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Invertible unital bimodules over rings with local units, and related exact sequences of groups, II ☆

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ABSTRACT

Let R be a ring with a set of local units, and a homomorphism of groups $\underline{\Theta} : \mathcal{G} \to \mathbf{Pic}(R)$ to the Picard group of R. We study under which conditions $\underline{\Theta}$ is determined by a factor map, and, henceforth, it defines a generalized crossed product with a same set of local units. Given a ring extension $R \subseteq S$ with the same set of local units and assuming that $\underline{\Theta}$ is induced by a homomorphism of groups $\mathcal{G} \to \mathbf{Inv}_R(S)$ to the group of all invertible R-sub-bimodules of S, then we construct an analogue of the Chase–Harrison–Rosenberg seven terms exact sequence of groups attached to the triple $(R \subseteq S, \underline{\Theta})$, which involves the first, the second and the third cohomology groups of $\mathcal{G}$ with coefficients in the group of all R-bilinear automorphisms of R. Our approach generalizes the works by Kanzaki and Miyashita in the unital case.

Introduction

For a Galois extension $R/\Bbbk$ of commutative rings with identity and finite Galois group G, Chase, Harrison and Rosenberg, gave in [5, Corollary 5.5] (see also [6]) the following seven terms exact sequence of groups

$$1 \longrightarrow H^1\big(G, \mathcal{U}(R)\big) \longrightarrow \mathbf{Pic}_{\Bbbk}(\Bbbk) \longrightarrow \mathbf{Pic}_R(R)^G \longrightarrow H^2\big(G, \mathcal{U}(R)\big) \longrightarrow B(R/\Bbbk)$$
$$\longrightarrow H^1\big(G, \mathbf{Pic}_R(R)\big) \longrightarrow H^3\big(G, \mathcal{U}(R)\big),$$

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where for a ring A with identity, $\mathbf{Pic}_A(A) = \{[P] \in \mathbf{Pic}(A) \mid ap = pa,\ \forall a \in A,\ p \in P\}$ is a sub-group of the Picard group $\mathbf{Pic}(A)$, and $\mathcal{U}(A)$ the group of units. The group $B(R/\Bbbk)$ is the Brauer group of Azumaya $\Bbbk$-algebras split by R, and the other terms are the usual cohomology groups of G with coefficients in abelian groups. As one can realize, the case of finite Galois extension of fields reduces to fundamental theorems in Galois cohomology of fields extensions, namely, the Hilbert's 90 Theorem, and the isomorphism of the Brauer group of a field with the second cohomology group.

The construction of this sequence of groups, as was given in [5,6], is ultimately based on the consideration of a certain spectral sequence already introduced by Grothendieck, and the generalized Amitsur cohomology. The same sequence was also constructed after that by Kanzaki [13, Theorem, p. 187], using only elementary methods that employ the novel notion of generalized crossed products. This notion has been the key success which allowed Miyashita to extend in [14, Theorem 2.12] the above framework to the context of a noncommutative unital ring extension $R \subseteq S$ with a homomorphism of groups to the group of all invertible sub-R-bimodules of S. In this setting the seven terms exact sequence takes of course a slightly different form, although the first, the fourth and the last terms remain the cohomology groups of the acting group with coefficients in the group of units of the base ring. The commutative Galois case is then recovered by first considering a tower $\Bbbk \subseteq R \subseteq S$, where $\Bbbk \subseteq R$ is a finite Galois extension and S is a fixed crossed product constructed from the Galois group; secondly by considering the homomorphism of groups which sends any element in the Galois group to it corresponding homogeneous component of S (components which belong to the group of all invertible sub-R-bimodules of S).

In this paper we construct an analogue of the above seven terms exact sequence in the context of rings with local units (see the definition below). The most common class of examples are the rings with enough orthogonal idempotents or Gabriel's rings which are defined from small additive categories. It is noteworthy that up to our knowledge there is no satisfactory notion in the literature of morphism of rings with local units that could be, for instance, extracted from a given additive functor between additive small categories. Here we restrict ourselves to the naive case of extensions of the form $R \subseteq S$ which have the same set of local units. This at least includes the situation where two small additive categories have the same set of objects and differ only in the size of the sets of morphisms, as it frequently happens in the theory of localization in abelian categories. Our methods are inspired from Miyashita's work [14] and use the results of our earlier work [9]. The present paper is in fact a continuation of [9]. It is worth noticing that, although the steps for constructing the seven terms sequence in our context are slightly parallel to [14], this construction is not an easy task since it has its own challenges and difficulties.

The paper is organized as follows. In Section 1 we give some preliminary results on similar and invertible unital bimodules. Most of results on similar unital bimodules are in fact the non-unital version of [11,12], except perhaps the construction of the twisting natural transformations and its weak associativity (Proposition 1.3 and Lemma 1.4). The generalized crossed product is introduced in Section 2, where we also give useful constructions on the normalized 2- and 3-cocycles with coefficient in the unit group of the base ring. Section 3 is devoted to the construction of an abelian group whose elements consist of isomorphic classes of generalized crossed products. We show that it contains a sub-group which is isomorphic to the 2-cohomology group (Proposition 3.2). Our main result is contained in Section 4, which contains a number of exact sequences of groups that culminate in the seven terms exact sequence that generalizes Chase–Harrison–Rosenberg's one (Theorem 4.12).

Notations and basic notions

In this paper ring means an associative and not necessarily unital ring. We denote by $\mathcal{Z}(R)$ (resp. $\mathcal{Z}(G)$) the center of a ring R (resp. of a group G), and if R is unital we will denote by $\mathcal{U}(R)$ the unit group of R, that is, the set of all invertible elements of R.

Let R be a ring and $\mathsf{E} \subset R$ a set of idempotent elements; R is said to be a *ring with set of local units* E, provided that for every finite subset $\{r_1, \dots, r_n\} \subset R$, there exists $e \in \mathsf{E}$ such that

$$er_i = r_i e = r_i, \quad \text{for every } i = 1, \dots, n.$$

In this way, given a finite set $\{r_1, \ldots, r_n\}$ of elements in R, we denote

$$Unit\{r_1, \ldots, r_n\} = \{e \in \mathsf{E} \mid er_i = r_i e = r_i, \text{ for every } i = 1, 2, \ldots, n\}.$$

Observe that our definition differs from that in [1, Definition 1.1], since we do not assume that the idempotents of E commute. In fact, our rings generalize those of [1] since, when the idempotents are commuting, it is enough to require that for each element $r \in R$ there exists $e \in \mathsf{E}$ such that $er = re = r$ (see [1, Lemma 1.2]). The Rees matrix rings considered in [2], are for instance, rings with a set of local units with noncommuting idempotents.

Let R be a ring with a set of local units E. A right R-module X is said to be *unital* provided one of the following equivalent conditions holds

(i) $X \otimes_R R \cong X$ via the right R-action on X,
(ii) $XR := \{\sum_{finite} x_i r_i \mid r_i \in R,\ x_i \in X\} = X$,
(iii) for every element $x \in X$, there exists an element $e \in \mathsf{E}$ such that $xe = x$.

Left unital R-modules are analogously defined. Obviously any right R-module X contains XR as the largest right unital R-submodule.

Let R and S be rings with a fixed set of local units, respectively, E and E'. In the sequel, an *extension of rings* $\phi : (R, \mathsf{E}) \to (S, \mathsf{E}')$ *with the same set of local units*, stand for an additive and multiplicative map ϕ which satisfies $\phi(\mathsf{E}) = \mathsf{E}'$. Note that since R and S have the same set of local units, any right (resp. left) unital S-module can be considered as right (resp. left) unital R-module by restricting scalars. Furthermore, for any right S-module X, we have the equality $XR = XS$.

1. Similar and invertible unital bimodules

Let R be a ring with a set of local units E. In the first part of this section we give some preliminary results on unital R-bimodules, which we will use frequently in the sequel. Most of them were already formulated by K. Hirata in [11] and [12] when R is unital. For the convenience of the reader we will give in our case complete proofs. The second part provides a relation between the automorphism group of an invertible unital R-bimodule and the automorphisms of the base ring R. This result will be also used in the forthcoming sections.

A *unital R-bimodule* is an R-bimodule which is left and right unital. In the sequel, the expression *R-bilinear map* means homomorphism of R-bimodules. Note that if M is a unital R-bimodule, then for every element $m \in M$, there exists a *two sided unit* e for m. That is, there exists $e \in \mathsf{E}$ such that $m = me = em$, see [7, p. 733]. If several R-bimodules are handled, say M, N, P with a finite number of elements $\{m_i, n_i, p_i\}_i$, $m_i \in M$, $n_i \in N$, $p_i \in P$, then we denote by $Unit\{m_i, n_i, p_i\}$ the set of common two sided units of the m_i's, n_i's and the p_i's. This means, that $e \in Unit\{m_i, n_i, p_i\} \subseteq \mathsf{E}$ if and only if

$$em_i = m_i = m_i e, \qquad en_i = n_i = n_i e, \qquad ep_i = p_i = p_i e, \quad \text{for every } i = 1, \ldots, n.$$

Given a unital R-bimodule M, we denote its invariant sub-bimodule, that is, the set of all R-bilinear maps from R to M, by

$$M^R := \mathrm{Hom}_{R\text{-}R}(R, M).$$

It is canonically a module over the commutative ring of all R-bimodule endomorphisms $\mathcal{Z} := \mathrm{End}_{R\text{-}R}(R)$. The $\mathcal{Z}$-action is given as follows: for every $z \in \mathcal{Z}$ and $f \in M^R$, we have

$$z.f : {}_R R_R \longrightarrow {}_R M_R \quad \big(r \longmapsto f\big(z(r)\big) = z(e) f(r) = f(r) z(e), \text{ where } e \in Unit\{r\}\big).$$

Lemma 1.1. *Let R be a ring with local units and M a unital R-bimodule such that $M \overset{\oplus}{\hookrightarrow} R^{(n)}$, that is, M is isomorphic as an R-bimodule to a direct summand of direct sum of n copies of R. Then*

(i) *M^R is a finitely generated and projective $\mathcal{Z}$-module*;
(ii) *there is an isomorphism of R-bimodules*

$$M \cong R \otimes_{\mathcal{Z}} M^R,$$

where the R-bimodule structure of the right hand term is induced by that of R. That is,

$$r(s \otimes_{\mathcal{Z}} f)t = (rst) \otimes_{\mathcal{Z}} f \quad \big(s \otimes_{\mathcal{Z}} f \in R \otimes_{\mathcal{Z}} M^R,\ r, t \in R\big).$$

Proof. Let us consider the following canonical R-bimodule homomorphisms:

$$\varphi_i : M \xhookrightarrow{\iota} R^{(n)} \xrightarrow{\pi_i} R, \qquad \psi_i : R \xhookrightarrow{\iota_i} R^{(n)} \xrightarrow{\pi} M, \quad i = 1, \dots, n,$$

where π_i and ι_i are the canonical projections and injections.

(i) It is clear that $\{(\psi_i, \psi_i^*)\}_i \in M^R \times \mathrm{Hom}_{\mathcal{Z}}(M^R, \mathcal{Z})$ is a finite dual $\mathcal{Z}$-basis for M^R, where each ψ_i^* is defined by the right composition with φ_i, that is, $\psi_i^*(f) = \varphi_i \circ f$, for every $f \in M^R$.

(ii) We know that there is a well defined R-bilinear map

$$\begin{aligned} R \otimes_{\mathcal{Z}} M^R &\xrightarrow{\ \eta\ } M \\ r \otimes_{\mathcal{Z}} f &\longmapsto f(r). \end{aligned}$$

We can easily show that

$$\begin{aligned} M &\xrightarrow{\ \eta^{-1}\ } R \otimes_{\mathcal{Z}} M^R \\ m &\longmapsto \textstyle\sum_i^n \varphi_i(m) \otimes_{\mathcal{Z}} \psi_i, \end{aligned}$$

is actually the inverse map of η. $\square$

We will follow Miyashita's notation concerning this class of R-bimodules. That is, if we have an R-bimodule M which is a direct summand as a bimodule of finitely many copies of another R-bimodule N, we shall denote this situation by $M|N$. It is clear that this relation is reflexive and transitive. Furthermore, it is compatible with the tensor product over R, in the sense that we have $M \otimes_R Q | N \otimes_R Q$ and $Q \otimes_R M | Q \otimes_R N$, for every unital R-bimodule Q, whenever $M|N$. In this way, we set

$$M \sim N \quad \text{if and only if} \quad M|N \text{ and } N|M, \tag{1}$$

and call M *is similar to* N. This in fact defines an equivalence relation which is, in the above sense, compatible with the tensor product over R.

Lemma 1.2. *Let R be a ring with local units and M, N are unital R-bimodules such that $M|R$ and $N|R$ as bimodules. Then*

(i) *for every finitely generated and projective $\mathcal{Z}$-module P, we have an isomorphism of $\mathcal{Z}$-modules*

$$(R \otimes_{\mathcal{Z}} P)^R \cong P;$$

(ii) *there is a natural* (*on both components*) *$\mathcal{Z}$-linear isomorphism*

$$(M \otimes_R N)^R \cong M^R \otimes_{\mathcal{Z}} N^R.$$

Proof. (i) Let $\{(p_i, p_i^*)\}_{1 \leqslant i \leqslant n}$ be a dual $\mathcal{Z}$-basis for P. Given an element $f \in (R \otimes_{\mathcal{Z}} P)^R$, we define a finite family of elements in $\mathcal{Z}$:

$$z_{f,i} := \left(R \otimes_{\mathcal{Z}} p_i^*\right) \circ f : R \longrightarrow R \otimes_{\mathcal{Z}} P \longrightarrow R \otimes_{\mathcal{Z}} \mathcal{Z} \cong R.$$

Now, an easy computation shows that the following maps are mutually inverse

$$\begin{array}{ccc} P & \longrightarrow & (R \otimes_{\mathcal{Z}} P)^R \\ p & \longmapsto & [r \longmapsto r \otimes_{\mathcal{Z}} p], \end{array} \qquad \begin{array}{ccc} (R \otimes_{\mathcal{Z}} P)^R & \longrightarrow & P \\ f & \longmapsto & \sum_i^n z_{f,i} p_i, \end{array}$$

which gives the stated isomorphism.

(ii) By Lemma 1.1(ii) we know that

$$M \cong R \otimes_{\mathcal{Z}} M^R \quad \text{and} \quad N \cong R \otimes_{\mathcal{Z}} N^R.$$

Therefore, we have

$$M \otimes_R N \cong R \otimes_{\mathcal{Z}} \left(M^R \otimes_{\mathcal{Z}} N^R\right).$$

Since by Lemma 1.1(i), $M^R \otimes_{\mathcal{Z}} N^R$ is a finitely generated and projective $\mathcal{Z}$-module, we obtain the desired isomorphism by applying item (i) just proved above. $\square$

Keep the notations of the first paragraph in the proof of Lemma 1.1.

Proposition 1.3. *Let R be a ring with local units and M, N are unital R-bimodules such that $M|R$ and $N|R$ as bimodules. Then there is a natural R-bilinear isomorphism $M \otimes_R N \cong N \otimes_R M$ given explicitly by*

$$\begin{array}{rcl} \mathsf{T}_{M,N} : M \otimes_R N & \xrightarrow{\ \cong\ } & N \otimes_R M \\ x \otimes_R y & \longmapsto & \sum_i^n \varphi_i(x) y \otimes_R \psi_i(e), \end{array}$$

where $e \in Unit\{x, y\}$.

Proof. By Lemmata 1.1 and 1.2, we have the following chain of R-bilinear isomorphisms

$$M \otimes_R N \xrightarrow{\ \cong\ } R \otimes_{\mathcal{Z}} M^R \otimes_{\mathcal{Z}} N^R \xrightarrow{\ \cong\ } R \otimes_{\mathcal{Z}} N^R \otimes_{\mathcal{Z}} M^R \xrightarrow{\ \cong\ } N \otimes_R \left(R \otimes_{\mathcal{Z}} M^R\right)$$
$$\xrightarrow{\ \cong\ } N \otimes_R M.$$

Writing down explicitly the composition of these isomorphisms, we obtain the stated one. $\square$

The natural transformation $\mathsf{T}_{-,-}$ is associative in the following sense.

Lemma 1.4. *Let R be ring with a local units. Consider X, Y, P, Q, U, V unital R-bimodules which satisfy $Q|R$, $(Y \otimes_R V)|R$, $(X \otimes_R U)|R$ and $(P \otimes_R Y)|R$, all as bimodules. Then the following diagram is commutative*:

$$\begin{tikzcd}
X \otimes_R U \otimes_R P \otimes_R Y \otimes_R V \otimes_R Q \arrow[r, "X\otimes U\otimes P\otimes \mathsf{T}_{Y\otimes V,Q}"] \arrow[d, "\mathsf{T}_{X\otimes U, P\otimes Y}\otimes V\otimes Q"'] & X \otimes_R U \otimes_R P \otimes_R Q \otimes_R Y \otimes_R V \arrow[d, "\mathsf{T}_{X\otimes U, P\otimes Q\otimes Y}\otimes V"] \\
P \otimes_R Y \otimes_R X \otimes_R U \otimes_R V \otimes_R Q \arrow[r, "P\otimes \mathsf{T}_{Y\otimes X\otimes U\otimes V,Q}"'] & P \otimes_R Q \otimes_R Y \otimes_R X \otimes_R U \otimes_R V.
\end{tikzcd}$$

Proof. Straightforward. □

Recall form [9, p. 227] that a unital R-bimodule X is said to be *invertible* if there exists another unital R-bimodule Y with two R-bilinear isomorphisms

$$X \otimes_R Y \xrightarrow[\cong]{\mathfrak{l}} R \xleftarrow[\cong]{\mathfrak{r}} Y \otimes_R X.$$

As was shown in [9], one can choose the isomorphisms $\mathfrak{l}, \mathfrak{r}$ such that

$$X \otimes_R \mathfrak{r} = \mathfrak{l} \otimes_R X, \qquad Y \otimes_R \mathfrak{l} = \mathfrak{r} \otimes_R Y.$$

In others words, such that $(\mathfrak{l}, \mathfrak{r}^{-1})$ is a Morita context from R to R. In this way, Y is isomorphic as an R-bimodule to the right unital part X^*R of the right dual R-module $X^* = \mathrm{Hom}_{-R}(X, R)$ of X. This isomorphism is explicitly given by

$$Y \xrightarrow{\cong} X^*R \quad \bigl(y \longmapsto \bigl[x \longmapsto \mathfrak{r}(y \otimes x)\bigr]\bigr).$$

It is worth mentioning that X is not necessarily finitely generated as right unital R-module; but a direct limit of finitely generated and projective right unital R-modules.

Given $e \in \mathsf{E}$ consider its decomposition with respect to the invertible unital R-bimodule X, that is,

$$e = \sum_{(e)} x_e y_e \quad \Bigl(\text{i.e. } e = \sum_{(e)} \mathfrak{l}(x_e \otimes_R y_e)\Bigr).$$

It is clear that the x_e's and the y_e's can be chosen such that

$$e x_e = x_e, \quad \text{and} \quad y_e = y_e e,$$

which means that e is a right unit for the y_e's and left unit for the x_e's.

Remark 1.5. It is noteworthy that a two sided unit of the set of elements $\{x_e, y_e\}$ does not need in general to coincide with e. However, any two sided unit $e_1 \in \mathit{Unit}\{x_e, y_e\}$ is also a unit for e. In other words, the elements e and $\sum_{(e)} x_e e y_e$ belong to the same unital ring eRe, but they are not necessarily equal. This makes a difference in technical difficulties compared with the case of unital rings.

The following lemma gives explicitly the relation between the automorphisms group of unital invertible R-bimodule and the unit group of the commutative unital ring $\mathcal{Z} = \mathrm{End}_{R\text{-}R}(R)$.

Lemma 1.6. *Let R be a ring with local units and X an invertible unital R-bimodule. Then there is an isomorphism of groups, from the group of R-bilinear automorphisms of X, $\mathbf{Aut}_{R\text{-}R}(X)$ to the unit group of $\mathcal{Z}$, defined by*

$$\begin{array}{ccc} \mathbf{Aut}_{R\text{-}R}(X) & \xrightarrow{\widetilde{(-)}} & \mathbf{Aut}_{R\text{-}R}(R) = \mathcal{U}(\mathcal{Z}) \\ \sigma & \longmapsto & \widetilde{\sigma} \end{array} \tag{2}$$

where for each $r \in R$ with unit $e \in Unit\{r\}$, we have

$$\widetilde{\sigma}(r) = \sum_{(e)} \sigma(x_e) y_e r = \sum_{(e)} r\sigma(x_e) y_e.$$

In particular, for every element $\mathfrak{t} \in X$ and $\sigma \in \mathbf{Aut}_{R\text{-}R}(X)$, we have

$$\sigma(\mathfrak{t}) = \widetilde{\sigma}(e)\mathfrak{t}, \quad \text{whenever } e \in Unit\{\mathfrak{t}\}.$$

Proof. Follows directly from [4, Lemma 2.1]. $\square$

The Picard group of R is denoted by $\mathbf{Pic}(R)$. It consists of all isomorphisms classes of invertible unital R-bimodules, with multiplication induced by the tensor product. As in the unital case [10, Theorem 2(i)], there is a canonical homomorphism of groups $\boldsymbol{\alpha} : \mathbf{Pic}(R) \to \mathbf{Aut}(\mathcal{U}(\mathcal{Z}))$. This homomorphism is explicitly given as follows. Given $[X] \in \mathbf{Pic}(R)$ and $u \in \mathcal{U}(\mathcal{Z})$, we consider the R-bilinear automorphism of X defined by

$$\sigma_u : X \longrightarrow X \quad \big(\mathfrak{t} \longmapsto \mathfrak{t}u(e),\ \mathfrak{t} \in X,\ \text{and } e \in Unit\{\mathfrak{t}\}\big).$$

We clearly have $\sigma_{uv} = \sigma_u \circ \sigma_v$, and so $\widetilde{\sigma_{uv}} = \widetilde{\sigma_u} \circ \widetilde{\sigma_v}$, for $u, v \in \mathcal{U}(\mathcal{Z})$. It follows from Lemma 1.6 that

$$\sigma_u(r\mathfrak{t}) = \widetilde{\sigma_u}(r)\mathfrak{t}, \quad \text{for every } \mathfrak{t} \in X,\ r \in R. \tag{3}$$

This equation implies that the element $\widetilde{\sigma_u} \in \mathcal{U}(\mathcal{Z})$ is independent from the choice of the representative X of the class $[X] \in \mathbf{Pic}(R)$. We have thus a well defined map

$$\boldsymbol{\alpha} : \mathbf{Pic}(R) \longrightarrow \mathbf{Aut}\big(\mathcal{U}(\mathcal{Z})\big), \quad [X] \longmapsto \boldsymbol{\alpha}_{[X]}(u) = \widetilde{\sigma_u}$$

which is a homomorphism of groups by (3). This is exactly the homomorphism induced by the map α of [4, p. 135]. We have the formula:

$$\boldsymbol{\alpha}_{[X]}(u)(r) = \widetilde{\sigma_u}(r) = \sum_{(e)} r x_e u(e_1) y_e \quad \left(\text{i.e. } \sum_{(e)} r\mathfrak{l}\big(x_e \otimes_R u(e_1) y_e\big)\right), \tag{4}$$

where $e \in Unit\{r\}$, $e_1 \in Unit\{x_e, y_e\}$, and as before $e = \sum_{(e)} x_e y_e$.

2. Generalized crossed products with local units

Let $\mathcal{G}$ be any group with neutral element 1. Consider a ring R with a set of local units E, and a group homomorphism $\underline{\Theta}: \mathcal{G} \to \mathbf{Pic}(R)$. This is equivalent to give sets of invertible R-bimodules and isomorphisms of bimodules

$$\{\Theta_x\colon x \in \mathcal{G}\}, \quad \left\{\mathcal{F}^{\Theta}_{x,y}: \Theta_x \otimes_R \Theta_y \xrightarrow{\cong} \Theta_{xy},\ x, y \in \mathcal{G}\right\},$$

where $\underline{\Theta}(x) = [\Theta_x]$ for every $x \in \mathcal{G}$. Consider the product on the direct sum $\bigoplus_{x\in\mathcal{G}} \Theta_x$ determined by the maps $\mathcal{F}^{\Theta}_{x,y}$. There is no reason to expect that this product, after the choice of the representatives Θ_x, will be associative. In fact, it becomes an associative multiplication if and only if the diagram

$$\begin{tikzcd}
\Theta_x \otimes_R \Theta_y \otimes_R \Theta_z \arrow[r, "\mathcal{F}^{\Theta}_{xy}\otimes\Theta_z"] \arrow[d, "\Theta_x\otimes\mathcal{F}^{\Theta}_{yz}"'] & \Theta_{xy} \otimes_R \Theta_z \arrow[d, "\mathcal{F}^{\Theta}_{xy,z}"] \\
\Theta_x \otimes_R \Theta_{yz} \arrow[r, "\mathcal{F}^{\Theta}_{x,yz}"] & \Theta_{xyz}
\end{tikzcd} \tag{5}$$

commutes for every $x, y, z \in \mathcal{G}$. Even in this case, it is not clear whether this multiplication has a set of local units. On the other hand, we know that there is an isomorphism of R-bimodules $\iota: R \to \Theta_1$, so a natural candidate for such a set should be $\mathsf{E}' = \iota(\mathsf{E})$. In fact, this is a set of local units if and only if, for every $x \in \mathcal{G}$, the following diagrams are commutative

$$\begin{tikzcd}
\Theta_x \otimes_R R \arrow[rr, "\simeq"] \arrow[dr, "\Theta_x\otimes\iota"'] & & \Theta_x, \\
& \Theta_x \otimes_R \Theta_1 \arrow[ur, "\mathcal{F}^{\Theta}_{x,1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
R \otimes_R \Theta_x \arrow[rr, "\simeq"] \arrow[dr, "\iota\otimes\Theta_x"'] & & \Theta_x. \\
& \Theta_1 \otimes_R \Theta_x \arrow[ur, "\mathcal{F}^{\Theta}_{1,x}"'] &
\end{tikzcd} \tag{6}$$

The previous discussion suggest then the following definition.

Definition 2.1. A *factor map* for the morphism $\underline{\Theta}: \mathcal{G} \to \mathbf{Pic}(R)$ consists of sets of invertible R-bimodules and isomorphisms of bimodules

$$\{\Theta_x\colon x \in \mathcal{G}\}, \quad \left\{\mathcal{F}^{\Theta}_{x,y}: \Theta_x \otimes_R \Theta_y \xrightarrow{\cong} \Theta_{xy},\ x, y \in \mathcal{G}\right\},$$

where $\underline{\Theta}(x) = [\Theta_x]$ for every $x \in \mathcal{G}$, and one isomorphism of R-bimodules $\iota: R \to \Theta_1$ such that the diagrams (5) and (6) are commutative.

Now we come back to our initial homomorphism of groups $\underline{\Theta}: \mathcal{G} \to \mathbf{Pic}(R)$ and its associated family of isomorphisms $\{\mathcal{F}^{\Theta}_{x,y}\}_{x,y,\in\mathcal{G}}$. Our next aim is to find conditions under which we can extract from these data a factor map for R and $\mathcal{G}$. First of all we should assume that the diagrams of (6) are commutative. We can define using the homomorphism of groups $\boldsymbol{\alpha}: \mathbf{Pic}(R) \to \mathbf{Aut}(\mathcal{U}(\mathcal{Z}))$ given by Eq. (4), a left $\mathcal{G}$-action on the group of units $\mathcal{U}(\mathcal{Z})$, where $\mathcal{Z} = \mathrm{End}_{R\text{-}R}(R)$. Written down explicitly the composition $\boldsymbol{\alpha} \circ \underline{\Theta}: \mathcal{G} \to \mathbf{Pic}(R) \to \mathbf{Aut}(\mathcal{U}(\mathcal{Z}))$ the outcome action is described as follows. Given a unit $e \in \mathsf{E}$, its decomposition with respect to the invertible R-bimodule Θ_x is written as $e = \sum_{(e)} x_e \bar{x}_e \in \iota^{-1}(\Theta_1) = R$. Using formula (4), the left $\mathcal{G}$-action on $\mathcal{U}(\mathcal{Z})$ is then defined by

$$\begin{aligned} \mathcal{G} \times \mathcal{U}(\mathcal{Z}) &\longrightarrow \mathcal{U}(\mathcal{Z}) \\ (x, u) &\longmapsto {}^{x}u = \Big[r \longmapsto \sum_{(e)} r x_e u(e_1) \bar{x}_e \Big], \end{aligned} \tag{7}$$

where $e \in Unit\{r\}$ and $e_1 \in Unit\{x_e, \bar{x}_e\}$. Recall that in this case $e_1 \in Unit\{e\}$, and so $e_1 \in Unit\{r\}$.

The abelian group of all n-cocycles with respect to this action is denoted by $Z^n_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$; while by $B^n_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$ we denote the n-coboundary sub-group. The n-cohomology group, is then denoted by $H^n_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$.

Remark 2.2. To each homomorphism of groups $\underline{\Theta} : \mathcal{G} \to \mathbf{Pic}(R)$, it corresponds as before a $\mathcal{G}$-group structure on $\mathcal{U}(\mathcal{Z})$, and so it leads to the cohomology groups $H^*_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$. It seems that this correspondence has not been systematically studied, even in the case of unital rings. However, as we will see, it is in fact the starting key point of any theory of generalized crossed products. Following the commutative and unital case, see Remark 2.13 below, here we will also extract our results from one fixed $\mathcal{G}$-action, this is to say fixed cohomology groups. For instance, one could start with the cohomology which corresponds to the trivial homomorphism of groups, that is, $\mathsf{I} : \mathcal{G} \to \mathbf{Pic}(R)$ sending $x \mapsto [R]$, of course here the cohomology is not at all trivial.

Here is a non-trivial example.

Example 2.3. (See [4].) Let R be a ring with E as a set of local units. Assume that there is a homomorphism of groups $\Phi : \mathcal{G} \to \mathbf{Aut}(R)$ to the group of ring automorphisms of R (we assume that $\Phi(x)(\mathsf{E}) = \mathsf{E}$, for every $x \in \mathcal{G}$). Composing with the canonical homomorphism $\mathbf{Aut}(R) \to \mathbf{Pic}(R)$ which sends $x \mapsto [R^x]$ (i.e. the R-bimodule whose underlying left R-module is ${}_R R$ and its right module structure is $r.s = rx(s)$), we obtain a homomorphism $\underline{\Theta} : \mathcal{G} \to \mathbf{Pic}(R)$. The corresponding family of maps $\{\mathcal{F}^{\Theta}_{x,y}\}_{x,y \in \mathcal{G}}$ is given by

$$\mathcal{F}^{\Theta}_{x,y} : R^x \otimes_R R^y \longrightarrow R^{xy} \quad \big(r \otimes_R s \longmapsto r x(s)\big).$$

Here of course the associativity and unitary properties of the multiplication of R, both imply that $\{\mathcal{F}^{\Theta}_{x,y}\}_{x,y \in \mathcal{G}}$ is actually a factor map for R and $\mathcal{G}$. A routine computation shows now that the cohomology $H^*_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$ which corresponds to this $\underline{\Theta}$ is exactly the one considered in [4, p. 135], since the $\mathcal{G}$-action (7) coincides for this case with that given in [4].

Lemma 2.4. *Keep the above $\mathcal{G}$-action on $\mathcal{U}(\mathcal{Z})$. Fix an element $x \in \mathcal{G}$.*

(1) *For every $\mathfrak{t} \in \Theta_x$ and $u \in \mathcal{U}(\mathcal{Z})$, we have*

$$\mathfrak{t} u(e) = {}^{x}u(e)\mathfrak{t}, \quad \textit{where } e \in Unit\{\mathfrak{t}\}. \tag{8}$$

(2) *Given similar bimodules $M \sim \Theta_x$; for every $\mathfrak{m} \in M$ and $u \in \mathcal{U}(\mathcal{Z})$, we have*

$$\mathfrak{m} u(e) = {}^{x}u(e)\mathfrak{m}, \quad \textit{with } e \in Unit\{\mathfrak{m}\}. \tag{9}$$

Proof. (1) This is Eq. (3).

(2) The identity (8) clearly lifts to finite direct sums $\Theta_x^{(n)}$, and, hence, for sub-bimodules M of $\Theta_x^{(n)}$. $\square$

Proposition 2.5. *Let R be a ring with a set of local units E and $\mathcal{G}$ any group. Assume that there is a homomorphism of groups $\underline{\Theta} : \mathcal{G} \to \mathbf{Pic}(R)$ whose family of maps $\{\mathcal{F}^{\Theta}_{x,y}\}_{x,y\in\mathcal{G}}$ satisfy the commutativity of diagrams (6). Denote by $\alpha_{x,y,z} : \Theta_{xyz} \to \Theta_{xyz}$ the isomorphisms that satisfy*

$$\alpha_{x,y,z} \circ \mathcal{F}^{\Theta}_{x,yz} \circ \big(\Theta_x \otimes_R \mathcal{F}^{\Theta}_{y,z}\big) = \mathcal{F}^{\Theta}_{xy,z} \circ \big(\mathcal{F}^{\Theta}_{x,y} \otimes_R \Theta_z\big). \tag{10}$$

Then $\widetilde{\alpha_{x,y,z}}$ defines a normalized element of $Z^3_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$, where $\widetilde{\alpha_{x,y,z}}$ is the image of $\alpha_{x,y,z}$ under the isomorphism of groups stated in Lemma 1.6. Furthermore, if $\widetilde{\alpha_{-,-,-}}$ is cohomologically trivial, that is, there is a normalized map $\sigma_{-,-} : \mathcal{G} \times \mathcal{G} \to \mathcal{U}(\mathcal{Z})$ such that

$$\widetilde{\alpha_{x,y,z}} = \sigma_{x,yz}{}^{x}\sigma_{y,z}\sigma^{-1}_{x,y}\sigma^{-1}_{xy,z},$$

for every $x, y, z \in \mathcal{G}$, then there is a factor map $\{\overline{\mathcal{F}}^{\Theta}_{x,y}\}_{x.y\in\mathcal{G}}$ for $\underline{\Theta}$, defined by

$$\begin{aligned} \overline{\mathcal{F}}^{\Theta}_{x,y} : \Theta_x \otimes_R \Theta_y &\longrightarrow \Theta_{xy} \\ \mathfrak{u}_x \otimes_R \mathfrak{u}_y &\longmapsto \sigma_{x,y}(e)\mathcal{F}^{\Theta}_{x,y}(\mathfrak{u}_x \otimes_R \mathfrak{u}_y), \end{aligned}$$

where $e \in Unit\{\mathfrak{u}_x, \mathfrak{u}_y\}$.

Proof. Consider $x, y, z, t \in \mathcal{G}$, and $\mathfrak{u}_x \in \Theta_x$, $\mathfrak{u}_y \in \Theta_y$, $\mathfrak{u}_z \in \Theta_z$ and $\mathfrak{u}_t \in \Theta_t$, denote by $\mathfrak{u}_x\mathfrak{u}_y$ the image of $\mathfrak{u}_x \otimes_R \mathfrak{u}_y$ under $\mathcal{F}^{\Theta}_{x,y}$. There are several isomorphisms from $\Theta_x \otimes_R \Theta_y \otimes_R \Theta_z \otimes_R \Theta_t$ to Θ_{xyzt} which are defined by the family $\mathcal{F}^{\Theta}_{-,-}$. So it will be convenient to adopt some notations. In order to do this, we rewrite the stated equation (10) satisfied by the $\alpha_{-,-,-}$'s, as follows

$$\alpha_{x,y,z}\big(\mathfrak{u}_x(\mathfrak{u}_y\mathfrak{u}_z)\big) = \big((\mathfrak{u}_x\mathfrak{u}_y)\mathfrak{u}_z\big).$$

Using Lemma 1.6, this is equivalent to

$$\beta_{x,y,z}(e)\big(\mathfrak{u}_x(\mathfrak{u}_y\mathfrak{u}_z)\big) = \big((\mathfrak{u}_x\mathfrak{u}_y)\mathfrak{u}_z\big),$$

where $e \in Unit\{\mathfrak{u}_x, \mathfrak{u}_y, \mathfrak{u}_z\}$ and $\beta_{-,-,-} := \widetilde{\alpha_{-,-,-}}$ is the image of $\alpha_{x,y,z}$ under the homomorphism of groups defined in Lemma 1.6. In this way, if we fix a unit $e \in Unit\{\mathfrak{u}_x, \mathfrak{u}_y, \mathfrak{u}_z, \mathfrak{u}_t\}$, then we have

$$\begin{aligned} \big((\mathfrak{u}_x\mathfrak{u}_y)\mathfrak{u}_z\big)\mathfrak{u}_t &= \beta_{x,y,z}(e)\big(\mathfrak{u}_x(\mathfrak{u}_y\mathfrak{u}_z)\big)\mathfrak{u}_t \\ &= \beta_{x,y,z}(e)\beta_{x,yz,t}(e)\big(\mathfrak{u}_x\big((\mathfrak{u}_y\mathfrak{u}_z)\mathfrak{u}_t\big)\big) \\ &= \beta_{x,y,z}(e)\beta_{x,yz,t}(e)\big(\mathfrak{u}_x\big(\beta_{y,z,t}(e)\big(\mathfrak{u}_y(\mathfrak{u}_z\mathfrak{u}_t)\big)\big)\big) \\ &= \beta_{x,y,z}(e)\beta_{x,yz,t}(e)\big(\mathfrak{u}_x\beta_{y,z,t}(e)\big(\mathfrak{u}_y(\mathfrak{u}_z\mathfrak{u}_t)\big)\big) \\ &\overset{(8)}{=} \beta_{x,y,z}(e)\beta_{x,yz,t}(e)\big({}^{x}\beta_{y,z,t}(e)\mathfrak{u}_x\big(\mathfrak{u}_y(\mathfrak{u}_z\mathfrak{u}_t)\big)\big) \\ &= \beta_{x,y,z}(e)\beta_{x,yz,t}(e){}^{x}\beta_{y,z,t}(e)\big(\mathfrak{u}_x\big(\mathfrak{u}_y(\mathfrak{u}_z\mathfrak{u}_t)\big)\big) \\ &= \beta_{x,y,z}(e)\beta_{x,yz,t}(e){}^{x}\beta_{y,z,t}(e)\beta^{-1}_{x,y,zt}(e)\big((\mathfrak{u}_x\mathfrak{u}_y)(\mathfrak{u}_z\mathfrak{u}_t)\big) \\ &= \beta_{x,y,z}(e)\beta_{x,yz,t}(e){}^{x}\beta_{y,z,t}(e)\beta^{-1}_{x,y,zt}(e)\beta^{-1}_{xy,z,t}(e)\big((\mathfrak{u}_x\mathfrak{u}_y)\mathfrak{u}_z\big)\mathfrak{u}_t. \end{aligned}$$

Given $h \in \mathsf{E}$, using the above notations, we can consider the following units

$$h = \sum_{(h)} x_h \bar{x}_h, \quad h_1 \in Unit\{x_h, \bar{x}_h\};$$

$$h_1 = \sum_{(h_1)} y_{h_1} \bar{y}_{h_1}, \quad h_2 \in Unit\{y_{h_1}, \bar{y}_{h_1}\};$$

$$h_2 = \sum_{(h_2)} z_{h_2} \bar{z}_{h_2}, \quad h_3 \in Unit\{z_{h_2}, \bar{z}_{h_2}\};$$

$$h_3 = \sum_{(h_3)} t_{h_3} \bar{t}_{h_3}, \quad h_4 \in Unit\{t_{h_3}, \bar{t}_{h_3}\}.$$

It is clear that h_4 is a unit for all the handled elements, in particular $h_4 \in Unit\{x_h, y_{h_1}, z_{h_2}, t_{h_3}\}$. According to the equality showed previously, for every set $\{x_h, y_{h_1}, z_{h_2}, t_{h_3}\}$, we have

$$\big((x_h y_{h_1}) z_{h_2}\big) t_{h_3} = \beta_{x,y,z}(h_4) \beta_{x,yz,t}(h_4)^x \beta_{y,z,t}(h_4) \beta^{-1}_{x,y,zt}(h_4) \beta^{-1}_{xy,z,t}(h_4) \big((x_h y_{h_1}) z_{h_2}\big) t_{h_3}.$$

Using the map $\mathcal{F}^{\Theta}_{t,t^{-1}}$, and summing up, we get

$$\sum_{(h_3)} \big((x_h y_{h_1}) z_{h_2}\big) t_{h_3} \bar{t}_{h_3} = \beta_{x,y,z}(h_4) \beta_{x,yz,t}(h_4)^x \beta_{y,z,t}(h_4) \beta^{-1}_{x,y,zt}(h_4) \beta^{-1}_{xy,z,t}(h_4) \big((x_h y_{h_1}) z_{h_2}\big) t_{h_3} \bar{t}_{h_3},$$

which means that

$$(x_h y_{h_1}) z_{h_2} = \beta_{x,y,z}(h_4) \beta_{x,yz,t}(h_4)^x \beta_{y,z,t}(h_4) \beta^{-1}_{x,y,zt}(h_4) \beta^{-1}_{xy,z,t}(h_4) (x_h y_{h_1}) z_{h_2}, \text{ equality in } \Theta_{xyz}.$$

Repeating thrice the same process, we end up with

$$\begin{aligned} h &= \beta_{x,y,z}(h_4) \beta_{x,yz,t}(h_4)^x \beta_{y,z,t}(h_4) \beta^{-1}_{x,y,zt}(h_4) \beta^{-1}_{xy,z,t}(h_4) h \\ &= \beta_{x,y,z}(h) \beta_{x,yz,t}(h)^x \beta_{y,z,t}(h) \beta^{-1}_{x,y,zt}(h) \beta^{-1}_{xy,z,t}(h), \end{aligned}$$

since $h_4 h = h h_4 = h$ and the involved maps are R-bilinear. Therefore, for any unit $h \in \mathsf{E}$, we have

$$\beta_{xy,z,t} \circ \beta_{x,y,zt}(h) = {}^x\beta_{y,z,t} \circ \beta_{x,yz,t} \circ \beta_{x,y,z}(h).$$

Since the $\beta_{-,-,-}$'s are R-bilinear, this equality implies the 3-cocycle condition, that is,

$$\beta_{xy,z,t} \circ \beta_{x,y,zt} = {}^x\beta_{y,z,t} \circ \beta_{x,yz,t} \circ \beta_{x,y,z}, \quad \text{for every } x, y, z, t \in \mathcal{G}.$$

A routine computation, using Eq. (8), shows the last statement. □

Corollary 2.6. *Let R be a ring with local units and $\mathcal{G}$ any group, and fix a factor map $\{\Theta_x\}_{x \in \mathcal{G}}$ for R and $\mathcal{G}$ with its associated cohomology groups. Assume that there is a family of invertible unital R-bimodules $\{\Omega_x\}_{x \in \mathcal{G}}$ such that $\Omega_x \sim \Theta_x$ (they are similar), for every $x \in \mathcal{G}$, with a family of R-bilinear isomorphisms*

$$\mathcal{F}_{x,y} : \Omega_x \otimes_R \Omega_y \longrightarrow \Omega_{xy}.$$

For $x, y, z \in \mathcal{G}$, we denote by $\alpha_{x,y,z} : \Omega_{xyz} \to \Omega_{xyz}$ the resulting isomorphisms which satisfy

$$\alpha_{x,y,z} \circ \mathcal{F}_{x,yz} \circ (\Omega_x \otimes_R \mathcal{F}_{y,z}) = \mathcal{F}_{xy,z} \circ (\mathcal{F}_{x,y} \otimes_R \Omega_z). \tag{11}$$

Then $\widetilde{\alpha_{x,y,z}}$ defines an element of $Z^3_\Theta(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$, where $\widetilde{\alpha_{x,y,z}}$ is the image of $\alpha_{x,y,z}$ under the isomorphism of groups stated in Lemma 1.6.

Proof. This is a direct consequence of Proposition 2.5 and Lemma 2.4(2). □

We need to clarify the situation when two different classes of representatives are chosen.

Proposition 2.7. *Let R be a ring with a set of local units E and $\mathcal{G}$ any group. Consider two factor maps $\{\mathcal{F}^\Theta_{x,y}\}_{x,y\in\mathcal{G}}$ and $\{\mathcal{F}^\Gamma_{x,y}\}_{x,y\in\mathcal{G}}$ such that, for every $x \in \mathcal{G}$, there exists an R-bilinear isomorphism $\mathsf{a}_x : \Gamma_x \xrightarrow{\cong} \Theta_x$. Denote by $\tau_{x,y}$, where $x, y \in \mathcal{G}$, the following R-bilinear isomorphism*

$$\tau_{x,y} : \Theta_{xy} \xrightarrow{{\mathcal{F}^{\Theta}_{x,y}}^{-1}} \Theta_x \otimes_R \Theta_y \xrightarrow{\cong} \Gamma_x \otimes_R \Gamma_y \xrightarrow{\mathcal{F}^\Gamma_{x,y}} \Gamma_{xy} \xrightarrow{\cong} \Theta_{xy}.$$

That is,

$$\tau_{x,y} \circ \mathcal{F}^\Theta_{x,y} \circ (\mathsf{a}_x \otimes_R \mathsf{a}_y) = \mathsf{a}_{xy} \circ \mathcal{F}^\Gamma_{x,y}.$$

Then $\widetilde{\tau_{x,y}}$ defines a normalized element of $Z^2_\Theta(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$, where $\widetilde{\tau_{x,y}}$ is the image of $\tau_{x,y}$ under the isomorphism of groups stated in Lemma 1.6.

Proof. Fix a set of elements $x, y, z \in \mathcal{G}$. Assume that we have

$$^{x}\widetilde{\tau_{y,z}}(h)\widetilde{\tau_{x,yz}}(h) = \widetilde{\tau_{x,y}}(h)\widetilde{\tau_{xy,z}}(h), \quad \text{for every unit } h \in \mathsf{E}. \tag{12}$$

So taking any element $r \in R$ with unit $f \in Unit\{r\}$, we obtain

$$\begin{aligned}
\widetilde{\tau_{x,yz}} \circ {}^{x}\widetilde{\tau_{y,z}}(r) &= \widetilde{\tau_{x,yz}} \circ {}^{x}\widetilde{\tau_{y,z}}(fr) \\
&= {}^{x}\widetilde{\tau_{y,z}}(f)\widetilde{\tau_{x,yz}}(r) \\
&= {}^{x}\widetilde{\tau_{y,z}}(f)\widetilde{\tau_{x,yz}}(f)r \\
&\overset{(12)}{=} \widetilde{\tau_{x,y}}(f)\widetilde{\tau_{xy,z}}(f)r \\
&= \widetilde{\tau_{xy,z}}\big(\widetilde{\tau_{x,y}}(f)r\big) \\
&= \widetilde{\tau_{xy,z}} \circ \widetilde{\tau_{x,y}}(r),
\end{aligned}$$

which shows that $\widetilde{\tau_{-,-}}$ is a 2-cocycle. In fact $\widetilde{\tau_{-,-}}$ is by definition a normalized 2-cocycle. Eq. (12) is fulfilled by following analogue steps as in the proof of Proposition 2.5. □

Proposition 2.8. *The hypothesis is that of Corollary 2.6. Assume further that there are two families of invertible unital R-bimodules $\{\Omega_x\}_{x\in\mathcal{G}}$ and $\{\Gamma_x\}_{x\in\mathcal{G}}$ as in Corollary 2.6 (i.e. $\Gamma_x \sim \Theta_x \sim \Omega_x$), together with families of R-bilinear isomorphisms*

$$\mathcal{F}^\Omega_{x,y} : \Omega_x \otimes_R \Omega_y \longrightarrow \Omega_{xy}, \qquad \mathcal{F}^\Gamma_{x,y} : \Gamma_x \otimes_R \Gamma_y \longrightarrow \Gamma_{xy}.$$

Assume also that there are R-bilinear isomorphisms $\mathsf{a}_x : \Omega_x \to \Gamma_x$, $x \in \mathcal{G}$. Consider the associated 3-cocycles $\beta^\Omega_{-,-,-}, \beta^\Gamma_{-,-,-} \in Z^3_\Theta(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$ given by Proposition 2.5. Then,

$$\beta^\Omega_{-,-,-} \circ \big(\beta^\Gamma_{-,-,-}\big)^{-1} \in B^2_\Theta\big(\mathcal{G}, \mathcal{U}(\mathcal{Z})\big).$$

That is, they are cohomologous, or $[\beta^\Omega_{-,-,-}] = [\beta^\Gamma_{-,-,-}]$ in $H^3_\Theta(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$.

Proof. Let us denote by $\alpha^{\Omega}_{-,-,-}$ and $\alpha^{\Gamma}_{-,-,-}$ the R-bilinear isomorphisms satisfying Eq. (11), respectively, for $\{\Omega_x\}_{x\in\mathcal{G}}$ and $\{\Gamma_x\}_{x\in\mathcal{G}}$. For any pair $x, y \in \mathcal{G}$, we consider the R-bilinear isomorphism $\mathsf{b}_{xy} : \Omega_{xy} \to \Omega_{xy}$ defined by the following equation

$$\mathsf{a}_{xy} \circ \mathsf{b}_{xy} \circ \mathcal{F}^{\Omega}_{x,y} = \mathcal{F}^{\Gamma}_{x,y} \circ (\mathsf{a}_x \otimes_R \mathsf{a}_y). \tag{13}$$

We denote by $\gamma_{x,y}$, $x, y \in \mathcal{G}$, the image of b_{xy} under the homomorphism of groups defined in Lemma 1.6.

Fix $x, y, z \in \mathcal{G}$, and consider $(\mathsf{t}_x, \mathsf{t}_y, \mathsf{t}_z) \in \Omega_x \times \Omega_y \times \Omega_z$ and $(\mathsf{u}_x, \mathsf{u}_y, \mathsf{u}_z) \in \Gamma_x \times \Gamma_y \times \Gamma_z$ with a common unit $e \in \mathit{Unit}\{\mathsf{t}_x, \mathsf{t}_y, \mathsf{t}_z, \mathsf{u}_x, \mathsf{u}_y, \mathsf{u}_z\}$. Using the notation of the proof of Proposition 2.5, we can write

$$\beta^{\Omega}_{x,y,z}(e)\mathsf{t}_x(\mathsf{t}_y\mathsf{t}_z) = (\mathsf{t}_x\mathsf{t}_y)\mathsf{t}_z, \tag{14}$$

$$\beta^{\Gamma}_{x,y,z}(e)\mathsf{u}_x(\mathsf{u}_y\mathsf{u}_z) = (\mathsf{u}_x\mathsf{u}_y)\mathsf{u}_z, \tag{15}$$

$$\gamma_{x,y}(e)\mathsf{a}_{xy}(\mathsf{t}_x\mathsf{t}_y) = \mathsf{a}_x(\mathsf{t}_x)\mathsf{a}_y(\mathsf{t}_y). \tag{16}$$

A routine computation as in the proof of Proposition 2.5 using Eqs. (14), (15) and (16), shows that, for every unit $h \in \mathsf{E}$, we have

$$\big(\beta^{\Omega}_{x,y,z}(h)\big)^{-1}\beta^{\Gamma}_{x,y,z}(h) = \big(\gamma_{x,yz}(h)\big)^{-1}\big({}^{x}\gamma_{y,z}(h)\big)^{-1}\gamma_{x,y}(h)\gamma_{xy,z}(h).$$

This implies that

$$\beta^{\Gamma}_{x,y,z} \circ \big(\beta^{\Omega}_{x,y,z}\big)^{-1} = \gamma_{xy,z} \circ \gamma_{x,y} \circ \big({}^{x}\gamma_{y,z}\big)^{-1} \circ (\gamma_{x,yz})^{-1}, \quad \text{in } \mathcal{U}(\mathcal{Z}),$$

which finishes the proof. □

Now we are able to give the right definition of generalized crossed product. We hope it will result cleaner than the one considered in [13,14] for rings with identity. In the sequel, R denotes a ring with a set of local units E.

Definition 2.9. Given a factor map as in Definition 2.1

$$\big\{\mathcal{F}^{\Theta}_{x,y} : \Theta_x \otimes_R \Theta_y \xrightarrow{\cong} \Theta_{xy},\ x, y \in \mathcal{G}\big\},$$

with an isomorphism of R-bimodules $\iota : R \to \Theta_1$, we define its associated *generalized crossed product* $\Delta(\Theta) = \bigoplus_{x\in\mathcal{G}} \Theta_x$ with multiplication

$$\theta_x\theta_y = \mathcal{F}^{\Theta}_{x,y}(\theta_x \otimes \theta_y), \quad \text{for } \theta_x \in \Theta_x,\ \theta_y \in \Theta_y.$$

It follows that Θ_1 is a subring of $\Delta(\Theta)$ and the map $\iota : R \to \Theta_1$ is a ring isomorphism. Therefore, $\iota(\mathsf{E})$ is a set of local units for Θ_1 that serves as well as a set of local units for $\Delta(\Theta)$. Normally, we will identify R with Θ_1, and, thus, E will be a set of local units for the generalized crossed product $\Delta(\Theta)$. Obviously, $\Delta(\Theta)$ is a $\mathcal{G}$-graded ring such that $\Theta_x\Theta_y = \Theta_{xy}$ for every $x, y \in \mathcal{G}$ (thus, it could be referred to as a *strongly graded* ring after [15]). For instance, if we take Θ as in Example 2.3, then $\Delta(\Theta)$ is the skew group ring $R * \mathcal{G}$, see [4].

Remark 2.10. A factor map as in Definition 2.1 determines a generalized crossed product $\Gamma = \bigoplus_{x\in\mathcal{G}} \Gamma_x$ with ring isomorphism $\iota : R \to \Gamma_1$ and multiplication given by $\gamma_x\gamma_y = \mathcal{F}^{\Gamma}_{x,y}(\gamma_x \otimes_R \gamma_y)$ for $\gamma_x \in \Gamma_x$, $\gamma_y \in \Gamma_y$. In other words, generalized crossed products and factor maps are just two ways to define the same mathematical object. A generalized crossed product Γ gives clearly a group homomorphism

$$\underline{\Gamma} : \mathcal{G} \longrightarrow \mathbf{Pic}(R) \quad \bigl(x \longmapsto [\Gamma_x]\bigr).$$

Whether a general group homomorphism $\mathcal{G} \to \mathbf{Pic}(R)$ gives some generalized crossed product is not so clear. However, as we have seen in Proposition 2.5, this is possible, if the underlying maps satisfy the commutativity of diagrams (6) and the associated 3-cocycle is trivial or at least cohomologically trivial.

Definition 2.11. A *morphism* of generalized crossed products $(\Delta(\Theta), \nu), (\Delta(\Gamma), \iota)$ is a graded ring homomorphism $f : \Delta(\Theta) \to \Delta(\Gamma)$ such that $f \circ \nu = \iota$. Equivalently, it consists of a set of homomorphisms of R-bimodules

$$\{f_x : \Theta_x \longrightarrow \Gamma_x \colon x \in \mathcal{G}\}$$

such that all the diagrams

$$\begin{tikzcd}
\Theta_x \otimes_R \Theta_y \arrow[r, "\mathcal{F}^{\Theta}_{x,y}"] \arrow[d, "f_x \otimes f_y"'] & \Theta_{xy} \arrow[d, "f_{xy}"] \\
\Gamma_x \otimes_R \Gamma_y \arrow[r, "\mathcal{F}^{\Gamma}_{x,y}"'] & \Gamma_{xy}
\end{tikzcd}$$

commute, and $f_1 \circ \nu = \iota$.

Let $R \subseteq S$ be an extension of rings with the same set of local units E. We denote as in [9] by $\mathbf{Inv}_R(S)$ the group of all invertible unital R-sub-bimodules of S. An R-sub-bimodule $X \subseteq S$ belongs to the group $\mathbf{Inv}_R(S)$ if and only if there exists an R-sub-bimodule $Y \subseteq S$ such that

$$XY = R = YX,$$

where the first and last terms are obviously defined by the multiplication of S. Clearly, there is a homomorphism of groups $\mu : \mathbf{Inv}_R(S) \to \mathbf{Pic}(R)$.

Remark 2.12. A generalized crossed product Γ of R with $\mathcal{G}$ gives in particular the ring extension $\iota : (R, E) \to (\Gamma, E')$ with local units. In this way, Γ_x is a unital R-sub-bimodule of Γ for every $x \in \mathcal{G}$, and the isomorphism $\iota : R \to \Gamma_1$ becomes R-bilinear. We will identify Γ_1 with R and E with E'. Let us denote by $\mathcal{F}^{\Gamma} : \Gamma \otimes_R \Gamma \to \Gamma$ the multiplication map of the ring Γ. It follows from [9, Lemma 1.1] that its restriction to $\Gamma_x \otimes_R \Gamma_y$ gives an R-bilinear isomorphism

$$\mathcal{F}^{\Gamma}_{xy} : \Gamma_x \otimes_R \Gamma_y \xrightarrow{\simeq} \Gamma_x\Gamma_y = \Gamma_{xy}$$

for every $x, y \in \mathcal{G}$, since, obviously, $\Gamma_x \in \mathbf{Inv}_R(\Gamma)$ for every $x \in \mathcal{G}$. Clearly, our generalized crossed product determines a factor map in the sense of Definition 2.1. On the other hand, the associated homomorphisms of groups $\underline{\Gamma}: \mathcal{G} \to \mathbf{Pic}(R)$ actually factors throughout the morphism μ. That is, we have a commutative diagram of groups

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "\underline{\Gamma}"] \arrow[dr, dashed, "\overline{\Gamma}"'] & & \mathbf{Pic}(R) \\ & \mathbf{Inv}_R(\Gamma). \arrow[ur, "\mu"'] & \end{tikzcd} \tag{17}$$

As we have seen in Remark 2.12, one can start with a fixed ring extension $R \subseteq S$ with the same set of local units E, and then work with the $\overline{\Gamma}$'s defined this time by $\overline{\Gamma}: \mathcal{G} \to \mathbf{Inv}_R(S)$ instead of the $\underline{\Gamma}$'s. In this case the situation becomes much more manageable, in the sense that each morphism $\overline{\Gamma}$ induces automatically a factor map and vice versa.

Remark 2.13. As was shown in [13, Remarks 1, 2 and 3], the above notion generalizes the classical notion of crossed product in the commutative Galois setting. Specifically, given a commutative Galois extension $\Bbbk \subseteq R$ with a finite Galois group $\mathcal{G}$, and consider the canonical homomorphism of groups $\Phi_0: \mathcal{G} \to \mathbf{Pic}_{\Bbbk}(R)$, where $\mathbf{Pic}_{\Bbbk}(R) = \{[P] \in \mathbf{Pic}(R) \mid kp = pk,\ \forall k \in \Bbbk,\ \ p \in P\}$, which sends $x \to [R^x]$ to the class of the R-bimodules R^x whose underlying left R-module is ${}_RR$ and its right R-module structure was twisted by the automorphism x, see Example 2.3. In this case, the over ring is chosen to be $S = \bigoplus_{x \in \mathcal{G}} \Phi_0(x)$ defined as the usual crossed product attached to Φ_0, which is not necessary trivial, that is, defined using any 2-cocycle. Thus, in the Galois theory of commutative rings, the homomorphism of groups which leads to the study of intermediate crossed products rings, is the obvious homomorphism of groups $\overline{\Theta}$ which satisfies $\Phi_0 = \mu \circ \overline{\Theta}$, where μ is as in diagram (17).

3. An abelian group of generalized crossed products

From now on, we fix a ring extension $R \subseteq S$ with the same set of local units E. Let $\mathcal{G}$ be any group and assume given a morphism of groups

$$\overline{\Theta}: \mathcal{G} \longrightarrow \mathbf{Inv}_R(S) \quad (x \longmapsto \Theta_x),$$

where we have used the notation of Remark 2.12. The multiplication of S induces then a factor map

$$\left\{\mathcal{F}^{\Theta}_{x,y}: \Theta_x \otimes_R \Theta_y \longrightarrow \Theta_{xy}:\ x, y \in \mathcal{G}\right\}.$$

Clearly $R = \Theta_1$, whence E is a set of local units for $\Delta(\Theta)$. We introduce in this section an abelian group $\mathcal{C}(\Theta/R)$ whose elements are the isomorphism classes of generalized crossed products whose homogeneous components are similar as R-bimodules to that of $\Delta(\Theta)$, see Section 2 for definition. Our definition is inspired from that of Miyashita [14, §2], however the proofs presented here are slightly different. This group will be crucial for the construction of the seven terms exact sequence of groups in the next section.

Let us consider thus the set $\mathcal{C}(\Theta/R)$ of isomorphism classes of generalized crossed products $[\Delta(\Gamma)]$ such that, for each $x \in \mathcal{G}$, we have $\Gamma_x \sim \Theta_x$, that is they are similar as R-bimodules, see Section 1.

Proposition 3.1. *Consider the set of isomorphisms classes of generalized crossed products $\mathcal{C}(\Theta/R)$ defined above. Then $\mathcal{C}(\Theta/R)$ has the structure of an abelian group. Moreover the subset $\mathcal{C}_0(\Theta/R)$ consisting of classes $[\Delta(\Lambda)]$ such that for every $x \in \mathcal{G}$, $\Lambda_x \cong \Theta_x$ as R-bimodules, is a sub-group of $\mathcal{C}(\Theta/R)$.*

Proof. Given two classes $[\Delta(\Omega)], [\Delta(\Gamma)] \in \mathcal{C}(\Theta/R)$, their multiplication is defined by

$$\big[\Delta(\Omega)\big]\big[\Delta(\Gamma)\big] = \bigg[\bigoplus_{x\in\mathcal{G}} \Omega_x \otimes_R \Theta_{x^{-1}} \otimes_R \Gamma_x\bigg].$$

The factor maps of its representative generalized crossed product are given by the following composition

$$\begin{tikzcd}
\Omega_x \otimes_R \Theta_{x^{-1}} \otimes_R \Gamma_x \otimes_R \Omega_y \otimes_R \Theta_{y^{-1}} \otimes_R \Gamma_y \arrow[dr, "\Omega_x \otimes \mathsf{T}_{\Theta_{x^{-1}}\otimes\Gamma_x, \Omega_y\otimes_R\Theta_{y^{-1}}} \otimes \Gamma_y"] \arrow[ddr, dashed, bend right, "\mathcal{F}^{\Omega\Gamma}_{x,y}"'] & \\
& \Omega_x \otimes_R \Omega_y \otimes_R \Theta_{y^{-1}} \otimes_R \Theta_{x^{-1}} \otimes_R \Gamma_x \otimes_R \Gamma_y \arrow[d, "\mathcal{F}^{\Omega}_{x,y} \otimes \mathcal{F}^{\Theta}_{y^{-1},x^{-1}} \otimes \mathcal{F}^{\Gamma}_{x,y}"] \\
& \Omega_{xy} \otimes_R \Theta_{(xy)^{-1}} \otimes_R \Gamma_{xy},
\end{tikzcd}$$

where $\mathsf{T}_{\Theta_{x^{-1}}\otimes\Gamma_x, \Omega_y\otimes_R\Theta_{y^{-1}}}$ is the twist R-bilinear map defined in Proposition 1.3. The associativity of the $\mathcal{F}^{\Omega\Gamma}_{-,-}$'s is easily deduced using Lemma 1.4. This multiplication is commutative since for any two classes $[\Delta(\Omega)], [\Delta(\Gamma)] \in \mathcal{C}(\Theta/R)$, we have a family of R-bilinear isomorphisms

$$\begin{tikzcd}
\Gamma_x \otimes_R \Theta_{x^{-1}} \otimes_R \Omega_x \arrow[r, "\cong"] \arrow[ddr, dashed, bend right, "\cong"'] & \Gamma_x \otimes_R \Theta_{x^{-1}} \otimes_R \Omega_x \otimes_R \Theta_{x^{-1}} \otimes_R \Theta_x \arrow[d, "\mathsf{T}\otimes\Theta_x"] \\
& \Omega_x \otimes_R \Theta_{x^{-1}} \otimes_R \Gamma_x \otimes_R \Theta_{x^{-1}} \otimes_R \Theta_x \arrow[d, "\cong"] \\
& \Omega_x \otimes_R \Theta_{x^{-1}} \otimes_R \Gamma_x,
\end{tikzcd}$$

which is easily shown to be compatible with both factors maps of Γ and Ω. Thus, it induces an isomorphism at the level of generalized crossed products.

It is clear that the class of $[\Delta(\Theta)]$ is the unit of this multiplication. The inverse of any class $[\Delta(\Omega)]$ is given by

$$\big[\Delta(\Omega)\big]^{-1} = \bigg[\bigoplus_{x\in\mathcal{G}} \Theta_x \otimes_R \Omega_{x^{-1}} \otimes_R \Theta_x\bigg].$$

Lastly, the fact that $\mathcal{C}_0(\Theta/R)$ is a sub-group of $\mathcal{C}(\Theta/R)$, can be immediately deduced from definitions. □

Proposition 3.2. *Consider the abelian group $\mathcal{C}_0(\Theta/R)$ of Proposition 3.1. Then there is an isomorphism of groups*

$$\mathcal{C}_0(\Theta/R) \cong H^2_{\Theta}\big(\mathcal{G}, \mathcal{U}(\mathcal{Z})\big).$$

Proof. The stated isomorphism is given by the following map:

$$\zeta : \mathcal{C}_0(\Theta/R) \longrightarrow H^2_{\Theta}\big(\mathcal{G}, \mathcal{U}(\mathcal{Z})\big) \quad \big(\big[\Delta(\Lambda)\big] \longmapsto [\widetilde{\tau}_{-,-}]\big),$$

where $\widetilde{\tau}_{-,-}$ is the normalized 2-cocycle defined in Proposition 2.7 and $[\widetilde{\tau}_{-,-}]$ denotes its equivalence class in $H^2_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$. First we need to show that this map is in fact well defined. To do so, we take two isomorphic generalized crossed products $\chi : \Delta(\Lambda) \xrightarrow{\cong} \Delta(\Sigma)$ which represent the same element in the sub-group $\mathcal{C}_0(\Theta/R)$, with R-bilinear isomorphisms $\mathsf{a}_x : \Lambda_x \to \Theta_x \leftarrow \Sigma_x : \mathsf{b}_x$, for all $x \in \mathcal{G}$. Then we check that the associated 2-cocycles are cohomologous. So let us denote these cocycles by $\widetilde{\tau}_{-,-}$ and $\widetilde{\gamma}_{-,-}$ associated, respectively, to $\Delta(\Lambda)$ and $\Delta(\Sigma)$. We need to show that $[\widetilde{\tau}_{-,-}] = [\widetilde{\gamma}_{-,-}]$. Recall from Proposition 2.7 that we have

$$\tau_{x,y} \circ \mathcal{F}^{\Theta}_{x,y} \circ (\mathsf{a}_x \otimes_R \mathsf{a}_y) = \mathsf{a}_{xy} \circ \mathcal{F}^{\Lambda}_{x,y}, \qquad \gamma_{x,y} \circ \mathcal{F}^{\Theta}_{x,y} \circ (\mathsf{b}_x \otimes_R \mathsf{b}_y) = \mathsf{b}_{xy} \circ \mathcal{F}^{\Sigma}_{x,y},$$

for every $x, y \in \mathcal{G}$, where $\mathcal{F}^{\Lambda}_{-,-}$ and $\mathcal{F}^{\Sigma}_{-,-}$ are, respectively, the factor maps relative to Λ and Σ. For each $x \in \mathcal{G}$, we set the following R-bilinear isomorphism

$$\beta_x : \Theta_x \xrightarrow{\mathsf{a}_x^{-1}} \Lambda_x \xrightarrow{\chi_x} \Sigma_x \xrightarrow{\mathsf{b}_x} \Theta_x,$$

where χ_x is the x-homogeneous component of the isomorphism χ. Then it is easily seen, using the fact that χ is morphism of generalized crossed products, that, for each pair of elements $x, y \in \mathcal{G}$, we have

$$\beta_{xy} \circ \tau_{x,y} \circ \mathcal{F}^{\Theta}_{x,y} = \gamma_{x,y} \circ \mathcal{F}^{\Theta}_{x,y} \circ (\beta_x \otimes_R \beta_y). \tag{18}$$

We claim that

$$\widetilde{\tau}_{x,y}(e)\widetilde{\beta}_{xy}(e) = \widetilde{\gamma}_{x,y}(e)\widetilde{\beta}_x(e)^x\widetilde{\beta}_y(e), \quad \text{for all } e \in \mathsf{E}. \tag{19}$$

So let $e \in \mathsf{E}$, and consider its decomposition $e = \sum_{(e)} x_e \bar{x}_e$ with respect to Θ_x, for a given $x \in \mathcal{G}$. Let $e_1 \in Unit\{x_e, \bar{x}_e\}$. Using Eq. (18), we get the following equality in S

$$\sum_{(e),(e_1)} \beta_{xy} \circ \tau_{x,y}(x_e y_{e_1})\bar{y}_{e_1}\bar{x}_e = \sum_{(e),(e_1)} \gamma_{x,y}\big(\beta_x(x_e)\beta_y(y_{e_1})\big)\bar{y}_{e_1}\bar{x}_e. \tag{20}$$

Routine computations show that,

$$\sum_{(e),(e_1)} \beta_{xy}\tau_{x,y}(x_e y_{e_1})\bar{y}_{e_1}\bar{x}_e = \widetilde{\tau}_{x,y}(e_1)\widetilde{\beta}_{xy}(e_1)e = \widetilde{\tau}_{x,y}(e)\widetilde{\beta}_{xy}(e), \quad \text{and}$$

$$\sum_{(e),(e_1)} \gamma_{xy}\big(\beta_x(x_e)\beta_y(y_{e_1})\big)\bar{y}_{e_1}\bar{x}_e \overset{(8)}{=} \widetilde{\gamma}_{x,y}(e_1)\widetilde{\beta}_x(e_1)^x\widetilde{\beta}_y(e_1)e = \widetilde{\gamma}_{x,y}(e)\widetilde{\beta}_x(e)^x\widetilde{\beta}_y(e),$$

which by (20) imply that

$$\widetilde{\tau}_{x,y}(e)\widetilde{\beta}_{xy}(e) = \widetilde{\gamma}_{x,y}(e)\widetilde{\beta}_{x}(e)^{x}\widetilde{\beta}_{y}(e),$$

which is the claimed equality. On the other hand we define

$$\begin{aligned} h : \mathcal{G} &\longrightarrow \mathcal{U}(\mathcal{Z}) = \mathbf{Aut}_{R\text{-}R}(R) \\ x &\longmapsto \Big(h_x : r \longmapsto \sum_{(e)} r\beta_x(x_e)\bar{x}_e\Big) \end{aligned}$$

where $e \in Unit\{r\}$, and $e = \sum_{(e)} x_e\bar{x}_e$ is the decomposition of e in $R = \Theta_x\Theta_{x^{-1}}$. It is clear form definitions that $\widetilde{\beta}_x = h_x$, for ever $x \in \mathcal{G}$. Taking an arbitrary element $r \in R$, we can show using Eq. (19) that

$$\widetilde{\tau}_{x,y} \circ h_{xy}(r) = \widetilde{\gamma}_{x,y} \circ {}^{x}h_y \circ h_x(r).$$

Therefore, $\widetilde{\tau}_{x,y} \circ h_{xy} = \widetilde{\gamma}_{x,y} \circ {}^{x}h_y \circ h_x$, which means that $\widetilde{\tau}_{-,-}$ and $\widetilde{\gamma}_{-,-}$ are cohomologous.

By the same way, we show that, if $\widetilde{\tau}_{-,-}$ and $\widetilde{\gamma}_{-,-}$ are cohomologous, then $\Delta(\Lambda/R)$ and $\Delta(\Sigma/R)$ are isomorphic as generalized crossed products, which means that the stated map ζ is injective. To show that this map is surjective, we start with a given normalized 2-cocycle $\sigma_{-,-} : \mathcal{G}^2 \to \mathcal{U}(\mathcal{Z})$, and consider the following R-bilinear automorphisms: for every $x \in \mathcal{G}$

$$\vartheta_x : \Theta_x \longrightarrow \Theta_x \quad \big(\mathfrak{u} \longmapsto \sigma_{x,x}(e)\mathfrak{u}\big),$$

where $e \in Unit\{\mathfrak{u}\}$. Now set $\Sigma_x = \Theta_x$ as an R-bimodule with the R-bilinear isomorphism $\vartheta_x : \Sigma_x \to \Theta_x$ previously defined. We define new factor maps relative to those Σ_x's, by

$$\mathcal{F}^{\Sigma}_{x,y} : \Sigma_x \otimes_R \Sigma_y \longrightarrow \Sigma_{xy} \quad \big(\mathfrak{u}_x \otimes_R \mathfrak{u}_y \longmapsto \sigma_{x,y}(e)\mathcal{F}^{\Theta}_{x,y}(\mathfrak{u}_x \otimes_R \mathfrak{u}_y)\big),$$

where $e \in Unit\{\mathfrak{u}_x, \mathfrak{u}_y\}$. The 2-cocycle condition on $\sigma_{-,-}$ gives the associativity of $\mathcal{F}^{\Sigma}_{-,-}$, while the normalized condition gives the unitary property. Thus, $\Delta(\Sigma/R)$ is a generalized crossed product whose isomorphic class belongs by definition to the sub-group $\mathcal{C}_0(\Theta/R)$.

Lastly, by a routine computation, using the definition of the twist natural map given in Proposition 1.3, we show that the map ζ is a homomorphism of groups. □

4. The Chase–Harrison–Rosenberg seven terms exact sequence

In this section we show the analogue of Chase–Harrison–Rosenberg's [5] seven terms exact sequence, generalizing by this the case of commutative Galois extensions with finite Galois group due of T. Kanzaki [13, Theorem, p. 187] and the noncommutative unital case treated by Y. Miyashita in [14, Theorem 2.12].

From now on we fix a ring extension $R \subseteq S$ with a same set of local units E, together with a morphism of groups $\overline{\Theta} : \mathcal{G} \to \mathbf{Inv}_R(S)$ and the associated generalized crossed product $\Delta(\Theta/R)$ having $\mathcal{F}^{\Theta}_{-,-}$ as a factor map relative to Θ.

We denote by $\mathbf{Pic}(R)$ and $\mathbf{Pic}(S)$, respectively, the Picard group of R and S. There is a canonical left $\mathcal{G}$-action on $\mathbf{Pic}(R)$ induced by the obvious homomorphism of groups $\mathbf{Inv}_R(S) \to \mathbf{Pic}(R)$ given explicitly by

$${}^{x}[P] = [\Theta_x \otimes_R P \otimes_R \Theta_{x^{-1}}], \quad \text{for every } x \in \mathcal{G}, \text{ and } [P] \in \mathbf{Pic}(R).$$

The corresponding $\mathcal{G}$-invariant sub-group is denoted by $\mathbf{Pic}(R)^{\mathcal{G}}$. We will consider the sub-group $\mathbf{Pic}_{\mathcal{Z}}(R)$ whose elements are represented by $\mathcal{Z}$-invariant R-bimodules, in the sense that

$$[P] \in \mathbf{Pic}_{\mathcal{Z}}(R) \quad \text{if and only if} \quad [P] \in \mathbf{Pic}(R), \quad \text{and} \quad \mathfrak{z}(e)pe' = ep\mathfrak{z}(e'), \quad \forall p \in P,$$

for every pair of units $e, e' \in \mathsf{E}$, and every element $\mathfrak{z} \in \mathcal{Z}$. In this way, we set

$$\mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}} := \mathbf{Pic}_{\mathcal{Z}}(R) \cap \mathbf{Pic}(R)^{\mathcal{G}}.$$

Recall from [9, Section 3] the group $\mathcal{P}(S/R)$:

$$\mathcal{P}(S/R) = \left\{ [P] \xRightarrow{[\phi]} [X] \,\middle|\, [P] \in \mathbf{Pic}(R),\ [X] \in \mathbf{Pic}(S),\ \text{and a map } \phi : {}_R P_R \longrightarrow {}_R X_R \right\}$$

where at least one of the maps $\overline{\phi}_l : P \otimes_R S \to X$, or $\overline{\phi}_r : S \otimes_R P \to X$, canonically attached to ϕ, is an isomorphisms of bimodules. For more details on the structure group of this set we refer to [9, Section 3]. Obviously there is a homomorphism of groups

$$\mathcal{O}_l : \mathcal{P}(S/R) \longrightarrow \mathbf{Pic}(R) \quad \left(\left([P] \xRightarrow{[\phi]} [X]\right) \longmapsto [P]\right). \tag{21}$$

This group also inherits the above $\mathcal{G}$-action. That is, there is a canonical left $\mathcal{G}$-action on this group which is induced by Θ and given as follows:

$${}^{x}\left([P] \xRightarrow{[\phi]} [X]\right) = \left([\Theta_x \otimes_R P \otimes_R \Theta_{x^{-1}}] \xRightarrow{[\psi]} [X]\right),$$

where $\psi := \Theta_x \otimes_R \phi \otimes_R \Theta_{x^{-1}}$ (composed with the multiplication of S). The sub-group $\mathcal{P}(S/R)^{\mathcal{G}}$ of $\mathcal{G}$-invariant elements of $\mathcal{P}(S/R)$ has a slightly simpler description:

Lemma 4.1. *Keeping the above notations, we have*

$$\mathcal{P}(S/R)^{\mathcal{G}} = \left\{ [P] \xRightarrow{[\phi]} [X] \in \mathcal{P}(S/R) \,\middle|\, \phi(P)\Theta_x = \Theta_x\phi(P),\ \textit{for every } x \in \mathcal{G} \right\},$$

where $\phi(P)\Theta_x$ and $\Theta_x\phi(P)$ stand for the obvious subsets of the S-bimodule X, e.g.

$$\phi(P)\Theta_x = \left\{ \sum_{\textit{finite}} \phi(p_i)\mathfrak{u}_x^i \,\middle|\, p_i \in P,\ \mathfrak{u}_x^i \in \Theta_x \right\}.$$

Proof. Is based up on two main facts. The first one is that the R-bilinear map $\phi : P \to X$ defining the given element $[P] \xRightarrow{[\phi]} [X] \in \mathcal{P}(S/R)$, is always injective, see [9, Lemma 3.1]. The second fact consists on the following equivalence: For a fixed $x \in \mathcal{G}$, saying that $\phi(P)\Theta_x = \Theta_x\phi(P)$ in ${}_S X_S$ is equivalent to say that there is an isomorphism $f_x : P \cong \Theta_x \otimes_R P \otimes_R \Theta_{x^{-1}}$ which completes the commutativity of the following diagram

$$\begin{tikzcd}
P \arrow[rrr, "\phi"] \arrow[d, dashed, "f_x"'] & & & X \arrow[d, equal] \\
\Theta_x \otimes_R P \otimes_R \Theta_{x^{-1}} \arrow[r, "\Theta_x \otimes \phi \otimes \Theta_{x^{-1}}"] \arrow[rrr, dashed, bend right, "\psi"'] & \Theta_x \otimes_R X \otimes_R \Theta_{x^{-1}} \arrow[r, hook] & S \otimes_R X \otimes_R S \arrow[r] & X.
\end{tikzcd}$$

The later means, by the definition of $\mathcal{P}(S/R)$, that

$$\big([P] \xRightarrow{[\phi]} [X]\big) = \big([\Theta_x \otimes P \otimes \Theta_{x^{-1}}] \xRightarrow{[\psi]} [X]\big). \qquad \square$$

4.1. *The first exact sequence*

Now we set

$$\mathcal{P}_{\mathcal{Z}}(S/R) = \big\{[P] \xRightarrow{[\phi]} [X] \in \mathcal{P}(S/R) \;\big|\; [P] \in \mathbf{Pic}_{\mathcal{Z}}(R)\big\},$$

and

$$\mathcal{P}_{\mathcal{Z}}(S/R)^{\mathcal{G}} = \mathcal{P}_{\mathcal{Z}}(S/R) \cap \mathcal{P}(S/R)^{\mathcal{G}}.$$

On the other hand, we consider the group $\mathbf{Aut}_{R\text{-}ring}(S)$ of all R-ring automorphisms of S (with same set of local units). It has the following sub-group

$$\mathbf{Aut}_{R\text{-}ring}(S)^{(\mathcal{G})} = \big\{f \in \mathbf{Aut}_{R\text{-}ring}(S) \;\big|\; f(\Theta_x) = \Theta_x, \text{ for all } x \in \mathcal{G}\big\}.$$

There are two interesting maps which will be used in the sequel: The first one is given by

$$\mathcal{E} : \mathbf{Aut}_{R\text{-}ring}(S) \longrightarrow \mathcal{P}(S/R) \quad \big(f \longmapsto \big([R] \xRightarrow{[\iota_f]} [S_f]\big)\big),$$

where ι_f is the inclusion $R \subset S_f$, and S_f is the S-bimodule induced from the automorphism f, that is, the left S-action is that of ${}_SS$, while the right one has been altered by f, that is, we have a new left S-action

$$s.s' = sf\big(s'\big), \quad \text{for all } s, s' \in S.$$

The second connects the group $\mathcal{U}(\mathcal{Z})$ with $\mathbf{Aut}_{R\text{-}ring}(S)$, and it is defined by

$$\mathcal{F} : \mathcal{U}(\mathcal{Z}) = \mathbf{Aut}_{R\text{-}R}(R) \longrightarrow \mathbf{Aut}_{R\text{-}ring}(S) \quad \big(\sigma \longmapsto \big[\mathcal{F}(\sigma) : s \longmapsto \sigma^{-1}(e)s\sigma(e)\big]\big), \tag{22}$$

where $e \in Unit\{s\}$, $s \in S$.

Proposition 4.2. *Keeping the previous notations, there is a commutative diagram whose rows are exact sequences of groups*

$$\begin{tikzcd}
\mathcal{U}(\mathcal{Z}) = \mathbf{Aut}_{R\text{-}R}(R) \arrow[r, "\mathscr{F}"] & \mathbf{Aut}_{R\text{-}ring}(S) \arrow[r, "\mathscr{E}"] & \mathcal{P}(S/R) \arrow[r, "\mathcal{O}_l"] & \mathbf{Pic}(R) \\
\mathcal{U}(\mathcal{Z}) = \mathbf{Aut}_{R\text{-}R}(R) \arrow[r] \arrow[u, equal] & \mathbf{Aut}_{R\text{-}ring}(S)^{(\mathcal{G})} \arrow[r] \arrow[u, hook] & \mathcal{P}_{\mathcal{Z}}(S/R)^{\mathcal{G}} \arrow[r] \arrow[u, hook] & \mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}}. \arrow[u, hook]
\end{tikzcd}$$

Proof. The exactness of the first row was proved in [9, Proposition 5.1]. The commutativity as well as the exactness of the second row, using routine computations, are immediately deduced from the definition of the involved maps. $\square$

The conditions under assumption, that is, the existence of $\overline{\Theta} : \mathcal{G} \to \mathbf{Inv}_R(S)$ a homomorphism of groups with a given extension of rings $R \subseteq S$ with the same set of local units E, are satisfied for the extension $R \subseteq \Delta(\Theta)$. Precisely, the map Θ factors thought out a homomorphism of groups $\overline{\Theta}' : \mathcal{G} \to \mathbf{Inv}_R(\Delta(\Theta))$ $(x \mapsto \Theta_x)$, as $R \subseteq \Delta(\Theta)$ is an extension of rings with the same set of local units E. Thus, applying Proposition 4.2 to this extension, we obtain the first statement of the following corollary.

Corollary 4.3. *Consider the generalized crossed product $\Delta := \Delta(\Theta)$ of S with $\mathcal{G}$. Then there is an exact sequence of groups*

$$1 \longrightarrow \mathcal{U}(\mathcal{Z}) \longrightarrow \mathbf{Aut}_{R\text{-}ring}(\Delta)^{(\mathcal{G})} \longrightarrow \mathcal{P}_{\mathcal{Z}}(\Delta/R)^{\mathcal{G}} \longrightarrow \mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}}.$$

In particular, we have the following exact sequence of groups

$$1 \longrightarrow H^1_{\Theta}\big(\mathcal{G}, \mathcal{U}(\mathcal{Z})\big) \xrightarrow{\mathscr{S}_1} \mathcal{P}_{\mathcal{Z}}(\Delta/R)^{\mathcal{G}} \xrightarrow{\mathscr{S}_2} \mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}}.$$

Proof. We only need to check the particular statement. So let $f \in \mathbf{Aut}_{R\text{-}ring}(\Delta)^{(\mathcal{G})}$, this gives a family of R-bilinear isomorphisms $f_x : \Theta_x \to \Theta_x$, $x \in \mathcal{G}$, which by Lemma 1.6 lead to a map

$$\sigma : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{Z}) \quad (x \longmapsto \widetilde{f}_x),$$

as $\Theta_x \in \mathbf{Inv}_R(S)$. An easy verification, using (8), shows that, for every unit $e \in \mathsf{E}$, we have

$$\widetilde{f}_{xy}(e) = \widetilde{f}_x(e)^x \widetilde{f}_y(e). \tag{23}$$

Therefore,

$$\sigma_{xy} = \sigma_x \circ {}^x\sigma_y, \quad \text{for every } x, y \in \mathcal{G}.$$

That is, σ is a normalized 1-cocycle, i.e. $\sigma \in Z^1_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$.

Conversely, given a normalized 1-cocycle $\gamma \in Z^1_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$, we consider the R-bilinear isomorphisms

$$g_x : \Theta_x \longrightarrow \Theta_x \quad \big(u \longmapsto \gamma_x(e)u\big), \text{ where } e \in Unit\{u\},\ u \in \Theta_x.$$

Clearly, the direct sum $g := \bigoplus_{x \in \mathcal{G}} g_x$ defines an automorphism of generalized crossed product. Hence, $g \in \mathbf{Aut}_{R\text{-}ring}(\Delta)^{(\mathcal{G})}$.

In conclusion we have constructed mutually inverse maps which in fact establish an isomorphism of groups

$$\mathbf{Aut}_{R\text{-}ring}(\Delta)^{(\mathcal{G})} \cong Z^1_{\Theta}\big(\mathcal{G}, \mathcal{U}(\mathcal{Z})\big). \tag{24}$$

Now let us consider an element $\mathfrak{u} \in \mathcal{U}(\mathcal{Z}) = \mathbf{Aut}_{R\text{-}R}(R)$, and its image $\mathcal{F}(\mathfrak{u})$ under the map $\mathcal{F}$ defined in (22) using the extension $R \subseteq \Delta$. Then, for every element $a \in \Theta_x$, we have

$$\begin{aligned} \mathcal{F}(\mathfrak{u})(a) &= \mathfrak{u}^{-1}(e)a\mathfrak{u}(e), \quad e \in Unit\{a\} \\ &\overset{(8)}{=} \mathfrak{u}^{-1}(e)\,{}^{x}\mathfrak{u}(e)a. \end{aligned}$$

Therefore, $\mathfrak{u}$ defines, under the isomorphism of (24), a 1-coboundary,

$$\mathcal{G} \longrightarrow \mathcal{U}(\mathcal{Z}) \quad \big(x \longmapsto {}^{x}\mathfrak{u} \circ \mathfrak{u}^{-1}\big).$$

We have then constructed an isomorphism of groups

$$\mathbf{Aut}_{R\text{-}ring}(\Delta)^{(\mathcal{G})}/\mathbf{Aut}_{R\text{-}R}(R) \cong H^1_{\Theta}\big(\mathcal{G}, \mathcal{U}(\mathcal{Z})\big).$$

The exactness follows now from the first statement which is a particular case of Proposition 4.2. □

4.2. The second exact sequence

For any element $[P] \in \mathbf{Pic}(R)$, we will denote by $[P^{-1}]$ its inverse element. Let us consider the following groups

$$\mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})} := \big\{[P] \in \mathbf{Pic}_{\mathcal{Z}}(R) \;\big|\; P \otimes_R \Theta_x \otimes_R P^{-1} \sim \Theta_x, \text{ for all } x \in \mathcal{G}\big\}.$$

A routine computation shows that this is a sub-group of $\mathbf{Pic}_{\mathcal{Z}}(R)$ which contains the sub-group of all $\mathcal{G}$-invariant elements. That is, there is a homomorphism of groups $\mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}} \to \mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})}$.

Given an element $[P] \in \mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})}$, set $\Omega_x := P \otimes_R \Theta_x \otimes_R P^{-1}$, for every $x \in \mathcal{G}$. Consider now the family of R-bilinear isomorphisms

$$\mathcal{F}^{\Omega}_{x,y} : \Omega_x \otimes_R \Omega_y \longrightarrow \Omega_{xy}$$

which is defined using the factor maps $\mathcal{F}^{\Theta}_{x,y}$ and the isomorphism $P \otimes_R P^{-1} \cong R$. This is clearly a factor map relative to the homomorphism of groups $\overline{\Omega} : \mathcal{G} \to \mathbf{Inv}_R(S)$ sending $x \mapsto \Omega_x$. This gives us a generalized crossed product $\Delta(\Omega)$, and in fact establishes a homomorphism of groups

$$\mathcal{L} : \mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})} \longrightarrow \mathcal{C}(\Theta/R),$$

where the right hand group was defined in Section 3. The proof of the following lemma is left to the reader.

Lemma 4.4. *Keep the above notations. There is a commutative diagram of groups*

$$\begin{tikzcd} \mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}} \arrow[r, "\mathcal{L}"] \arrow[d, hook] & \mathcal{C}_0(\Theta/R) \arrow[d, hook] \\ \mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})} \arrow[r, "\mathcal{L}"] & \mathcal{C}(\Theta/R). \end{tikzcd}$$

Now we can show our second exact sequence.

Proposition 4.5. *Let $\Delta(\Theta)$ be a generalized crossed product of S with $\mathcal{G}$. Then there is an exact sequence of groups*

$$\mathcal{P}_{\mathcal{Z}}\big(\Delta(\Theta)/R\big)^{\mathcal{G}} \xrightarrow{\ \mathcal{S}_2\ } \mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}} \xrightarrow{\ \mathcal{S}_3\ } \mathcal{C}_0(\Theta/R),$$

where $\mathcal{S}_2$ and $\mathcal{S}_3$ are, respectively, the restrictions of the map $\mathcal{O}_l$ given in Eq. (21) attached to the extension $R \subseteq \Delta(\Theta)$, and of the map defined in Lemma 4.4.

Proof. Let $[P] \in \mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}}$ such that $\mathcal{S}_3([P]) = [\Delta(\Omega)] = [\Delta(\Theta)]$, where for every $x \in \mathcal{G}$, $\Omega_x = P \otimes_R \Theta_x \otimes_R P^{-1}$. This means that we have an isomorphism $\Delta(\Omega) \cong \Delta(\Theta)$ of generalized crossed products, and so a family of R-bilinear isomorphisms $\iota_x : \Theta_x \otimes_R P \cong P \otimes_R \Theta_x$, $x \in \mathcal{G}$. We then obtain an R-bilinear isomorphism

$$\iota = \bigoplus_{x \in \mathcal{G}} \iota_x : \Delta(\Theta) \otimes_R P \longrightarrow P \otimes_R \Delta(\Theta),$$

which in fact satisfies the conditions stated in [8, Eqs. (5.1) and (5.2), p. 161]. This implies that $\Delta(\Theta) \otimes_R P$ admits a structure of $\Delta(\Theta)$-bimodule whose underlying left structure is given by that of ${}_{\Delta(\Theta)}\Delta(\Theta)$. In this way, the previous map ι becomes an isomorphism of $(\Delta(\Theta), R)$-bimodules, and so we have

$$\begin{aligned}
\big(\Delta(\Theta) \otimes_R P\big) \otimes_{\Delta(\Theta)} \big(P^{-1} \otimes_R \Delta(\Theta)\big) &\cong \big(\Delta(\Theta) \otimes_R P\big) \otimes_{\Delta(\Theta)} \big(\Delta(\Theta) \otimes_R P^{-1}\big) \\
&\cong \Delta(\Theta) \otimes_R P \otimes_R P^{-1} \\
&\cong \Delta(\Theta),
\end{aligned}$$

an isomorphism of $\Delta(\Theta)$-bimodules. Whence the isomorphic class of $\Delta(\Theta) \otimes_R P$ belongs to $\mathbf{Pic}(\Delta(\Theta))$. We have then constructed an element $[P] \xRightarrow{[\phi]} [\Delta(\Theta) \otimes_R P]$ of the group $\mathcal{P}(\Delta(\Theta)/R)$, where ϕ is the obvious map

$$\phi : P \longrightarrow \Delta(\Theta) \otimes_R P \quad (p \longmapsto e \otimes_R p),$$

where $e \in Unit\{p\}$. It is clear that, for every $x \in \mathcal{G}$, the equality $\phi(P)\Theta_x = \Theta_x\phi(P)$ holds true in $\Delta(\Theta) \otimes_R P$. Thus, $[P] \xRightarrow{[\phi]} [\Delta(\Theta) \otimes_R P] \in \mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{(\mathcal{G})}$ and

$$\mathcal{S}_2\big([P] \xRightarrow{[\phi]} \big[\Delta(\Theta) \otimes_R P\big]\big) = [P].$$

This shows the inclusion $\mathrm{Ker}(\mathcal{S}_3) \subseteq \mathrm{Im}(\mathcal{S}_2)$.

Conversely, let $[Q] \in \mathrm{Im}(\mathcal{S}_2)$, that is, there exists an element $[Q] \xRightarrow{[\psi]} [Y] \in \mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{\mathcal{G}}$. We need to compute the image $\mathcal{S}_3([Q])$. From the choice of $[Q]$ we know that $[Q] \in \mathbf{Pic}_{\mathcal{Z}}(R)$ and that, for every $x \in \mathcal{G}$, $\psi(Q)\Theta_x = \Theta_x\psi(Q)$ in the $\Delta(\Theta)$-bimodule Y. As in the proof of Lemma 4.1, the later means that, there are R-bilinear isomorphisms:

$$f_x : Q \otimes_R \Theta_x \xrightarrow{\ \cong\ } \Theta_x \otimes_R Q, \quad \text{for every } x \in \mathcal{G},$$

which convert commutative the following diagrams

$$
\begin{tikzcd}
Q \otimes_R \Theta_x \otimes_R \Theta_y \arrow[rr, "Q\otimes \mathcal{F}^{\Theta}_{x,y}"] \arrow[d, "f_x \otimes \Theta_y"'] & & Q \otimes_R \Theta_{xy} \arrow[d, "f_{xy}"] \arrow[dr, hook] & \\
 & & & Y. \\
\Theta_x \otimes_R Q \otimes_R \Theta_y \arrow[dr, "\Theta_x \otimes f_y"'] & & \Theta_{xy} \otimes_R Q \arrow[ur, hook] & \\
 & \Theta_x \otimes_R \Theta_y \otimes_R Q \arrow[ur, "\mathcal{F}^{\Theta}_{x,y}\otimes Q"'] & &
\end{tikzcd}
$$

Coming back to the image $\mathcal{S}_3([Q]) := [\Delta(\Gamma)]$, we know that for every $x \in \mathcal{G}$, we have $\Gamma_x = Q \otimes_R \Theta_x \otimes_R Q^{-1} \cong \Theta_x$, via the f_x's. The above commutative diagram shows that these isomorphisms are compatible with the factor maps $\mathcal{F}^{\Gamma}_{x,y}$ and $\mathcal{F}^{\Theta}_{x,y}$. Therefore, $\Delta(\Theta) \cong \Delta(\Gamma)$ as generalized crossed products, and so $\mathcal{S}_3([Q]) = [\Delta(\Theta)]$ the neutral element of the group $\mathcal{C}_0(\Theta/R)$. $\square$

4.3. *Comparison with a previous alternative exact sequence*

Let us assume here that $\underline{\Theta} : \mathcal{G} \xrightarrow{\sigma} \mathbf{Aut}(R) \to \mathbf{Pic}(R)$, as in Example 2.3. We consider the extension $R \subset R\mathcal{G}$, where $S := R\mathcal{G} = \Delta(\Theta)$ is the skew group ring of R by $\mathcal{G}$. Thus $\overline{\Theta} : \mathcal{G} \to \mathbf{Inv}_R(S)$ factors though out μ as in diagram (17). In what follows, we want to explain the relation between the five terms exact sequence given in [4, Theorem 2.8], and the resulting one by combining Proposition 4.5 and Corollary 4.3, that is the sequence

$$
\begin{aligned}
1 \longrightarrow H^1_{\Theta}\big(\mathcal{G}, \mathcal{U}(\mathcal{Z})\big) \xrightarrow{\ \mathcal{S}_1\ } \mathcal{P}_{\mathcal{Z}}\big(\Delta(\Theta)/R\big)^{\mathcal{G}} \xrightarrow{\ \mathcal{S}_2\ } \mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}} \\
\xrightarrow{\ \mathcal{S}_3\ } H^2_{\Theta}\big(\mathcal{G}, \mathcal{U}(\mathcal{Z})\big) \cong \mathcal{C}_0(\Theta/R).
\end{aligned}
$$

As we argued in Example 2.3, the cohomology $H^*_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$ coincides with that considered in [4]. Thus, following the notation of [4], the result [4, Theorem 2.8] says that

$$
1 \longrightarrow H^1_{\Theta}\big(\mathcal{G}, \mathcal{U}(\mathcal{Z})\big) \xrightarrow{\ \varphi_\sigma\ } \mathcal{G}\text{-}\mathbf{Pic}(R) \xrightarrow{\ F_\sigma\ } \mathbf{Pic}(R)^{\mathcal{G}} \xrightarrow{\ \Phi_\sigma\ } H^2_{\Theta}\big(\mathcal{G}, \mathcal{U}(\mathcal{Z})\big)
$$

is an exact sequence (of pointed sets). It is not difficult to see that the restriction of Φ_σ to $\mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}}$ gives exactly $\mathcal{S}_3$. On the other hand, for any element $[P] \overset{[\phi]}{\Longrightarrow} [X] \in \mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{\mathcal{G}}$, we will define an object (P, τ) where $\tau : \mathcal{G} \to \mathbf{Aut}_{\mathbb{Z}}(P)$ (to the group of additive automorphisms of P) is a $\mathcal{G}$-homomorphism [4, Definition 2.3]. As in the proof of Lemma 4.1, we know that there is a family of R-bilinear isomorphisms $\mathfrak{j}_x : R^x \otimes_R P \to P \otimes_R R^x$, $x \in \mathcal{G}$, so we define τ by the following composition

$$
\tau_x : {}^{x^{-1}}P \xrightarrow{\ \cong\ } R^x \otimes_R P \xrightarrow{\ \mathfrak{j}_x\ } P \otimes_R R^x \xrightarrow{\ \cong\ } P^x,
$$

here P^y denotes the R-bimodule constructed from P by twisting its right module structure using the automorphism y, and the same construction is used on the left. To check that τ is a homomorphism of groups, one uses the equality

$$\mathrm{j}_{xy} \circ \left(\mathcal{F}^{\Theta}_{x,y} \otimes_R P\right) = \left(P \otimes_R \mathcal{F}^{\Theta}_{x,y}\right) \circ \left(\mathrm{j}_x \otimes_R R^y\right) \circ \left(R^x \otimes_R \mathrm{j}_y\right), \quad \text{for all } x, y \in \mathcal{G}.$$

Now, if we take another representative, that is, if we assume that

$$[P] \xRightarrow{[\phi]} [X] = \left[P'\right] \xRightarrow{[\phi']} \left[X'\right] \in \mathcal{P}_{\mathcal{Z}}\left(\Delta(\Theta)/R\right)^{\mathcal{G}},$$

then, by the definition of the group $\mathcal{P}(\Delta(\Theta)/R)$, see [9, Section 3], there are bilinear isomorphisms $f : P \to P'$ and $g : X \to X'$ such that

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ P' \arrow[r, "\phi'"'] & X' \end{tikzcd}$$

commutes. In this way we have also the following commutative diagram

$$\begin{tikzcd} R^x \otimes_R P \arrow[r, "\mathrm{j}_x"] \arrow[d, "R^x \otimes f"'] & P \otimes_R R^x \arrow[d, "f \otimes R^x"] \\ R^x \otimes_R P' \arrow[r, "\mathrm{j}'_x"'] & P' \otimes_R R^x, \end{tikzcd}$$

which says that $\tau'_x \circ f = f \circ \tau_x$. This implies that $[P, \tau] = [P', \tau']$ in the group $\mathcal{G}$-**Pic**(R). We then have constructed a monomorphism of groups $\mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{\mathcal{G}} \to \mathcal{G}$-**Pic**$(R)$ such that the restriction of the map F_σ to $\mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{\mathcal{G}}$ is exactly $\mathcal{S}_2$. We leave to the reader to check that we furthermore obtain a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & H^1_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z})) \arrow[r, "\mathcal{S}_1"] \arrow[d, equal] & \mathcal{P}_{\mathcal{Z}}(\Delta(\Theta)/R)^{\mathcal{G}} \arrow[r, "\mathcal{S}_2"] \arrow[d, hook] & \mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}} \arrow[r, "\mathcal{S}_3"] \arrow[d, hook] & H^2_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z})) \arrow[d, equal] \\ 1 \arrow[r] & H^1_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z})) \arrow[r, "\varphi_\sigma"] & \mathcal{G}\text{-}\mathbf{Pic}(R) \arrow[r, "F_\sigma"] & \mathbf{Pic}(R)^{\mathcal{G}} \arrow[r, "\Phi_\sigma"] & H^2_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z})). \end{tikzcd}$$

4.4. The third exact sequence

We define the group $\mathcal{B}(\Theta/R)$ as the quotient group

$$\mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})} \xrightarrow{\mathcal{L}} \mathcal{C}(\Theta/R) \longrightarrow \mathcal{B}(\Theta/R) \longrightarrow 1, \tag{25}$$

where $\mathcal{L}$ is the map defined in Lemma 4.4.

The third exact sequence of groups is stated in the following

Proposition 4.6. *Let $\Delta(\Theta)$ be a generalized crossed product of S with $\mathcal{G}$. Then there is an exact sequence of groups*

$$\mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}} \xrightarrow{\mathcal{S}_3} \mathcal{C}_0(\Theta/R) \xrightarrow{\mathcal{S}_4} \mathcal{B}(\Theta/R),$$

where $\mathcal{S}_3$ and $\mathcal{S}_4$ are constructed from the sequence (25) and the commutativity of the diagram stated in Lemma 4.4.

Proof. The inclusion $\mathrm{Im}(\mathcal{S}_3) \subseteq \mathrm{Ker}(\mathcal{S}_4)$ is by construction trivial. Now, let $[\Delta(\Gamma)] \in \mathcal{C}_0(\Theta/R)$ such that $\mathcal{S}_4([\Delta(\Gamma)]) = 1$. By the definition of the group $\mathcal{B}(\Theta/R)$, there exists $[P] \in \mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})}$ such that $\mathcal{L}([P]) = [\Delta(\Gamma)]$. Therefore, we obtain a chain of R-bilinear isomorphisms:

$$\Theta_x \cong \Gamma_x \cong P \otimes_R \Theta_x \otimes_R P^{-1}, \quad \text{for all } x \in \mathcal{G},$$

which means that $[P] \in \mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}}$, and so $[\Delta(\Gamma)] = \mathcal{S}_3([P])$ which completes the proof. □

Remark 4.7. In the case of a unital commutative Galois extension $\Bbbk \subseteq R$ with a finite Galois group, see for instance [3, p. 396], and taking the extension $R \subseteq S$ and Θ as in Remark 2.13. The homomorphism $\mathcal{S}_4$ coincides, up to the isomorphism of Proposition 3.2, with the homomorphism of groups defined in [3, Theorem A12, p. 406] and where the group $\mathcal{B}(\Theta/R)$ coincides with the Brauer group of the $\Bbbk$-Azumaya algebras split by R.

4.5. *The fourth exact sequence*

Consider the following sub-group of the Picard group $\mathbf{Pic}(R)$:

$$\mathbf{Pic}_0(R) = \big\{[P] \in \mathbf{Pic}(R) \bigm| P \sim R, \text{ as bimodules}\big\}.$$

As we have seen in Proposition 1.3, this is an abelian group. Clearly it inherits the $\mathcal{G}$-module structure of $\mathbf{Pic}(R)$: The action is given by

$${}^{x}[P] = [\Theta_x \otimes_R P \otimes_R \Theta_{x^{-1}}], \quad \text{for every } x \in \mathcal{G}.$$

Lemma 4.8. *Keep the above notations. Then the map*

$$\begin{aligned} \mathcal{C}(\Theta/R) &\xrightarrow{\ \zeta\ } Z^1\big(\mathcal{G}, \mathbf{Pic}_0(R)\big) \\ \big[\Delta(\Gamma)\big] &\longmapsto \big[x \longmapsto [\Gamma_x][\Theta_{x^{-1}}]\big] \end{aligned}$$

defines a homomorphism of groups. Furthermore, there is an exact sequence of groups

$$1 \longrightarrow \mathcal{C}_0(\Theta/R) \longrightarrow \mathcal{C}(\Theta/R) \xrightarrow{\zeta} Z^1\big(\mathcal{G}, \mathbf{Pic}_0(R)\big).$$

Proof. The first claim is immediate from the definitions, as well as the inclusion $\mathcal{C}_0(\Theta/R) \subseteq \mathrm{Ker}(\zeta)$. Conversely, given an element $[\Delta(\Gamma)] \in \mathcal{C}(\Theta/R)$ such that $\zeta([\Delta(\Gamma)]) = 1$. This implies that for every $x \in \mathcal{G}$, there is an isomorphism of R-bimodule $\Gamma_x \otimes_R \Theta_{x^{-1}} \cong R$. Hence, $\Gamma_x \cong \Theta_x$, for every $x \in \mathcal{G}$, which means that $[\Delta(\Gamma)] \in \mathcal{C}_0(\Theta/R)$. $\square$

We define another abelian group, which in the commutative Galois case [13], coincides with the first cohomology group of $\mathcal{G}$ with coefficients in $\mathbf{Pic}_0(R)$. It is defined by the following commutative diagram

$$\begin{tikzcd}
\mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})} \arrow[rr, dashed] \arrow[dr, "\mathscr{L}"'] & & Z^1(\mathcal{G}, \mathbf{Pic}_0(R)) \arrow[r] & \overline{H}^1(\mathcal{G}, \mathbf{Pic}_0(R)) \arrow[r] & 1 \\
& \mathcal{C}(\Theta/R) \arrow[ur, "\zeta"'] & & &
\end{tikzcd}$$

whose row is an exact sequence. Our fourth exact sequence is given by the following.

Proposition 4.9. *Let $\Delta(\Theta)$ be a generalized crossed product of S with $\mathcal{G}$. Then there is an exact sequence of groups*

$$\mathcal{C}_0(\Theta/R) \xrightarrow{\mathcal{S}_4} \mathcal{B}(\Theta/R) \xrightarrow{\mathcal{S}_5} \overline{H}^1\big(\mathcal{G}, \mathbf{Pic}_0(R)\big).$$

Proof. First we need to define $\mathcal{S}_5$. We construct it as the map which completes the commutativity of the diagram

$$\begin{tikzcd}
& \mathcal{C}_0(\Theta/R) \arrow[d, hook] \arrow[dr, "\mathcal{S}_4"] & & \\
\mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})} \arrow[r, "\mathscr{L}"] & \mathcal{C}(\Theta/R) \arrow[r, "\mathscr{L}^c"] \arrow[d, "\zeta"'] & \mathcal{B}(\Theta/R) \arrow[r] \arrow[d, dashed, "\mathcal{S}_5"] & 1 \\
& Z^1(\mathcal{G}, \mathbf{Pic}_0(R)) \arrow[r] & \overline{H}^1(\mathcal{G}, \mathbf{Pic}_0(R)) \arrow[r] & 1,
\end{tikzcd}$$

whose first row is exact.

The fact that $\mathrm{Im}(\mathcal{S}_4) \subseteq \mathrm{Ker}(\mathcal{S}_5)$ is easily deduced from the previous diagram and Lemma 4.8. Conversely, let $\Xi = \mathscr{L}^c([\Delta(\Gamma)])$, for some $[\Delta(\Gamma)] \in \mathcal{C}(\Theta/R)$, be an element in $\mathcal{B}(\Theta/R)$ (here f^c denotes the cokernel map of f), such that $\mathcal{S}_5(\Xi) = \mathcal{S}_5 \circ \mathscr{L}^c([\Delta(\Gamma)]) = 1$. Then $(\zeta\mathscr{L})^c \circ \zeta([\Delta(\Gamma)]) = 1$, which implies that $\zeta([\Delta(\Gamma)]) \in \zeta(\mathscr{L}(\mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})}))$. Therefore, there exists an element $[P] \in \mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})}$ such that $[\Delta(\Gamma)]\mathscr{L}([P])^{-1} \in \mathrm{Ker}(\zeta) = \mathcal{C}_0(\Theta/R)$ by Lemma 4.8. Now, we have

$$\begin{aligned}
\mathcal{S}_4\big(\big[\Delta(\Gamma)\big]\mathscr{L}\big([P]\big)^{-1}\big) &= \mathcal{S}_4\big(\big[\Delta(\Gamma)\big]\mathscr{L}\big(\big[P^{-1}\big]\big)\big) \\
&= \mathscr{L}^c\big(\big[\Delta(\Gamma)\big]\big)\mathscr{L}^c\mathscr{L}\big(\big[P^{-1}\big]\big) \\
&= \mathscr{L}^c\big(\big[\Delta(\Gamma)\big]\big) = \Xi,
\end{aligned}$$

and this shows that $\Xi \in \mathrm{Im}(\mathcal{S}_4)$. $\square$

4.6. The fifth exact sequence and the main theorem

Keep from Section 4.5 the definition of the group $\mathbf{Pic}_0(R)$ with its structure of $\mathcal{G}$-module. Before stating the fifth sequence, we will need first to give a homomorphism of group from the group $\overline{H}^1(\mathcal{G}, \mathbf{Pic}_0(R))$ to the third cohomology group $H^3_\Theta(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$.

Given a normalized 1-cocycle $g \in Z^1(\mathcal{G}, \mathbf{Pic}_0(R))$ and put $g_x = [\nabla_x]$. Then, for every pair of elements $x, y \in \mathcal{G}$, one can easily shows that

$$ {}^x g_y[\Theta_x] = [\Theta_x] g_y. \tag{26}$$

Now, for every $x \in \mathcal{G}$, we set

$$[U_x] := g_x[\Theta_x], \quad \text{in } \mathbf{Pic}(R).$$

By the cocycle condition, we obtain

$$\begin{aligned} [U_{xy}] &= g_{xy}[\Theta_{xy}] \\ &= g_x {}^x g_y [\Theta_x][\Theta_y] \\ &\overset{(26)}{=} g_x[\Theta_x] g_y [\Theta_y] \\ &= [U_x][U_y]. \end{aligned}$$

This means that there are R-bilinear isomorphisms

$$\mathcal{F}^g_{x,y} : U_x \otimes_R U_y \longrightarrow U_{xy}, \tag{27}$$

with $\mathcal{F}^g_{1,x} = id = \mathcal{F}^g_{x,1}$ for every $x \in \mathcal{G}$. By Proposition 2.5, we have a 3-cocycle in $Z^3_\Theta(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$ attached to these maps $\mathcal{F}^g_{-,-}$, which we denote by $\alpha^g_{-,-,-}$. If there is another class $[V_x] \in \mathbf{Pic}(R)$ such that $[V_x] = g_x[\Theta_x]$, for every $x \in \mathcal{G}$, then the families of invertible R-bimodules $\{V_x\}_{x\in\mathcal{G}}$ and $\{U_x\}_{x\in\mathcal{G}}$ satisfy the conditions of Proposition 2.8. Therefore, the associated 3-cocycles are cohomologous. This means that the correspondence $g \mapsto [\alpha^g_{-,-,-}] \in H^3_\Theta(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$ is a well defined map. Henceforth, we have a well defined homomorphism of groups

$$\begin{array}{ccc} Z^1(\mathcal{G}, \mathbf{Pic}_0(R)) & \xrightarrow{\ \mathcal{S}_{13}\ } & H^3_\Theta(\mathcal{G}, \mathcal{U}(\mathcal{Z})) \\ g & \longmapsto & [\alpha^g_{-,-,-}]. \end{array} \tag{28}$$

The proof of the fact that $\mathcal{S}_{13}$ is a multiplicative map uses Lemma 1.1(ii) and the twisted natural transformation of Proposition 1.3. Complete and detailed steps of this proof are omitted.

Proposition 4.10. *Let $R \subseteq S$ be an extension of rings with the same set of local units, and $\Delta(\Theta)$ a generalized crossed product of S with $\mathcal{G}$. Then the homomorphism of formula (28) satisfies $\mathcal{S}_{13} \circ \zeta = [1]$, where ζ is the homomorphism of Lemma 4.8. Furthermore, there is a commutative diagram of groups*

$$\begin{tikzcd}
\mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})} \arrow[r, "(\zeta\circ\mathscr{L})"] & Z^1(\mathcal{G}, \mathbf{Pic}_0(R)) \arrow[r] \arrow[d, "\mathcal{S}_{13}"'] & \overline{H}^1(\mathcal{G}, \mathbf{Pic}_0(R)) \arrow[r] \arrow[dl, dashed, "\mathcal{S}_6"] & 1 \\
& H^3_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z})) & &
\end{tikzcd}$$

with exact row.

Proof. We need to check that $\mathcal{S}_{13} \circ \zeta([\Delta(\Gamma)]) = [1]$, for any element $[\Delta(\Gamma)] \in \mathcal{C}(\Theta/R)$. Set $f = \zeta([\Delta(\Gamma)])$, so we have $f_x[\Theta_x] = [\Gamma_x]$, for every $x \in \mathcal{G}$. Thus, the corresponding maps $\mathcal{F}^{f}_{-,-}$ of formula (27), are exactly given by the factor maps $\mathcal{F}^{\Gamma}_{-,-}$ of $\Delta(\Gamma)$ relative to Γ. They define an associative multiplication on $\Delta(\Gamma)$, and so induce a trivial 3-cocycle, which means that $\mathcal{S}_{13}(f) = [1]$. The last statement is now clear. □

At this level we are ending up with the following commutative diagram

$$\begin{tikzcd}
& \mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}} \arrow[r, "\mathcal{S}_3"] \arrow[d, hook] & \mathcal{C}_0(\Theta/R) \arrow[d, hook] \arrow[dr, "\mathcal{S}_4"] & & \\
1 \arrow[r] & \mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})} \arrow[r, "\mathscr{L}"] \arrow[d, equal] & \mathcal{C}(\Theta/R) \arrow[r, "\mathscr{L}^c"] \arrow[d, "\zeta"'] & \mathcal{B}(\Theta/R) \arrow[r] \arrow[d, "\mathcal{S}_5"] & 1 \\
& \mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})} \arrow[r, "(\zeta\circ\mathscr{L})"] & Z^1(\mathcal{G}, \mathbf{Pic}_0(R)) \arrow[r, "(\zeta\circ\mathscr{L})^c"] \arrow[d, "\mathcal{S}_{13}"] & \overline{H}^1(\mathcal{G}, \mathbf{Pic}_0(R)) \arrow[r] \arrow[dl, "\mathcal{S}_6"] & 1. \\
& & H^3_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z})) & &
\end{tikzcd}$$

Proposition 4.11. *Let $\Delta(\Theta)$ be a generalized crossed product of S with $\mathcal{G}$. Then there is an exact sequence of groups*

$$\mathcal{B}(\Theta/R) \xrightarrow{\mathcal{S}_5} \overline{H}^1\big(\mathcal{G}, \mathbf{Pic}_0(R)\big) \xrightarrow{\mathcal{S}_6} H^3_{\Theta}\big(\mathcal{G}, \mathcal{U}(\mathcal{Z})\big).$$

Proof. It is clear from the above diagram that $\mathrm{Im}(\mathcal{S}_5) \subseteq \mathrm{Ker}(\mathcal{S}_6)$ since by Proposition 4.10 $\mathcal{S}_{13} \circ \zeta = [1]$. Conversely, let us consider a class $[h] \in \overline{H}^1(\mathcal{G}, \mathbf{Pic}_0(R))$ such that $\mathcal{S}_6([h]) = 1$, for some element $h \in Z^1(\mathcal{G}, \mathbf{Pic}_0(R))$. By the same diagram, we also have that the class $\mathcal{S}_{13}(h) = [1] \in H^3_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$. This says that $\mathcal{S}_{13}(h) = [\beta]$, for some $\beta \in B^3_{\Theta}(\mathcal{G}, \mathcal{U}(\mathcal{Z}))$, and so there exists $\sigma_{-,-} : \mathcal{G} \times \mathcal{G} \to \mathcal{U}(\mathcal{Z})$ such that

$$\beta_{x,y,z} = \sigma_{x,yz}\, {}^{x}\sigma_{y,z}\sigma^{-1}_{x,y}\sigma^{-1}_{xy,z},$$

for every $x, y, z \in \mathcal{G}$. Now, for each $x \in \mathcal{G}$, we put $h_x[\Theta_x] = [\Omega_x]$, and consider $\mathcal{F}^{h}_{-,-}$ the associated family of maps as in (27), $\mathcal{F}^{h}_{x,y} : \Omega_x \otimes_R \Omega_y \to \Omega_{xy}$, where $x, y \in \mathcal{G}$. By definition the $\beta_{-,-,-}$'s satisfy

$$\beta_{x,y,z} \circ \mathcal{F}^{h}_{x,yz} \circ \big(\mathcal{F}^{h}_{x,y} \otimes_R \Omega_z\big) = \mathcal{F}^{h}_{xy,z} \circ \big(\Omega_x \otimes_R \mathcal{F}^{h}_{y,z}\big),$$

for every $x, y, z \in \mathcal{G}$. In this way, there are factor maps $\mathcal{F}^{\Omega}_{-,-}$ relative to Ω, defined by

$$\begin{aligned} \mathcal{F}^{\Omega}_{x,y} : \Omega_x \otimes_R \Omega_y &\longrightarrow \Omega_{xy} \\ \omega_x \otimes_R \omega_y &\longmapsto \sigma_{x,y}(e)\mathcal{F}^h_{x,y}(\omega_x \otimes_R \omega_y), \end{aligned}$$

where $e \in Unit\{\omega_x, \omega_y\}$. A routine computation shows that $\Delta(\Omega)$ is actually a generalized crossed product of S with $\mathcal{G}$ with factor maps $\mathcal{F}^{\Omega}_{-,-}$ relative to Ω.

On the other hand, since for each $x \in \mathcal{G}$, $h_x \in \mathbf{Pic}_0(R)$, we have $\Omega_x \sim \Theta_x$. Thus $[\Delta(\Omega)] \in \mathcal{C}(\Theta/R)$ with $\zeta([\Delta(\Omega)]) = h$. Whence

$$(\zeta\mathcal{L})^c(h) = [h] = (\zeta\mathcal{L})^c \circ \zeta\big(\big[\Delta(\Omega)\big]\big) = \mathcal{S}_5\big(\mathcal{L}^c\big(\big[\Delta(\Omega)\big]\big)\big),$$

which implies that $[h] \in \operatorname{Im}(\mathcal{S}_5)$, and this completes the proof. □

The following diagram summarizes the information we have shown so far.

$$\begin{tikzcd}
H^1_{\Theta}(\mathcal{G},\mathcal{U}(\mathcal{Z})) \arrow[r, "\mathcal{S}_1"] & \mathcal{P}_{\mathcal{Z}}(\Delta/R) \arrow[r, "\mathcal{S}_2"] & \mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}} \arrow[r, "\mathcal{S}_3"] \arrow[d, hook] & H^2_{\Theta}(\mathcal{G},\mathcal{U}(\mathcal{Z})) \arrow[d, hook] \arrow[dr, "\mathcal{S}_4"] & \\
 & & \mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})} \arrow[r, "\mathcal{L}"] \arrow[d, equal] & \mathcal{C}(\Theta/R) \arrow[r, "\mathcal{L}^c"] \arrow[d, "\zeta"'] & \mathcal{B}(\Theta/R) \arrow[d, "\mathcal{S}_5"] \\
 & & \mathbf{Pic}_{\mathcal{Z}}(R)^{(\mathcal{G})} \arrow[r, "(\zeta\circ\mathcal{L})"] & Z^1(\mathcal{G}, \mathbf{Pic}_0(R)) \arrow[r, "(\zeta\circ\mathcal{L})^c"] \arrow[d, "\mathcal{S}_{13}"] & \overline{H}^1(\mathcal{G}, \mathbf{Pic}_0(R)). \arrow[dl, "\mathcal{S}_6"] \\
 & & & H^3_{\Theta}(\mathcal{G},\mathcal{U}(\mathcal{Z})) &
\end{tikzcd}$$

We are now in position to announce our main result.

Theorem 4.12. *Let $R \subseteq S$ be an extension of rings with the same set of local units, and $\Delta(\Theta)$ a generalized crossed product of S with a group $\mathcal{G}$. Then there is an exact sequence of groups*

$$1 \longrightarrow H^1_{\Theta}\big(\mathcal{G},\mathcal{U}(\mathcal{Z})\big) \xrightarrow{\mathcal{S}_1} \mathcal{P}_{\mathcal{Z}}(\Delta/R)^{\mathcal{G}} \xrightarrow{\mathcal{S}_2} \mathbf{Pic}_{\mathcal{Z}}(R)^{\mathcal{G}} \xrightarrow{\mathcal{S}_3} H^2_{\Theta}\big(\mathcal{G},\mathcal{U}(\mathcal{Z})\big)$$
$$\xrightarrow{\mathcal{S}_4} \mathcal{B}(\Theta/R) \xrightarrow{\mathcal{S}_5} \overline{H}^1\big(\mathcal{G}, \mathbf{Pic}_0(R)\big) \xrightarrow{\mathcal{S}_6} H^3_{\Theta}\big(\mathcal{G},\mathcal{U}(\mathcal{Z})\big).$$

Proof. This is a direct consequence of Corollary 4.3 and Propositions 4.5, 4.6, 4.9, and 4.11. □

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