{C}\otimes_A W \to \widetilde{W}$ are 2-cells, then the horizontal composition $\mathfrak{a} \otimes \mathfrak{b}$ is given by

$$
\begin{array}{ccc}
\mathfrak{D} \otimes_B \Sigma \otimes_A W & \xrightarrow{\Delta_{\mathfrak{D}}\otimes_B\Sigma\otimes_A W} \mathfrak{D} \otimes_B \mathfrak{D} \otimes_B \Sigma \otimes_A W \xrightarrow{\mathfrak{D}\otimes_B\mathfrak{a}\otimes_A W} & \mathfrak{D} \otimes_B \widetilde{\Sigma} \otimes_A W \\
& & \downarrow{\widetilde{\mathfrak{s}}\otimes_A W} \\
& & \widetilde{\Sigma} \otimes_A \mathfrak{C} \otimes_A W \\
& & \downarrow{\widetilde{\Sigma}\otimes_A \mathfrak{b}} \\
& & \widetilde{\Sigma} \otimes_A \widetilde{W}
\end{array}
$$

(diagonal arrow labeled $\mathfrak{a}\otimes\mathfrak{b}$ from top-left to bottom-right)

Every 1-cell in REM(Bim) defines a functor between categories of right comodules. This statement is made explicit in the following

Proposition 2.1. *Let* $(\Sigma, \mathfrak{s})$ *be a* 1-*cell from* $(\mathfrak{C} : A)$ *to* $(\mathfrak{D} : B)$ *in the bicategory* REM(Bim). *There is a functor* $\Sigma_\circ : \mathscr{M}^{\mathfrak{D}} \longrightarrow \mathscr{M}^{\mathfrak{C}}$ *sending*

$$
((M, \varrho^M) \mapsto (M\otimes_B \Sigma, \varrho^{M\otimes_B \Sigma} = (M\otimes_B \mathfrak{s}) \circ (\varrho^M\otimes_B \Sigma)), \quad (f \mapsto f\otimes_B \Sigma).
$$

In particular $\mathfrak{D}\otimes_B \Sigma$ *admits the structure of the* $(\mathfrak{D}, \mathfrak{C})$*-bicomodule.*

Proof. For any object $(\mathfrak{C} : A)$, the category $_{(k:k)}\mathscr{R}_{(\mathfrak{C}:A)}$ is isomorphic to the category $\mathscr{M}^{\mathfrak{C}}$ of right $\mathfrak{C}$-comodules. The functor $\Sigma_\circ$ is then identified with the horizontal composition functor—$\otimes(\Sigma, \mathfrak{s}):_{(k:k)}\mathscr{R}_{(\mathfrak{D}:B)} \longrightarrow {}_{(k:k)}\mathscr{R}_{(\mathfrak{C}:A)}$. View $\mathfrak{D}\otimes_B \Sigma$ as a left $\mathfrak{D}$-comodule with the coaction $^{\mathfrak{D}\otimes_B\Sigma}\varrho = \Delta_{\mathfrak{D}}\otimes_B \Sigma$, and compute

$$
\begin{aligned}
(\mathfrak{D}\otimes_B\varrho^{\mathfrak{D}\otimes_B\Sigma}) \circ (\Delta_{\mathfrak{D}}\otimes_B\Sigma) &= (\mathfrak{D}\otimes_B\mathfrak{D}\otimes_B\mathfrak{s}) \circ (\mathfrak{D}\otimes_B\Delta_{\mathfrak{D}}\otimes_B\Sigma) \circ (\Delta_{\mathfrak{D}}\otimes_B\Sigma) \\
&= (\mathfrak{D}\otimes_B\mathfrak{D}\otimes_B\mathfrak{s}) \circ (\Delta_{\mathfrak{D}}\otimes_B\mathfrak{D}\otimes_B\Sigma) \circ (\Delta_{\mathfrak{D}}\otimes_B\Sigma) \\
&= (\Delta_{\mathfrak{D}}\otimes_B\Sigma\otimes_A\mathfrak{C}) \circ (\mathfrak{D}\otimes_B\mathfrak{s}) \circ (\Delta_{\mathfrak{D}}\otimes_B\Sigma) \\
&= (\Delta_{\mathfrak{D}}\otimes_B\Sigma\otimes_A\mathfrak{C}) \circ \varrho^{\mathfrak{D}\otimes_B\Sigma}.
\end{aligned}
$$

Hence $^{\mathfrak{D}\otimes_B\Sigma}\varrho$ is right $\mathfrak{C}$-colinear, i.e. $\mathfrak{D}\otimes_B\Sigma$ is a $(\mathfrak{D}, \mathfrak{C})$-bicomodule, as stated. $\square$

As observed in [21, p. 249], 2-cells in a bicategory of monads can be defined in a *reduced* or an *unreduced* form. This bijective correspondence can be dualised to bicategories of comonads, and, in the case of REM(Bim), can be interpreted in terms of bicomodules.

Proposition 2.2. *Let $(\Sigma, \mathfrak{s})$ and $(\widetilde{\Sigma}, \widetilde{\mathfrak{s}})$ be two 1-cells from $(\mathfrak{C} : A)$ to $(\mathfrak{D} : B)$ in the bicategory* REM(Bim). *There is a bijection*

$$_{(\mathfrak{D}:B)}\mathscr{R}_{(\mathfrak{C}:A)}\left((\Sigma, \mathfrak{s}), (\widetilde{\Sigma}, \widetilde{\mathfrak{s}})\right) \simeq \operatorname{Hom}^{\mathfrak{D},\mathfrak{C}}(\mathfrak{D}\otimes_B\Sigma, \mathfrak{D}\otimes_B\widetilde{\Sigma}),$$

where $\mathfrak{D}\otimes_B\Sigma$ and $\mathfrak{D}\otimes_B\widetilde{\Sigma}$ are $(\mathfrak{D}, \mathfrak{C})$-bicomodules by Proposition 2.1. *Explicitly,*

$$(\mathfrak{a} \longmapsto (\mathfrak{D}\otimes_B\mathfrak{a}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma)), \quad ((\varepsilon_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}) \circ f \longleftarrow\!\mid f).$$

Proof. We only need to prove that the mutually inverse maps are well-defined. For a 2-cell $\mathfrak{a} : \mathfrak{D}\otimes_B\Sigma \to \widetilde{\Sigma}$, the left $\mathfrak{D}$-colinearity of $((\mathfrak{D}\otimes_B\mathfrak{a}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma))$ follows by the following simple calculation that uses the coassociativity of $\varDelta_{\mathfrak{D}}$:

$$(\varDelta_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}) \circ (\mathfrak{D}\otimes_B\mathfrak{a}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma) = (\mathfrak{D}\otimes_B\mathfrak{D}\otimes_B\mathfrak{a}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\mathfrak{D}\otimes_B\Sigma) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma)$$
$$= (\mathfrak{D}\otimes_B((\mathfrak{D}\otimes_B\mathfrak{a}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma))) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma).$$

Using the above calculation (to derive the second equality) and the fact that $\mathfrak{a}$ is a 2-cell (to derive the third equality), one computes

$$\varrho^{\mathfrak{D}\otimes_B\widetilde{\Sigma}} \circ (\mathfrak{D}\otimes_B\mathfrak{a}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma) = (\mathfrak{D}\otimes_B\widetilde{\mathfrak{s}}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}) \circ (\mathfrak{D}\otimes_B\mathfrak{a}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma)$$
$$= (\mathfrak{D}\otimes_B(\widetilde{\mathfrak{s}} \circ (\mathfrak{D}\otimes_B\mathfrak{a}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma))) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma)$$
$$= (\mathfrak{D}\otimes_B\mathfrak{a}\otimes_A\mathfrak{C}) \circ (\mathfrak{D}\otimes_B\mathfrak{D}\otimes_B\mathfrak{s}) \circ (\mathfrak{D}\otimes_B\varDelta_{\mathfrak{D}}\otimes_B\Sigma) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma)$$
$$= (\mathfrak{D}\otimes_B\mathfrak{a}\otimes_A\mathfrak{C}) \circ (\mathfrak{D}\otimes_B\mathfrak{D}\otimes_B\mathfrak{s}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\mathfrak{D}\otimes_B\Sigma) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma)$$
$$= (\mathfrak{D}\otimes_B\mathfrak{a}\otimes_A\mathfrak{C}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma\otimes_A\mathfrak{C}) \circ (\mathfrak{D}\otimes_B\mathfrak{s}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma)$$
$$= (((\mathfrak{D}\otimes_B\mathfrak{a}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma))\otimes_A\mathfrak{C}) \circ \varrho^{\mathfrak{D}\otimes_B\Sigma},$$

thus proving the right $\mathfrak{C}$-colinearity of $((\mathfrak{D}\otimes_B\mathfrak{a}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma))$.

    Conversely, consider a $(\mathfrak{D}, \mathfrak{C})$-bicolinear map $f : \mathfrak{D}\otimes_B\Sigma \to \mathfrak{D}\otimes_B\widetilde{\Sigma}$. We need to check that the bilinear map $\mathfrak{a} = (\varepsilon_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}) \circ f : \mathfrak{D}\otimes_B\Sigma \to \widetilde{\Sigma}$ is a 2-cell in REM(Bim). Using the $\mathfrak{D}$-colinearity of f, we compute

$$\widetilde{\mathfrak{s}} \circ (\mathfrak{D}\otimes_B\mathfrak{a}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma) = \widetilde{\mathfrak{s}} \circ (\mathfrak{D}\otimes_B\varepsilon_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}) \circ (\mathfrak{D}\otimes_B f) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma)$$
$$= \widetilde{\mathfrak{s}} \circ (\mathfrak{D}\otimes_B\varepsilon_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}) \circ f = \widetilde{\mathfrak{s}} \circ f.$$

On the other hand,

$$(\mathfrak{a}\otimes_A\mathfrak{C}) \circ (\mathfrak{D}\otimes_B\mathfrak{s}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma)$$
$$= (\varepsilon_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}\otimes_A\mathfrak{C}) \circ (f\otimes_A\mathfrak{C}) \circ (\mathfrak{D}\otimes_B\mathfrak{s}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\Sigma)$$
$$= (\varepsilon_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}\otimes_A\mathfrak{C}) \circ (\mathfrak{D}\otimes_B\widetilde{\mathfrak{s}}) \circ (\varDelta_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}) \circ f = \widetilde{\mathfrak{s}} \circ f,$$

where we have used the $\mathbb{C}$-colinearity of *f*. Therefore,

$$\widetilde{\mathfrak{s}} \circ (\mathfrak{D} \otimes_B \mathfrak{a}) \circ (\varDelta_{\mathfrak{D}} \otimes_B \varSigma) = (\mathfrak{a} \otimes_A \mathbb{C}) \circ (\mathfrak{D} \otimes_B \mathfrak{s}) \circ (\varDelta_{\mathfrak{D}} \otimes_B \varSigma),$$

i.e. $\mathfrak{a}$ is a 2-cell as required. $\quad\square$

Remark 2.3. Morphisms between corings over different base rings have a natural meaning in REM(Bim). Given an algebra morphism $\alpha : B \to A$, any 1-cell from $(\mathbb{C} : A)$ to $(\mathfrak{D} : B)$ of the form $(A, \mathfrak{s})$, defines a morphism of corings $(\gamma, \alpha) : (\mathfrak{D} : B) \to (\mathbb{C} : A)$ with

$$\gamma : \mathfrak{D} \xrightarrow{\;\mathfrak{D} \otimes_B \alpha\;} \mathfrak{D} \otimes_B A \xrightarrow{\;\mathfrak{s}\;} A \otimes_A \mathbb{C} \simeq \mathbb{C}.$$

Conversely, any coring morphism $(\gamma, \alpha) : (\mathfrak{D} : B) \to (\mathbb{C} : A)$ entails a 1-cell $(A, \mathfrak{s})$ from $(\mathbb{C} : A)$ to $(\mathfrak{D} : B)$ with

$$\mathfrak{s} : \mathfrak{D} \otimes_B A \xrightarrow{\;\gamma \otimes_B A\;} \mathbb{C} \otimes_B A \xrightarrow{\;\iota\;} \mathbb{C} \simeq A \otimes_A \mathbb{C},$$

where ι is the multiplication map.

2.2. The locally finite duality

To study functors between categories of comodules, it is convenient to introduce a different bicategory of corings. Thus we define fREM(Bim) as a bicategory obtained from REM(Bim) by restricting the class of 1-cells to those that are finitely generated and projective as right modules. Explicitly, fREM(Bim) has the same objects and 2-cells as REM(Bim), while a 1-cell $(\varSigma, \mathfrak{s})$ from $(\mathbb{C} : A)$ to $(\mathfrak{D} : B)$ in REM(Bim) is a 1-cell in fREM(Bim) provided $\varSigma_A$ is a finitely generated and projective module. The hom-category consisting of 1-cells from $(\mathbb{C} : A)$ to $(\mathfrak{D} : B)$ and their 2-cells in the bicategory fREM(Bim) is denoted by $_{(\mathfrak{D}:B)}\mathscr{R}^{\mathsf{f}}_{(\mathbb{C}:A)}$.

In order to understand better the meaning of fREM(Bim) for REM(Bim) we need to study the *left* bicategory of corings LEM(Bim) $:=$ EM(Bim^{co})^{coop} (cf. [21, p. 249]). LEM(Bim) has corings as objects. 1-cells from $(\mathbb{C} : A)$ to $(\mathfrak{D} : B)$ are pairs $(\varXi, \mathfrak{x})$ consisting of an (A, B)-bimodule $\varXi$ and an (A, B)-bilinear map $\mathfrak{x} : \varXi \otimes_B \mathfrak{D} \to \mathbb{C} \otimes_A \varXi$, which is compatible with the comultiplications and counits of both $\mathfrak{D}$ and $\mathbb{C}$. The identity 1-cell associated to $(\mathbb{C} : A)$ is the pair $(A, \mathbb{C})$. If $(\varXi, \mathfrak{x})$ and $(\widetilde{\varXi}, \widetilde{\mathfrak{x}})$ are 1-cells from $(\mathbb{C} : A)$ to $(\mathfrak{D} : B)$, then a 2-cell is an (A, B)-bilinear map $\mathfrak{a} : \varXi \otimes_B \mathfrak{D} \to \widetilde{\varXi}$, which is compatible with $\mathfrak{x}$, $\widetilde{\mathfrak{x}}$ and the comultiplication of $\mathfrak{D}$.

The category consisting of all 1 and 2-cells form $(\mathbb{C} : A)$ to $(\mathfrak{D} : B)$ in LEM(Bim) is denoted by $_{(\mathfrak{D}:B)}\mathscr{L}_{(\mathbb{C}:A)}$. Furthermore, $_{(\mathfrak{D}:B)}\mathscr{L}^{\mathsf{f}}_{(\mathbb{C}:A)}$ denotes the full subcategory of $_{(\mathfrak{D}:B)}\mathscr{L}_{(\mathbb{C}:A)}$ whose objects are 1-cells $(\varXi, \mathfrak{x})$ such that $_A\varXi$ is a finitely generated and projective left module. We use the notation fLEM(Bim) for the bicategory induced by the hom-categories $_{(\mathfrak{D}:B)}\mathscr{L}^{\mathsf{f}}_{(\mathbb{C}:A)}$.

Recall that a *duality* between categories is an equivalence of categories via contravariant functors. The standard duality between left and right finitely generated and projective modules induces a duality between the hom-categories of fLEM(Bim) and fREM(Bim).

Lemma 2.4. *Let* $(\mathfrak{C} : A)$ *and* $(\mathfrak{D} : B)$ *be corings. For any object* $(\Sigma, \mathfrak{s})$ *in* $_{(\mathfrak{D}:B)}\mathscr{R}^{\mathsf{f}}_{(\mathfrak{C}:A)}$, *define* $\mathfrak{s}_* : \Sigma^* \otimes_B \mathfrak{D} \to \mathfrak{C} \otimes_A \Sigma^*$ *by*

$$s^* \otimes_B d \longmapsto \sum_i ((s^* \otimes_A \mathfrak{C}) \circ \mathfrak{s}(d \otimes_B e_i)) \otimes_A e_i^*.$$

For any morphism $\mathfrak{a} : \mathfrak{D} \otimes_B \Sigma \to \widetilde{\Sigma}$ *in* $_{(\mathfrak{D}:B)}\mathscr{R}^{\mathsf{f}}_{(\mathfrak{C}:A)}$, *define* $\mathfrak{a}_* : (\widetilde{\Sigma})^* \otimes_B \mathfrak{D} \to \Sigma^*$ *by*

$$\widetilde{s}^* \otimes_B d \longmapsto \sum_i \widetilde{s}^*(\mathfrak{a}(d \otimes_B e_i)) e_i^*.$$

The functor

$$\mathscr{D}((\mathfrak{D} : B), (\mathfrak{C} : A)) : \; _{(\mathfrak{D}:B)}\mathscr{R}^{\mathsf{f}}_{(\mathfrak{C}:A)} \longrightarrow \; _{(\mathfrak{D}:B)}\mathscr{L}^{\mathsf{f}}_{(\mathfrak{C}:A)},$$

given by $((\Sigma, \mathfrak{s}) \mapsto (\Sigma^*, \mathfrak{s}_*))$ *and* $(\mathfrak{a} \mapsto \mathfrak{a}_*)$ *is a duality of categories.*

Proof. The maps $\mathfrak{s}_*, \mathfrak{a}_*$ are well-defined because, for all $b \in B$, $\sum_i be_i \otimes_A e_i^* = \sum_i e_i \otimes_A e_i^* b$ and the canonical element $\sum_i e_i \otimes_A e_i^*$ is basis-independent. The quasi-inverse contravariant functor is analogously constructed (its action is denoted by the asterisk on the left). For every object $(\Sigma, \mathfrak{s}) \in \; _{(\mathfrak{D}:B)}\mathscr{R}^{\mathsf{f}}_{(\mathfrak{C}:A)}$, we need to show that the evaluating isomorphism $ev :$ $\Sigma \simeq {}^*(\Sigma^*)$ is an isomorphism in $_{(\mathfrak{D}:B)}\mathscr{R}^{\mathsf{f}}_{(\mathfrak{C}:A)}$. This follows from the (easily checked) commutativity of the following diagram

$$
\begin{array}{ccc}
\mathfrak{D} \otimes_B \Sigma & \xrightarrow{\quad \mathfrak{s} \quad} & \Sigma \otimes_A \mathfrak{C} \\
{\scriptstyle \mathfrak{D} \otimes_B ev} \downarrow & & \downarrow {\scriptstyle ev \otimes_A \mathfrak{C}} \\
\mathfrak{D} \otimes_B {}^*(\Sigma^*) & \xrightarrow{\quad {}^*(\mathfrak{s}_*) \quad} & {}^*(\Sigma^*) \otimes_A \mathfrak{C}.
\end{array}
$$

Therefore, $(\Sigma, \mathfrak{s}) \simeq ({}^*(\Sigma^*), {}_*(\mathfrak{s}_*))$ in $_{(\mathfrak{D}:B)}\mathscr{R}^{\mathsf{f}}_{(\mathfrak{C}:A)}$ by Proposition 2.2. The proof is completed by routine computations that use the dual basis criterion. $\square$

Observe that the compatibility of the functors $\mathscr{D}(-, -)$ of Lemma 2.4 with the horizontal and vertical compositions is guaranteed by the functoriality of tensor products. In other words, functors $\mathscr{D}(-, -)$ give local equivalences between the bicategories fREM(Bim) and $(\mathsf{fLEM(Bim)})^{co}$. Since both bicategories have the same objects, we obtain the following:

Proposition 2.5. *The functors* $\mathscr{D}(-, -)$ *of Lemma* 2.4 *establish a biequivalence of bicategories* fREM(Bim) *and* $(\mathsf{fLEM(Bim)})^{co}$.

3. Base ring extensions by modules

Base ring extensions of corings described in the introduction correspond to extensions of base rings, i.e. to ring maps. Since the work of Sugano [29], it has become clear that a

more general and unifying framework for studying ring extensions is provided by bimodules rather than ring maps. In this section, we describe base extension of corings provided by a bimodule and also study properties of modules reflected by base extensions of corings.

3.1. Definition of a base ring extension by a bimodule

The basic idea of the construction of a base ring extension of corings by a bimodule hinges on the relationship between comatrix corings and Sweedler's corings.

Theorem 3.1. *Given a coring* $(\mathfrak{D} : B)$, *let* Σ *be a* (B, A)-*bimodule that is finitely generated and projective as a right A-module. Then the A-bimodule*

$$\Sigma[\mathfrak{D}] := \Sigma^* \otimes_B \mathfrak{D} \otimes_B \Sigma$$

is an A-coring with the comultiplication

$$\Delta_{\Sigma[\mathfrak{D}]} : \Sigma[\mathfrak{D}] \to \Sigma[\mathfrak{D}] \otimes_A \Sigma[\mathfrak{D}],$$
$$s^* \otimes_B d \otimes_B s \mapsto \sum_i s^* \otimes_B d_{(1)} \otimes_B e_i \otimes_A e_i^* \otimes_B d_{(2)} \otimes_B s,$$

and the counit

$$\varepsilon_{\Sigma[\mathfrak{D}]} : \Sigma[\mathfrak{D}] \to A, \quad s^* \otimes_B d \otimes_B s \mapsto s^*(\varepsilon_{\mathfrak{D}}(d)s).$$

The coring $\Sigma[\mathfrak{D}]$ *is called a* base ring extension of a coring by a module.

Proof. A B-coring $\mathfrak{D}$ induces a comonad $(F = -\otimes_B \mathfrak{D}, \delta = -\otimes_B \Delta_{\mathfrak{D}}, \xi = -\otimes_B \varepsilon_{\mathfrak{D}})$ in $\mathcal{M}_B$. On the other hand, Σ induces an adjunction

$$S = -\otimes_A \Sigma : \mathcal{M}_B \rightleftarrows \mathcal{M}_A : T = -\otimes_A \Sigma^*$$

with unit ζ and counit χ, given by, for all $X \in \mathcal{M}_B$, $Y \in \mathcal{M}_A$,

$$\zeta_X : X \longrightarrow X \otimes_B \Sigma \otimes_A \Sigma^*, \quad \chi_Y : Y \otimes_A \Sigma^* \otimes_B \Sigma \longrightarrow Y,$$
$$x \mapsto \sum_i x \otimes_B e_i \otimes_A e_i^*, \quad y \otimes_A s^* \otimes_B s \mapsto y s^*(s). \tag{3.1}$$

By [16, Theorem 4.2] (cf. the dual version of [10, Proposition 2.3]), these data give rise to a comonad in $\mathcal{M}_A$,
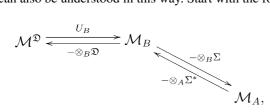
$$(SFT, SF\zeta_{FT} \circ S\delta_T, \chi \circ S\xi_T).$$

Therefore, $SFT(A) = A \otimes_A \Sigma^* \otimes_B \mathfrak{D} \otimes_B \Sigma \simeq \Sigma^* \otimes_B \mathfrak{D} \otimes_B \Sigma$ is an A-coring. The resulting comultiplication and counit come out as stated. $\square$

Examples of base ring extensions of a coring by a module include both base ring extensions and comatrix corings. In the former case, given an algebra map $\alpha : B \to A$, one views A as a (B, A)-bimodule via α and multiplication in A, i.e. $baa' = \alpha(b)aa'$, for all $a, a' \in A$, $b \in B$. Obviously, A is a finitely generated and projective right A-module, and

the identification of A with its dual A^* immediately shows that if $\Sigma = A$ then $\Sigma[\mathfrak{D}] \simeq A_\alpha[\mathfrak{D}]$. To obtain a comatrix coring $\Sigma^* \otimes_B \Sigma$, take $\mathfrak{D}$ to be the trivial B-coring B.

Remark 3.2. By [11, Example 5.1], comatrix corings can be understood as the coendomorphism corings of suitable quasi-finite bicomodules. The coring introduced in Theorem 3.1 can also be understood in this way. Start with the following adjunctions

$$
\mathcal{M}^{\mathfrak{D}} \underset{-\otimes_B \mathfrak{D}}{\overset{U_B}{\rightleftarrows}} \mathcal{M}_B \underset{-\otimes_A \Sigma^*}{\overset{-\otimes_B \Sigma}{\rightleftarrows}} \mathcal{M}_A,
$$

where U_B is the forgetful functor. The counit and unit of the composite adjunction

$$
-\otimes_A (\Sigma^* \otimes_B \mathfrak{D}) : \mathcal{M}_A \rightleftarrows \mathcal{M}^{\mathfrak{D}} : (-\otimes_B \Sigma) \circ U_B \tag{3.2}
$$

are given, respectively, for all $Y \in \mathcal{M}_A$ and $X \in \mathcal{M}^{\mathfrak{D}}$,

$$
\psi_Y : U_B(Y \otimes_A \Sigma^* \otimes_B \mathfrak{D}) \otimes_B \Sigma \longrightarrow Y,
$$
$$
\sum_\alpha y^\alpha \otimes_A s_\alpha^* \otimes_B d^\alpha \otimes_B u_\alpha \longmapsto \sum_\alpha y^\alpha s_\alpha^* (\varepsilon_{\mathfrak{D}}(d^\alpha) u_\alpha) \tag{3.3}
$$

and

$$
\eta_X : X \longrightarrow (U_B(X) \otimes_B \Sigma) \otimes_A \Sigma^* \otimes_B \mathfrak{D},
$$
$$
x \longmapsto \sum_{i,(x)} x_{(0)} \otimes_B e_i \otimes_A e_i^* \otimes_B x_{(1)}. \tag{3.4}
$$

Since $(-\otimes_B \Sigma) \circ U_B$ is the left adjoint to $-\otimes_A (\Sigma^* \otimes_B \mathfrak{D})$, $\Sigma^* \otimes_B \mathfrak{D}$ is an $(A, \mathfrak{D})$-quasi-finite $(A, \mathfrak{D})$-bicomodule, where A is considered trivially as an A-coring (cf. [13, Definition 4.1]). The corresponding co-hom functor comes out as $h_{\mathfrak{D}}(\Sigma^* \otimes_B \mathfrak{D}, -) = U_B(-) \otimes_B \Sigma$. Therefore, the discussion of [11, Section 5] implies that the coendomorphism A-coring associated to $\Sigma^* \otimes_B \mathfrak{D}$ is

$$
e_{\mathfrak{D}}(\Sigma^* \otimes_B \mathfrak{D}) = h_{\mathfrak{D}}(\Sigma^* \otimes_B \mathfrak{D}, \Sigma^* \otimes_B \mathfrak{D}) = \Sigma^* \otimes_B \mathfrak{D} \otimes_B \Sigma.
$$

This is precisely the coring constructed in Theorem 3.1, and the forms of the unit and the counit of the defining adjunction imply immediately the stated form of the comultiplication and counit.

The constructions given in Theorem 3.1 are functorial. The following proposition summarises basic properties of these functors.

Proposition 3.3. *Let Σ be a (B, A)-bimodule that is finitely generated and projective as a right A-module.*

(1) *The assignment*

$$\mathfrak{D} \mapsto \Sigma[\mathfrak{D}], \quad f \mapsto \Sigma^* \otimes_B f \otimes_B \Sigma$$

defines a functor $\Sigma[-] : B\text{-}\mathbf{Crg} \to A\text{-}\mathbf{Crg}$, *which commutes with colimits.*
(2) *For any* (C, B)-*bimodule* Ξ *that is finitely generated and projective as a right* B-*module,*

$$\Sigma[-] \circ \Xi[-] \simeq (\Xi \otimes_B \Sigma)[-]$$

as functors.

Proof. (1) A straightforward calculation that uses the definitions of the comultiplication and counit in $\Sigma[\mathfrak{D}]$ and the fact that f is a morphism of B-corings, confirms that $\Sigma^* \otimes_B f \otimes_B \Sigma$ is a morphism of A-corings. Since the compositions in B-**Crg** and A-**Crg** are the same as composition of k-modules, $\Sigma[-]$ is a functor. That $\Sigma[-]$ commutes with colimits is an easy consequence of the fact that tensor products commute with colimits, and that the colimit of an inductive system of B-corings is already computed in the category of B-bimodules.

(2) For any C-coring $\mathfrak{D}$, consider the (A, A)-bimodule isomorphism

$$\phi_{\Sigma, \Xi, \mathfrak{D}} : \Sigma^* \otimes_B \Xi^* \otimes_C \mathfrak{D} \otimes_C \Xi \otimes_B \Sigma \to (\Xi \otimes_B \Sigma)^* \otimes_C \mathfrak{D} \otimes_C (\Xi \otimes_B \Sigma),$$

$$s^* \otimes_B x^* \otimes_C d \otimes_C x \otimes_B s \mapsto \overline{(s^* \otimes_B x^*)} \otimes_C d \otimes_C x \otimes_B s,$$

where $\overline{(-)} : \Sigma^* \otimes_B \Xi^* \to (\Xi \otimes_B \Sigma)^*$ is the (A, C)-bilinear isomorphism sending $s^* \otimes_B x^*$ to $x \otimes_B s \mapsto \overline{(s^* \otimes_B x^*)}(x \otimes_B s) = s^*(x^*(x)s)$. We need to prove that $\phi_{\Sigma, \Xi, \mathfrak{D}}$ is an A-coring map, natural in $\mathfrak{D}$. Write $V = \Xi \otimes_B \Sigma$. Clearly V is a (C, A)-bimodule that is finitely generated and projective as a right A-module. Let $\{f_k, f_k^*\}$ be a dual basis for Ξ_B and let $\{e_i, e_i^*\}$ be a dual basis for Σ_A. Then $\{f_k \otimes_B e_i, \overline{(e_i^* \otimes_B f_k^*)}\}_{i,k}$ is a finite dual basis for V_A. For all $s^* \in \Sigma^*$, $x^* \in \Xi^*$, $x \in \Xi$, $s \in \Sigma$ and $d \in \mathfrak{D}$,

$$\varepsilon_{V[\mathfrak{D}]}(\phi_{\Sigma, \Xi, \mathfrak{D}}(s^* \otimes_B x^* \otimes_C d \otimes_C x \otimes_B s)) = \overline{(s^* \otimes_B x^*)}(\varepsilon_{\mathfrak{D}}(d) x \otimes_B s)$$
$$= s^*(x^*(\varepsilon_{\mathfrak{D}}(d)x)s)$$
$$= \varepsilon_{\Sigma[\Xi[\mathfrak{D}]]}(s^* \otimes_B x^* \otimes_C d \otimes_C x \otimes_B s).$$

Furthermore,

$$\Delta_{V[\mathfrak{D}]}(\phi_{\Sigma, \Xi, \mathfrak{D}}(s^* \otimes_B x^* \otimes_C d \otimes_C x \otimes_B s))$$
$$= \sum_{i,k} \overline{(s^* \otimes_B x^*)} \otimes_C d_{(1)} \otimes_C (f_k \otimes_B e_i) \otimes_A \overline{(e_i^* \otimes_B f_k^*)} \otimes_C d_{(2)} \otimes_C (x \otimes_B s)$$

$$= (\phi_{\Sigma, \Xi, \mathfrak{D}} \otimes_A \phi_{\Sigma, \Xi, \mathfrak{D}}) \left(\sum_{i,k} s^* \otimes_B x^* \otimes_C d_{(1)} \otimes_C f_k \otimes_B e_i \otimes_A \right.$$

$$\left. e_i^* \otimes_B f_k^* \otimes_C d_{(2)} \otimes_C x \otimes_B s \right)$$

$$= (\phi_{\Sigma, \Xi, \mathfrak{D}} \otimes_A \phi_{\Sigma, \Xi, \mathfrak{D}}) \circ \Delta_{\Sigma[\Xi[\mathfrak{D}]]}(s^* \otimes_B x^* \otimes_C d \otimes_C x \otimes_B s).$$

This proves that

$$\varepsilon_{(\Xi\otimes_B\Sigma)[\mathfrak{D}]} \circ \phi_{\Sigma,\Xi,\mathfrak{D}} = \varepsilon_{\Sigma[\Xi[\mathfrak{D}]]} \quad \text{and}$$

$$\Delta_{(\Xi\otimes_B\Sigma)[\mathfrak{D}]} \circ \phi_{\Sigma,\Xi,\mathfrak{D}} = \left(\phi_{\Sigma,\Xi,\mathfrak{D}}\otimes_A\phi_{\Sigma,\Xi,\mathfrak{D}}\right) \circ \Delta_{\Sigma[\Xi[\mathfrak{D}]]},$$

i.e. that for all C-corings $\mathfrak{D}$, the (A, A)-bimodule isomorphism $\phi_{\Sigma,\Xi,\mathfrak{D}}$ is a morphism of A-corings. Since a bijective morphism of corings is an isomorphism, all the $\phi_{\Sigma,\Xi,\mathfrak{D}}$ are isomorphisms of A-corings and their explicit forms immediately imply that they are natural in $\mathfrak{D}$. □

In particular, Proposition 3.3 leads to

Corollary 3.4. *Let Σ be a (B, A)-bimodule that is finitely generated and projective as a right A-module and let $\mathfrak{D}$ be a B-coring.*

(1) *The map*

$$\varepsilon_{\mathfrak{D},\Sigma} : \Sigma[\mathfrak{D}] \to \Sigma^*\otimes_B\Sigma, \quad s^*\otimes_Bd\otimes_Bs \mapsto s^*\otimes_B\varepsilon_{\mathfrak{D}}(d)s,$$

is a morphism of A-corings.
(2) *The maps*

$$\mathfrak{s} : \mathfrak{D}\otimes_B\Sigma \to \Sigma\otimes_A\Sigma^*\otimes_B\Sigma, \quad d\otimes_Bs \mapsto \sum_i e_i\otimes_Ae_i^*\otimes_B\varepsilon_{\mathfrak{D}}(d)s,$$

and

$$\mathfrak{s}' : \mathfrak{D}\otimes_B\Sigma \to \Sigma\otimes_A\Sigma[\mathfrak{D}], \quad d\otimes_Bs \mapsto \sum_i e_i\otimes_Ae_i^*\otimes_Bd\otimes_Bs,$$

define, respectively, 1-cells $(\Sigma, \mathfrak{s})$ and $(\Sigma, \mathfrak{s}')$ from $(\Sigma^*\otimes_B\Sigma : A)$ to $(\mathfrak{D} : B)$ and from $(\Sigma[\mathfrak{D}] : A)$ to $(\mathfrak{D} : B)$ in the bicategory fREM(Bim).
(3) *The maps*

$$\mathfrak{t} : \Sigma^*\otimes_B\mathfrak{D} \to \Sigma^*\otimes_B\Sigma\otimes_A\Sigma^*, \quad s^*\otimes_Bd \mapsto \sum_i s^*\varepsilon_{\mathfrak{D}}(d)\otimes_Be_i\otimes_Ae_i^*,$$

and

$$\mathfrak{t}' : \Sigma^*\otimes_B\mathfrak{D} \to \Sigma[\mathfrak{D}]\otimes_A\Sigma^*, \quad s^*\otimes_Bd \mapsto \sum_i s^*\otimes_Bd\otimes_Be_i\otimes_Ae_i^*,$$

define, respectively, 1-cells $(\Sigma^*, \mathfrak{t})$ and $(\Sigma^*, \mathfrak{t}')$ from $(\Sigma^*\otimes_B\Sigma : A)$ to $(\mathfrak{D} : B)$ and from $(\Sigma[\mathfrak{D}] : A)$ to $(\mathfrak{D} : B)$ in the bicategory fLEM(Bim).

Proof. (1) View B as a trivial B-coring. Then $\varepsilon_{\mathfrak{D}} : \mathfrak{D} \to B$ is a morphism of B-corings. Note that $\varepsilon_{\mathfrak{D},\Sigma} = \Sigma[\varepsilon_{\mathfrak{D}}]$, hence it is a morphism of A-corings. The proofs of statements (2) and (3) use the dual basis criterion, and are left to the reader. □

Given a comatrix coring $\Sigma^*\otimes_B\Sigma$, Σ is a right comodule and Σ^* is a left comodule of $\Sigma^*\otimes_B\Sigma$ (cf. [11, p. 891]). This can be extended to the following observation.

Lemma 3.5. *Given a coring* $(\mathfrak{D} : B)$, *let* Σ *be a* (B, A)-*bimodule that is finitely generated and projective as a right* A-*module and let* $\Sigma[\mathfrak{D}]$ *be the associated base ring extension coring. Then*:

(1) $\mathfrak{D}\otimes_B\Sigma$ *is a* $(\mathfrak{D}, \Sigma[\mathfrak{D}])$-*bicomodule with the left coaction* $^{\mathfrak{D}\otimes_B\Sigma}\varrho = \varDelta_{\mathfrak{D}}\otimes_B\Sigma$ *and the right coaction*

$$\varrho^{\mathfrak{D}\otimes_B\Sigma} : \mathfrak{D}\otimes_B\Sigma \to \mathfrak{D}\otimes_B\Sigma\otimes_A\Sigma[\mathfrak{D}],$$
$$d\otimes_B s \mapsto \sum_i d_{(1)}\otimes_B e_i\otimes_A e_i^*\otimes_B d_{(2)}\otimes_B s.$$

(2) $\Sigma^*\otimes_B\mathfrak{D}$ *is a* $(\Sigma[\mathfrak{D}], \mathfrak{D})$-*bicomodule with the right coaction* $\varrho^{\Sigma^*\otimes_B\mathfrak{D}} = \Sigma^*\otimes_B\varDelta_{\mathfrak{D}}$ *and the left coaction*

$$^{\Sigma^*\otimes_B\mathfrak{D}}\varrho : \Sigma^*\otimes_B\mathfrak{D} \to \Sigma[\mathfrak{D}]\otimes_A\Sigma^*\otimes_B\mathfrak{D},$$
$$s^*\otimes_B d \mapsto \sum_i s^*\otimes_B d_{(1)}\otimes_B e_i\otimes_A e_i^*\otimes_B d_{(2)}.$$

Proof. This is a direct consequence of Proposition 2.1 (and its left-handed version) and Corollary 3.4. $\square$

3.2. Properties reflected by base ring extensions

In [5] it has been studied how properties of bimodules are reflected by the associated finite comatrix corings. The aim of this section is to study such reflection of properties by the base ring extensions of corings by modules and thus to generalise the main results of [5].

Following [29], a (B, A)-bimodule Σ is said to be a *separable bimodule*, if the evaluation map

$$\Sigma\otimes_A {}^*\Sigma \to B, \quad s\otimes_A s^* \mapsto s^*(s)$$

is a split epimorphism of (B, B)-bimodules. Furthermore, recall from [1] and [17] that Σ is said to be a *Frobenius bimodule* if $_B\Sigma$ and Σ_A are finitely generated projective modules, and $\Sigma^* \simeq {}^*\Sigma$ as (A, B)-bimodules.

An A-coring $\mathfrak{C}$ is said to be a *Frobenius coring* if the forgetful functor $U_A : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is a Frobenius functor in the sense of [7,8] (cf. [3,4] for more details and other equivalent characterisations of a Frobenius coring). An A-coring $\mathfrak{C}$ is said to be a *cosplit coring* if and only if $\varepsilon_{\mathfrak{C}}$ is a split epimorphism of (A, A)-bimodules. Equivalently, $\mathfrak{C}$ is a cosplit coring if there exists an invariant element $c \in \mathfrak{C}$ (i.e. $ac = ca$, for all $a \in A$) such that $\varepsilon_{\mathfrak{C}}(c) = 1_A$. An A-coring $\mathfrak{C}$ is called a *coseparable coring* if and only if $\varDelta_{\mathfrak{C}}$ is a split monomorphism of $\mathfrak{C}$-bicomodules (cf. [15,14] for more details on coseparable corings).

Take a (B, A)-bimodule Σ that is finitely generated and projective as a right A-module, and consider its right endomorphism ring $S = \text{End}_{-A}(\Sigma)$. Then there is a canonical isomorphism of (S, S)-bimodules $\Sigma\otimes_A\Sigma^* \simeq S$ given by $\Sigma\otimes_A\Sigma^* \ni s\otimes_A s^* \to [s' \mapsto ss^*(s')]$. Its inverse is given by $f \mapsto \sum_i f(e_i)\otimes_A e_i^*$.

Proposition 3.6. *Let $\mathfrak{D}$ be a B-coring and Σ a (B, A)-bimodule such that Σ_A is a finitely generated projective module.*

(1) *If $_B\Sigma_A$ is a separable bimodule and $\mathfrak{D}$ is a coseparable B-coring, then $\Sigma[\mathfrak{D}]$ is a coseparable A-coring.*

(2) *If $_A\Sigma_B^*$ is a separable bimodule and $\mathfrak{D}$ is a cosplit B-coring, then $\Sigma[\mathfrak{D}]$ is a cosplit A-coring. Conversely, if $\Sigma[\mathfrak{D}]$ is a cosplit A-coring, then $_A\Sigma_B^*$ is a separable bimodule.*

(3) *If $_B\Sigma_A$ is a Frobenius bimodule and $\mathfrak{D}$ is a Frobenius B-coring, then $\Sigma[\mathfrak{D}]$ is a Frobenius A-coring.*

Proof. (1) By [18, Theorem 3.1(1)], the ring extension $B \to S$ is split. Let $\kappa : S \to B$ be a B-bilinear splitting map. Since 1_S is sent to $\sum_i e_i \otimes_A e_i^*$ by the isomorphism $S \simeq \Sigma \otimes_A \Sigma^*$, $\kappa\left(\sum_i e_i \otimes_A e_i^*\right) = 1_B$. Let $\nabla_{\mathfrak{D}} : \mathfrak{D} \otimes_B \mathfrak{D} \to \mathfrak{D}$ be a $\mathfrak{D}$-bicolinear map such that $\nabla_{\mathfrak{D}} \circ \varDelta_{\mathfrak{D}} = \mathfrak{D}$. Consider an A-bilinear map,

$$\nabla_{\Sigma[\mathfrak{D}]} : \Sigma[\mathfrak{D}] \otimes_A \Sigma[\mathfrak{D}] \xrightarrow{\hspace{3cm}} \Sigma[\mathfrak{D}],$$

$$s^* \otimes_B d \otimes_B s \otimes_A \tilde{s}^* \otimes_B d' \otimes_B \tilde{s} \longmapsto s^* \otimes_B \nabla_{\mathfrak{D}}(d\kappa(s \otimes_A \tilde{s}^*) \otimes_B d') \otimes_B \tilde{s}.$$

Take any $s^* \in \Sigma^*$, $s \in \Sigma$ and $d \in \mathfrak{D}$ and compute

$$
\begin{aligned}
\nabla_{\Sigma[\mathfrak{D}]} \circ \varDelta_{\Sigma[\mathfrak{D}]}(s^* \otimes_B d \otimes_B s) &= \nabla_{\Sigma[\mathfrak{D}]}\left(\sum_i s^* \otimes_B d_{(1)} \otimes_B e_i \otimes_A e_i^* \otimes_B d_{(2)} \otimes_B s\right) \\
&= \sum_i s^* \otimes_B \nabla_{\mathfrak{D}}(d_{(1)}\kappa(e_i \otimes_A e_i^*) \otimes_B d_{(2)}) \otimes_B s \\
&= \sum_i s^* \otimes_B \nabla_{\mathfrak{D}}(d_{(1)} \otimes_B d_{(2)}) \otimes_B s \\
&= s^* \otimes_B d \otimes_B s.
\end{aligned}
$$

Thus, $\nabla_{\Sigma[\mathfrak{D}]} \circ \varDelta_{\Sigma[\mathfrak{D}]} = \Sigma[\mathfrak{D}]$, as required. We need to show that $\nabla_{\Sigma[\mathfrak{D}]}$ is a $\Sigma[\mathfrak{D}]$-bicolinear map. Take any $s^*, \tilde{s}^* \in \Sigma^*$, $d, d' \in \mathfrak{D}$ and $s, \tilde{s} \in \Sigma$, and compute

$$
\begin{aligned}
\varDelta_{\Sigma[\mathfrak{D}]} &\circ \nabla_{\Sigma[\mathfrak{D}]}(s^* \otimes_B d \otimes_B s \otimes_A \tilde{s}^* \otimes_B d' \otimes_B \tilde{s}) \\
&= \sum_i s^* \otimes_B \nabla_{\mathfrak{D}}(d\kappa(s \otimes_A \tilde{s}^*) \otimes_B d'_{(1)}) \otimes_B e_i \otimes_A e_i^* \otimes_B d'_{(2)} \otimes_B \tilde{s} \\
&= (\nabla_{\Sigma[\mathfrak{D}]} \otimes_A \Sigma[\mathfrak{D}])\left(\sum_i s^* \otimes_B d \otimes_B s \otimes_A \tilde{s}^* \otimes_B d'_{(1)} \otimes_B e_i \otimes_A e_i^* \otimes_B d'_{(2)} \otimes_B \tilde{s}\right) \\
&= (\nabla_{\Sigma[\mathfrak{D}]} \otimes_A \Sigma[\mathfrak{D}]) \circ (\Sigma[\mathfrak{D}] \otimes_A \varDelta_{\Sigma[\mathfrak{D}]})(s^* \otimes_B d \otimes_B s \otimes_A \tilde{s}^* \otimes_B d' \otimes_B \tilde{s}).
\end{aligned}
$$

To derive this equality we have used the right $\mathfrak{D}$-colinearity of $\nabla_{\mathfrak{D}}$. This proves that $\nabla_{\Sigma[\mathfrak{D}]}$ is a right $\Sigma[\mathfrak{D}]$-colinear map. Similarly, one uses the left $\mathfrak{D}$-colinearity of $\nabla_{\mathfrak{D}}$ to obtain the left $\Sigma[\mathfrak{D}]$-colinearity of $\nabla_{\Sigma[\mathfrak{D}]}$. Thus we conclude that $\Sigma[\mathfrak{D}]$ is a coseparable coring as claimed.

(2) Consider the composition of A-bilinear maps $\theta : A \to \Sigma^* \otimes_B {}^*(\Sigma^*) \simeq \Sigma^* \otimes_B \Sigma$, where the first map is a splitting of the evaluation map that is provided by the separability of $_A\Sigma_B^*$. Put $\theta(1_A) = \sum_k s_k^* \otimes_B s_k$. Note that $\theta(1_A)$ is an invariant element of the

A-bimodule $\Sigma^*\otimes_B\Sigma$ that in addition satisfies $\sum_k s_k^*(s_k) = 1_A$. By hypothesis, $\mathfrak{D}$ is a cosplit B-coring, hence there exists an invariant element $d \in \mathfrak{D}$ such that $\varepsilon_{\mathfrak{D}}(d) = 1_B$. Combining d with $\theta(1_A)$ one obtains an invariant element $\sum_k s_k^*\otimes_B d\otimes_B s_k \in \Sigma[\mathfrak{D}]$, such that $\varepsilon_{\Sigma[\mathfrak{D}]}\left(\sum_k s_k^*\otimes_B d\otimes_B s_k\right) = 1_A$.

Conversely, if $\Sigma[\mathfrak{D}]$ is a cosplit A-coring, and $\sum_l s_l^*\otimes_B d_l\otimes_B s_l$ is the corresponding invariant element, then $\sum_l s_l^*\otimes_B \varepsilon_{\mathfrak{D}}(d_l)s_l \in \Sigma^*\otimes_B\Sigma$ gives a section for the evaluation $\Sigma^*\otimes_B {}^*(\Sigma^*) \to A$.

(3) Suppose that Σ is a Frobenius (B, A)-bimodule and let $\gamma : \Sigma^* \to {}^*\Sigma$ be the corresponding (A, B)-bilinear isomorphism. Let T be the opposite ring to the left dual ring ${}^*\mathfrak{D}$. By [3, Theorem 4.1], there is an isomorphism $\sigma : \mathfrak{D} \to T$ of (B, T)-bimodules and ${}_B\mathfrak{D}$ is a finitely generated and projective module. Consider an A-bilinear map $\beta = ({}^*\Sigma\otimes_B\sigma\otimes_B\Sigma) \circ (\gamma\otimes_B\mathfrak{D}\otimes_B\Sigma)$. This leads to an (A, B)-bimodule isomorphism

$$\zeta : \Sigma^*\otimes_B\mathfrak{D}\otimes_B\Sigma \xrightarrow{\beta} {}^*\Sigma\otimes_B {}^*\mathfrak{D}\otimes_B\Sigma \xrightarrow{\simeq} \mathrm{Hom}_{B-}(\mathfrak{D}\otimes_B\Sigma, \Sigma),$$

where the last isomorphism is a consequence of the fact that both ${}_B\mathfrak{D}$ and ${}_B\Sigma$ are finitely generated and projective modules. Note that ${}_A\Sigma[\mathfrak{D}]$ is a finitely generated and projective module. In view of [3, Theorem 4.1], it suffices to show that $\Sigma[\mathfrak{D}] \simeq R$ as (A, R)-bimodules, where R is the opposite ring of the left dual ring ${}^*\Sigma[\mathfrak{D}]$. Consider the following chain of isomorphisms

$$\Sigma^*\otimes_B\mathfrak{D}\otimes_B\Sigma \xrightarrow{\zeta} \mathrm{Hom}_{B-}(\mathfrak{D}\otimes_B\Sigma, \Sigma) \xrightarrow{\simeq} \mathrm{Hom}_{B-}(\mathfrak{D}\otimes_B\Sigma, \mathrm{Hom}_{A-}(\Sigma^*, A))$$
$$\xrightarrow{\simeq} \mathrm{Hom}_{A-}(\Sigma^*\otimes_B\mathfrak{D}\otimes_B\Sigma, A) = R,$$

where the second isomorphism follows from the fact that Σ_A is a finitely generated and projective module. One easily checks that all these isomorphisms are (A, R)-bimodule maps and thus their composition provides one with the required map. $\quad\square$

Remark 3.7. There are properties of a B-coring $\mathfrak{D}$ that are directly reflected in A-coring $\Sigma[\mathfrak{D}]$ without any assumption on bimodule Σ. For instance, if $\mathfrak{D}$ is a cosemisimple B-coring [12], then by [11, Theorem 4.4], $\mathfrak{D}$ decomposes into a direct sum of comatrix corings, $\mathfrak{D} \simeq \oplus_{\Xi\in\Lambda}\Xi^*\otimes_{D_\Xi}\Xi$, where each of the Ξ is a finitely generated and projective right B-module and each D_Ξ is a division subring of the endomorphism ring $\mathrm{End}_{-B}(\Xi)$. Thus, $\mathfrak{D} \simeq \oplus_{\Xi\in\Lambda}\Xi[D_\Xi]$ as B-corings. Applying the functor $\Sigma[-]$ to this isomorphism and using Proposition 3.3, one obtains an isomorphism of A-corings

$$\Sigma[\mathfrak{D}] \simeq \oplus_{\Xi\in\Lambda}\Sigma[-] \circ \Xi[D_\Xi]$$
$$\simeq \oplus_{\Xi\in\Lambda}(\Xi\otimes_B\Sigma)[D_\Xi].$$

Therefore, [11, Theorem 4.4] implies that $\Sigma[\mathfrak{D}]$ is a cosemisimple A-coring.

4. Module-morphisms and push-out and pull-back functors

Following the general strategy of replacing algebra maps by bimodules we introduce the notion of a *module-morphism* of corings and study properties of associated functors between categories of comodules.

4.1. The categories of module-morphisms and module-representations

Definition 4.1. Let $(\mathbb{C} : A)$ and $(\mathfrak{D} : B)$ be corings. A $((\mathfrak{D} : B), (\mathbb{C} : A))$-*module-morphism* is a pair $\mathbf{\Sigma} = (\Sigma, \sigma)$, where Σ is a (B, A)-bimodule that is finitely generated and projective as a right A-module and $\sigma : \Sigma[\mathfrak{D}] \to \mathbb{C}$ is a morphism of A-corings. A $((\mathfrak{D} : B), (\mathbb{C} : A))$-module-morphism $\mathbf{\Sigma}$ is often denoted by $_{(\mathfrak{D}:B)}\mathbf{\Sigma}_{(\mathbb{C}:A)}$.

Example 4.2. (1) Any morphism of corings $(\gamma, \alpha) : (\mathfrak{D} : B) \to (\mathbb{C} : A)$ gives rise to a module-morphism $(A, \tilde{\gamma})$, where A is a left B-module via α and $\tilde{\gamma} : A \otimes_B \mathfrak{D} \otimes_B A \to \mathbb{C}$ is the induced map $a \otimes_B d \otimes_B a' \mapsto a\gamma(d)a'$.

(2) Let $\mathbb{C}$ be an A-coring and let M be a right $\mathbb{C}$-comodule that is finitely generated and projective as a right A-module. Let B be the endomorphism ring $B = \mathrm{End}^{-\mathbb{C}}(M)$ so that M is a (B, A)-bimodule. Take any B-coring $\mathfrak{D}$. Then

$$\mathrm{can}_{\mathfrak{D},M} : M^* \otimes_B \mathfrak{D} \otimes_B M \to \mathbb{C}, \quad m^* \otimes_B d \otimes_B m \mapsto \sum m^*(\varepsilon_{\mathfrak{D}}(d)m_{(0)})m_{(1)},$$

is an A-coring map. Thus $(M, \mathrm{can}_{\mathfrak{D},M})$ is a $((\mathfrak{D} : B), (\mathbb{C} : A))$-module-morphism.

Proof. Example (1) follows immediately from the definition of a morphism of corings. To check (2) simply note that $\mathrm{can}_{\mathfrak{D},M} = \mathrm{can}_M \circ \varepsilon_{\mathfrak{D},M}$, where $\varepsilon_{\mathfrak{D},M}$ is the A-coring morphism in Corollary 3.4 and $\mathrm{can}_M := \mathrm{can}_{B,M}$ is an A-coring morphism by [6, 18.26] and [11, Proposition 2.7]. Thus $\mathrm{can}_{\mathfrak{D},M}$ is an A-coring morphism as required. $\quad\square$

Corollary 3.4 leads to the following interpretation of module-morphisms in terms of 1-cells in the bicategory fREM(Bim),

Lemma 4.3. $((\mathfrak{D} : B), (\mathbb{C} : A))$-*module-morphisms are in bijective correspondence with 1-cells from* $(\mathbb{C} : A)$ *to* $(\mathfrak{D} : B)$ *in* fREM(Bim).

Proof. The correspondence follows from the natural isomorphism

$$\mathrm{Hom}_{A,A}(\Sigma^* \otimes_B \mathfrak{D} \otimes_B \Sigma, \mathbb{C}) \simeq \mathrm{Hom}_{B,A}(\mathfrak{D} \otimes_B \Sigma, \Sigma \otimes_A \mathbb{C}). \tag{4.1}$$

Explicitly, if (Σ, σ) is a $((\mathfrak{D} : B), (\mathbb{C} : A))$-module-morphism, then $(\Sigma, \mathfrak{s}_\sigma)$, with

$$\mathfrak{s}_\sigma : \mathfrak{D} \otimes_B \Sigma \to \Sigma \otimes_A \mathbb{C}, \quad d \otimes_B s \mapsto \sum_i e_i \otimes_A \sigma(e_i^* \otimes_B d \otimes_B s) \tag{4.2}$$

is a 1-cell in fREM(Bim). Conversely, given a 1-cell $(\Sigma, \mathfrak{s})$ in fREM(Bim), define $\sigma_\mathfrak{s}$ as the composition

$$\sigma_\mathfrak{s} : \Sigma^* \otimes_B \mathfrak{D} \otimes_B \Sigma \xrightarrow{\Sigma^* \otimes_B \mathfrak{s}} \Sigma^* \otimes_B \Sigma \otimes_A \mathbb{C} \xrightarrow{\varepsilon_{\Sigma[B]} \otimes_A \mathbb{C}} A \otimes_A \mathbb{C} \simeq \mathbb{C}.$$

Then $(\Sigma, \sigma_\mathfrak{s})$ is a module-morphism. That this correspondence is well-defined and bijective can be checked directly by using the properties of dual bases. $\quad\square$

Remark 4.4. Put together, Lemmas 4.3 and 2.4 establish a bijective correspondence between module-morphisms and 1-cells in the bicategory fLEM(Bim). For a module-morphism $_{(\mathfrak{D}:B)}\mathbf{\Sigma}_{(\mathbb{C}:A)}$, define the (A, B)-bilinear map

$$\mathsf{t}_\sigma : \Sigma^* \otimes_B \mathfrak{D} \to \mathbb{C} \otimes_A \Sigma^*, \quad s^* \otimes_B d \mapsto \sum_i \sigma(s^* \otimes_B d \otimes_B e_i) \otimes_A e_i^*. \tag{4.3}$$

Then $(\Sigma^*, \mathsf{t}_\sigma)$ is a 1-cell from $(\mathbb{C} : A)$ to $(\mathfrak{D} : B)$ in fLEM(Bim). In the converse direction, an object $(\Xi, \mathsf{x}) \in {}_{(\mathfrak{D}:B)}\mathscr{L}^{\mathsf{f}}_{(\mathbb{C}:A)}$ induces a module-morphism $({}^*\Xi, \sigma_\mathsf{x})$, where

$$\sigma_\mathsf{x} : ({}^*\Xi)^* \otimes_B \mathfrak{D} \otimes_B {}^*\Xi \simeq \Xi \otimes_B \mathfrak{D} \otimes_B {}^*\Xi \xrightarrow{\mathsf{t} \otimes_B {}^*\Xi} \mathbb{C} \otimes_A \Xi \otimes_B {}^*\Xi \xrightarrow{\mathbb{C} \otimes_A \varepsilon \,{}^*\Xi[B]} \mathbb{C} \otimes_A A \simeq \mathbb{C}.$$

In view of Proposition 2.2 and Lemma 4.3, for any module-morphism $_{(\mathfrak{D}:B)}\mathbf{\Sigma}_{(\mathbb{C}:A)}$, the (B, A)-bimodule $\mathfrak{D} \otimes_B \Sigma$ is a $(\mathfrak{D}, \mathbb{C})$-bicomodule with the left $\mathfrak{D}$-coaction $\Delta_\mathfrak{D} \otimes_B \Sigma$ and the right $\mathbb{C}$-coaction $\varrho^{\Sigma_\circ(\mathfrak{D})}$ given by Proposition 2.1. A *map of* $((\mathfrak{D} : B), (\mathbb{C} : A))$-*module-morphisms* $\mathbf{\Sigma} \to \widetilde{\mathbf{\Sigma}}$ is defined as a $(\mathfrak{D}, \mathbb{C})$-bicomodule map $\mathfrak{D} \otimes_B \Sigma \to \mathfrak{D} \otimes_B \widetilde{\Sigma}$. The category of $((\mathfrak{D} : B), (\mathbb{C} : A))$-module-morphisms (with arrows given by module-morphism maps and composed as bicolinear maps) is denoted by $_{(\mathfrak{D}:B)}\mathscr{M}_{(\mathbb{C}:A)}$.

Proposition 4.5. *The category of* $((\mathfrak{D} : B), (\mathbb{C} : A))$-*module-morphisms* $_{(\mathfrak{D}:B)}\mathscr{M}_{(\mathbb{C}:A)}$ *is isomorphic to the category* $_{(\mathfrak{D}:B)}\mathscr{R}^{\mathsf{f}}_{(\mathbb{C}:A)}$.

Proof. On objects, mutually inverse functors are given by the bijective correspondence of Lemma 4.3. The actions on morphisms are given by the bijections of the hom-sets stated in Proposition 2.2. □

The duality in Lemma 2.4 and observations made in Remark 4.4 allow one to construct a category dual to $_{(\mathfrak{D}:B)}\mathscr{M}_{(\mathbb{C}:A)}$. The resulting category generalises that of representations of a coring in a coring (cf. [6, 24.3]). By Remark 4.4, any module-morphism $_{(\mathfrak{D}:B)}\mathbf{\Sigma}_{(\mathbb{C}:A)}$ can be viewed as an object in $_{(\mathfrak{D}:B)}\mathscr{L}^{\mathsf{f}}_{(\mathbb{C}:A)}$. The hom-set bijections

$$\mathrm{Hom}_{A,A}(\Sigma^* \otimes_B \mathfrak{D} \otimes_B \widetilde{\Sigma}, A) \simeq \mathrm{Hom}_{A,B}(\Sigma^* \otimes_B \mathfrak{D}, \widetilde{\Sigma}^*), \tag{4.4}$$

allow for identification of morphisms in $_{(\mathfrak{D}:B)}\mathscr{L}^{\mathsf{f}}_{(\mathbb{C}:A)}$ with A-bimodule maps

$$f : \Sigma^* \otimes_B \mathfrak{D} \otimes_B \widetilde{\Sigma} \to A, \tag{4.5}$$

such that for all $s^* \in \Sigma^*$, $\tilde{s} \in \widetilde{\Sigma}$ and $d \in \mathfrak{D}$,

$$\sum_j f(s^* \otimes_B d_{(1)} \otimes_B \tilde{e}_j) \tilde{\sigma}(\tilde{e}_j^* \otimes_B d_{(2)} \otimes_B \tilde{s})$$

$$= \sum_i \sigma(s^* \otimes_B d_{(1)} \otimes_B e_i) f(e_i^* \otimes_B d_{(2)} \otimes_B \tilde{s}). \tag{4.6}$$

where $\{e_i, e_i^*\}$ is a dual basis of Σ and $\{\tilde{e}_j, \tilde{e}_j^*\}$ is a dual basis of $\widetilde{\Sigma}$. The composition of f in Eq. (4.5) with $g : \widetilde{\Sigma}^* \otimes_B \mathfrak{D} \otimes_B \widehat{\Sigma} \to A$, transferred from that in $_{(\mathfrak{D}:B)}\mathscr{L}^{\mathsf{f}}_{(\mathfrak{C}:A)}$, comes out as

$$g \diamond f : \Sigma^* \otimes_B \mathfrak{D} \otimes_B \widehat{\Sigma} \to A,$$
$$s^* \otimes_B d \otimes_B \hat{s} \mapsto \sum_j f(s^* \otimes_B d_{(1)} \otimes_B \tilde{e}_j)g(\tilde{e}_j^* \otimes_B d_{(2)} \otimes_B \hat{s}). \tag{4.7}$$

In view of Lemma 2.4 and Proposition 4.5, the category $\mathbf{Rep}^{\mathrm{M}}(\mathfrak{D} : B|\mathfrak{C} : A)$ with objects module-morphisms and arrows given as maps f satisfying condition (4.6) and composed according to (4.7), is dual (in fact, anti-isomorphic) to $_{(\mathfrak{D}:B)}\mathscr{M}_{(\mathfrak{C}:A)}$. Since any morphism can be viewed as a module-morphism as in Example 4.2, the category $\mathbf{Rep}(\mathfrak{D} : B|\mathfrak{C} : A)$ of representations of $(\mathfrak{D} : B)$ in $(\mathfrak{C} : A)$ (cf. [6, 24.3]) is a full subcategory of $\mathbf{Rep}^{\mathrm{M}}(\mathfrak{D} : B|\mathfrak{C} : A)$.

4.2. Push-out and pull-back functors

Since any module-morphism $_{(\mathfrak{D}:B)}\Sigma_{(\mathfrak{C}:A)}$ induces a 1-cell in the bicategory $\mathsf{fREM}(\mathsf{Bim})$ by Lemma 4.3, Proposition 2.1 yields the existence of the functor $\Sigma_\circ : \mathscr{M}^{\mathfrak{D}} \to \mathscr{M}^{\mathfrak{C}}$, $(M, \varrho^M) \mapsto (M \otimes_B \Sigma, \varrho^{M \otimes_B \Sigma})$. In terms of σ, the coaction is derived from Eq. (4.2), and reads

$$\varrho^{\Sigma_\circ(M)} : M \otimes_B \Sigma \to M \otimes_B \Sigma \otimes_A \mathfrak{C},$$
$$m \otimes_B s \mapsto \sum_i m_{(0)} \otimes_B e_i \otimes_A \sigma(e_i^* \otimes_B m_{(1)} \otimes_B s). \tag{4.8}$$

The functor $\Sigma_\circ$ is called a (*right*) *push-out functor*. Taking into account that σ is a morphism of corings and the observations made in Remark 3.2, it is reasonable to compare the cotensor functor induced by the quasi-finite bicomodule $\mathfrak{D} \otimes_B \Sigma$ and the right push-out functor $\Sigma_\circ$. This leads to the following:

Proposition 4.6. *If* $_{(\mathfrak{D}:B)}\Sigma_{(\mathfrak{C}:A)}$ *is a module-morphism, then* $\mathfrak{D} \otimes_B \Sigma$ *induces a functor* $-\square_{\mathfrak{D}}(\mathfrak{D} \otimes_B \Sigma) : \mathscr{M}^{\mathfrak{D}} \to \mathscr{M}^{\mathfrak{C}}$ *which is naturally isomorphic to* $\Sigma_\circ$.

Proof. Start with an arbitrary right $\mathfrak{D}$-comodule M and consider the following commutative diagram

$$0 \longrightarrow M \square_{\mathfrak{D}}(\mathfrak{D} \otimes_B \Sigma) \xrightarrow{\omega^k_{M,\mathfrak{D} \otimes_B \Sigma}} M \otimes_B (\mathfrak{D} \otimes_B \Sigma) \xrightarrow{\omega_{M,\mathfrak{D} \otimes_B \Sigma}} M \otimes_B \mathfrak{D} \otimes_B (\mathfrak{D} \otimes_B \Sigma)$$

with the maps f, $\varrho^M \otimes_B \Sigma$, $M \otimes_B \varepsilon_{\mathfrak{D}} \otimes_B \Sigma$ and $M \otimes_B \Sigma$.
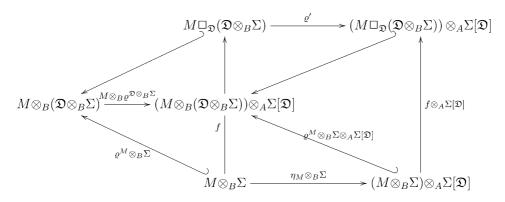
where f is uniquely determined by the universal property of the kernel $\omega^k_{M,\mathfrak{D}\otimes_B\Sigma}$ in the category of right A-modules. Clearly, $(M\otimes_B\varepsilon_{\mathfrak{D}}\otimes_B\Sigma)\circ\omega^k_{M,\mathfrak{D}\otimes_B\Sigma}$ is the inverse map of f, and so f is an isomorphism of right A-modules. Consider the composition map

$$
\begin{array}{ccc}
M\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B\Sigma) & \overset{\varrho'}{\dashrightarrow} & (M\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B\Sigma))\otimes_A\Sigma[\mathfrak{D}] \\
{\scriptstyle f^{-1}}\downarrow & & \uparrow{\scriptstyle f\otimes_A\Sigma[\mathfrak{D}]} \\
M\otimes_B\Sigma & \xrightarrow{\eta_M\otimes_B\Sigma} & (M\otimes_B\Sigma)\otimes_A\Sigma[\mathfrak{D}]
\end{array}
$$

where η_- is the natural transformation in Eq. (3.4) defined at M by

$$
\eta_M(m\otimes_B s) = \sum_i m_{(0)}\otimes_B e_i\otimes_A e_i^*\otimes_B m_{(1)},
$$

for every $m\in M$, $s\in\Sigma$. Since the coaction $\varrho^{\mathfrak{D}\otimes_B\Sigma}$ of Lemma 3.5 satisfies $\eta_{M\otimes_B\mathfrak{D}}\otimes_B\Sigma = M\otimes_B\varrho^{\mathfrak{D}\otimes_B\Sigma}$, we obtain the following commutative diagram

The right $\Sigma[\mathfrak{D}]$-coaction $M\otimes_B\varrho^{\mathfrak{D}\otimes_B\Sigma}$ induces the structure of a right $\Sigma[\mathfrak{D}]$-comodule on both $M\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B\Sigma)$ and $M\otimes_B\Sigma$. The coactions are given, respectively, by ϱ' and $\eta_M\otimes_B\Sigma$. Furthermore, the A-module isomorphism $M\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B\Sigma) \simeq M\otimes_B\Sigma$ becomes an isomorphism of right $\Sigma[\mathfrak{D}]$-comodules. Applying the induction functor associated to the underlying morphism of corings σ to this $\Sigma[\mathfrak{D}]$-comodule isomorphism, we obtain an isomorphism of right $\mathfrak{C}$-comodules which gives the desired natural isomorphism. $\square$

By Remark 4.4 and the left-handed versions of both Proposition 2.1 and Proposition 4.6, a module-morphism $_{(\mathfrak{D}:B)}\Sigma_{(\mathfrak{C}:A)}$ induces a (left) *push-out functor*

$$
{}_\circ\Sigma : {}^{\mathfrak{D}}\mathscr{M} \to {}^{\mathfrak{C}}\mathscr{M}, \quad N \mapsto (\Sigma^*\otimes_B\mathfrak{D})\square_{\mathfrak{D}}N \simeq \Sigma^*\otimes_B N.
$$

Explicitly, using Eq. (4.3), the left $\mathfrak{C}$-coaction reads, for all $s^*\in\Sigma^*$ and $n\in N$,

$$
{}_{{}_\circ\Sigma(N)}\varrho(s^*\otimes_B n) = \sum_i \sigma(s^*\otimes_B n_{(-1)}\otimes_B e_i)\otimes_A e_i^*\otimes_B n_{(0)}. \tag{4.9}
$$

In particular, in view of Lemma 3.5, $_\circ\mathbf{\Sigma}(\mathfrak{D})$ is a $(\mathfrak{C}, \mathfrak{D})$-bicomodule. We say that a module-morphism $_{(\mathfrak{D}:B)}\mathbf{\Sigma}_{(\mathfrak{C}:A)}$ is *right pure* if for all right $\mathfrak{C}$-comodules L the coaction equalising map

$$\omega_{L,\,_\circ\mathbf{\Sigma}(\mathfrak{D})} = \varrho^L\otimes_A {_\circ\mathbf{\Sigma}}(\mathfrak{D}) - L\otimes_A {^{\circ\mathbf{\Sigma}(\mathfrak{D})}}\varrho : L\otimes_A {_\circ\mathbf{\Sigma}}(\mathfrak{D}) \to L\otimes_A\mathfrak{C}\otimes_A {_\circ\mathbf{\Sigma}}(\mathfrak{D})$$

is a $\mathfrak{D}\otimes_B\mathfrak{D}$-pure morphism of right B-modules. Obviously, if $\mathfrak{D}$ is a flat left B-module, then every right B-module map is $\mathfrak{D}\otimes_B\mathfrak{D}$-pure, hence any module-morphism $_{(\mathfrak{D}:B)}\mathbf{\Sigma}_{(\mathfrak{C}:A)}$ is right pure in this case.

If $_{(\mathfrak{D}:B)}\mathbf{\Sigma}_{(\mathfrak{C}:A)}$ is right pure then for all right $\mathfrak{C}$-comodules N, $N\square_{\mathfrak{C}}{_\circ\mathbf{\Sigma}}(\mathfrak{D})$ is a right $\mathfrak{D}$-comodule (cf. [6, 22.3]). Thus any right pure module-morphism $_{(\mathfrak{D}:B)}\mathbf{\Sigma}_{(\mathfrak{C}:A)}$ gives rise to a functor

$$\mathbf{\Sigma}^\circ : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}, \qquad N \mapsto N\square_{\mathfrak{C}}{_\circ\mathbf{\Sigma}}(\mathfrak{D}).$$

The $\mathfrak{D}$-coaction on $\mathbf{\Sigma}^\circ(N)$, $\varrho^{\mathbf{\Sigma}^\circ(N)} : N\square_{\mathfrak{C}}(\Sigma^*\otimes_B\mathfrak{D}) \to (N\square_{\mathfrak{C}}(\Sigma^*\otimes_B\mathfrak{D}))\otimes_B\mathfrak{D}$ explicitly reads:

$$\varrho^{\mathbf{\Sigma}^\circ(N)}\left(\sum_\alpha n^\alpha\otimes_A s_\alpha^*\otimes_B d^\alpha\right) = \sum_\alpha n^\alpha\otimes_A s_\alpha^*\otimes_B d_{(1)}^\alpha\otimes_B d_{(2)}^\alpha.$$

The functor $\mathbf{\Sigma}^\circ$ is called a (*right*) *pull-back functor* associated to a right pure module-morphism $_{(\mathfrak{D}:B)}\mathbf{\Sigma}_{(\mathfrak{C}:A)}$.

In a similar way one defines a left pure morphism and a (left) pull-back functor. There is an obvious left-right symmetry, hence we restrict ourselves to right pure morphisms and (right) pull-back functors.

Theorem 4.7. *For any right pure module-morphism $_{(\mathfrak{D}:B)}\mathbf{\Sigma}_{(\mathfrak{C}:A)}$, the pull-back functor $\mathbf{\Sigma}^\circ$ is the right adjoint to the push-out functor $\mathbf{\Sigma}_\circ$.*

Proof. First we construct the unit of the adjunction. For any $M \in \mathcal{M}^{\mathfrak{D}}$, consider a k-linear map

$$\eta_M : M \to \mathbf{\Sigma}^\circ(\mathbf{\Sigma}_\circ(M)) = (M\otimes_B\Sigma)\square_{\mathfrak{C}}(\Sigma^*\otimes_B\mathfrak{D}),$$
$$m \mapsto \sum_i m_{(0)}\otimes_B e_i\otimes_A e_i^*\otimes_B m_{(1)}.$$

Clearly, the map η_M is well-defined and it immediately follows from the definitions of the coactions on $\mathbf{\Sigma}_\circ(M)$ and $_\circ\mathbf{\Sigma}(\mathfrak{D})$ that the image of η_M is in the required cotensor product. The way in which the definition of the map η_M depends upon the coaction ϱ^M ensures that η_M is a right B-module map (since ϱ^M is such a map). One easily checks that η_M is also a morphism in $\mathcal{M}^{\mathfrak{D}}$. Next, take any $f : M \to M'$ in $\mathcal{M}^{\mathfrak{D}}$ and compute for any $m \in M$,

$$\eta_{M'}(f(m)) = \sum_i f(m)_{(0)}\otimes_B e_i\otimes_A e_i^*\otimes_B f(m)_{(1)}$$
$$= \sum_i f(m_{(0)})\otimes_B e_i\otimes_A e_i^*\otimes_B m_{(1)} = \mathbf{\Sigma}_\circ(\mathbf{\Sigma}^\circ(f))(\eta_M(m)).$$

The second equality follows, since f is a morphism of right $\mathfrak{D}$-comodules. Thus we have constructed a natural map $\eta : I_{\mathcal{M}^{\mathfrak{D}}} \to \Sigma_\circ \Sigma^\circ$ that will be shown to be the unit of the adjunction.

Now, for any $N \in \mathcal{M}^{\mathfrak{C}}$, consider a right A-module map

$$\psi_N : \Sigma_\circ(\Sigma^\circ(N)) = (N\square_{\mathfrak{C}}(\Sigma^*\otimes_B\mathfrak{D}))\otimes_B\Sigma \to N,$$
$$\sum_\alpha n^\alpha\otimes_A s_\alpha^*\otimes_B d^\alpha\otimes_B s \mapsto \sum_\alpha n^\alpha s_\alpha^*(\varepsilon_{\mathfrak{D}}(d^\alpha)s).$$

First we need to check whether the map ψ_N is a morphism in the category of right $\mathfrak{C}$-comodules. Take any $x = \sum_\alpha n^\alpha\otimes_A s_\alpha^*\otimes_B d^\alpha\otimes_B s \in (N\square_{\mathfrak{C}}(\Sigma^*\otimes_B\mathfrak{D}))\otimes_B\Sigma$. Then,

$$
\begin{aligned}
\sum \psi_N(x_{(0)})\otimes_B x_{(1)} &= \sum_{i,\alpha} \psi_N(n^\alpha\otimes_A s_\alpha^*\otimes_B d_{(1)}^\alpha\otimes_B e_i)\otimes_A\sigma(e_i^*\otimes_B d_{(2)}^\alpha\otimes_B s)\\
&= \sum_{i,\alpha} n^\alpha s_\alpha^*(\varepsilon_{\mathfrak{D}}(d_{(1)}^\alpha)e_i)\otimes_A\sigma(e_i^*\otimes_B d_{(2)}^\alpha\otimes_B s)\\
&= \sum_{i,\alpha} n^\alpha\otimes_A s_\alpha^*(e_i)\sigma(e_i^*\otimes_B d^\alpha\otimes_B s)\\
&= \sum_\alpha n^\alpha\otimes_A\sigma(s_\alpha^*\otimes_B d^\alpha\otimes_B s).
\end{aligned}
$$

The final equality is a consequence of the left A-linearity of σ and the dual basis property. Since $\sum_\alpha n^\alpha\otimes_A s_\alpha^*\otimes_B d^\alpha \in N\square_{\mathfrak{C}}(\Sigma^*\otimes_B\mathfrak{D})$,

$$\sum_\alpha n_{(0)}^\alpha\otimes_A n_{(1)}^\alpha\otimes_A s_\alpha^*\otimes_B d^\alpha = \sum_{\alpha,i} n^\alpha\otimes_A\sigma(s_\alpha^*\otimes_B d_{(1)}^\alpha\otimes_B e_i)\otimes_A e_i^*\otimes_B d_{(2)}^\alpha,$$

hence

$$\sum_\alpha n_{(0)}^\alpha\otimes_A n_{(1)}^\alpha\otimes_A s_\alpha^*\varepsilon_{\mathfrak{D}}(d^\alpha) = \sum_{\alpha,i} n^\alpha\otimes_A\sigma(s_\alpha^*\otimes_B d^\alpha\otimes_B e_i)\otimes_A e_i^*.$$

Using this equality, the A-linearity of σ and the properties of a dual basis, we can compute

$$
\begin{aligned}
\sum \psi_N(x_{(0)})\otimes_B x_{(1)} &= \sum_\alpha n^\alpha\otimes_A\sigma(s_\alpha^*\otimes_B d^\alpha\otimes_B s)\\
&= \sum_{\alpha,i} n^\alpha\otimes_A\sigma(s_\alpha^*\otimes_B d^\alpha\otimes_B e_i e_i^*(s))\\
&= \sum_{\alpha,i} n^\alpha\otimes_A\sigma(s_\alpha^*\otimes_B d^\alpha\otimes_B e_i)e_i^*(s)\\
&= \sum_\alpha n_{(0)}^\alpha\otimes_A n_{(1)}^\alpha s_\alpha^*(\varepsilon_{\mathfrak{D}}(d^\alpha)s)\\
&= \sum \psi_N(x)_{(0)}\otimes_A\psi_N(x)_{(1)}.
\end{aligned}
$$

Therefore, ψ_N is a right $\mathfrak{C}$-comodule map as required. Thus for any right $\mathfrak{C}$-comodule N we have constructed a morphism ψ_N in $\mathcal{M}^{\mathfrak{C}}$. Noting that any right $\mathfrak{C}$-comodule map is

necessarily a right A-module map, one easily checks that the map ψ_N is natural in N, i.e. the collection of all the ψ_N defines a morphism of functors $\psi : \Sigma_\circ \Sigma^\circ \to I_{\mathscr{M}^{\mathfrak{C}}}$.

The verification that η and ψ are the unit and counit respectively, i.e. that for all $M \in \mathscr{M}^{\mathfrak{D}}$ and $N \in \mathscr{M}^{\mathfrak{C}}$, $\psi_{\Sigma_\circ(M)} \circ \Sigma_\circ(\eta_M) = \Sigma_\circ(M)$ and $\Sigma^\circ(\psi_N) \circ \eta_{\Sigma^\circ(N)} = \Sigma^\circ(N)$, is a straightforward application of the properties of a dual basis, and is left to the reader. $\square$

Remark 4.8. One easily checks that the unit and counit defined in the proof of Theorem 4.7 are induced, respectively, by the unit and counit of the adjunction (3.2) (i.e. by the natural transformations (3.4) and (3.3)). If a module-morphism $_{(\mathfrak{D}:B)}\Sigma_{(\mathfrak{C}:A)}$ is right pure, then Theorem 4.7 asserts that this last adjunction is extended to the category of right $\mathfrak{C}$-comodules. That is, Theorem 4.7 and its proof can be seen as an example and also an application of the statements and the proof of [13, Proposition 4.2(1)].

In case $_{(\mathfrak{D}:B)}\Sigma_{(\mathfrak{C}:A)}$ corresponds to a morphism of corings, i.e. there is an algebra map $B \to A$, $\Sigma = A$ and $\sigma : A \otimes_B \mathfrak{D} \otimes_B A \to \mathfrak{C}$ is an A-coring map (cf. Example 4.2), the functor $\Sigma_\circ$ is the induction functor and Σ° is the ad-induction functor introduced in [13]. In this case Theorem 4.7 reduces to [13, Proposition 5.4].

Theorem 4.7 tantamounts to the existence of isomorphisms, for all $M \in \mathscr{M}^{\mathfrak{D}}$ and $N \in \mathscr{M}^{\mathfrak{C}}$,

$$\Omega_{M,N} : \mathrm{Hom}^{-\mathfrak{C}}(M \otimes_B \Sigma, N) \to \mathrm{Hom}^{-\mathfrak{D}}(M, N\square_{\mathfrak{C}}(\Sigma^* \otimes_B \mathfrak{D})), \qquad (4.10)$$

for a right pure module-morphism $_{(\mathfrak{D}:B)}\Sigma_{(\mathfrak{C}:A)}$. Explicitly, these isomorphisms read, for all $\phi \in \mathrm{Hom}^{-\mathfrak{C}}(M \otimes_B \Sigma, N)$ and $m \in M$,

$$\Omega_{M,N}(\phi)(m) = \sum_i \phi(m_{(0)} \otimes_B e_i) \otimes_A e_i^* \otimes_B m_{(1)}. \qquad (4.11)$$

To write out the inverse of $\Omega_{M,N}$ explicitly, take any $\tilde{\phi} \in \mathrm{Hom}^{-\mathfrak{D}}(M, N\square_{\mathfrak{C}}(\Sigma^* \otimes_B \mathfrak{D}))$ and $m \in M$, and write $\tilde{\phi}(m) = \sum \tilde{\phi}(m)^{[1]} \otimes_A \tilde{\phi}(m)^{[2]} \otimes_B \tilde{\phi}(m)^{[3]}$. Then, for all $s \in \Sigma$,

$$\Omega_{M,N}^{-1}(\tilde{\phi})(m \otimes_B s) = \sum \tilde{\phi}(m)^{[1]} \tilde{\phi}(m)^{[2]}(\varepsilon_{\mathfrak{D}}(\tilde{\phi}(m)^{[3]})s). \qquad (4.12)$$

A module-morphism $_{(\mathfrak{D}:B)}\Sigma_{(\mathfrak{C}:A)}$ is assumed to be right pure to assure that for all right $\mathfrak{C}$-comodules N, $\Sigma^\circ(N)$ is a right $\mathfrak{D}$-comodule. On the other hand, for any module-morphism there exist right $\mathfrak{C}$-comodules N, such that $\Sigma^\circ(N)$ is a right $\mathfrak{D}$-comodule. For any such comodule the isomorphisms in Eqs. (4.10)–(4.12) are well-defined. This observation leads to the following:

Corollary 4.9. *For any module-morphism $_{(\mathfrak{D}:B)}\Sigma_{(\mathfrak{C}:A)}$ and any right $\mathfrak{D}$-comodule M,*

$$\mathrm{Hom}^{-\mathfrak{D}}(M, \Sigma^* \otimes_B \mathfrak{D}) \simeq \mathrm{Hom}^{-\mathfrak{C}}(M \otimes_B \Sigma, \mathfrak{C}) \simeq \mathrm{Hom}_{-A}(M \otimes_B \Sigma, A)$$
$$= (M \otimes_B \Sigma)^*,$$

as k-modules.

Proof. Note that $\Sigma^\circ(\mathfrak{C}) = \mathfrak{C}\Box_{\mathfrak{C}}(\Sigma^*\otimes_B\mathfrak{D}) \simeq \Sigma^*\otimes_B\mathfrak{D}$, so that it is a right $\mathfrak{D}$-comodule. Thus $\Omega_{M,\mathfrak{C}}$: $\mathrm{Hom}^{-\mathfrak{C}}(M\otimes_B\Sigma, \mathfrak{C}) \to \mathrm{Hom}^{-\mathfrak{D}}(M, \Sigma^*\otimes_B\mathfrak{D})$ are well-defined isomorphisms. In view of Eqs. (4.11)–(4.12), and the identification $\mathfrak{C}\Box_{\mathfrak{C}}(\Sigma^*\otimes_B\mathfrak{D}) \simeq \Sigma^*\otimes_B\mathfrak{D}$, these isomorphisms and their inverses come out explicitly as, for all $m \in M$ and $\phi \in \mathrm{Hom}^{-\mathfrak{D}}(M, \Sigma^*\otimes_B\mathfrak{D})$,

$$\Omega_{M,\mathfrak{C}}(\phi)(m) = \sum_i \varepsilon_{\mathfrak{C}}(\phi(m_{(0)}\otimes_B e_i))e_i^*\otimes_B m_{(1)},$$

and for all $\tilde{\phi} \in \mathrm{Hom}^{-\mathfrak{C}}(M\otimes_B\Sigma, \mathfrak{C}), m \in M$ and $s \in \Sigma$,

$$\Omega_{M,\mathfrak{C}}^{-1}(\tilde{\phi})(m\otimes_B s) = \sigma(\tilde{\phi}(m)\otimes_B s).$$

The second isomorphism follows from $\mathrm{Hom}^{-\mathfrak{C}}(N, \mathfrak{C}) \simeq \mathrm{Hom}_{-A}(N, A)$, which holds for any right $\mathfrak{C}$-comodule N (and is given by the counit with the inverse provided by the coaction). $\square$

Corollary 4.10. *For any module-morphism $_{(\mathfrak{D}:B)}\Sigma_{(\mathfrak{C}:A)}$, the endomorphism ring of the right $\mathfrak{D}$-comodule $\Sigma^*\otimes_B\mathfrak{D}$ is isomorphic to the right dual ring of the A-coring $\Sigma[\mathfrak{D}]$, i.e. $\mathrm{End}^{-\mathfrak{D}}(\Sigma^*\otimes_B\mathfrak{D}) \simeq (\Sigma^*\otimes_B\mathfrak{D}\otimes_B\Sigma)^*$, as rings.*

Proof. Setting $M = \Sigma^*\otimes_B\mathfrak{D}$ in Corollary 4.9, one obtains an isomorphism of k-modules

$$\Theta : \mathrm{End}^{-\mathfrak{D}}(\Sigma^*\otimes_B\mathfrak{D}) \to (\Sigma^*\otimes_B\mathfrak{D}\otimes_B\Sigma)^*,$$
$$\phi \mapsto [s^*\otimes_B d\otimes_B s \mapsto \varepsilon_{\mathfrak{C}}(\sigma(\phi(s^*\otimes_B d)\otimes_B s))].$$

We need to check if this isomorphism is an algebra map. For any $\phi \in \mathrm{End}^{-\mathfrak{D}}(\Sigma^*\otimes_B\mathfrak{D})$, $d \in \mathfrak{D}$ and $s^* \in \Sigma^*$, write $\phi(s^*\otimes_B d) = \sum \phi(s^*\otimes_B d)^{[1]}\otimes_B\phi(s^*\otimes_B d)^{[2]} \in \Sigma^*\otimes_B\mathfrak{D}$. Since σ is a morphism of A-corings,

$$\varepsilon_{\mathfrak{C}}(\sigma(\phi(s^*\otimes_B d)\otimes_B s)) = \varepsilon_{\Sigma[\mathfrak{D}]}(\phi(s^*\otimes_B d)\otimes_B s) = \sum \phi(s^*\otimes_B d)^{[1]}(\varepsilon_{\mathfrak{D}}(\phi(s^*\otimes_B d)^{[2]}s)).$$

In particular Θ maps the identity morphism in $\Sigma^*\otimes_B\mathfrak{D}$ to the counit $\varepsilon_{\Sigma[\mathfrak{D}]}$. Furthermore, since ϕ is a right $\mathfrak{D}$-comodule map

$$\phi(s^*\otimes_B d_{(1)})\otimes_B d_{(2)} = \sum \phi(s^*\otimes_B d)^{[1]}\otimes_B\phi(s^*\otimes_B d)^{[2]}_{(1)}\otimes_B\phi(s^*\otimes_B d)^{[2]}_{(2)},$$

so that

$$\sum \phi(s^*\otimes_B d_{(1)})^{[1]}\varepsilon_{\mathfrak{D}}(\phi(s^*\otimes_B d_{(1)})^{[2]})\otimes_B d_{(2)} = \phi(s^*\otimes_B d).$$

Taking this into account we can compute, for all $\phi, \phi' \in \mathrm{End}^{-\mathfrak{D}}(\Sigma^* \otimes_B \mathfrak{D}), d \in \mathfrak{D}, s \in \Sigma$ and $s^* \in \Sigma^*$,

$$
\begin{aligned}
(\Theta(\phi') * \Theta(\phi))(s^* \otimes_B d \otimes_B s) &= \sum_i \Theta(\phi')(\Theta(\phi)(s^* \otimes_B d_{(1)} \otimes_B e_i)e_i^* \otimes_B d_{(2)} \otimes_B s) \\
&= \sum_i \Theta(\phi')(\phi(s^* \otimes_B d_{(1)})^{[1]} \\
&\quad \times (\varepsilon_{\mathfrak{D}}(\phi(s^* \otimes_B d_{(1)})^{[2]})e_i)e_i^* \otimes_B d_{(2)} \otimes_B s) \\
&= \sum_i \Theta(\phi')(\phi(s^* \otimes_B d)^{[1]}(e_i) \\
&\quad \times e_i^* \otimes_B \phi(s^* \otimes_B d)^{[2]} \otimes_B s) \\
&= \sum \Theta(\phi')(\phi(s^* \otimes_B d)^{[1]} \otimes_B \phi(s^* \otimes_B d)^{[2]} \otimes_B s) \\
&= \sum \Theta(\phi')(\phi(s^* \otimes_B d) \otimes_B s) \\
&= \varepsilon_{\mathfrak{C}}(\sigma(\phi'(\phi(s^* \otimes_B d))) \otimes_B s)) \\
&= \Theta(\phi' \circ \phi)(s^* \otimes_B d \otimes_B s).
\end{aligned}
$$

Thus we conclude that Θ is an algebra map as required. $\quad\square$

In case $\mathfrak{D} = B$, Corollary 4.10 implies that the ring of endomorphisms of the right B-module Σ^* is isomorphic to the right dual ring of the corresponding comatrix coring. This is a right-handed version of one of the assertions in [11, Proposition 2.1].

4.3. Natural isomorphisms between push-out functors

The horizontal composition in bicategory fREM(Bim) induces the *cotensor product* of module-morphisms. Start with morphisms $\phi : \mathbf{\Sigma} \to \widetilde{\mathbf{\Sigma}}$ in $_{(\mathfrak{D}:B)}\mathscr{M}_{(\mathfrak{C}:A)}$ and $\psi : \mathbf{\Xi} \to \widetilde{\mathbf{\Xi}}$ in $_{(\mathfrak{C}:C)}\mathscr{M}_{(\mathfrak{D}:B)}$, i.e. $\phi : \mathfrak{D} \otimes_B \Sigma \to \mathfrak{D} \otimes_B \widetilde{\Sigma}$ is a $(\mathfrak{D}, \mathfrak{C})$-bicomodule map and $\psi : \mathfrak{E} \otimes_C \Xi \to \mathfrak{E} \otimes_C \widetilde{\Xi}$ is a $(\mathfrak{E}, \mathfrak{D})$-bicomodule map. Therefore, we can consider the $(\mathfrak{E}, \mathfrak{C})$-bicomodule map

$$
\psi \square_{\mathfrak{D}} \phi : (\mathfrak{E} \otimes_C \Xi) \square_{\mathfrak{D}} (\mathfrak{D} \otimes_B \Sigma) \to (\mathfrak{E} \otimes_C \widetilde{\Xi}) \square_{\mathfrak{D}} (\mathfrak{D} \otimes_B \widetilde{\Sigma}).
$$

Using the isomorphisms $\mathfrak{E} \otimes_C \Xi \otimes_B \Sigma \simeq (\mathfrak{E} \otimes_C \Xi) \square_{\mathfrak{D}} (\mathfrak{D} \otimes_B \Sigma)$ and $(\mathfrak{E} \otimes_C \widetilde{\Xi}) \square_{\mathfrak{D}} (\mathfrak{D} \otimes_B \widetilde{\Sigma}) \simeq \mathfrak{E} \otimes_C \widetilde{\Xi} \otimes_B \widetilde{\Sigma}$ we thus arrive at an $(\mathfrak{E}, \mathfrak{C})$-bicomodule map, hence the map of module-morphisms,

$$
\psi \square_{\mathfrak{D}} \phi : \mathfrak{E} \otimes_C \Xi \otimes_B \Sigma \to \mathfrak{E} \otimes_C \widetilde{\Xi} \otimes_B \widetilde{\Sigma}.
$$

The morphisms in the category of module-morphisms detect relationships between push-out functors. More precisely, one can formulate the following

Proposition 4.11. *There is a bijective correspondence between morphisms in the category of module-morphisms and natural transformations between corresponding push-out functors. This correspondence is compatible with horizontal compositions.*

Proof. Let $_{(\mathfrak{D}:B)}\Sigma_{(\mathbb{C}:A)}$, $_{(\mathfrak{D}:B)}\widetilde{\Sigma}_{(\mathbb{C}:A)}$ be module-morphisms and suppose that $f: \Sigma_\circ \to \widetilde{\Sigma}_\circ$ is a natural map. This means that for all $M \in \mathcal{M}^{\mathfrak{D}}$, there is a right $\mathbb{C}$-comodule map $f_M : M\otimes_B\Sigma \to M\otimes_B\widetilde{\Sigma}$. Consider morphisms in $\mathcal{M}^{\mathfrak{D}}$, $\varrho^M : M \to M\otimes_B\mathfrak{D}$ and, for all $m \in M$, $\ell_m : \mathfrak{D} \to M\otimes_B\mathfrak{D}$, $d \mapsto m\otimes_B d$. The naturality of f implies that, for all $m \in M$, $d \in \mathfrak{D}$ and $s \in \Sigma$,

$$f_{M\otimes_B\mathfrak{D}}(m\otimes_B d\otimes_B s) = m\otimes_B f_{\mathfrak{D}}(d\otimes_B s),$$
$$f_{M\otimes_B\mathfrak{D}} \circ (\varrho^M\otimes_B\Sigma) = (\varrho^M\otimes_B\widetilde{\Sigma}) \circ f_M.$$

Put together, this means that

$$(\varrho^M\otimes_B\widetilde{\Sigma}) \circ f_M = (M\otimes_B f_{\mathfrak{D}}) \circ (\varrho^M\otimes_B\Sigma). \tag{4.13}$$

If $M = \mathfrak{D}$ Eq. (4.13) implies that $f_{\mathfrak{D}}$ is a left $\mathfrak{D}$-comodule map, hence it is a $(\mathfrak{D}, \mathbb{C})$-bicomodule map. Recall that $M\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B\Sigma) \simeq M\otimes_B\Sigma$ by $M\otimes_B\varepsilon_{\mathfrak{D}}\otimes_B\Sigma$ and $\varrho^M\otimes_B\Sigma$. Thus, applying $M\otimes_B\varepsilon_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}$ to Eq. (4.13) we obtain

$$f_M = (M\otimes_B\varepsilon_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}) \circ (M\otimes_B f_{\mathfrak{D}}) \circ (\varrho^M\otimes_B\Sigma) \simeq M\square_{\mathfrak{D}} f_{\mathfrak{D}}.$$

Hence the required bijective correspondence is provided by

$$\mathrm{Nat}(\Sigma_\circ, \widetilde{\Sigma}_\circ) \ni f \mapsto f_{\mathfrak{D}} \in \mathrm{Hom}^{\mathfrak{D},\mathbb{C}}(\mathfrak{D}\otimes_B\widetilde{\Sigma}, \mathfrak{D}\otimes_B\Sigma),$$

with the inverse, for all $M \in \mathcal{M}^{\mathfrak{D}}$,

$$\mathrm{Hom}^{\mathfrak{D},\mathbb{C}}(\mathfrak{D}\otimes_B\widetilde{\Sigma}, \mathfrak{D}\otimes_B\Sigma) \ni \phi \mapsto (M\otimes_B\varepsilon_{\mathfrak{D}}\otimes_B\widetilde{\Sigma}) \circ (M\otimes_B\phi) \circ (\varrho^M\otimes_B\Sigma)$$
$$\simeq M\square_{\mathfrak{D}}\phi.$$

Note that this is natural in M by the functoriality of the cotensor product. Clearly, this bijective correspondence is compatible with compositions. $\square$

Corollary 4.12. *Let* $_{(\mathfrak{D}:B)}\Sigma_{(\mathbb{C}:A)}$, $_{(\mathfrak{D}:B)}\widetilde{\Sigma}_{(\mathbb{C}:A)}$ *be* $((\mathfrak{D} : B), (\mathbb{C} : A))$-*module-morphisms. Then the following statements are equivalent*:

(a) *push-out functors* $\Sigma_\circ$ *and* $\widetilde{\Sigma}_\circ$ *are naturally isomorphic to each other.*
(b) $_{(\mathfrak{D}:B)}\Sigma_{(\mathbb{C}:A)} \simeq {}_{(\mathfrak{D}:B)}\widetilde{\Sigma}_{(\mathbb{C}:A)}$ *in* $\mathbf{Rep}^{\mathrm{M}}(\mathfrak{D} : B|\mathbb{C} : A)$.
(c) $_{(\mathfrak{D}:B)}\Sigma_{(\mathbb{C}:A)} \simeq {}_{(\mathfrak{D}:B)}\widetilde{\Sigma}_{(\mathbb{C}:A)}$ *in* $_{(\mathfrak{D}:B)}\mathcal{M}_{(\mathbb{C}:A)}$.

Proof. This follows immediately from Proposition 4.11 and the duality between $\mathbf{Rep}^{\mathrm{M}}(\mathfrak{D} : B|\mathbb{C} : A)$ and $_{(\mathfrak{D}:B)}\mathcal{M}_{(\mathbb{C}:A)}$. $\square$

5. Equivalences induced by push-out and pull-back functors

In this section we study when a push-out functor is an equivalence. In terms of non-commutative algebraic geometry, this is a problem of determining which changes of covers are admissible (a change of cover of a non-commutative space should not change the space, i.e. the associated category of sheaves). We then proceed to study the generalised descent associated to a morphism of corings.

5.1. Criteria for an equivalence

Theorem 5.1. *Let* $\Sigma = (\Sigma, \sigma)$ *be a* $((\mathfrak{D} : B), (\mathfrak{C} : A))$*-module-morphism and assume that* $_B\mathfrak{D}$ *is flat.*

(1) *If* σ *is an isomorphism of A-corings and* $\mathfrak{D} \otimes_B \Sigma$ *is a coflat left* $\mathfrak{D}$*-comodule, then* $_A\mathfrak{C}$ *is flat and the pull-back functor* Σ° *is full and faithful.*
(2) *If the pull-back functor* Σ° *is full and faithful then* σ *is an isomorphism of A-corings.*

Proof. First note that since $_B\mathfrak{D}$ is flat, Σ is a right pure module-morphism so that the pull-back functor is well-defined. Furthermore, $\mathcal{M}^{\mathfrak{D}}$ is a Grothendieck category.

(1) If $\mathfrak{D} \otimes_B \Sigma$ is a coflat left $\mathfrak{D}$-comodule, then the cotensor functor $-\square_{\mathfrak{D}}(\mathfrak{D} \otimes_B \Sigma)$ is exact. Since $M\square_{\mathfrak{D}}(\mathfrak{D} \otimes_B \Sigma) \simeq M \otimes_B \Sigma$, every short exact sequence $0 \to M \to M' \to M'' \to 0$ of right $\mathfrak{D}$-comodules yields an exact sequence $0 \to M \otimes_B \Sigma \to M' \otimes_B \Sigma \to M'' \otimes_B \Sigma \to 0$.

Consider an exact sequence $0 \to V \to V' \to V'' \to 0$ of right A-modules. Since $_A\Sigma^*$ and $_B\mathfrak{D}$ are flat, the above sequence yields an exact sequence

$$0 \to V \otimes_A \Sigma^* \otimes_B \mathfrak{D} \to V' \otimes_A \Sigma^* \otimes_B \mathfrak{D} \to V'' \otimes_A \Sigma^* \otimes_B \mathfrak{D} \to 0$$

of right $\mathfrak{D}$-comodules. The coflatness of $\mathfrak{D} \otimes_B \Sigma$ then produces an exact sequence

$$0 \to V \otimes_A \Sigma^* \otimes_B \mathfrak{D} \otimes_B \Sigma \to V' \otimes_A \Sigma^* \otimes_B \mathfrak{D} \otimes_B \Sigma \to V'' \otimes_A \Sigma^* \otimes_B \mathfrak{D} \otimes_B \Sigma \to 0.$$

Hence $\Sigma[\mathfrak{D}]$ is a flat left A-module, and since $\sigma : \Sigma[\mathfrak{D}] \to \mathfrak{C}$ is an isomorphism of A-bimodules, also $_A\mathfrak{C}$ is flat.

For any right $\mathfrak{C}$-comodule N, consider the following commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 & \longrightarrow & (N\square_{\mathfrak{C}}(\Sigma^* \otimes_B \mathfrak{D})) \otimes_B \Sigma & \longrightarrow & N \otimes_A \Sigma^* \otimes_B \mathfrak{D} \otimes_B \Sigma & \longrightarrow & N \otimes_A \mathfrak{C} \otimes_A \Sigma^* \otimes_B \mathfrak{D} \otimes_B \Sigma \\
& & \downarrow{\scriptstyle \psi_N} & & \downarrow{\scriptstyle N \otimes_A \sigma} & & \downarrow{\scriptstyle N \otimes_A \mathfrak{C} \otimes_A \sigma} \\
0 & \longrightarrow & N & \xrightarrow{\varrho^N} & N \otimes_A \mathfrak{C} & \xrightarrow{N \otimes_A \Delta_{\mathfrak{C}} - \varrho^N \otimes_A \mathfrak{C}} & N \otimes_A \mathfrak{C} \otimes_A \mathfrak{C}.
\end{array}
$$

The top row is the defining sequence of a cotensor product tensored with $_B\Sigma$, hence it is exact (for $\mathfrak{D} \otimes_B \Sigma$ is coflat). The bottom row is exact by the coassociativity of the coaction. Since σ is an isomorphism of right $\mathfrak{C}$-comodules, so are the second and third vertical maps. This implies that the counit of the adjunction ψ_N is an isomorphism of right $\mathfrak{C}$-comodules. Thus the pull-back functor Σ° is full and faithful.

(2) If Σ° is a full and faithful functor, the counit ψ_N is an isomorphism of right $\mathfrak{C}$-comodules for any $N \in \mathcal{M}^{\mathfrak{C}}$. In particular, $\psi_{\mathfrak{C}} : (\mathfrak{C}\square_{\mathfrak{C}}(\Sigma^* \otimes_B \mathfrak{D})) \otimes_B \Sigma \to \mathfrak{C}$ is an isomorphism of right $\mathfrak{C}$-comodules. Using the definitions of the left coaction $^{\circ\Sigma(\mathfrak{C})}\varrho$ in Eq. (4.9) and of $\psi_{\mathfrak{C}}$ from the proof of Theorem 4.7, we can compute, for all $s \in \Sigma, d \in \mathfrak{D}$ and $s^* \in \Sigma^*$,

$$\psi_{\mathfrak{C}}((\circ^{\Sigma(\mathfrak{C})}\varrho \otimes_B \Sigma)(s^* \otimes_B d \otimes_B s)) = \psi_{\mathfrak{C}}\left(\sum_i \sigma(s^* \otimes_B d_{(1)} \otimes_B e_i) \otimes_A e_i^* \otimes_B d_{(2)} \otimes_B s\right)$$

$$= \sum_i \sigma(s^* \otimes_B d_{(1)} \otimes_B e_i) e_i^*(\varepsilon_{\mathfrak{D}}(d_{(2)})s)$$

$$= \sum_i \sigma(s^* \otimes_B d \otimes_B e_i) e_i^*(s) = \sigma(s^* \otimes_B d \otimes_B s).$$

Since $\circ^{\Sigma(\mathfrak{C})}\varrho \otimes_B \Sigma : \Sigma^* \otimes_B \mathfrak{D} \otimes_B \Sigma \to (\mathfrak{C}\square_{\mathfrak{C}}(\Sigma^* \otimes_B \mathfrak{D})) \otimes_B \Sigma$ is an isomorphism, σ is a composition of isomorphisms, hence also an isomorphism (of A-corings). $\square$

Theorem 5.2. *Let* $\Sigma = (\Sigma, \sigma)$ *be a* $((\mathfrak{D} : B), (\mathfrak{C} : A))$*-module-morphism and assume that* $_B\mathfrak{D}$ *is flat. The following statements are equivalent*:

(a) σ *is an isomorphism of A-corings and $\mathfrak{D}\otimes_B\Sigma$ is a faithfully coflat left $\mathfrak{D}$-comodule*;.
(b) $_A\mathfrak{C}$ *is flat and Σ° is an equivalence of categories with the inverse $\Sigma_\circ$.*

Proof. (a) $\Rightarrow$ (b) By Theorem 5.1, $\mathfrak{C}$ is a flat left A-module. We need to show that for all $M \in \mathcal{M}^{\mathfrak{D}}$, $N \in \mathcal{M}^{\mathfrak{C}}$, a morphism $\phi \in \mathrm{Hom}^{-\mathfrak{C}}(M \otimes_B \Sigma, N)$ is an isomorphism if and only if $\Omega_{M,N}(\phi) \in \mathrm{Hom}^{-\mathfrak{D}}(M, N\square_{\mathfrak{C}}(\Sigma^* \otimes_B \mathfrak{D}))$ is an isomorphism. Here $\Omega_{M,N}$ is the adjunction isomorphism given in Eq. (4.11). Observe that there is an isomorphism

$$\theta_N : N \to (N\square_{\mathfrak{C}}(\Sigma^* \otimes_B \mathfrak{D}))\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B\Sigma),$$

obtained as the following composition of isomorphisms:

$$N \xrightarrow{\varrho^N} N\square_{\mathfrak{C}}\mathfrak{C} \xrightarrow{N\square_{\mathfrak{C}}\sigma^{-1}} N\square_{\mathfrak{C}}(\Sigma^* \otimes_B \mathfrak{D}\otimes_B \Sigma)$$

$$\xrightarrow{N\square_{\mathfrak{C}}(\Sigma^* \otimes_B \Delta_{\mathfrak{D}} \otimes_B \Sigma)} N\square_{\mathfrak{C}}((\Sigma^* \otimes_B \mathfrak{D})\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B \Sigma))$$

$$\longrightarrow (N\square_{\mathfrak{C}}(\Sigma^* \otimes_B \mathfrak{D}))\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B \Sigma).$$

The last isomorphism is a consequence of the fact that $\mathfrak{D}\otimes_B\Sigma$ is a (faithfully) coflat left $\mathfrak{D}$-comodule. In this way we are led to the following commutative diagram

$$
\begin{array}{ccc}
M\otimes_B\Sigma & \xrightarrow{\varrho^M \otimes_B \Sigma} & M\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B\Sigma) \\
\phi \downarrow & & \downarrow \Omega_{M,N}(\phi)\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B\Sigma) \\
N & \xrightarrow{\theta_N} & (N\square_{\mathfrak{C}}(\Sigma^* \otimes_B \mathfrak{D}))\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B\Sigma).
\end{array}
$$

Since the rows are isomorphisms and $\mathfrak{D}\otimes_B\Sigma$ is a faithfully coflat left $\mathfrak{D}$-comodule, the map ϕ is an isomorphism if and only if $\Omega_{M,N}(\phi)$ is an isomorphism. Thus Σ° is an equivalence as required.

(b) $\Rightarrow$ (a) Since $_A\mathfrak{C}$ and $_B\mathfrak{D}$ are flat, both $\mathcal{M}^{\mathfrak{D}}$ and $\mathcal{M}^{\mathfrak{C}}$ are Abelian categories, and kernels (and cokernels) are computed in Abelian groups. The functor $\Sigma_\circ = -\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B\Sigma)$ is an equivalence, hence it reflects and preserves exact sequences. In view of the fact that a sequence in $\mathcal{M}^{\mathfrak{C}}$ is exact if and only if it is exact as a sequence of Abelian groups, this implies that $\mathfrak{D}\otimes_B\Sigma$ is a faithfully coflat left $\mathfrak{D}$-comodule. $\square$

Corollary 5.3. *Let $\mathfrak{D}$ be a B-coring, and Σ a (B, A)-bimodule such that Σ_A is a finitely generated and projective module. If $_B\mathfrak{D}$ is a flat module and $_B\Sigma$ is a faithfully flat module, then*

$$-\otimes_B \Sigma : \mathscr{M}^{\mathfrak{D}} \longrightarrow \mathscr{M}^{\Sigma[\mathfrak{D}]}$$

is an equivalence of categories with the inverse

$$-\square_{\Sigma[\mathfrak{D}]}(\Sigma^* \otimes_B \mathfrak{D}) : \mathscr{M}^{\Sigma[\mathfrak{D}]} \longrightarrow \mathscr{M}^{\mathfrak{D}}.$$

In particular, if $B \to A$ is a left faithfully flat ring extension (i.e. $_B A$ is a faithfully flat module), $_B\Sigma_A = {}_B A_A^{(n)}$ for a positive integer n, and $_B\mathfrak{D}$ and $\mathfrak{D}_B$ are flat modules, then the functors $-\otimes_B A^{(n)} : \mathscr{M}^{\mathfrak{D}} \to \mathscr{M}^{A^{(n)}[\mathfrak{D}]}$ and $A^{(n)}\otimes_B - : {}^{\mathfrak{D}}\mathscr{M} \to {}^{A^{(n)}[\mathfrak{D}]}\mathscr{M}$ are equivalences (i.e. $\mathfrak{D}$ is Morita-Takeuchi equivalent to $A^{(n)}[\mathfrak{D}]$).

Proof. In the notation of Theorem 5.2, take $\mathfrak{C} = \Sigma[\mathfrak{D}]$ and σ the identity map. Observe that since, for all right $\mathfrak{D}$-comodules M, $M\square_{\mathfrak{D}}(\mathfrak{D}\otimes_B \Sigma) \simeq M\otimes_B \Sigma$, the fact that Σ is a faithfully flat left B-module implies that $\mathfrak{D}\otimes_B \Sigma$ is a faithfully coflat left $\mathfrak{D}$-comodule. Finally, in this case $-\otimes_B \Sigma$ is the push-out functor and $-\square_{\Sigma[\mathfrak{D}]}(\Sigma^* \otimes_B \mathfrak{D})$ is the pull-back functor, hence the assertions follow from Theorem 5.2.   □

5.2. Generalised descent

To any ring extension $\alpha : B \to A$ one can associate a category of *right descent data* **Desc**$_\alpha$ defined in [24, 3.3] (cf. [9,19]). As observed in [3, Example 1.2], the category **Desc**$_\alpha$ is isomorphic to the category of right comodules over the canonical Sweedler A-coring $A\otimes_B A$. This isomorphism allows one to formulate a generalised descent theorem [3, Theorem 5.6]. In this subsection we introduce the category of descent data relative to a coring morphism, and give a descent theorem in this general case.

Let $(\gamma, \alpha) : (\mathfrak{D} : B) \to (\mathfrak{C} : A)$ be a coring morphism, and consider the base ring extension coring $A_\alpha[\mathfrak{D}] = A\otimes_B \mathfrak{D}\otimes_B A$. There are two coring morphisms associated to (γ, α),

$$\widetilde{\gamma} : A_\alpha[\mathfrak{D}] \longrightarrow \mathfrak{C}, \quad (\widetilde{\alpha}, \alpha) : (\mathfrak{D} : B) \longrightarrow (A_\alpha[\mathfrak{D}] : A),$$
$$a\otimes_B d\otimes_B a' \longmapsto a\gamma(d)a', \quad (d, b) \longmapsto (1\otimes_B d\otimes_B 1, \alpha(b)),$$

such that

$$\begin{array}{ccc} \mathfrak{D} & \xrightarrow{\ \widetilde{\alpha}\ } & A_\alpha[\mathfrak{D}] \xrightarrow{\ \varepsilon_{\mathfrak{D}, A_\alpha}\ } A\otimes_B A \\ & {}^{\gamma}\searrow & \downarrow{\scriptstyle \widetilde{\gamma}} \\ & & \mathfrak{C} \end{array}$$

$\hspace{11cm}$(5.1)

is a commutative diagram of coring morphisms (cf. [6, Section 24]).

A *descent datum* associated to a coring morphism $(\gamma, \alpha) : (\mathfrak{D} : B) \to (\mathfrak{C} : A)$ is a pair (X, ρ'_X) consisting of a right A-module X and a right A-linear map $\rho'_X : X \to X\otimes_B \mathfrak{D}\otimes_B A$

(here X is considered as a right B-module by restriction of scalars) rendering commutative the following diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{\quad\quad\quad\quad X \quad\quad\quad\quad} & X \\
\downarrow{\scriptstyle \rho'_X} & & \uparrow{\scriptstyle \sigma^l_X} \\
X \otimes_B \mathfrak{D} \otimes_B A & \xrightarrow{X \otimes_B \varepsilon_{\mathfrak{D}} \otimes_B A} X \otimes_B B \otimes_B A \xrightarrow{\ \simeq\ } & X \otimes_B A
\end{array}
$$

and

$$
\begin{array}{ccc}
X & \xrightarrow{\quad\quad\quad\quad \rho'_X \quad\quad\quad\quad} & X \otimes_B \mathfrak{D} \otimes_B A \\
\downarrow{\scriptstyle \rho'_X} & & \downarrow{\scriptstyle \rho'_X \otimes_B \mathfrak{D} \otimes_B A} \\
X \otimes_B \mathfrak{D} \otimes_B A \xrightarrow{X \otimes_B \Delta_{\mathfrak{D}} \otimes_B A} X \otimes_B \mathfrak{D} \otimes_B \mathfrak{D} \otimes_B A \xrightarrow{X \otimes_B \nu_{\mathfrak{D},\mathfrak{D}} \otimes_B A} & X \otimes_B \mathfrak{D} \otimes_B A_\alpha[\mathfrak{D}].
\end{array}
$$

Here $\sigma^l_- : -\otimes_B A \to -$ is the natural transformation defined by right multiplication, and $\nu_{-,-} : -\otimes_B - \to -\otimes_B A \otimes_B -$ is the canonical natural transformation. A *morphism* of descent data is a right A-linear map $f : (X, \rho_X) \to (Z, \rho'_Z)$ such that

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Z \\
\downarrow{\scriptstyle \rho'_X} & & \downarrow{\scriptstyle \rho'_Z} \\
X \otimes_B \mathfrak{D} \otimes_B A & \xrightarrow{f \otimes_B \mathfrak{D} \otimes_B A} & Z \otimes_B \mathfrak{D} \otimes_B A
\end{array}
$$

is a commutative diagram. Descent data associated to a coring morphism (γ, α) and their morphisms form a category called the category of *right descent data associated to (γ, α)* and denoted by $\mathbf{Desc}_{(\gamma,\alpha)}$.

Lemma 5.4. *Let $(\gamma, \alpha) : (\mathfrak{D} : B) \to (\mathfrak{C} : A)$ be a morphism of corings. Then $\mathbf{Desc}_{(\gamma,\alpha)}$ is isomorphic to $\mathscr{M}^{A_\alpha[\mathfrak{D}]}$.*

Proof. The isomorphism of categories is provided by the following two functors. Given any right $A_\alpha[\mathfrak{D}]$-comodule X, define a right A-linear map

$$
\rho'_X : X \xrightarrow{\varrho^X} X \otimes_A A \otimes_B \mathfrak{D} \otimes_B A \xrightarrow{\ \simeq\ } X \otimes_B \mathfrak{D} \otimes_B A.
$$

It is clear that (X, ρ'_X) is an object of $\mathbf{Desc}_{(\gamma,\alpha)}$. Obviously, any right $A_\alpha[\mathfrak{D}]$-colinear map induces a morphism in $\mathbf{Desc}_{(\gamma,\alpha)}$. Conversely, let (X, ρ'_X) be an object of $\mathbf{Desc}_{(\gamma,\alpha)}$. Then X is a right $A_\alpha[\mathfrak{D}]$-comodule with the coaction

$$
\varrho^X : X \xrightarrow{\rho'_X} X \otimes_B \mathfrak{D} \otimes_B A \xrightarrow{\ \simeq\ } X \otimes_A A \otimes_B \mathfrak{D} \otimes_B A.
$$

Clearly any arrow of $\mathbf{Desc}_{(\gamma,\alpha)}$ induces a right $A_\alpha[\mathfrak{D}]$-colinear map. Finally, a straightforward computation shows that the constructed functors are mutually inverse. $\square$
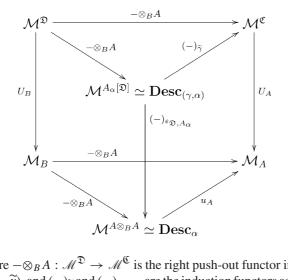
Corollary 5.5. *Let* $(\gamma, \alpha) : (\mathfrak{D} : B) \to (\mathfrak{C} : A)$ *be a coring morphism. If* $_B\mathfrak{D}$ *is a flat module and* $_B A$ *is a faithfully flat module, then*

$$\mathscr{M}^{\mathfrak{D}} \xrightarrow{\quad -\otimes_B A \quad} \mathscr{M}^{A_\alpha[\mathfrak{D}]} \equiv \mathbf{Desc}_{(\gamma, \alpha)}$$

is an equivalence of categories.

Proof. This corollary is a straightforward consequence of Lemma 5.4 and Corollary 5.3. □

The following commutative diagram of functors summarises and combines the old and the new situations

Here $-\otimes_B A : \mathscr{M}^{\mathfrak{D}} \to \mathscr{M}^{\mathfrak{C}}$ is the right push-out functor induced by the module-morphism $(A_\alpha, \widetilde{\gamma})$, and $(-)_{\widetilde{\gamma}}$ and $(-)_{\varepsilon_{\mathfrak{D}, A_\alpha}}$ are the induction functors associated to the coring morphisms $\widetilde{\gamma}$ and $\varepsilon_{\mathfrak{D}, A_\alpha}$, respectively (the latter defined in diagram (5.1)).

Acknowledgements

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