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Cohomology for bicomodules: Separable and Maschke functors

by

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Abstract

We introduce the category of bicomodules for a comonad on a Grothendieck category whose underlying functor is right exact and preserves direct sums. We characterize comonads with a separable forgetful functor by means of cohomology groups using cointegrations into bicomodules. We present two applications: the characterization of coseparable corings stated in [14], and the characterization of coseparable coalgebra coextensions stated in [19].

Key Words: Comonad, bicomodule, Separable functor, Maschke functor

Mathematics Subject Classification 1991: 16W30

Introduction

In [16] D. W. Jonah studied the second and the third cohomology groups of coalgebras defined in a, not necessary abelian, multiplicative category (see also [2]). M. Kleiner gave in [17] a cohomological characterization of separable algebras using integrations. Another approach via derivations was given by M. Barr and G. Rinehart in [3]. This last one has been dualised to the case of coseparable coalgebras by Doi [7]. Nakajima [19] showed that Doi's result can be extended to coalgebra extensions (or co-extension) with a co-commutative base coalgebra. In [14], F. Guzman used Jonah's methods to generalize Doi's characterization for corings over an arbitrary base-ring and unified this with a dualisation of Kleiner's approach of cointegrations. This gives rise to a nice characterization of coseparable corings in terms of cohomology, derived functors and both cointegrations and coderivations. Unfortunately this last characterization can not be applied to coalgebra co-extensions, and Nakajima's results are not recovered.

The common framework behind Guzman's and Nakajima's approach is the fact that both coseparable corings and coseparable coalgebra co-extensions can be interpreted as comonads with a separable forgetful functor (in the sense of [20], see below). In all situations discussed before, the multiplicative base-category was additive with cokernels and arbitrary direct sums, and the (co)monad functor preserved cokernels and direct sums. In the present paper we will approach the problem by this comonad point of view. We work with a comonad over a

Grothendieck category (not necessary multiplicative) whose underlying functor fits the above mentioned class of functors. These functors were studied in relation with corings in [12], see also [11] and references cited there. We will present a generalisation of Guzman's characterization in this situation, and as a particular application we also give, under different assumption, Nakajima's result.

We will start by defining the category of bicomodules over a comonad as in [16], and we consider its universal cogenerator [9] (i.e. the universal adjunction defining the comonadic structure) in order to prove that the forgetful functor in this universal adjunction is separable ([20], see below) if and only if the forgetful functor in bicomodules is Maschke ([6], see below) if and only if the comultiplication splits in the category of bicomodules. This will be the main result of section 1 (Theorem 1.6) (see [1] for a different approach). In section 2 we define cointegrations and coderivations, we also establish, as in [14], an isomorphism between the abelian group of cointegrations into a comonad and the group of all coderivations. This will serve to show that the comultiplication splits as a morphism of bicomodules if and only if the universal cointegration is inner if and only if the universal coderivation is inner (Corollary 2.4). Section 3 is devoted to the relative cohomology for bicomodules defined as in [10, 16] using a relative resolution with respect to the injective class of sequences in the category of bicomodules which are cosplit after forgetting the left coaction. Up to isomorphisms, cointegrations appear as 1-cocycles and inner cointegrations as 1-coboundaries. The relative injectivity is thus interpreted by the fact that all into-cointegrations are inner. This happens for all bicomodules if and only if the comultiplication splits in the category of bicomodules (Theorem 3.5). The last section presents two applications of this last Theorem, the first one makes use of the comonad defined by tensor product over algebras [14], and the second uses cotensor product over coalgebras over fields [19]. Since in recent years it became clear that corings and comodules provide a general framework to study entwining structures and entwined modules and by this all sorts of (relative) Hopf-modules (we refer to [5] for a profound overview), separability properties of these structures are covered by our theory as special cases.

NOTATIONS AND BASIC NOTIONS: Given any Hom-set category $\mathscr{A}$, the notation $X \in \mathscr{A}$ means that X is an object of $\mathscr{A}$. The identity morphism of X will be denoted by X itself. The set of all morphisms $f : X \to X'$ in $\mathscr{A}$, is denoted by $\mathrm{Hom}_{\mathscr{A}}\left(X, X'\right)$. The identity functor of $\mathscr{A}$ will be denoted by $\mathbb{1}_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$. A natural transformation between two functors $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$, is denoted by $\beta_{-} : \mathcal{F} \to \mathcal{G}$. If $\mathcal{H} : \mathscr{B} \to \mathscr{C}$, and $\mathcal{I} : \mathscr{D} \to \mathscr{A}$ are other functors, then $\beta_{\mathcal{I}(-)}$ (or $\beta_{\mathcal{I}}$) denotes the natural transformation defined at each object $Z \in \mathscr{D}$ by $\beta_{\mathcal{I}(Z)} : \mathcal{F}\mathcal{I}(Z) \to \mathcal{G}\mathcal{I}(Z)$, while $\mathcal{H}\beta_{-}$ (or $\mathcal{H}\beta$) denotes the natural transformation

defined at each object $X \in \mathscr{A}$ by $\mathcal{H}(\beta_X) : \mathcal{HF}(X) \to \mathcal{HG}(X)$.
Any covariant functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ leads to a (bi)functor

$$\mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(-),\mathcal{F}(-)) : \mathscr{A}^{op} \times \mathscr{A} \to \mathscr{S}et.$$

In particular, the identical functor $\mathbb{1}_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ gives rise to

$$\mathrm{Hom}_{\mathscr{A}}(-,-) : \mathscr{A}^{op} \times \mathscr{A} \to \mathscr{S}et.$$

So we find a natural transformation induced by $\mathcal{F}$,

$$\mathscr{F} : \mathrm{Hom}_{\mathscr{A}}(-,-) \to \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(-),\mathcal{F}(-));$$

defined by $\mathscr{F}_{X,X'}(f) = \mathcal{F}(f)$, for any arrow $f : X \to X'$ in $\mathscr{A}$. Recall from [21] (see [20] for the original definition) that the functor $\mathcal{F}$ is called *separable* if and only if $\mathscr{F}$ has a left inverse, i.e. there exists a natural transformation

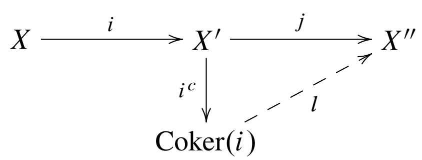
$$\mathscr{P} : \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(-),\mathcal{F}(-)) \to \mathrm{Hom}_{\mathscr{A}}(-,-)$$

such that $\mathscr{P} \circ \mathscr{F} = \mathbb{1}_{\mathrm{Hom}_{\mathscr{A}}(-,-)}$. If in addition $\mathcal{F}$ has a right adjoint functor $\mathcal{G} : \mathscr{B} \to \mathscr{A}$ with unit $\eta_- : \mathbb{1}_{\mathscr{A}} \to \mathcal{GF}$, then it is well known from [21], that $\mathcal{F}$ is separable if and only if there exists a natural transformation $\mu : \mathcal{GF} \to \mathbb{1}_{\mathscr{A}}$ such that $\mu \circ \eta = \mathbb{1}_{\mathscr{A}}$.
Let $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ be again a covariant functor. Recall from [6], that an object $M \in \mathscr{A}$ is called *relative injective* (or $\mathcal{F}$*-injective*) if and only if for every morphism $i : X \to X'$ in $\mathscr{A}$, such that $\mathcal{F}(i) : \mathcal{F}(X) \to \mathcal{F}(X')$ has a left inverse j in $\mathcal{B}$ (i.e. $\mathcal{F}(i)$ is a split monomorphism or just split-mono) and for every $f : X \to M$ in $\mathscr{A}$ we can find a morphism $g : X' \to M$ in $\mathscr{A}$ such that $g \circ i = f$. The functor $\mathcal{F}$ is said to be *a Maschke functor* if every object of $\mathscr{A}$ is $\mathcal{F}$-injective. If in addition $\mathcal{F}$ has a right adjoint functor $\mathcal{G} : \mathscr{B} \to \mathscr{A}$ with unit $\eta_- : \mathbb{1}_{\mathscr{A}} \to \mathcal{GF}$, then, by [6, Theorem 3.4], an object $M \in \mathscr{A}$ is $\mathcal{F}$-injective if and only if η_M has a left inverse. In particular $\mathcal{F}$ is a Maschke functor if and only if for every object $M \in \mathscr{A}$, η_M has a left inverse.

Assume that a preadditive category $\mathscr{A}$ is given. Following [10, pages 3-4], a sequence

$$E : X \xrightarrow{\ i\ } X \xrightarrow{\ j\ } X''$$

(i.e. $j \circ i = 0$) is said to be *co-exact* if i has a cokernel and if in the commutative diagram



l is a monomorphism. If in addition l is a split-mono, then E is said to be *cosplit*. The *exact* and *split* sequence are dually defined by using kernels. The notions of sequence, coexact sequence, cosplit sequence,... are extended to long diagrams simply by applying them to each consecutive pair of morphisms. One can prove that the above notions of exact and coexact sequences coincide with the usual meaning of exact sequences in abelian categories. In case of a diagram of the form

$$E' : 0 \longrightarrow X \longrightarrow X' \longrightarrow X'' \longrightarrow 0$$

(i.e. short sequence) in the category $\mathscr{A}$, we have by [16, Lemma 2.1], that E' is cosplit if and only if it is split.

Next, we recall from [10, I.2] the notions of closed and injective classes. Let $\mathscr{E}$ be a class of sequences in $\mathscr{A}$, then an object $X \in \mathscr{A}$ is said to be $\mathscr{E}$*-injective* if $\mathrm{Hom}_{\mathscr{A}}(E,X)$ is an exact sequence of abelian groups, for every sequence E in $\mathscr{E}$. The class of all $\mathscr{E}$-injective objects is denoted by $\mathscr{I}_{\mathscr{E}}$. Conversely, given $\mathscr{I}$ a class of objects of $\mathscr{A}$, a sequence E of morphism of $\mathscr{A}$ is said to be $\mathscr{I}$*-exact* if $\mathrm{Hom}_{\mathscr{A}}(E,Y)$ is an exact sequence of an abelian groups, for every object Y in $\mathscr{I}$. The class of all $\mathscr{I}$-exact sequences is denoted by $\mathscr{E}_{\mathscr{I}}$. A class of sequences $\mathscr{E}$ in $\mathscr{A}$ is said to be *closed* whenever $\mathscr{E}$ coincides with $\mathscr{E}_{\mathscr{I}_{\mathscr{E}}}$. An *injective class* is a closed class of sequences $\mathscr{E}$ such that, for every morphism $X \to X'$, there exists a morphism $X' \to Y$ with $Y \in \mathscr{I}_{\mathscr{E}}$ and with $X \to X' \to Y$ in $\mathscr{E}$. A closed and projective classes are dually defined.

If in addition the category $\mathscr{A}$ possesses cokernels, then one can check that the class $\mathscr{E}_0$ of all cosplit sequences form an injective class and $\mathscr{I}_{\mathscr{E}_0}$ is exactly the class of all objects of $\mathscr{A}$. Given any adjunction $\mathscr{F} : \mathscr{A} \rightleftarrows \mathscr{B} : \mathscr{G}$ where $\mathscr{F}$ is left adjoint functor to $\mathscr{G}$ (we use the notation $\mathscr{F} \dashv \mathscr{G}$), and a class of sequences $\mathscr{E}'$ in $\mathscr{B}$, denote by $\mathscr{E} = \mathscr{F}^{-1}(\mathscr{E}')$ the class of sequences E in $\mathscr{A}$ such that $\mathscr{F}(E)$ is in $\mathscr{E}'$. The dual of Eilenberg-Moore's Theorem [10, Theorem 2.1, page 15] stated in [16, Theorem 2.9] asserts that $\mathscr{E}$ is an injective class whenever $\mathscr{E}'$ is.

1. Bicomodules and Separability

Let $\mathcal{A}$ and $\mathcal{B}$ two Grothendieck categories, we denote by $\overline{\mathrm{Funct}}(\mathcal{A},\mathcal{B})$ the class of all (additive) covariant functors $F : \mathcal{A} \to \mathcal{B}$ such that F preserves cokernels and commutes with direct sums. Thus F commutes with inductive limits. By [8, Lemma 5.1], the natural transformations between two objects of the class $\overline{\mathrm{Funct}}(\mathcal{A},\mathcal{B})$ form a set. Henceforth, $\overline{\mathrm{Funct}}(\mathcal{A},\mathcal{B})$ is a Hom-set category (or Set-category).

A *comonad* on a category $\mathcal{A}$ is a three-tuple $\mathbf{F} = (F,\delta,\xi)$ consisting of an endofunctor $F : \mathcal{A} \to \mathcal{A}$ and two natural transformations $\delta : F \to F^2 = F \circ F$ and

$\xi : F \to \mathbb{1}_{\mathcal{A}}$ such that

$$\delta_F \circ \delta = F\delta \circ \delta \quad \text{and} \quad F\xi \circ \delta = \xi_F \circ \delta = F, \tag{1.1}$$

where we denote the identical natural transformation $F \to F$ again by F. A *comonad homomorphism* $\phi : (F,\delta,\xi) \to (F',\delta',\xi')$ is a natural transformation $\phi : F \to F'$ such that $\xi' \circ \phi = \xi$ and $\delta' \circ \phi = \varrho_\phi \circ \delta$, where ϱ_ϕ is the natural transformation defined by $(\varrho_\phi)_- = F'\phi_- \circ \phi_{F(-)} = \phi_{F'(-)} \circ F\phi_-$, see [4].

It is well known from [15, 9, 18], that any adjunction $S : \mathcal{B} \rightleftarrows \mathcal{A} : T$ with $S \dashv T$, leads to a comonad on $\mathcal{A}$ given by the three-tuple $(ST, S\eta_T, \zeta)$, where $\eta : \mathbb{1}_{\mathcal{B}} \to TS$ and $\zeta : ST \to \mathbb{1}_{\mathcal{A}}$ are, respectively, the unit and the counit of this adjunction.

Let $\mathbf{F} = (F,\delta,\xi)$ be a comonad on $\mathcal{A}$ with $F \in \overline{\text{Funct}}(\mathcal{A},\mathcal{A})$. We define the category of $(\mathcal{B},\mathbf{F})$-*bicomodules* ${}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}}$ by the following data:

- *Objects*: A $(\mathcal{B},\mathbf{F})$-bicomodule is a pair $(\mathsf{M},\mathfrak{m})$ consisting of a functor $\mathsf{M} \in \overline{\text{Funct}}(\mathcal{B}, \mathcal{A})$ and natural transformation $\mathfrak{m} : \mathsf{M} \to F\mathsf{M}$ satisfying

$$\delta_{\mathsf{M}} \circ \mathfrak{m} = F\mathfrak{m} \circ \mathfrak{m}, \quad \xi_{\mathsf{M}} \circ \mathfrak{m} = \mathsf{M}. \tag{1.2}$$

- *Morphisms*: A morphism $\mathsf{f} : (\mathsf{M},\mathfrak{m}) \to (\mathsf{M}',\mathfrak{m}')$ is a natural transformation $\mathsf{f} : \mathsf{M} \to \mathsf{M}'$ satisfying

$$\mathfrak{m}' \circ \mathsf{f} = F\mathsf{f} \circ \mathfrak{m}. \tag{1.3}$$

It is easily seen that $(F\mathsf{M},\delta_{\mathsf{M}})$ is an object of the category ${}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}}$, for every object $\mathsf{M} \in \overline{\text{Funct}}(\mathcal{B},\mathcal{A})$. This in fact establishes a functor $\mathscr{F} : \overline{\text{Funct}}(\mathcal{B},\mathcal{A}) \to {}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}}$ with a left adjoint the forgetful functor $\mathscr{O} : {}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}} \to \overline{\text{Funct}}(\mathcal{B},\mathcal{A})$.

Similarly, we can define the category of $(\mathbf{F},\mathcal{B})$-bicomodules denoted by ${}^{\mathbf{F}}\mathscr{M}_{\mathcal{B}}$, using this time the objects of the category $\overline{\text{Funct}}(\mathcal{A},\mathcal{B})$.

Remark 1.1 Given any adjunction $\mathsf{M} : \mathcal{B} \rightleftarrows \mathcal{A} : \mathsf{N}$ such that $\mathsf{M} \dashv \mathsf{N}$ with counit ζ and unit η, then [13, Proposition 1.1] establishes one-to-one correspondences between natural transformations $\mathfrak{m} : \mathsf{M} \to F\mathsf{M}$ satisfying equation (1.2) and homomorphisms of comonads from $(\mathsf{MN},\mathsf{M}\eta_{\mathsf{N}},\zeta)$ to $\mathbf{F}$, and natural transformations $\mathfrak{s} : \mathsf{N} \to \mathsf{N}F$ satisfying the dual version of equation (1.2). When N and M are both right exact and preserve direct sums, then the previous correspondence can be interpreted in our terminology as follows: There are bijections between the $(\mathbf{F},\mathcal{B})$-bicomodule structures on M, the $(\mathcal{B},\mathbf{F})$-bicomodule structures on N, and the homomorphisms of comonads from $(\mathsf{MN},\mathsf{M}\eta_{\mathsf{N}},\zeta)$ to $\mathbf{F}$.

Take now $\mathbf{G} = (G,\vartheta,\varsigma)$ another comonad on $\mathcal{B}$ with $G \in \overline{\text{Funct}}(\mathcal{B},\mathcal{B})$, we define the category of $(\mathbf{G},\mathbf{F})$-*bicomodules* ${}^{\mathbf{G}}\mathscr{M}^{\mathbf{F}}$ as follows:

- *Objects*: A $(\mathbf{G},\mathbf{F})$ bicomodule is a three-tuple $(\mathsf{M},\mathfrak{m},\mathfrak{n})$ consisting of a functor $\mathsf{M} \in \overline{\text{Funct}}(\mathcal{B},\mathcal{A})$ and two natural transformations $\mathfrak{m} : \mathsf{M} \to F\mathsf{M}$, $\mathfrak{n} : \mathsf{M} \to \mathsf{M}G$ such that $(\mathsf{M},\mathfrak{m}) \in {}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}}$ and $(\mathsf{M},\mathfrak{n}) \in {}^{\mathbf{G}}\mathscr{M}_{\mathcal{A}}$, that is

$$\delta_{\mathsf{M}} \circ \mathfrak{m} = F\mathfrak{m} \circ \mathfrak{m}, \xi_{\mathsf{M}} \circ \mathfrak{m} = \mathsf{M} \,\text{and}\, \mathsf{M}\vartheta \circ \mathfrak{n} = \mathfrak{n}_G \circ \mathfrak{n}, \mathsf{M}\varsigma \circ \mathfrak{n} = \mathsf{M} \qquad (1.4)$$

with compatibility condition

$$\mathfrak{m}_G \circ \mathfrak{n} = F\mathfrak{n} \circ \mathfrak{m}. \qquad (1.5)$$

In other words $\mathfrak{m}$ is a morphism of ${}^{\mathbf{G}}\mathscr{M}_{\mathcal{A}}$, equivalently, $\mathfrak{n}$ is a morphism of ${}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}}$, where $(F\mathsf{M}, F\mathfrak{n}) \in {}^{\mathbf{G}}\mathscr{M}_{\mathcal{A}}$ and $(\mathsf{M}G, \mathfrak{m}_G) \in {}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}}$.

- *Morphisms*: A morphism $\mathsf{f} : (\mathsf{M},\mathfrak{m},\mathfrak{n}) \to (\mathsf{M}',\mathfrak{m}',\mathfrak{n}')$ is a natural transformation $\mathsf{f} : \mathsf{M} \to \mathsf{M}'$ such that $\mathsf{f} : (\mathsf{M},\mathfrak{m}) \to (\mathsf{M}',\mathfrak{m}')$ is a morphism of ${}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}}$ and $\mathsf{f} : (\mathsf{M},\mathfrak{n}) \to (\mathsf{M}',\mathfrak{n}')$ is a morphism of ${}^{\mathbf{G}}\mathscr{M}_{\mathcal{A}}$, that is

$$\mathfrak{n}' \circ \mathsf{f} = \mathsf{f}_G \circ \mathfrak{n} \,\text{and}\, \mathfrak{m}' \circ \mathsf{f} = F\mathsf{f} \circ \mathfrak{m}. \qquad (1.6)$$

It is clear that ${}^{\mathbb{1}_{\mathcal{B}}}\mathscr{M}^{\mathbf{F}} = {}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}}$ and ${}^{\mathbf{G}}\mathscr{M}^{\mathbb{1}_{\mathcal{A}}} = {}^{\mathbf{G}}\mathscr{M}_{\mathcal{A}}$, where $\mathbb{1}_{\mathcal{A}}$ and $\mathbb{1}_{\mathcal{B}}$ are endowed with a trivial comonad structure.

Remark 1.2 It is easily seen that $\overline{\text{Funct}}(\mathcal{A},\mathcal{A})$ is a strict monoidal category (or multiplicative category), taking the composition of functors as the tensor product and $\mathbb{1}_{\mathcal{A}}$ as the identity object. To any coalgebra in a monoidal category one can associate in a canonical way a category of bicomodules, see [16, Section 1]. If we consider $\mathbf{F}$ as a coalgebra in $\overline{\text{Funct}}(\mathcal{A},\mathcal{A})$, then the category of $(\mathbf{F},\mathbf{F})$-bicomodules as defined above coincides exactly with this canonical one. However, if we consider $(\mathbf{G},\mathbf{F})$-bicomodules and thus the base-category is changed, the monoidal arguments fail. In that case one must consider the 2-category of Grothendieck categories (0-cells), additive functors that preserve cokernels and commute with direct sums (1-cells), and natural transformations (2-cells). Observe that $\mathbf{F}$ and $\mathbf{G}$ are comonads inside this 2-category (see [11] for elementary treatment).

By the observation that the bicomodules as introduced above coincide with certain 1-cells in a 2-category, we can state the following well known lemma.

Lemma 1.3 *Let $\mathcal{A}$ (respectively $\mathcal{B}$) be a Grothendieck category, and $\mathbf{F} = (F,\delta,\xi)$ (respectively $\mathbf{G} = (G,\vartheta,\varsigma)$) a comonad on $\mathcal{A}$ (respectively on $\mathcal{B}$) whose underlying functor F (respectively G) is right exact and commutes with direct sums. The category of $(\mathbf{G},\mathbf{F})$-bicomodules ${}^{\mathbf{G}}\mathscr{M}^{\mathbf{F}}$ is a preadditive category with cokernels and arbitrary direct sums.*

Consider the categories of bicomodules ${}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}}$ and ${}^{\mathbf{G}}\mathscr{M}^{\mathbf{F}}$. There are two functors connecting those categories. The left forgetful functor $\mathscr{S} : {}^{\mathbf{G}}\mathscr{M}^{\mathbf{F}} \to {}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}}$, which sends any $(\mathbf{G},\mathbf{F})$-bicomodule $(\mathsf{M},\mathfrak{m},\mathfrak{n})$ to the $(\mathcal{B},\mathbf{F})$-bicomodule $(\mathsf{M},\mathfrak{m})$ and which is identical on the morphisms. Secondly, the functor $\mathscr{T} : {}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}} \to {}^{\mathbf{G}}\mathscr{M}^{\mathbf{F}}$ which sends $(\mathsf{M}',\mathfrak{m}') \to (\mathsf{M}'G,\mathfrak{m}'_G,\mathsf{M}'\vartheta)$ and $\mathsf{f} \to \mathsf{f}_G$. These functors form an adjunction, more precisely we have

Lemma 1.4 *For every pair of objects* $\Big((\mathsf{N},\mathfrak{r},\mathfrak{s}),(M,\mathfrak{m})\Big)$ *of* ${}^{\mathbf{G}}\mathscr{M}^{\mathbf{F}} \times {}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}}$*, there is a natural transformation*

$$\begin{array}{ccc}
\mathrm{Hom}_{{}^{\mathbf{G}}\mathscr{M}^{\mathbf{F}}}\Big((\mathsf{N},\mathfrak{r},\mathfrak{s}),\mathscr{T}(\mathsf{M},\mathfrak{m})\Big) & \xrightarrow{\Phi_{\mathsf{N},\mathsf{M}}} & \mathrm{Hom}_{{}_{\mathcal{B}}\mathscr{M}^{\mathbf{F}}}\Big(\mathscr{S}(\mathsf{N},\mathfrak{r},\mathfrak{s}),(\mathsf{M},\mathfrak{m})\Big) \\
\mathsf{f} & \longmapsto & \mathsf{M}\varsigma \circ \mathsf{f} \\
\mathsf{g}_G \circ \mathfrak{s} & \longleftarrow\!\shortmid & \mathsf{g}.
\end{array}$$

That is $\mathscr{S}$ *is a left adjoint functor to* $\mathscr{T}$.

Let $\mathcal{X}$ be the discrete one-object category, then the category ${}_{\mathcal{X}}\mathscr{M}^{\mathbf{F}}$ can be described as follows. A functor $\mathsf{X} : \mathcal{X} \to \mathcal{A}$ is completely determined by the image X of the single object in $\mathcal{X}$. A natural transformation $\mathfrak{x} : \mathsf{X} \to F\mathsf{X}$ is completely determined by a morphism $d^X : X \to F(X)$. In this way, we can identify an object in ${}_{\mathcal{X}}\mathscr{M}^{\mathbf{F}}$ with a pair (X,d^X) consisting of an object $X \in \mathcal{A}$ and a morphism $d^X : X \to F(X)$ satisfying

$$\delta_X \circ d^X = F(d^X)\circ d^X, \quad \xi_X \circ d^X = X.$$

Similarly, a morphism $f : (X,d^X) \to (X',d^{X'})$ in ${}_{\mathcal{X}}\mathscr{M}^{\mathbf{F}}$ is completely determined by a morphism $f : X \to X'$ of $\mathcal{A}$ such that

$$d^{X'} \circ f = F(f)\circ d^X.$$

Under this identification, we will denote this category by $\mathcal{A}^{\mathbf{F}}$. Denote by $\mathbf{S} : \mathcal{A}^{\mathbf{F}} \to \mathcal{A}$ the forgetful functor and $\mathbf{T} : \mathcal{A} \to \mathcal{A}^{\mathbf{F}}, \mathbf{T}(Y) = (F(Y),\delta_Y)$, $\mathbf{T} = F(f)$, for every object Y and morphism f of $\mathcal{A}$. Then we obtain an adjunction $\mathbf{S} \dashv \mathbf{T}$, with $\mathbf{ST} = F$ satisfying a universal property, see [9, Theorem 2.2].

Remark 1.5 It is well known that $\mathcal{A}^{\mathbf{F}}$ is an additive category with direct sums and cokernels, admitting $(F(U),\delta_U)$ as a sub-generator, whenever U is a generator of $\mathcal{A}$. However, $\mathcal{A}^{\mathbf{F}}$ is not necessarily a Grothendieck category. But, if we assume that F is an exact functor and that $\mathcal{A}$ possesses a generating set of finitely generated objects, then one can easily check that $\mathcal{A}^{\mathbf{F}}$ becomes a Grothendieck category.

The main result of this section is the following

Theorem 1.6 *Let $\mathcal{A}$ be a Grothendieck category. Consider a comonad $\mathbf{F} = (F,\delta,\xi)$ on $\mathcal{A}$ whose functor F preserves cokernels and commutes with direct sums. The following statements are equivalent*

(i) $\mathbf{S} : \mathcal{A}^{\mathbf{F}} \longrightarrow \mathcal{A}$ *is a separable functor;*

(ii) $\mathscr{S} : {}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}} \longrightarrow {}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$ *is a Maschke functor;*

(iii) $\delta : (F,\delta,\delta) \longrightarrow (F^2,\delta_F,F\delta)$ *is a split monomorphism in the category* ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$.

Proof: $(i) \Rightarrow (iii)$. The unit of the adjunction $\mathbf{S} \dashv \mathbf{T}$ is given by

$$\eta_{(X,d^X)} : (X,d^X) \xrightarrow{\quad d^X \quad} \mathbf{TS}(X,d^X) = (F(X),\delta_X) \tag{1.7}$$

for every object (X,d^X) of $\mathcal{A}^{\mathbf{F}}$. By hypothesis there is a natural transformation $\psi : \mathbf{TS} \to \mathbb{1}_{\mathcal{A}^{\mathbf{F}}}$ such that $\psi \circ \eta = \mathbb{1}_{\mathcal{A}^{\mathbf{F}}}$. Let us denote by $\nabla : F^2 \to F$ the natural transformation given by the collection of morphisms $\nabla_X = \mathbf{S}(\psi_{(F(X),\delta_X)})$, where X runs through the class of objects of $\mathcal{A}$. By construction $\nabla \circ \delta = F$ and $\nabla : (F^2,\delta_F) \to (F,\delta)$ is a morphism of the category ${}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$. Since ψ is a natural transformation and $\delta_X : (F(X),\delta_X) \to (F^2(X),\delta_{F(X)})$ is morphism in $\mathcal{A}^{\mathbf{F}}$, we have the following commutative diagram

$$\begin{array}{ccc} F^3 & \xrightarrow{\mathbf{S}\psi_{F^2}} & F^2 \\ {\scriptstyle F\delta}\uparrow & & \uparrow{\scriptstyle \delta} \\ F^2 & \xrightarrow{\mathbf{S}\psi_F} & F \end{array}$$

Therefore $\delta \circ \nabla = \nabla_F \circ F\delta$, which means that $\nabla : (F^2,\delta_F,F\delta) \to (F,\delta,\delta)$ is a morphism in the category ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$. Thus δ is a split monomorphism of the category ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$.

$(iii) \Rightarrow (ii)$. Let us denote by $\Lambda : (F^2,\delta_F,F\delta) \to (F,\delta,\delta)$ the left inverse of $\delta : (F,\delta,\delta) \to (F^2,\delta_F,F\delta)$, i.e. $\Lambda \circ \delta = F$, in the category ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$. Let $(\mathsf{M},\mathfrak{m},\mathfrak{n})$ be any $\mathbf{F}$-bicomodule. The unit of the adjunction $\mathscr{S} \dashv \mathscr{T}$ stated in Lemma 1.4, at this bicomodule is given by

$$\Theta_{(\mathsf{M},\mathfrak{m},\mathfrak{n})} : (\mathsf{M},\mathfrak{m},\mathfrak{n}) \xrightarrow{\quad \mathfrak{n} \quad} \mathscr{T} \circ \mathscr{S}(\mathsf{M},\mathfrak{m},\mathfrak{n}) = (MF,\mathfrak{m}_F,M\delta). \tag{1.8}$$

Consider the natural transformation defined by the following composition

$$\upsilon : \mathsf{M}F \xrightarrow{\mathfrak{n}_F} \mathsf{M}F^2 \xrightarrow{\mathsf{M}\Lambda} \mathsf{M}F \xrightarrow{\mathsf{M}\xi} \mathsf{M}\,.$$

It is easily seen that $\upsilon \circ \mathfrak{n} = \mathsf{M}$. The implication will be established if we show that υ is a morphism in the category of bicomodules ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$. We can compute

$$\begin{aligned}
\mathfrak{m} \circ \upsilon &= \mathfrak{m} \circ \mathsf{M}\xi \circ \mathsf{M}\Lambda \circ \mathfrak{n}_F \\
&= F\mathsf{M}\xi \circ \mathfrak{m}_F \circ \mathsf{M}\Lambda \circ \mathfrak{n}_F, && \mathfrak{m}_- \text{ is natural} \\
&= F\mathsf{M}\xi \circ F\mathsf{M}\Lambda \circ \mathfrak{m}_{F^2} \circ \mathfrak{n}_F, && \mathfrak{m}_- \text{ is natural} \\
&= F\mathsf{M}\xi \circ F\mathsf{M}\Lambda \circ F\mathfrak{n}_F \circ \mathfrak{m}_F, && \text{by (1.5)} \\
&= F\Big(\mathsf{M}\xi \circ \mathsf{M}\Lambda \circ \mathfrak{n}_F\Big) \circ \mathfrak{m}_F \\
&= F\upsilon \circ \mathfrak{m}_F,
\end{aligned}$$

which proves that υ is a morphism in ${}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$. On the other hand, we have

$$\begin{aligned}
\mathfrak{n} \circ \upsilon &= \mathfrak{n} \circ \mathsf{M}\xi \circ \mathsf{M}\Lambda \circ \mathfrak{n}_F \\
&= \mathsf{M}F\xi \circ \mathfrak{n}_F \circ \mathsf{M}\Lambda \circ \mathfrak{n}_F, && \mathfrak{n}_- \text{ is natural} \\
&= \mathsf{M}F\xi \circ \mathsf{M}F\Lambda \circ \mathfrak{n}_{F^2} \circ \mathfrak{n}_F, && \mathfrak{n}_- \text{ is natural} \\
&= \mathsf{M}F\xi \circ \mathsf{M}F\Lambda \circ \mathsf{M}\delta_F \circ \mathfrak{n}_F, && \text{by (1.4)} \\
&= \mathsf{M}F\xi \circ \mathsf{M}\delta \circ \mathsf{M}\Lambda \circ \mathfrak{n}_F, && \text{by (1.6)} \\
&= \mathsf{M}\Lambda \circ \mathfrak{n}_F,
\end{aligned}$$

and

$$\begin{aligned}
\upsilon_F \circ \mathsf{M}\delta &= \mathsf{M}\xi_F \circ \mathsf{M}\Lambda_F \circ \mathfrak{n}_{F^2} \circ \mathsf{M}\delta \\
&= \mathsf{M}\xi \circ \mathsf{M}\Lambda_F \circ \mathsf{M}F\delta \circ \mathfrak{n}_F, && \mathfrak{n}_- \text{ is natural} \\
&= \mathsf{M}\xi_F \circ \mathsf{M}(\Lambda_F \circ F\delta) \circ \mathfrak{n}_F \\
&= \mathsf{M}\xi_F \circ \mathsf{M}\delta \circ \mathsf{M}\Lambda \circ \mathfrak{n}_F, && \text{by (1.6)} \\
&= \mathsf{M}\Lambda \circ \mathfrak{n}_F.
\end{aligned}$$

Therefore $\upsilon_F \circ \mathsf{M}\delta = \mathfrak{n} \circ \upsilon$ and υ is a morphism of $\mathbf{F}$-bicomodules. Hence $\mathscr{S}$ is a Maschke functor.
$(ii) \Rightarrow (i)$. Given an $\mathbf{F}$-bicomodule $(\mathsf{M},\mathfrak{m},\mathfrak{n})$, we denote by

$$\Gamma_{(\mathsf{M},\mathfrak{m},\mathfrak{n})} : \mathscr{T}\mathscr{S}(\mathsf{M},\mathfrak{m},\mathfrak{n}) = (\mathsf{M}F,\mathfrak{m}_F,\mathsf{M}\delta) \longrightarrow (\mathsf{M},\mathfrak{m},\mathfrak{n})$$

the splitting morphism of $\Theta_{(\mathsf{M},\mathfrak{m},\mathfrak{n})}$ in the category of $\mathbf{F}$-bicomodules. Here Θ_- is the unit of the adjunction $\mathscr{S} \dashv \mathscr{T}$. Since (F,δ,δ) is $\mathbf{F}$-bicomodule, we put $\gamma := \Gamma_{(F,\delta,\delta)}$, thus $\gamma \circ \delta = F$. For any object (X,d^X) of the category $\mathcal{A}^{\mathbf{F}}$, we consider the composition

$$\phi_{(X,d^X)} : F(X) \xrightarrow{F(d^X)} F^2(X) \xrightarrow{\gamma_X} F(X) \xrightarrow{\xi_X} X\ .$$

We claim that ϕ_- is a natural transformation which satisfies $\phi_- \circ \eta_- = \mathbb{1}_{\mathcal{A}^{\mathbf{F}}}$, where η_- is the unit of the adjunction $\mathbf{S} \dashv \mathbf{T}$ given in (1.7). First of all, we have

$$\begin{aligned}
\phi_{(X,d^X)} \circ \eta_{(X,d^X)} &= \xi_X \circ \gamma_X \circ F(d^X) \circ d^X \\
&= \xi_X \circ \gamma_X \circ \delta_X \circ d^X \\
&= \xi_X \circ d^X \;=\; (X,d^X),
\end{aligned}$$

for every object (X,d^X) of $\mathcal{A}^{\mathbf{F}}$. To see that $\phi_{(X,d^X)}$ is a morphism in $\mathcal{A}^{\mathbf{F}}$, we can compute on one hand

$$\begin{aligned}
d^X \circ \phi_{(X,d^X)} &= d^X \circ \xi_X \circ \gamma_X \circ F(d^X) \\
&= \xi_{F(X)} \circ F(d^X) \circ \gamma_X \circ F(d^X), && \xi_- \text{ is natural} \\
&= \xi_{F(X)} \circ \gamma_{F(X)} \circ F^2(d^X) \circ F(d^X), && \gamma_- \text{ is natural} \\
&= \xi_{F(X)} \circ \gamma_{F(X)} \circ F(\delta_X) \circ F(d^X) \\
&= \xi_{F(X)} \circ \delta_X \circ \gamma_X \circ F(d^X), && \text{by (1.6) applied to } \gamma \\
&= \gamma_X \circ F(d^X)
\end{aligned}$$

and secondly,

$$\begin{aligned}
F\phi_{(X,d^X)} \circ \delta_X &= F\xi_X \circ F\gamma_X \circ F^2(d^X) \circ \delta_X \\
&= F\xi_X \circ F\gamma_X \circ \delta_{F(X)} \circ F(d^X), && \delta_- \text{ is natural} \\
&= F\xi_X \circ \delta_X \circ \gamma_X \circ F(d^X), && \text{by (1.6) applied to } \gamma \\
&= \gamma_X \circ F(d^X).
\end{aligned}$$

Therefore, $F\phi_{(X,d^X)} \circ \delta_X = d^X \circ \phi_{(X,d^X)}$. Lastly, if we consider a morphism $f : (X,d^X) \to (Y,d^Y)$ in $\mathcal{A}^{\mathbf{F}}$, then

$$\begin{aligned}
f \circ \phi_{(X,d^X)} &= f \circ \xi_X \circ \gamma_X \circ F(d^X) \\
&= \xi_Y \circ F(f) \circ \gamma_X \circ F(d^X), && \xi_- \text{ is natural} \\
&= \xi_Y \circ \gamma_Y \circ F^2(f) \circ F(d^X), && \gamma_- \text{ is natural} \\
&= \xi_Y \circ \gamma_Y \circ F(d^Y) \circ F(f) \\
&= \phi_{(Y,d^Y)} \circ F(f),
\end{aligned}$$

which shows that ϕ_- is a natural transformation. □

In view of Remark 1.2, condition (iii) in Theorem 1.6 means that $\mathbf{F}$ is coseparable as a coalgebra in the monoidal category $\overline{\mathrm{Funct}}(\mathcal{A},\mathcal{A})$ (see [1, 2]).

2. Coderivations and Cointegrations

Let $\mathbf{F} = (F,\delta,\xi)$ be a comonad on $\mathcal{A}$ with underlying functor $F \in \overline{\mathrm{Funct}}(\mathcal{A},\mathcal{A})$. Consider a bicomodule $(\mathsf{M},\mathfrak{m},\mathfrak{n}) \in {}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$. A *coderivation* from M to F is a natural transformation $\mathsf{g} : \mathsf{M} \longrightarrow F$ such that

$$\delta \circ \mathsf{g} = F\mathsf{g} \circ \mathfrak{m} + \mathsf{g}_F \circ \mathfrak{n}. \tag{2.1}$$

The set of all coderivations from $(\mathsf{M},\mathfrak{m},\mathfrak{n})$ is an additive group which we denote by $\mathrm{Coder}(\mathsf{M},F)$. Notice that every coderivation $\mathsf{g} \in \mathrm{Coder}(\mathsf{M},F)$ satisfies the equality $\xi \circ \mathsf{g} = 0$. A coderivation $\mathsf{g} \in \mathrm{Coder}(\mathsf{M},F)$ is said to be *inner* if there exists a natural transformation $\lambda : \mathsf{M} \to \mathbb{1}_{\mathcal{A}}$ such that

$$\mathsf{g} = \lambda_F \circ \mathfrak{n} - F\lambda \circ \mathfrak{m}. \tag{2.2}$$

The sub-group of all inner coderivations will be denoted by $\mathrm{InCoder}(\mathsf{M},F)$.

Let $(\mathsf{M},\mathfrak{m},\mathfrak{n})$ and $(\mathsf{M}',\mathfrak{m}',\mathfrak{n}')$ be two $\mathbf{F}$-bicomodules. A *left cointegration* from $(\mathsf{M},\mathfrak{m},\mathfrak{n})$ into $(\mathsf{M}',\mathfrak{m}',\mathfrak{n}')$ is a natural transformation $\mathsf{h} : \mathsf{M} \to \mathsf{M}'F$ which satisfies

$$\mathfrak{m}'_F \circ \mathsf{h} = F\mathsf{h} \circ \mathfrak{m}, \quad \mathsf{M}'\delta \circ \mathsf{h} = \mathfrak{n}'_F \circ \mathsf{h} + \mathsf{h}_F \circ \mathfrak{n}. \tag{2.3}$$

The first equality means that $\mathsf{h} : \mathscr{S}(\mathsf{M},\mathfrak{m},\mathfrak{n}) = (\mathsf{M},\mathfrak{m}) \to \mathscr{S}\mathscr{T}\mathscr{S}(\mathsf{M},\mathfrak{m},\mathfrak{n}) = (\mathsf{M}'F,\mathfrak{m}'_F)$ is a morphism in the category ${}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$. Right cointegrations are defined in a similar way. Since we are only concerned with the left ones, we will not mention the word “left” before cointegration. The additive group of all cointegrations from $(\mathsf{M},\mathfrak{m},\mathfrak{n})$ into $(\mathsf{M}',\mathfrak{m}',\mathfrak{n}')$ will be denoted by $\mathrm{Coint}(\mathsf{M},\mathsf{M}')$. A cointegration $\mathsf{h} \in \mathrm{Coint}(\mathsf{M},\mathsf{M}')$ is said to be *inner* if there exists a natural transformation $\varphi : \mathsf{M} \to \mathsf{M}'$ which satisfies

$$\mathfrak{m}' \circ \varphi = F\varphi \circ \mathfrak{m}, \quad \mathsf{h} = \varphi_F \circ \mathfrak{n} - \mathfrak{n}' \circ \varphi. \tag{2.4}$$

The first equality means that $\varphi : (\mathsf{M},\mathfrak{m}) \to (\mathsf{M}',\mathfrak{m}')$ is a morphism in the category ${}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$. The sub-group of all inner cointegrations will be denoted by $\mathrm{InCoint}(\mathsf{M},\mathsf{M}')$. The following Proposition was first stated for bimodule over ring extension in [17] and for bicomodules over corings in [14]. For the sake of completeness, we give the proof.

Proposition 2.1 *For any* $\mathbf{F}$*-bicomodule* $(\mathsf{M},\mathfrak{m},\mathfrak{n})$*, there is a natural isomorphism of additive groups*

$$\begin{array}{ccc} \mathrm{Coint}(\mathsf{M},F) & \xrightarrow{\ \sim\ } & \mathrm{Coder}(\mathsf{M},F) \\ \mathsf{h} & \longmapsto & \xi_F \circ \mathsf{h} \\ F\mathsf{g} \circ \mathfrak{m} & \longleftarrow & \mathsf{g} \end{array}$$

whose restriction to the inner sub-groups gives again an isomorphism

$$\mathrm{InCoint}(\mathsf{M},F) \cong \mathrm{InCoder}(\mathsf{M},F).$$

Proof: We only show that the mutually inverse maps are well defined. Let $\mathsf{h} \in \mathrm{Coint}(\mathsf{M}, F)$, and put $\mathsf{g} := \xi_F \circ \mathsf{h}$. We have

$$\begin{aligned} \delta \circ g &= \delta \circ \xi_F \circ \mathsf{h} \\ &= \xi_{F^2} \circ F\delta \circ \mathsf{h}, \qquad \delta_- \text{ is natural} \\ &= \xi_{F^2} \circ \Big(\delta_F \circ \mathsf{h} + \mathsf{h}_F \circ \mathfrak{n}\Big) \\ &= (\xi_F \circ \delta)_F \circ \mathsf{h} + \xi_{F^2} \circ \mathsf{h}_F \circ \mathfrak{n} \\ &= \mathsf{h} + \xi_{F^2} \circ \mathsf{h}_F \circ \mathfrak{n} \end{aligned}$$

and

$$F\xi_F \circ F\mathsf{h} \circ \mathfrak{m} + \xi_{F^2} \circ \mathsf{h}_F \circ \mathfrak{n} = F\xi_F \circ \delta_F \circ \mathsf{h} + \xi_{F^2} \circ \mathsf{h}_F \circ \mathfrak{n} = \mathsf{h} + \xi_{F^2} \circ \mathsf{h}_F \circ \mathfrak{n}.$$

That is $\mathsf{g} \in \mathrm{Coder}(\mathsf{M}, F)$. Conversely, given $\mathsf{g} \in \mathrm{Coder}(\mathsf{M}, F)$, we put $\mathsf{h} = F\mathsf{g} \circ \mathfrak{m}$. We find

$$\begin{aligned} \delta_F \circ \mathsf{h} &= \delta_F \circ F\mathsf{g} \circ \mathfrak{m} \\ &= F^2\mathsf{g} \circ \delta_{\mathsf{M}} \circ \mathfrak{m}, \qquad \delta_- \text{ is natural} \\ &= F^2\mathsf{g} \circ F\mathfrak{m} \circ \mathfrak{m}, \qquad \text{by (1.4)} \\ &= F\mathsf{h} \circ \mathfrak{m}, \end{aligned}$$

which shows the first equality of equation (2.3). Now,

$$\begin{aligned} F\delta \circ \mathsf{h} &= F\delta \circ F\mathsf{g} \circ \mathfrak{m} \\ &= F(\delta \circ \mathsf{g}) \circ \mathfrak{m} \\ &= F\Big(F\mathsf{g} \circ \mathfrak{m} + \mathsf{g}_F \circ \mathfrak{n}\Big) \circ \mathfrak{m} \\ &= F^2\mathsf{g} \circ F\mathfrak{m} \circ \mathfrak{m} + F\mathsf{g}_F \circ F\mathfrak{n} \circ \mathfrak{m}, \qquad \text{by (1.4) and (1.5)} \\ &= F^2\mathsf{g} \circ \delta_{\mathsf{M}} \circ \mathfrak{m} + F\mathsf{g}_F \circ \mathfrak{m}_F \circ \mathfrak{n} \\ &= \delta_F \circ F\mathsf{g} \circ \mathfrak{m} + \Big(F\mathsf{g} \circ \mathfrak{m}\Big)_F \circ \mathfrak{n}, \qquad \delta_- \text{ is natural} \\ &= \delta_F \circ \mathsf{h} + \mathsf{h}_F \circ \mathfrak{n} \end{aligned}$$

which proves that $\mathsf{h} = F\mathsf{g} \circ \mathfrak{m} \in \mathrm{Coint}(\mathsf{M}, F)$. □

Following [14], we will give in the next step the notion of universal cointegration and that of universal coderivation.

Given $(\mathsf{M}, \mathfrak{m}, \mathfrak{n})$ any $\mathbf{F}$-bicomodule, consider the $\mathbf{F}$-bicomodule $(\mathsf{M}F, \mathfrak{m}_F, \mathsf{M}\delta)$, which is the image of $(\mathsf{M}, \mathfrak{m}, \mathfrak{n})$ under the functor $\mathscr{T}\mathscr{S}$. We call it the bicomodule induced by M. Since $\mathfrak{n} : (\mathsf{M},\mathfrak{m},\mathfrak{n}) \to (\mathsf{M}F,\mathfrak{m}_F,\mathsf{M}\delta)$ is a morphism of **F**-bicomodules, we obtain by Lemma 1.3 the following sequence of **F**-bicomodules

$$0 \longrightarrow (\mathsf{M},\mathfrak{m},\mathfrak{n}) \xrightarrow{\mathfrak{n}} (\mathsf{M}F,\mathfrak{m}_F,\mathsf{M}\delta) \xrightarrow{\mathfrak{n}^c} (\mathscr{K}(\mathsf{M}),\mathfrak{u},\mathfrak{v}) \longrightarrow 0\,, \qquad (2.5)$$

where $(\mathscr{K}(\mathsf{M}),\mathfrak{u},\mathfrak{v})$ and $\mathfrak{n}^c$ denote the cokernel of $\mathfrak{n}$ in the category ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$. Notice, that this is still a cokernel in the category ${}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$, after forgetting by $\mathscr{S}$. Consider the natural transformation

$$\mathsf{w}' := \mathsf{M}F - \mathfrak{n} \circ \mathsf{M}\xi : \mathsf{M}F \longrightarrow \mathsf{M}F.$$

It is easily checked that $\mathfrak{m}_F \circ \mathsf{w}' = F\mathsf{w}' \circ \mathfrak{m}_F$, thus w' is a morphism in the category ${}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$. Also, w' satisfies $\mathsf{w}' \circ \mathfrak{n} = 0$. So, by the universal property of cokernels, there exists a morphism in the category ${}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$, $\mathsf{w} : (\mathscr{K}(\mathsf{M}),\mathfrak{v}) \to (\mathsf{M}F,\mathfrak{m}_F)$ which makes the following diagram commutative

$$\begin{array}{ccccc} \mathsf{M} & \xrightarrow{\quad\mathfrak{n}\quad} & \mathsf{M}F & \xrightarrow{\quad\mathfrak{n}^c\quad} & \mathscr{K}(\mathsf{M}) \\ & & {\scriptstyle \mathsf{w}'}\downarrow & \swarrow{\scriptstyle \mathsf{w}} & \\ & & \mathsf{M}F & & \end{array} \qquad (2.6)$$

Thus $\mathsf{w} \circ \mathfrak{n}^c = \mathsf{w}'$, and so $\mathfrak{n}^c \circ \mathsf{w} \circ \mathfrak{n}^c = \mathfrak{n}^c$. Hence $\mathfrak{n}^c \circ \mathsf{w} = \mathscr{K}(\mathsf{M})$, since $\mathfrak{n}^c$ is an epimorphism.

A *universal cointegration into* M is a cointegration u from $\mathscr{K}(\mathsf{M})$ into M such that every cointegration into M factors through u. That is, u satisfies the following universal property: for every **F**-bicomodule $(\mathsf{M}',\mathfrak{m}',\mathfrak{n}')$ and every cointegration $\mathsf{h} \in \mathrm{Coint}(\mathsf{M}',\mathsf{M})$, there exists a morphism of **F**-bicomodules $\mathsf{f} : (\mathsf{M}',\mathfrak{m}',\mathfrak{n}') \to (\mathscr{K}(\mathsf{M}),\mathfrak{u},\mathfrak{v})$ such that $\mathsf{h} = \mathsf{u} \circ \mathsf{f}$.

Proposition 2.2 *The morphism* w *constructed in diagram* (2.6) *is a universal cointegration into* M. *Moreover, the following conditions are equivalent*

(i) The sequence

$$0 \longrightarrow (\mathsf{M},\mathfrak{m},\mathfrak{n}) \xrightarrow{\mathfrak{n}} (\mathsf{M}F,\mathfrak{m}_F,\mathsf{M}\delta) \xrightarrow{\mathfrak{n}^c} (\mathscr{K}(\mathsf{M}),\mathfrak{u},\mathfrak{v}) \longrightarrow 0$$

splits in the category of bicomodules ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$*;*

(ii) The universal cointegration $\mathsf{w} : \mathscr{K}(\mathsf{M}) \to \mathsf{M}F$ *is inner.*

Proof: For the first statement, it is enough to show that w' is cointegration into M, since $\mathfrak{n}^c$ is an epimorphism. By construction w' satisfies the first equality in (2.3). The second equality in (2.3), is obtained as follows

$$\mathsf{M}\delta \circ \mathsf{w}' = \mathsf{M}\delta - \mathsf{M}\delta \circ \mathfrak{n} \circ \mathsf{M}\xi = \mathsf{M}\delta - \mathfrak{n}_F \circ \mathfrak{n} \circ \mathsf{M}\xi$$

and

$$\begin{aligned} \mathfrak{n}_F \circ \mathsf{w}' + \mathsf{w}'_F \circ \mathsf{M}\delta &= \mathfrak{n}_F - \mathfrak{n}_F \circ \mathfrak{n} \circ \mathsf{M}\xi + \mathsf{M}\delta - \mathfrak{n}_F \circ \mathsf{M}\xi_F \circ \mathsf{M}\delta \\ &= \mathsf{M}\delta - \mathfrak{n}_F \circ \mathfrak{n} \circ \mathsf{M}\xi \\ &= \mathsf{M}\delta \circ \mathsf{w}'. \end{aligned}$$

The fact that w is universal follows from the following isomorphism of additive groups

$$\begin{array}{ccc} \mathrm{Hom}_{{}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}}\Big(\mathsf{M}', \mathscr{K}(\mathsf{M})\Big) & \xrightarrow{\ \sim\ } & \mathrm{Coint}(\mathsf{M}',\mathsf{M}) \\ \varphi & \longmapsto & \mathsf{w} \circ \varphi \\ \mathfrak{n}^c \circ \mathsf{h} & \longleftarrow\!\shortmid & \mathsf{h} \end{array} \tag{2.7}$$

whose proof is an easy computation. Now we check the stated equivalence.
$(i) \Rightarrow (ii)$. Let us denote by $\lambda : (\mathscr{K}(\mathsf{M}), \mathfrak{u}, \mathfrak{v}) \to (\mathsf{M}F, \mathfrak{m}_F, \mathsf{M}\delta)$ the right inverse of $\mathfrak{n}^c$ in the category ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$, i.e. $\mathfrak{n}^c \circ \lambda = \mathscr{K}(\mathsf{M})$. Define the composition

$$\varphi : \mathscr{K}(\mathsf{M}) \xrightarrow{\ \lambda\ } \mathsf{M}F \xrightarrow{\ \mathsf{M}\xi\ } \mathsf{M}\,.$$

Then we have

$$\mathfrak{m} \circ \varphi = \mathfrak{m} \circ \mathsf{M}\xi \circ \lambda = F\mathsf{M}\xi \circ \mathfrak{m}_F \circ \lambda = F\mathsf{M}\xi \circ F\lambda \circ \mathfrak{u} = F\Big(\mathsf{M}\xi \circ \lambda\Big) \circ \mathfrak{u} = F\varphi \circ \mathfrak{u},$$

which entails that φ is a morphism in ${}_{A}\mathscr{M}^{\mathbf{F}}$. The cointegration w is inner by φ. Namely,

$$\begin{aligned} \varphi_F \circ \mathfrak{v} - \mathfrak{n} \circ \varphi &= \mathsf{M}\xi_F \circ \lambda_F \circ \mathfrak{v} - \mathfrak{n} \circ \mathsf{M}\xi \circ \lambda \\ &= \mathsf{M}\xi_F \circ \mathsf{M}\delta \circ \lambda - \mathfrak{n} \circ \mathsf{M}\xi \circ \lambda \\ &= \lambda - \mathfrak{n} \circ \mathsf{M}\xi \circ \lambda \\ &= \Big(\mathsf{M}F - \mathfrak{n} \circ \mathsf{M}\xi\Big) \circ \lambda \\ &= \mathsf{w} \circ \mathfrak{n}^c \circ \lambda = \mathsf{w}. \end{aligned}$$

$(ii) \Rightarrow (i)$. Suppose that there exists a morphism $\beta : \mathscr{K}(\mathsf{M}) \to \mathsf{M}$ in ${}_{A}\mathscr{M}^{\mathbf{F}}$ such that $\mathsf{w} = \beta_F \circ \mathfrak{v} - \mathfrak{n} \circ \beta$. Consider the natural transformation

$$\Gamma : \mathscr{K}(\mathsf{M}) \xrightarrow{\ \mathfrak{v}\ } \mathscr{K}(\mathsf{M})F \xrightarrow{\ \beta_F\ } \mathsf{M}F\,.$$

Then we find $\mathfrak{n}^c \circ \Gamma = \mathfrak{n}^c \circ \beta_F \circ \mathfrak{v} = \mathfrak{n}^c \circ \mathsf{w} + \mathfrak{n}^c \circ \mathfrak{n} \circ \beta = \mathfrak{n}^c \circ \mathsf{w} = \mathscr{K}(\mathsf{M})$. Furthermore, Γ is a morphism in the category of bicomodules ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$, as the following commutative diagrams show

$$\begin{array}{ccccc} \mathscr{K}(\mathsf{M}) & \xrightarrow{\mathfrak{v}} & \mathscr{K}(\mathsf{M})F & \xrightarrow{\beta_F} & \mathsf{M}F \\ {\scriptstyle \mathfrak{u}}\downarrow & & {\scriptstyle \mathfrak{u}_F}\downarrow & & \downarrow{\scriptstyle \mathfrak{m}_F} \\ F\mathscr{K}(F) & \xrightarrow[F\mathfrak{v}]{} & F\mathscr{K}(\mathsf{M})F & \xrightarrow[F\beta_F]{} & F\mathsf{M}F \end{array} \qquad \begin{array}{ccccc} \mathscr{K}(\mathsf{M}) & \xrightarrow{\mathfrak{v}} & \mathscr{K}(\mathsf{M})F & \xrightarrow{\beta_F} & \mathscr{K}(\mathsf{M})F \\ {\scriptstyle \mathfrak{v}}\downarrow & & {\scriptstyle \mathscr{K}(\mathsf{M})\delta}\downarrow & & \downarrow{\scriptstyle \mathsf{M}\delta} \\ \mathscr{K}(\mathsf{M})F & \xrightarrow[\mathfrak{v}_F]{} & \mathscr{K}(\mathsf{M})F^2 & \xrightarrow[\beta_{F^2}]{} & \mathsf{M}F^2. \end{array}$$

Therefore the listed sequence splits in the category ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$. □

From now on w denotes the universal cointegration into the **F**-bicomodule (F,δ,δ). That is $\mathsf{w} : \mathscr{K}(F) \to F^2$ with properties $\mathsf{w} \circ \delta^c = F^2 - \delta \circ F\xi$ and $\delta^c \circ \mathsf{w} = \mathscr{K}(F)$, where

$$0 \longrightarrow (F,\delta,\delta) \xrightarrow{\delta} (F^2,\delta_F,F\delta) \xrightarrow{\delta^c} (\mathscr{K}(F),\mathfrak{u},\mathfrak{v}) \longrightarrow 0$$

is the canonical sequence. Consider the natural transformation $\mathsf{d} : \mathscr{K}(F) \to F$ defined by $\mathsf{d} := F\xi \circ \mathsf{w} - \xi_F \circ \mathsf{w}$.

Lemma 2.3 *The morphism* d *is a coderivation with the following universal property. For every* **F***-bicomodule* $(\mathsf{M},\mathfrak{m},\mathfrak{n})$ *and every coderivation* $\mathsf{g} \in \mathrm{Coder}(\mathsf{M},F)$*, there exists a natural transformation* $g' : \mathsf{M} \to \mathscr{K}(F)$ *such that* $\mathsf{d} \circ g' = \mathsf{g}$.

Proof: First, observe that

$$\mathsf{d} \circ \delta^c = F\xi - \xi_F$$

as $\mathsf{w} \circ \delta^c = F^2 - \delta \circ F\xi$. Now, since δ^c is an epimorphism, in order to get that d is a coderivation, it is enough to check that $\mathsf{e} := F\xi - \xi_F$ is a coderivation and in fact we have

$$\begin{aligned} F\mathsf{e} \circ \delta_F + \mathsf{e}_F \circ F\delta &= F^2\xi \circ \delta_F - (F\xi \circ \delta)_F + F(\xi_F \circ \delta) - \xi_{F^2} \circ F\delta \\ &= F^2\xi \circ \delta_F - \xi_{F^2} \circ F\delta \\ &= \delta \circ F\xi - \delta \circ \xi_F = \delta \circ \mathsf{e}. \end{aligned}$$

Take $\mathsf{g} \in \mathrm{Coder}(\mathsf{M},F)$. By Proposition 2.1, $F\mathsf{g} \circ \mathfrak{m} \in \mathrm{Coint}(\mathsf{M},F)$ so we can apply Proposition 2.2 to obtain a morphism of **F**-bicomodules $f : \mathsf{M} \to \mathscr{K}(F)$ such that

$F\mathsf{g} \circ \mathfrak{m} = \mathsf{w} \circ f$. We have

$$\begin{aligned} \mathsf{d} \circ f &= \mathsf{e} \circ \mathsf{w} \circ f \\ &= \mathsf{e} \circ F\mathsf{g} \circ \mathfrak{m} \\ &= F\xi \circ F\mathsf{g} \circ \mathfrak{m} - \xi_F \circ F\mathsf{g} \circ \mathfrak{m} \\ &= F(\xi \circ \mathsf{g}) \circ \mathfrak{m} - \xi_F \circ (\delta \circ \mathsf{g} - \mathsf{g}_F \circ \mathfrak{n}) \\ &= F(\xi \circ \mathsf{g}) \circ \mathfrak{m} - \xi_F \circ \delta \circ \mathsf{g} - \xi_F \circ \mathsf{g}_F \circ \mathfrak{n} \\ &= F(\xi \circ \mathsf{g}) \circ \mathfrak{m} - \mathsf{g} - (\xi \circ \mathsf{g})_F \circ \mathfrak{n}. \end{aligned}$$

Once observed that $\xi \circ \mathsf{g} = 0$ (since $\mathsf{g} \in \mathrm{Coder}(M, F)$), it is clear that we can set $g' = -f$. □

Corollary 2.4 *Let* $\mathbf{F} = (F, \delta, \xi)$ *be a comonad on a Grothendieck category* $\mathcal{A}$ *such that* F *is right exact and commutes with direct sums. Consider the universal cointegration* w *and the universal coderivation* d *associated to the* $\mathbf{F}$*-bicomodule* (F, δ, δ)*. The following conditions are equivalent*

(i) The sequence

$$0 \longrightarrow F \xrightarrow{\delta} F^2 \xrightarrow{\delta^c} \mathscr{K}(F) \longrightarrow 0$$

is a split sequence in the category of bicomodules ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$*;*

(ii) the universal cointegration w *is inner;*

(iii) the universal coderivation d *is inner.*

Proof: The equivalence $(i) \Leftrightarrow (ii)$ is consequence of Proposition 2.2. Let us check the equivalence between (ii) and (iii).
$(ii) \Rightarrow (iii)$. We know there exists a morphism $\varphi : \mathscr{K}(F) \to F$ in ${}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$ such that $\mathsf{w} = \varphi_F \circ \mathfrak{v} - \delta \circ \varphi$. We have

$$\begin{aligned} \xi_F \circ \mathsf{w} &= \xi_F \circ \varphi_F \circ \mathfrak{v} - \varphi = \Big(\xi \circ \varphi\Big)_F \circ \mathfrak{v} - \varphi, \\ F\xi \circ \mathsf{w} &= F\xi \circ \varphi_F \circ \mathfrak{v} - \varphi. \end{aligned}$$

Hence

$$\begin{aligned} \mathsf{d} &= F\xi \circ \varphi_F \circ \mathfrak{v} - \varphi - \Big(\xi \circ \varphi\Big)_F \circ \mathfrak{v} + \varphi \\ &= \varphi \circ \mathscr{K}(F)\xi \circ \mathfrak{v} - \Big(\xi \circ \varphi\Big)_F \circ \mathfrak{v} \\ &= \varphi - \Big(\xi \circ \varphi\Big)_F \circ \mathfrak{v}. \end{aligned}$$

But $F\left(\xi \circ \varphi\right) \circ \mathfrak{u} = \varphi$, as φ is a morphism in ${}_{A}\mathscr{M}^{\mathbf{F}}$, which proves that d is inner by $-\xi \circ \varphi$.
$(iii) \Rightarrow (ii)$. Let us denote by $\lambda : \mathscr{K}(F) \to \mathbb{1}_{\mathcal{A}}$ the natural transformation which satisfies $\mathsf{d} = \lambda_F \circ \mathfrak{v} - F\lambda \circ \mathfrak{u}$. We define the map ψ as the following composition $\psi = F\lambda \circ \mathfrak{u} : \mathscr{K}(F) \to F\mathscr{K}(F) \to F$. This ψ satisfies

$$\delta \circ \psi = \delta \circ F\lambda \circ \mathfrak{u} = F^2\lambda \circ \delta_{\mathscr{K}(F)} \circ \mathfrak{u} = F^2\lambda \circ F\mathfrak{u} \circ \mathfrak{u} = F\psi \circ \mathfrak{u},$$

that is, ψ is a morphism in ${}_{A}\mathscr{M}^{\mathbf{F}}$. The universal cointegration is inner by $-\psi$, as the following computations show

$$\begin{aligned}
\psi_F \circ \mathfrak{v} - \delta \circ \psi &= \left(F\lambda \circ \mathfrak{u}\right)_F \circ \mathfrak{v} - \delta \circ F\lambda \circ \mathfrak{u} \\
&= F\lambda_F \circ \mathfrak{u}_F \circ \mathfrak{v} - \delta \circ F\lambda \circ \mathfrak{u} \\
&= F\lambda_F \circ F\mathfrak{v} \circ \mathfrak{u} - \delta \circ F\lambda \circ \mathfrak{u} \\
&= F\left(\lambda_F \circ \mathfrak{v}\right) \circ \mathfrak{u} - \delta \circ F\lambda \circ \mathfrak{u} \\
&= F\left(\left(F\xi - \xi_F\right) \circ \mathsf{w} + F\lambda \circ \mathfrak{u}\right) \circ \mathfrak{u} - \delta \circ F\lambda \circ \mathfrak{u} \\
&= F^2\xi \circ F\mathsf{w} \circ \mathfrak{u} - F\xi_F \circ F\mathsf{w} \circ \mathfrak{u} + F^2\lambda \circ F\mathfrak{u} \circ \mathfrak{u} - \delta \circ F\lambda \circ \mathfrak{u} \\
&= F^2\xi \circ \delta_F \circ \mathsf{w} - F\xi_F \circ \delta_F \circ \mathsf{w} + \delta \circ F\lambda \circ \mathfrak{u} - \delta \circ F\lambda \circ \mathfrak{u} \\
&= F^2\xi \circ \delta_F \circ \mathsf{w} - \left(F\xi \circ \delta\right)_F \circ \mathsf{w} \\
&= \delta \circ F\xi \circ \mathsf{w} - \mathsf{w} = \left(\delta \circ F\xi - F^2\right) \circ \mathsf{w} = -\mathsf{w} \circ \delta^c \circ \mathsf{w} = -\mathsf{w}.
\end{aligned}$$

□

3. Cohomology For Bicomodules

The following lemma, which will be used in the sequel, was in part proven in [6, Theorem 3.4]. For sake of completeness we will give a detailed proof.

Lemma 3.1 *Let $\mathscr{A}$ and $\mathscr{B}$ be two preadditive categories with cokernels, and $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ a covariant functor with right adjoint functor $\mathscr{G} : \mathscr{B} \to \mathscr{A}$. Denote by χ and θ respectively, the counit and unit of this adjunction. Let $\mathscr{E}_0$ be the injective class of all cosplit sequences in $\mathscr{B}$, and put $\mathscr{E} = \mathscr{F}^{-1}(\mathscr{E}_0)$. For every object $M \in \mathscr{A}$, the following conditions are equivalent*

(i) M is $\mathscr{F}$-injective;

(ii) M is $\mathscr{E}$-injective;

(iii) the unit at M, $\theta_M : M \to \mathscr{G}\mathscr{F}(M)$, is a split-mono in $\mathscr{A}$.

In particular every object of the form $\mathscr{G}(N)$ is $\mathscr{E}$-injective, for every object $N \in \mathscr{B}$. Moreover the functor $\mathscr{F}$ is Maschke if and only if the class of $\mathscr{E}$-injective objects coincides with the class of all objects of $\mathscr{A}$.

Proof: $(i) \Rightarrow (iii)$. We know by adjunction properties that $\chi_{\mathscr{F}(M)} \circ \mathscr{F}(\theta_M) = \mathscr{F}(M)$. Since M is $\mathscr{F}$-injective, θ_M has a left inverse.
$(iii) \Rightarrow (ii)$. Let us denote by $\gamma : \mathscr{G}\mathscr{F}(M) \to M$ the left inverse of θ_M. For any sequence

$$E : X \xrightarrow{\ i\ } X' \xrightarrow{\ j\ } X''$$

in $\mathscr{E}$, we need to prove that its corresponding sequence of abelian groups

$$\mathrm{Hom}_{\mathscr{A}}(X'', M) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(X', M) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(X, M)$$

is exact (in the usual sense). Given such E in $\mathscr{E}$, we have a commutative diagram in $\mathscr{B}$

$$\begin{array}{ccccc} \mathscr{F}(X) & \xrightarrow{\mathscr{F}(i)} & \mathscr{F}(X') & \xrightarrow{\mathscr{F}(j)} & \mathscr{F}(X'') \\ & & {\scriptstyle \mathscr{F}(i)^c}\big\downarrow & \nearrow_{l} & \\ & & \mathrm{Coker}(\mathscr{F}(i)) & & \end{array}$$

where l is split as monomorphism by l'. Let $\tau : X' \to M$ be a morphism in $\mathscr{A}$, such that $\tau \circ i = 0$. Then there exists a morphism $g : \mathrm{Coker}(\mathscr{F}(i)) \to \mathscr{F}(M)$ of $\mathscr{B}$ such that $g \circ \mathscr{F}(i)^c = \mathscr{F}(\tau)$. This leads to the composition

$$\begin{array}{ccc} X'' & \overset{\alpha}{\dashrightarrow} & M \\ {\scriptstyle \theta_{X''}}\big\downarrow & & \big\uparrow{\scriptstyle \gamma} \\ \mathscr{G}\mathscr{F}(X'') & \xrightarrow{\mathscr{G}(g \circ l')} & \mathscr{G}\mathscr{F}(M) \end{array}$$

The morphism α satisfies

$$\begin{aligned} \alpha \circ j &= \gamma \circ \mathscr{G}(g \circ l') \circ \theta_{X''} \circ j \\ &= \gamma \circ \mathscr{G}(g \circ l') \circ \mathscr{G}\mathscr{F}(j) \circ \theta_{X'} \\ &= \gamma \circ \mathscr{G}\Big(g \circ l' \circ \mathscr{F}(j)\Big) \circ \theta_{X'} \\ &= \gamma \circ \mathscr{G}\Big(g \circ l' \circ l \circ \mathscr{F}(i)^c\Big) \circ \theta_{X'} \\ &= \gamma \circ \mathscr{G}\Big(g \circ \mathscr{F}(i)^c\Big) \circ \theta_{X'} \\ &= \gamma \circ \mathscr{G}\mathscr{F}(\tau) \circ \theta_{X'} \\ &= \gamma \circ \theta_M \circ \tau = \tau \end{aligned}$$

which proves the exactness of the sequence of abelian groups.
$(ii) \Rightarrow (i)$ Let $i : X \to X'$ be a morphism of $\mathscr{A}$ such that $\mathscr{F}(i)$ has a left inverse. The later condition means that $0 \longrightarrow \mathscr{F}(X) \xrightarrow{\mathscr{F}(i)} \mathscr{F}(X')$ is a cosplit sequence in $\mathscr{B}$. Thus $0 \longrightarrow X \xrightarrow{i} X'$ is a sequence in $\mathscr{E}$. Therefore, the corresponding sequence of abelian groups

$$\mathrm{Hom}_{\mathscr{A}}\Big(X', M\Big) \longrightarrow \mathrm{Hom}_{\mathscr{A}}\Big(X, M\Big) \longrightarrow 0$$

is exact. Whence $\mathrm{Hom}_{\mathscr{A}}\Big(i, M\Big)$ is surjective and so M is $\mathscr{F}$-injective. □

Consider in the category of bicomodules ${}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$ the class $\mathscr{E}_0$ of all co-split sequences. This is an injective class, as ${}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$ is an additive category with cokernels. As we have mentioned, the corresponding class of $\mathscr{E}_0$-injective objects coincides with the class of all objects of ${}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$. Denote by $\mathscr{E} := \mathscr{S}^{-1}\Big(\mathscr{E}_0\Big)$ the class of sequences E in the category ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$ such that $\mathscr{S}(E)$ is a sequence in $\mathscr{E}_0$, as we have pointed out $\mathscr{E}$ is also an injective class.

Proposition 3.2 *Let* $(\mathsf{M}, \mathfrak{m}, \mathfrak{n})$ *be an* $\mathbf{F}$*-bicomodule. The following statements are equivalent*

(i) $(\mathsf{M}, \mathfrak{m}, \mathfrak{n})$ *is* $\mathscr{E}$*-injective;*

(ii) $(\mathsf{M}, \mathfrak{m}, \mathfrak{n})$ *is* $\mathscr{S}$*-injective;*

(iii) the unit $\Theta_{(\mathsf{M}, \mathfrak{m}, \mathfrak{n})}$ *of the adjunction* $\mathscr{S} \dashv \mathscr{T}$ *at* $(\mathsf{M}, \mathfrak{m}, \mathfrak{n})$*, stated in* (1.8)*, is a split monomorphism.*

In particular every bicomodule of the form $\mathscr{T}(\mathsf{N}, \mathfrak{r})$ *is* $\mathscr{E}$*-injective, for every bicomodule* $(\mathsf{N}, \mathfrak{r}) \in {}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$*, and so is every induced* $\mathbf{F}$*-bicomodule* $\mathscr{T}\mathscr{S}(\mathsf{M}, \mathfrak{m}, \mathfrak{n}) = (\mathsf{M}F, \mathfrak{m}_F, \mathsf{M}\delta)$.

Proof: Follows immediately from Lemma 3.1. □

Fix a comonad $\mathbf{F} = (F, \delta, \xi)$ on a Grothendieck category $\mathcal{A}$ with $F \in \overline{\mathrm{Funct}}(\mathcal{A}, \mathcal{A})$. For every $\mathbf{F}$-bicomodule $(\mathsf{M}, \mathfrak{m}, \mathfrak{n})$ and each $i \geq 1$, we consider the i^{-th} induced $\mathbf{F}$-bicomodule $(\mathsf{M}F^i, \mathfrak{m}_{F^i}, \mathsf{M}F^{i-1}\delta)$.

Proposition 3.3 *Let* $(\mathsf{M}, \mathfrak{m}, \mathfrak{n})$ *be any* $\mathbf{F}$*-bicomodule. The following sequence in the category of* $\mathbf{F}$*-bicomodules*

$$0 \longrightarrow \mathsf{M} \xrightarrow{\mathfrak{n}} \mathsf{M}F \xrightarrow{\mathfrak{d}^0} \mathsf{M}F^2 \xrightarrow{\mathfrak{d}^1} \cdots \longrightarrow \mathsf{M}F^{n+1} \xrightarrow{\mathfrak{d}^n} \mathsf{M}F^{n+2} \longrightarrow \cdots \quad (3.1)$$

where $\mathfrak{d}^0 = \mathsf{M}\delta - \mathfrak{n}_F$ *and recursively*

$$\mathfrak{d}^{n+1} = \mathfrak{d}^n_F + (-1)^{n+1}\mathsf{M}F^{n+1}\delta, \quad n = 0, 1, 2, \ldots \quad (3.2)$$

defines an $\mathscr{E}$-injective resolution for $(\mathsf{M},\mathfrak{m},\mathfrak{n})$.

Proof: Let us denote by $E(\mathsf{M})$ the sequence defined in (3.1). One can easily check that the family of morphisms

$$\mathsf{u}_n := (-1)^{n+1}\mathsf{M}F^n\xi : \mathsf{M}F^{n+1} \longrightarrow \mathsf{M}F^n$$

in ${}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$, defines a contracting homotopy for $\mathscr{S}(E(\mathsf{M}))$. This implies by [16, Lemma 2.4] that $\mathscr{S}(E(\mathsf{M}))$ is sequence in $\mathscr{E}_0$. Hence $E(\mathsf{M})$ is in $\mathscr{E}$. □

Let $(\mathsf{N},\mathfrak{r},\mathfrak{s})$ be another $\mathbf{F}$-bicomodule and denote by $\mathrm{Ext}_{\mathscr{E}}\Big(N,M\Big)$ the homology of the complex

$$0 \longrightarrow \mathrm{Hom}_{{}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}}\Big(\mathsf{N},\mathsf{M}F\Big) \longrightarrow \mathrm{Hom}_{{}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}}\Big(\mathsf{N},\mathsf{M}F^2\Big) \longrightarrow \cdots \tag{3.3}$$

obtained by applying the functor $\mathrm{Hom}_{{}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}}\Big(\mathsf{N},-\Big)$ to the $\mathscr{E}$-injective resolution of M given in (3.1). Using the natural isomorphism stated in Lemma 1.4, we can show that the complex (3.3) is isomorphic to

$$0 \longrightarrow \mathrm{Hom}_{{}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}}\Big(\mathsf{N},\mathsf{M}\Big) \xrightarrow{\partial^0} \mathrm{Hom}_{{}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}}\Big(\mathsf{N},\mathsf{M}F\Big) \xrightarrow{\partial^1} \cdots \tag{3.4}$$

where

$$\begin{aligned}
\partial^0(\mathsf{f}) &= \mathsf{f}_F\circ\mathfrak{s} - \mathfrak{n}\circ\mathsf{f},\\
\partial^1(\mathsf{f}) &= \mathsf{M}\delta\circ\mathsf{f} - \mathsf{f}_F\circ\mathfrak{s} - \mathfrak{n}_F\circ\mathsf{f},\\
\partial^n(\mathsf{f}) &= \sum_{i=0}^{n-1}(-1)^i\mathsf{M}F^i\delta_{F^{n-i-1}}\circ\mathsf{f} + (-1)^n\mathsf{f}_F\circ\mathfrak{s} - \mathfrak{n}_{F^n}\circ\mathsf{f}, \quad n=2,3,\ldots
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\mathrm{Ker}(\partial^1) &= \{\mathsf{f} : (\mathsf{N},\mathfrak{r}) \to (\mathsf{M}F,\mathfrak{m}_F) | \mathsf{M}\delta\circ\mathsf{f} = \mathsf{f}_F\circ\mathfrak{s} + \mathfrak{n}_F\circ\mathsf{f}\}\\
\mathrm{Im}(\partial^0) &= \{\mathsf{f} : (\mathsf{N},\mathfrak{r}) \to (\mathsf{M}F,\mathfrak{m}_F) | \mathsf{f} = \varphi_F\circ\mathfrak{s} - \mathfrak{n}\circ\varphi, \text{for some}\, \varphi : (\mathsf{N},\mathfrak{r}) \to (\mathsf{M},\mathfrak{m})\}.
\end{aligned}$$

That is the 1-cocycle are cointegrations and the 1-coboundaries are inner cointegrations. Thus

$$\mathrm{Ext}^1_{\mathscr{E}}\Big(N,M\Big) \cong \mathrm{Coint}(\mathsf{N},\mathsf{M})/\mathrm{InCoint}(\mathsf{N},\mathsf{M}). \tag{3.5}$$

The pair $(\mathscr{T}\mathscr{S},\Theta_-)$ form a resolvent pair in the sense of [16, Proposition 2.10] (dual to [10, Corollary 2.3, page 16]) for the injective class $\mathscr{E}$. Since ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$ has cokernels, [16, Lemma 2.11] implies that the cokernels constructed in (2.5) lead to a functor

$$\mathscr{K} : {}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}} \to {}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}},$$

and a natural transformation

$$\mathscr{T}\mathscr{S} \to \mathscr{K}.$$

Furthermore, $\mathscr{K}(E)$ is a sequence in $\mathscr{E}$, whenever E is a sequence in $\mathscr{E}$. By the isomorphism given in (2.7), we find that $\mathrm{Hom}_{{}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}}\Big(\mathsf{N}, \mathscr{K}(E)\Big) \cong \mathrm{Coint}(\mathsf{N}, E)$ is an exact sequence of abelian groups, for every $\mathscr{E}$-projective **F**-bicomodule N and every sequence E in $\mathscr{E}$. On the other hand, given an $\mathscr{E}$-injective **F**-bicomodule M, then $\mathscr{K}(\mathsf{M})$ is clearly $\mathscr{E}$-injective. Thus $\mathrm{Coint}(E, \mathsf{M})$, which by (2.7) is isomorphic to $\mathrm{Hom}_{{}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}}\Big(E, \mathscr{K}(\mathsf{M})\Big)$, is an exact sequence of abelian groups. This proves that the $\mathscr{E}$-derived functor of the bifunctor $\mathrm{Coint}(-,-)$ can be constructed. For two **F**-bicomodules N and M, let $H^*(\mathsf{N}, \mathsf{M})$ be this $\mathscr{E}$-derived functor which can be computed using the $\mathscr{E}$-injective resolution given in Proposition 3.3. Using this time the natural isomorphisms of (2.7) and the fact that $\mathscr{T}\mathscr{S}(\mathsf{M})$ are $\mathscr{E}$-injective for every **F**-bicomodule M, we can easily show that

$$\mathrm{Ext}^n_{\mathscr{E}}\Big(\mathsf{N}, \mathscr{K}(\mathsf{M})\Big) \cong H^n(\mathsf{N}, \mathsf{M}), \qquad n \geq 0 \tag{3.6}$$

$$\mathrm{Ext}^{n+1}_{\mathscr{E}}\Big(\mathsf{N}, \mathsf{M}\Big) \cong \mathrm{Ext}^n_{\mathscr{E}}\Big(\mathsf{N}, \mathscr{K}(\mathsf{M})\Big), \qquad n \geq 1. \tag{3.7}$$

By both Propositions 3.2 and 2.2, and the isomorphisms given in (3.5), (3.6), and (3.7), we have

Corollary 3.4 *For a* **F***-bicomodule* $(\mathsf{M}, \mathfrak{m}, \mathfrak{n})$*, the following conditions are equivalent*

(i) M *is* $\mathscr{E}$*-injective;*

(ii) M *is* $\mathscr{S}$*-injective;*

(iii) the sequence

$$0 \longrightarrow \mathsf{M} \xrightarrow{\ \mathfrak{n}\ } \mathsf{M}F \xrightarrow{\ \mathfrak{n}^c\ } \mathscr{K}(\mathsf{M}) \longrightarrow 0$$

splits in the category of bicomodules ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$*;*

(iv) the universal cointegration from $\mathscr{K}(\mathsf{M})$ *into* M *is inner;*

(v) every cointegration into M *is inner.*

Now we can formulate a characterization of comonads with a separable forgetful functor by means of the cohomology groups of their bicomodules.

Theorem 3.5 *Let $\mathcal{A}$ be a Grothendieck category and $\mathbf{F} = (F,\delta,\xi)$ a comonad on $\mathcal{A}$ with universal cogenerator the adjunction $\mathbf{S} : \mathcal{A}^{\mathbf{F}} \rightleftarrows \mathcal{A} : \mathbf{T}$. If F is right exact and preserves direct sums, then the following statements are equivalent*

(i) $\mathbf{S} : \mathcal{A}^{\mathbf{F}} \to \mathcal{A}$ *is a separable functor;*

(ii) $\mathscr{S} : {}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}} \to {}_{\mathcal{A}}\mathscr{M}^{\mathbf{F}}$ *is a Maschke functor;*

(iii) $\delta : F \to F^2$ *is a split monomorphism in the category of bicomodules* ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$*;*

(iv) (F,δ,δ) *is* $\mathscr{E}$*-injective* $\mathbf{F}$*-bicomodule;*

(v) the universal coderivation from $\mathscr{K}(F)$ *into* F *is inner;*

(vi) every coderivation into F *is inner;*

(vii) all cointegrations between $\mathbf{F}$*-bicomodules are inner;*

(viii) $\mathrm{Ext}^n_{\mathscr{E}}(-,-) = 0$ *for all* $n \geq 1$*;*

(ix) $H^n(\mathsf{N},F) = 0$ *for all* $\mathbf{F}$*-bicomodule* N *and all* $n \geq 1$.

Proof: Corollary 3.4, Proposition 2.1, and properties of Ext give the following equivalences $(ii) \Leftrightarrow (vii)$, $(ii) \Leftrightarrow (viii)$, $(iv) \Leftrightarrow (ix)$, $(iv) \Leftrightarrow (vi)$. Proposition 3.2 gives the equivalence $(iv) \Leftrightarrow (iii)$, and lastly both Theorem 1.6 and Corollary 2.4 give the equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (v)$. □

4. Applications

We present in this section two different applications of Theorem 3.5. The first one is devoted to coseparable corings [14], where of course the comonad is defined by the tensor product over algebra. The second deals with the coalgebra co-extensions over fields, and the comonad is defined using cotensor product. Here we obtain Nakajima's results [19] without requiring the co-commutativity of the base coalgebra. This condition is however replaced, in our case, by assuming that the extended coalgebra is left co-flat.

4.1. Coseparable corings

Let $\mathbb{K}$ be commutative ring with 1. In what follows all algebras are $\mathbb{K}$-algebras, and all bimodules over algebras are assumed to be central $\mathbb{K}$-bimodules. Let R be an algebra. An *R-coring* [22] is a three-tuple $(\mathfrak{C},\Delta,\varepsilon)$ consisting of an R-bimodule $\mathfrak{C}$ and two R-bilinear maps

$$\Delta : \mathfrak{C} \to \mathfrak{C} \otimes_R \mathfrak{C} \quad \text{and} \quad \varepsilon : \mathfrak{C} \to R,$$

known as the comultiplication and the counit, which satisfy

$$(\mathfrak{C} \otimes_R \Delta) \circ \Delta = (\Delta \otimes_R \mathfrak{C}) \circ \Delta, \quad (\mathfrak{C} \otimes_R \varepsilon) \circ \Delta = \mathfrak{C} = (\varepsilon \otimes_R \mathfrak{C}) \circ \Delta.$$

In this sub-section the unadorned symbol $- \otimes -$ between R-bimodules and R-bilinear maps denotes the tensor product $- \otimes_R -$. We denote as usual by ${}^{\mathfrak{C}}\mathscr{M}^{\mathfrak{C}}$ the category of $\mathfrak{C}$-bicomodules. The objects are three-tuples $(M, \varrho_M, \lambda_M)$ consisting of an R-bimodule M and two R-bilinear maps $\varrho_M : M \to M \otimes \mathfrak{C}$ (right $\mathfrak{C}$-coaction), $\lambda_M : M \to \mathfrak{C} \otimes M$ (left $\mathfrak{C}$-coaction) satisfying

$$(\mathfrak{C} \otimes \lambda_M) \circ \lambda_M = (\Delta \otimes M) \circ \lambda_M, \quad (\varepsilon \otimes M) \circ \lambda_M = M;$$
$$(\varrho_M \otimes \mathfrak{C}) \circ \varrho_M = (M \otimes \Delta) \circ \varrho_M, \quad (M \otimes \varepsilon) \circ \varrho_M = M;$$
$$(\mathfrak{C} \otimes \varrho_M) \circ \lambda_M = (\lambda_M \otimes \mathfrak{C}) \circ \varrho_M.$$

It is clear that $\mathbf{F} := (F, \delta, \xi)$ where $F = - \otimes \mathfrak{C} : \mathscr{M}_R \to \mathscr{M}_R$, $\delta = - \otimes \Delta$, and $\xi = - \otimes \varepsilon$, is a comonad on the category of right R-modules $\mathscr{M}_R$, with $F \in \overline{\text{Funct}}(\mathscr{M}_R, \mathscr{M}_R)$.

Given any $\mathbf{F}$-bicomodule $(\mathsf{M}, \mathfrak{m}, \mathfrak{n})$ we can use Watts's Theorem [23] to find a natural isomorphism

$$\daleth^{\mathsf{M}}_{-} : \mathsf{M} \longrightarrow - \otimes \mathsf{M}(R) \tag{4.1}$$

satisfying $(- \otimes \psi_R) \circ \daleth^{\mathsf{M}}_{-} = \daleth^{\mathsf{M}'}_{-} \circ \psi$ for every natural transformation $\psi : \mathsf{M} \to \mathsf{M}'$ with $(\mathsf{M}', \mathfrak{m}', \mathfrak{n}')$ is another $\mathbf{F}$-bicomodule. With the help of this natural isomorphism we can establish a functor

$$\begin{aligned} \mathscr{G} : {}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}} &\longrightarrow {}^{\mathfrak{C}}\mathscr{M}^{\mathfrak{C}} \\ (\mathsf{M}, \mathfrak{m}, \mathfrak{n}) &\longrightarrow \Big(\mathsf{M}(R), \varrho_{\mathsf{M}(R)}, \lambda_{\mathsf{M}(R)}\Big) \\ \mathsf{f} &\longrightarrow \mathsf{f}_R \end{aligned}$$

where the $\mathfrak{C}$-coactions are defined by $\varrho_{\mathsf{M}(R)} = \mathfrak{m}_R$ and $\lambda_{\mathsf{M}(R)} = \daleth^{\mathsf{M}}_{F(R)} \circ \mathfrak{n}_R$.

Conversely, given any $\mathfrak{C}$-bicomodule $(M, \varrho_M, \lambda_M)$, we clearly obtain an $\mathbf{F}$-bicomodule defined by the three-tuple $\Big(- \otimes M, - \otimes \varrho_M, \big(\daleth^{\mathsf{M}}_{F}\big)^{-1} \circ (- \otimes \lambda_M)\Big)$. This in fact entails an inverse functor, up to the natural isomorphisms $\daleth^{-}_{-}$, to the functor $\mathscr{G}$. Henceforth, $\mathscr{G}$ is an equivalence of categories ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$ and ${}^{\mathfrak{C}}\mathscr{M}^{\mathfrak{C}}$. It is then obvious that δ is a split-mono in the category of $\mathbf{F}$-bicomodules if and only if Δ is a split-mono in the category of $\mathfrak{C}$-bicomodules. It is well known (see [5]) that this later condition is satisfied if and only if the right coaction forgetful functor is separable.

Consider two $\mathfrak{C}$-bicomodules $(M, \varrho_M, \lambda_M)$ and $(N, \varrho_N, \lambda_N)$. Following [14], a R-bilinear map $g : M \to \mathfrak{C}$ is said to be *coderivation* if it satisfies

$$\Delta \circ g = (g \otimes \mathfrak{C}) \circ \varrho_M + (\mathfrak{C} \otimes g) \circ \lambda_M.$$

The coderivation g is said to be an *inner coderivation* if there exists a R-bilinear map $\gamma : M \to R$ such that $g = (\mathfrak{C} \otimes \gamma) \circ \lambda_M - (\gamma \otimes \mathfrak{C}) \circ \varrho_M$. We denote by $\text{Coder}_{\mathfrak{C}}(M, A)$

the abelian group of all coderivations from M to $\mathfrak{C}$. A *(left) cointegration* from N into M is an R-bilinear morphism $f : N \to \mathfrak{C} \otimes M$ such that

$$(\Delta \otimes \mathfrak{C}) \circ f \;=\; (\mathfrak{C} \otimes \lambda_M) \circ f + (\mathfrak{C} \otimes f) \circ \lambda_N.$$

The cointegration f is said to be an *inner cointegration* if there exists an R-bilinear map $\varphi : N \to M$ satisfying

$$\varrho_M \circ \varphi \;=\; (\varphi \otimes \mathfrak{C}) \circ \varrho_N, \text{and } f \;=\; (\mathfrak{C} \otimes \varphi) \circ \lambda_N - \lambda_M \circ \varphi.$$

The abelian group of all cointegrations from N into M will be denoted by $\mathrm{Coint}_{\mathfrak{C}}(N, M)$.

Cointegrations and coderivations in both categories of bicomodules ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$ and ${}^{\mathfrak{C}}\mathscr{M}^{\mathfrak{C}}$ are connected by the following isomorphisms of abelian groups

$$\begin{array}{ccc}
\mathrm{Coder}(\mathsf{M}, F) & \xrightarrow{\;\cong\;} & \mathrm{Coder}_{\mathfrak{C}}\Big(\mathsf{M}(R), \mathfrak{C}\Big) \\
\mathsf{g} & \longmapsto & \mathsf{g}_R \\
(- \otimes g) \circ \daleth^{\mathsf{M}}_{-} & \longleftarrow\!\shortmid & g
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Coint}(\mathsf{N}, \mathsf{M}) & \xrightarrow{\;\cong\;} & \mathrm{Coint}_{\mathfrak{C}}\Big(\mathsf{N}(R), \mathsf{M}(R)\Big) \\
\mathsf{f} & \longmapsto & \daleth^{\mathsf{M}}_{F(R)} \circ \mathsf{f}_R \\
\Big(\daleth^{\mathsf{M}}_{F(-)}\Big)^{-1} \circ (- \otimes f) \circ \daleth^{\mathsf{N}}_{-} & \longleftarrow\!\shortmid & f
\end{array}$$

where the isomorphism $F(R) \cong \mathfrak{C}$ was used as isomorphism of R-corings. The restrictions of the above isomorphisms to the sub-groups of inner coderivations or inner cointegrations, are also isomorphisms.

Applying Theorem 3.5 to this situation, we obtain

Corollary 4.1 ([14, Theorem 3.10]) *For any R-coring $(\mathfrak{C}, \Delta, \varepsilon)$, the following statements are equivalent*

(i) The forgetful functor $\mathbf{S} : \mathscr{M}^{\mathfrak{C}} \to \mathscr{M}_R$ from the category of right $\mathfrak{C}$-comodules to the category of right R-modules is a separable functor;

(ii) the forgetful functor ${}^{\mathfrak{C}}\mathscr{M}^{\mathfrak{C}} \to {}^{\mathfrak{C}}\mathscr{M}_R$ is a Maschke functor;

(iii) the short exact sequence

$$0 \longrightarrow \mathfrak{C} \xrightarrow{\;\Delta\;} \mathfrak{C} \otimes \mathfrak{C} \xrightarrow{\;\Delta^c\;} \Omega(\mathfrak{C}) \longrightarrow 0$$

splits in the category of bicomodules ${}^{\mathfrak{C}}\mathscr{M}^{\mathfrak{C}}$;

(iv) $\mathfrak{C}$ *is* $\mathscr{E}$*-injective, where* $\mathscr{E}$ *is the injective class in* ${}^{\mathfrak{C}}\mathscr{M}^{\mathfrak{C}}$ *whose sequences split in the category of* R*-bimodules* ${}_R\mathscr{M}_R$*;*

(v) *the universal coderivation from* $\Omega(\mathfrak{C})$ *into* $\mathfrak{C}$ *is inner;*

(vi) *every coderivation into* $\mathfrak{C}$ *is inner;*

(vii) *all cointegrations between* $\mathfrak{C}$*-bicomodules are inner;*

(viii) $\mathrm{Ext}^n_{\mathscr{E}}(-,-) = 0$ *for all* $n \geq 1$*;*

(ix) $H^n(N,\mathfrak{C}) = 0$ *for all* $\mathfrak{C}$*-bicomodule* N *and all* $n \geq 1$.

4.2. Coseparable coalgebras co-extension

In what follows $\mathbb{K}$ is assumed to be a field. The unadorned symbol $\otimes$ between $\mathbb{K}$-vector spaces means the tensor product $\otimes_{\mathbb{K}}$. Let A, C are two $\mathbb{K}$-coalgebras, and consider $\phi : A \to C$ a morphism of $\mathbb{K}$-coalgebras. This defines an adjunction

$$-\square_C A : \mathscr{M}^C \rightleftarrows \mathscr{M}^A : \mathscr{O}$$

between the categories of right comodules with $-\square_C A$ right adjoint to $\mathscr{O}$, and where $-\square_C-$ is the co-tensor product over C. In the remainder, we denote this bi-functor by $-\square- := -\square_C-$. Notice that $-\square-$ is associative (up to natural isomorphism), as C is a $\mathbb{K}$-coalgebra and $\mathbb{K}$ is a field. From now on, we assume that $-\square A : \mathscr{M}^C \to \mathscr{M}^A$ is right exact, and thus exact. Put $F := \mathscr{O}(-\square A) : \mathscr{M}^C \to \mathscr{M}^C$, since $\mathscr{M}^C$ is a Grothendieck category we can construct the category $\overline{\mathrm{Funct}}(\mathscr{M}^C, \mathscr{M}^C)$, and we have in this case that $F \in \overline{\mathrm{Funct}}(\mathscr{M}^C, \mathscr{M}^C)$. Let us denote by $\overline{\Delta} : A \to A\square A$ the resulting map from the universal property of kernels. This is in fact an A-bicomodule map, and thus a C-bicomodule map by applying $\mathscr{O}$. Furthermore, we have

$$\begin{aligned} (A\square\overline{\Delta}) \circ \overline{\Delta} &= (\overline{\Delta}\square A) \circ \overline{\Delta} \\ (\phi\square A) \circ \overline{\Delta} &= (A\square\phi) \circ \overline{\Delta} \;=\; A\,(\mathrm{uptoisomorphism}). \end{aligned}$$

Using these equalities, on can easily check that there is a comonad $\mathbf{F} := (F,\delta,\xi)$ on the category of right C-comodules $\mathscr{M}^C$, where δ and ξ are defined by the following commutative diagrams of natural transformations

$$\begin{array}{ccc} -\square A & \xrightarrow{-\square\overline{\Delta}} & -\square A\square A, \\ \| & & \| \\ F & \overset{\delta}{\dashrightarrow} & F^2 \end{array} \qquad \begin{array}{ccc} -\square A & \xrightarrow{-\square\phi} & -\square C. \\ \| & & \downarrow \cong \\ F & \overset{\xi}{\dashrightarrow} & \mathbb{1}_{\mathscr{M}^C} \end{array}$$

Given $(\mathsf{M}, \mathfrak{m}, \mathfrak{n})$ any $\mathbf{F}$-bicomodule, we know that $\mathsf{M} : \mathscr{M}^C \to \mathscr{M}^C$ is right exact and preserves direct sums. If M is assumed to be left exact, then by [12, Theorem 3.5], $\mathsf{M}(C) := M$ is a C-bicomodule, and there is a natural isomorphism

$$\Upsilon^{\mathsf{M}}_{-} : \mathsf{M} \xrightarrow{\ \cong\ } -\square M, \tag{4.2}$$

which satisfies $(-\square \beta_C) \circ \Upsilon^{\mathsf{M}}_{-} = \Upsilon^{\mathsf{N}}_{-} \circ \beta$, for every natural transformation $\beta : \mathsf{M} \to \mathsf{N}$ with $\mathsf{N} \in \overline{\mathrm{Funct}}(\mathscr{M}^C, \mathscr{M}^C)$ and N an exact functor. The natural transformation $\mathfrak{m}$ and $\mathfrak{n}$ induces by this isomorphism a structure of A-bicomodule on M. The right and left A-coactions are given by

$$\begin{array}{ccc} M & \xrightarrow{\mathfrak{m}_C} & M\square A \\ & \searrow^{\varrho_M} & \downarrow \mathfrak{eq}^k_{M,A} \\ & & M \otimes A \end{array} \qquad \begin{array}{ccccc} M & \xrightarrow{\mathfrak{n}_C} & \mathsf{M}F(C) & \xrightarrow[\cong]{\Upsilon^{\mathsf{M}}_{F(C)}} & F(C)\square M \cong A\square M \\ & & \searrow^{\lambda_M} & & \downarrow \mathfrak{eq}^k_{A,M} \\ & & & & A \otimes M \end{array}$$

where, for every right C-comodule X and left C-comodule Y, $\mathfrak{eq}^k_{X,Y}$ denotes the equalizer map. That is the kernel of the map $\mathfrak{eq}_{X,Y} : X \otimes Y \Longrightarrow X \otimes C \otimes Y$ defined by $\mathfrak{eq}_{X,Y} = \varrho_X \otimes Y - X \otimes \lambda_Y$. The counitary conditions of these new A-coactions are easily seen, while the co-associatively and compatibility conditions need a routine and long computations using properties of cotensor product over coalgebras over fields.

Let us denote by ${}^{\mathbf{F}}\mathsf{E}^{\mathbf{F}}$ the full subcategory of ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$ whose objects are $\mathbf{F}$-bicomodules $(\mathsf{M}, \mathfrak{m}, \mathfrak{n})$ such that $\mathsf{M} : \mathscr{M}^C \to \mathscr{M}^C$ is an exact functor which commutes with direct sums. For instance, (F, δ, ξ) is by hypothesis an object of this category.

The above arguments establish in fact a functor from the subcategory of $\mathbf{F}$-bicomodules ${}^{\mathbf{F}}\mathsf{E}^{\mathbf{F}}$ to the category of A-bicomodules sending

$$\mathscr{F} : {}^{\mathbf{F}}\mathsf{E}^{\mathbf{F}} \longrightarrow {}^A\mathscr{M}^A, \Big(\big(\mathsf{M}, \mathfrak{m}, \mathfrak{n}\big) \to \big(M, \varrho_M, \lambda_M\big) \Big), \big(\mathsf{f} \to \mathsf{f}_C\big). \tag{4.3}$$

For every $\mathbf{F}$-bicomodule $\mathsf{M} \in {}^{\mathbf{F}}\mathsf{E}^{\mathbf{F}}$, it is clear that $\mathscr{F}(\mathsf{M}) = M$ is a co-flat left C-comodule.

Conversely, given any A-bicomodule $(N, \varrho_N, \lambda_N)$ such that the underlying left C-comodule ${}_C N$ is co-flat, then we have a functor $-\square N : \mathscr{M}^C \to \mathscr{M}^C$ which is exact and preserves direct sums together with two natural transformations

$$-\square N \xrightarrow{-\square \lambda'_N} -\square A \square N, \qquad -\square N \xrightarrow{-\square \varrho'_N} -\square N \square A,$$

where λ'_N and ϱ'_N are C-bicolinear defined by universal property

$$\begin{array}{ccc} N & \xrightarrow{\lambda_N} & A \otimes N, \\ & \searrow^{\lambda'_N} & \uparrow \mathfrak{eq}^k_{A,N} \\ & & A \square N \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{\varrho_N} & N \otimes A. \\ & \searrow^{\varrho'_N} & \uparrow \mathfrak{eq}^k_{N,A} \\ & & N \square A \end{array}$$

By definition and the properties of cotensor product λ'_N and ϱ'_N satisfy the following equalities

$$\begin{aligned} (\overline{\Delta}\square N) \circ \lambda'_N &= (A\square\lambda'_N) \circ \lambda'_N, \quad (\phi\square N) \circ \lambda'_N = N \text{ (uptoisomorphism)} \\ (N\square\overline{\Delta}) \circ \varrho'_N &= (\varrho'_N\square A) \circ \varrho'_N, \quad (N\square\phi) \circ \varrho'_N = N \text{ (uptoisomorphism)} \\ (A\square\varrho'_N) \circ \lambda'_N &= (\lambda'_N\square A) \circ \varrho'_N. \end{aligned}$$

Consider the obtained three-tuple $(\mathsf{N},\mathfrak{r},\mathfrak{s})$, where $\mathsf{N} := -\square N : \mathscr{M}^{C} \to \mathscr{M}^{C}$ is a functor, and $\mathfrak{r} := -\square\varrho'_N : \mathsf{N} \to F\mathsf{N}$, $\mathfrak{s} := -\square\lambda'_N : \mathsf{N} \to \mathsf{N}F$ are two natural transformation. Since N is assumed to be co-flat left C-comodule, the previous equalities show that $(\mathsf{N},\mathfrak{r},\mathfrak{s})$ is actually an object of the category ${}^{\mathbf{F}}\mathsf{E}^{\mathbf{F}}$, whose image by $\mathscr{F}$ is isomorphic to the initial A-bicomodule (N,ϱ_N,λ_N), via the natural isomorphisms Υ^{-}_{-}. Now, given an A-bicolinear morphism $g : (N,\varrho_N,\lambda_N) \to (N',\varrho_{N'},\lambda_{N'})$, we get an $\mathbf{F}$-bicomodule morphism $\mathsf{g} := -\square g : \mathsf{N} \to \mathsf{N}'$. This shows that the above constructions are in fact functorial.

In conclusion, we have shown that the functor $\mathscr{F}$ defined in (4.3), establishes an equivalence of categories between ${}^{\mathbf{F}}\mathsf{E}^{\mathbf{F}}$ and ${}^{A}\mathscr{C}^{A}$, where the later is the full subcategory of the category of A-bicomodules ${}^{A}\mathscr{M}^{A}$ whose objects are co-flat left C-comodules after forgetting the right C-coaction.

Recall from [19] that A is said to be a *separable C-coalgebra* if the A-bicolinear map $\overline{\Delta} : A \to A\square A$ is a split-mono in the category of A-bicomodules. By [12, Theorem 5.6] this is equivalent to say that the forgetful functor $\mathscr{O}$ is a separable functor. Using the equivalence of categories established above, it is easy to check that δ is a split-mono in ${}^{\mathbf{F}}\mathsf{E}^{\mathbf{F}}$ (or equivalently in ${}^{\mathbf{F}}\mathscr{M}^{\mathbf{F}}$) if and only if $\overline{\Delta}$ is a split-mono in ${}^{A}\mathscr{C}^{A}$ (or equivalently in ${}^{A}\mathscr{M}^{A}$).

Given two A-bicomodules (M,ϱ_M,λ_M) and (N,ϱ_N,λ_N), a C-bicolinear map $g : M \to A$ is said to be *C-coderivation* if its satisfies

$$\overline{\Delta} \circ g = (g\square A) \circ \varrho'_M + (A\square g) \circ \lambda'_M.$$

The C-coderivation g is said to be an *inner C-coderivation* if there exists a C-bicolinear map $\gamma : M \to C$ such that $g = (A\square\gamma) \circ \lambda'_M - (\gamma\square A) \circ \varrho'_M$. We denote by $\mathrm{Coder}_C(M,A)$ the abelian group of all C-derivations from M to A. A *(left) C-cointegration* from N into M is a morphism of $C-A$-bicomodules $f : N \to A\Box M$ such that

$$(\overline{\Delta}\Box A)\circ f \,=\, (A\Box\lambda'_M)\circ f + (A\Box f)\circ\lambda'_N.$$

The C-cointegration f is said to be an *inner C-cointegration* if there exists a C-bicolinear map $\varphi : N \to M$ satisfying

$$\varrho'_M\circ\varphi \,=\, (\varphi\Box A)\circ\varrho'_N, \text{and } f \,=\, (A\Box\varphi)\circ\lambda'_N - \lambda'_M\circ\varphi.$$

The abelian group of all C-cointegration from N into M will be denoted by $\mathrm{Coint}_C(N,\, M)$.

Let $(\mathsf{M},\mathfrak{m},\mathfrak{n})$ and $(\mathsf{N},\mathfrak{r},\mathfrak{s})$ be two $\mathbf{F}$-bicomodules in ${}^{\mathbf{F}}\mathsf{E}^{\mathbf{F}}$ and consider their associated A-bicomodule via the above equivalence of categories $\mathscr{F}$:

$$\Big(\mathsf{M}(C) := M, \varrho_M, \lambda_M\Big) \quad \text{and} \quad \Big(\mathsf{N}(C) := N, \varrho'_N, \lambda'_N\Big).$$

We have an abelian group isomorphism

$$\begin{array}{ccc} \mathrm{Coder}(\mathsf{M},F) & \xrightarrow{\;\cong\;} & \mathrm{Coder}_C\Big(\mathsf{M}(C),A\Big) \\ \mathsf{g} & \longmapsto & \iota_A\circ\mathsf{g}_C \\ (-\Box g)\circ\Upsilon^{\mathsf{M}}_{-} & \longleftarrow\!\shortmid & g \end{array}$$

where $\iota_- : C\Box- \to \mathbb{1}_{\mathscr{M}^C}$ is the obvious natural isomorphism. The isomorphism of cointegrations groups is given by

$$\begin{array}{ccc} \mathrm{Coint}(\mathsf{N},\mathsf{M}) & \xrightarrow{\;\cong\;} & \mathrm{Coint}_C\Big(\mathsf{N}(C),\mathsf{M}(C)\Big) \\ \mathsf{f} & \longmapsto & (\iota_A\Box\mathsf{M}(C))\circ\Upsilon^{\mathsf{M}}_{F(C)}\circ\mathsf{f}_C \\ \Big(\Upsilon^{\mathsf{M}}_{F(-)}\Big)^{-1}\circ(-\Box f)\circ\Upsilon^{\mathsf{N}}_{-} & \longleftarrow\!\shortmid & f. \end{array}$$

Of course the restrictions of those isomorphisms to the sub-groups of inner cointegrations or inner coderivations are also group isomorphisms. Applying now Theorem 3.5, we arrive to the following

Corollary 4.2 (compare with [19, Theorem 1.2]) *Let $\phi : A \to C$ be a morphism of $\mathbb{K}$-coalgebras over a field $\mathbb{K}$. Assume that ${}_CA$ is a co-flat left C-comodule. The following statements are equivalent*

(i) A is a separable C-coalgebra;

(ii) for any A-bicomodule M such that ${}_C M$ is co-flat, every C-coderivation from M to A is inner;

(iii) for any pair of A-bicomodules M and N such that ${}_C M$ and ${}_C N$ are co-flat, every C-cointegration from M into N is inner.

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