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Prime and Primitive Ideals of a Class of Iterated Skew Polynomial Rings

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INTRODUCTION

Hodges and Levasseur described the primitive spectra of quantized coordinate rings of SL_3 [14], and then of SL_n [15]. These results established a close connection between primitive ideals, torus action, and Poisson geometry. The proofs relied on explicit computations involving generators and relations.

Subsequently, Joseph generalized the Hodges–Levasseur program to semisimple algebraic quantum groups [17]. Hodges *et al.* then expanded Joseph's work to include certain multiparameter deformations [16]. These papers rely less on concrete calculations and more on deeper, more conceptual, techniques.

It is a natural and important question, then, as to how the preceding theory might apply to other algebras, particularly other algebras arising in the study of quantum groups.

Goodearl and Letzter established parallel results for certain iterated skew polynomial rings [13], with application to quantum Weyl algebras and to some quantum coordinate rings.

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There have also been several papers (including [22–26]) by S. Q. Oh and collaborators that further the above program for certain other quantum coordinate rings.

The present paper fits into the above framework by, first, extending the work of Oh on primitive spectra [22, 26] to the prime spectra of the iterated Ore extensions introduced in [23], and second, providing a more detailed version of the Goodearl–Letzter study for these cases [13].

Goodearl and Letzter's general framework is to consider some group $\mathscr{H}$ acting as automorphisms on a ring R which give the set $\mathscr{H}$-Spec(R) consisting of all $\mathscr{H}$-prime ideals of R. The *$\mathscr{H}$-stratification* of the prime spectrum Spec(R) is then defined as

$$\text{(1)} \qquad \operatorname{Spec}(R) = \biguplus_{j \in \mathscr{H}\text{-}\operatorname{Spec}(R)} \operatorname{Spec}_J(R),$$

where each stratum $\operatorname{Spec}_J(R)$ consists of those prime ideals P of R such that $\bigcap_{h\in\mathscr{H}} h(P) = J$.

In the case that $\mathscr{H}$ is a torus of rank r acting rationally on a noetherian algebra R over an infinite field $\Bbbk$ (see [13] for details), the strata $\operatorname{Spec}_J(R)$ corresponding to completely prime $\mathscr{H}$-invariant ideals J of R are described in [13, Theorem 6.6] as follows.

(a) For each completely prime $\mathscr{H}$-invariant ideal J of R, there exists an Ore set $\mathscr{E}_J$ in the algebra R/J such that the localization map $R \to R/J \to R_J = (R/J)[\mathscr{E}_J^{-1}]$ induces a homeomorphism of $\operatorname{Spec}_J(R)$ onto $\operatorname{Spec}(R_J)$.

(b) Contraction and extension induce mutually inverse homeomorphisms between $\operatorname{Spec}(R_J)$ and $\operatorname{Spec}(Z(R_J))$, where $Z(R_J)$ is the centre of R_J.

(c) $Z(R_J)$ is a commutative Laurent polynomial ring over an extension of $\Bbbk$, in r or fewer indeterminates.

The foregoing description of the $\mathscr{H}$-strata applies to iterated Ore extensions of $\Bbbk$ under suitable conditions [13, Sect. 4]. For some quantized coordinate rings, the aforementioned general stratification of the prime spectrum can be worked out in detail. This non-trivial research has been done for the coordinate algebras of quantum symplectic spaces $\mathscr{O}_q(\mathfrak{sp}\Bbbk^{2\times n})$ in [8]. These algebras belong to the class of algebras $R_n^{(C,\Lambda)}(\Bbbk)$ introduced in [23], which also includes the coordinate rings $\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times n})$ of quantum euclidean spaces and the quantum Weyl algebras $A_n^{(\mathbf{q},\Lambda)}(\Bbbk)$.

The aim of this note is to give a detailed description of the prime spectra of the $\Bbbk$-algebras $R_n^{(C,\Lambda)}(\Bbbk)$ (see Definition 1.1), where $C = (c_1, \ldots, c_n, d, \lambda, u)$ is an element of $(\Bbbk^\times)^{n+2} \times \Bbbk$, such that $d = 1$ if $u \neq 0$ and $\Lambda = (\lambda_{ij})$, $\lambda_{ii} = 1$ is a multiplicatively anti-symmetric matrix with entries in $\Bbbk^\times$.

We first define a rational action of the torus $\mathscr{H} = (\Bbbk^\times)^r$, where $r = n$ if $u \neq 0$ and $r = n+1$ if $u = 0$, on the $\Bbbk$-algebra $R_n^{(C,\Lambda)}(\Bbbk)$ for an infinite base field $\Bbbk$, and we show that [13, Theorem 6.6] applies to $R_n^{(C,\Lambda)}(\Bbbk)$ for any $\mathscr{H}$-prime ideal. The algebra $R_n^{(C,\Lambda)}(\Bbbk)$ is filtered with a finite-dimensional filtration with semi-commutative associated graded algebra, (see, e.g., [4, Section 3; 5]) which implies, by [19, Theorem 3.8], that $R_n^{(C,\Lambda)}(\Bbbk)$ satisfies the Nullstellensatz over an arbitrary field $\Bbbk$, so [13, Corollary 6.9] applies to $R_n^{(C,\Lambda)}(\Bbbk)$.

In a second step we give a more explicit description of the $\mathscr{H}$-stratification of the spectra of $R_n^{(C,\Lambda)}(\Bbbk)$ in the following aspects.

(1) We prove that the $\mathscr{H}$-prime ideals are just the ideals generated by the admissible sets in the sense of [22]. More explicitly, consider the finite subset $\wp_n$ of $R_n^{(C,\Lambda)}(\Bbbk)$ as defined later in (6). The map $J \mapsto J \cap \wp_n$ gives a bijection between the $\mathscr{H}$-prime ideals of $R_n^{(C,\Lambda)}(\Bbbk)$ and the admissible subsets of $\wp_n$ (Proposition 2.10).

(2) For each $\mathscr{H}$-prime ideal J, let $T = J \cap \wp_n$ the corresponding admissible set. We give explicitly a McConnell–Pettit $\Bbbk$-algebra $\mathbf{P}(Q_T)$, which is strictly contained in R_J, such that the Jth stratum is described as

$$\mathrm{Spec}_J(R_n^{(C,\Lambda)}(\Bbbk)) = \{P \in \mathrm{Spec}(R_n^{(C,\Lambda)}(\Bbbk)) \mid P \cap \wp_n = T\},$$

and it is homeomorphic to the spectrum of $\mathbf{P}(Q_T)$ (Theorem 3.4).

(3) By using [12], we obtain that each stratum is homeomorphic to the spectrum of the centre $Z(\mathbf{P}(Q_T))$ of $\mathbf{P}\ (Q_T)$ for a suitable admissible set T. In the particular case $R_n^{(C,\Lambda)}(\Bbbk) = \mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times n})$, we give an explicit method to compute the number of indeterminates in the Laurent polynomial ring $Z(\mathbf{P}(Q_T))$ over $\Bbbk$, for any admissible set T (Corollary 4.8).

The first obstacle is to prove the nice properties of the ideals generated by the admissible sets as in [22]. Using well known results from [11], we prove that each admissible set T generates a polynormal prime ideal $\langle T \rangle$. We compute the Gelfand–Kirillov dimension of the factor algebras $R_n^{(C,\Lambda)}(\Bbbk)/\langle T \rangle$ by using Gröbner–Basis techniques; see [3]. An explicit homomorphism Φ_T connecting $R_n^{(C,\Lambda)}(\Bbbk)$ and the McConnell–Pettit algebra $\mathbf{P}(Q_T)$ is given. Such a mapping was used by Rigal in the case of quantum Weyl algebras [27] (see also [26] for a similar morphism in the quantum euclidean case).

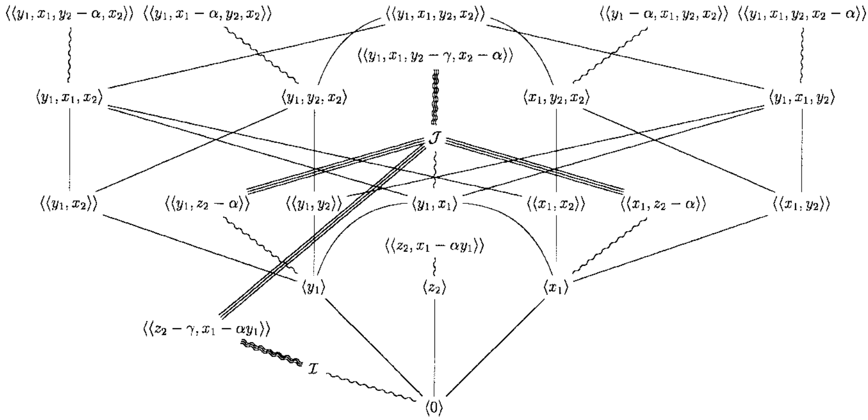

FIG. 1. The prime spectrum of $\mathcal{O}_q(\mathfrak{o}\mathbb{k}^{2\times 2})$ ($\mathbb{k}$ is algebraically closed, $\alpha, \gamma \in \mathbb{k}^{\times}$).

Our methods allow us to give an effective description (modulo Commutative Algebra) of $\mathrm{Spec}(\mathcal{O}_q(\mathfrak{o}\mathbb{k}^{2\times n}))$ for each given n (Corollary 4.9). This is possible because each prime ideal in the stratum $\mathrm{Spec}_T(\mathcal{O}_q(\mathfrak{o}\mathbb{k}^{2\times n}))$ is recognized as the inverse image under the algebra homomorphism Φ_T. In the algebraically closed case, we give an effective method to compute the primitive ideals of $\mathcal{O}_q(\mathfrak{o}\mathbb{k}^{2\times n})$. As an illustration, we compute $\mathrm{Spec}(\mathcal{O}_q(\mathfrak{o}\mathbb{k}^{2\times 2}))$ and $\mathrm{Prim}(\mathcal{O}_q(\mathfrak{o}\mathbb{k}^{2\times 2}))$ (see Fig. 1). Using the epimorphism defined in [23, Example 5], we determine the prime spectrum of $\mathcal{O}_q(\mathfrak{o}\mathbb{k}^{2n+1})$ (q has a square root in $\mathbb{k}$), and we compute $\mathrm{Spec}(\mathcal{O}_q(\mathfrak{o}\mathbb{k}^{3}))$ as an example (see Fig. 2).

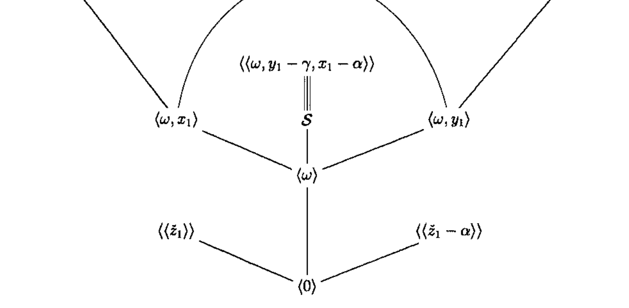

FIG. 2. The prime spectrum of $\mathcal{O}_q(\mathfrak{o}\mathbb{k}^{3})$ ($\mathbb{k}$ is algebraically closed, $\alpha, \gamma \in \mathbb{k}^{\times}$).

1. DEFINITION AND BASIC PROPERTIES

Throughout this note we will consider different quantum spaces, so we will use some convenient notation. Let $\Lambda = (\lambda_{ij})$ be a $p \times p$ matrix with entries in $\Bbbk$ such that $\lambda_{ii} = 1$ and $\lambda_{ji} = \lambda_{ji}^{-1}$. Consider the $\Bbbk$-algebra $\Bbbk_\Lambda[t_1, \dots, t_p]$ generated by $t_1, \dots, t_p$ subject to the relations $t_i t_j = \lambda_{ij} t_j t_i$. This is called the *coordinate algebra of the p-dimensional quantum affine space* associated to Λ and it is the iterated Ore extension

$$\Bbbk_\Lambda[t_1, \dots, t_p] = \Bbbk[t_1][t_2; \sigma_2] \cdots [t_p; \sigma_p], \tag{2}$$

where $\sigma_i(t_j) = \lambda_{ij} t_j$ for every $1 \leq j < i \leq p$. This $\Bbbk$-algebra is a noetherian domain, and its skew field of fractions is denoted by $\Bbbk_\Lambda(t_1, \dots, t_p)$. A useful intermediate algebra is the McConnell–Pettit algebra $\mathbf{P}(\Lambda) = \Bbbk_\Lambda[t_1^{\pm}, \dots, t_p^{\pm}]$ (see [20]).

DEFINITION 1.1. Let $\Bbbk$ be a field and let n be a strictly positive integer. Let $C = (c_1, c_2, \dots, c_n, d, \lambda, u)$ be an element of $(\Bbbk^\times)^{n+2} \times \Bbbk$ with $d = 1$ if $u \neq 0$. Consider a multiplicatively anti-symmetric matrix $\Lambda = (\lambda_{ji})_{1 \leq i < j \leq n}$ with entries in $\Bbbk^\times$ such that $\lambda_{ii} = 1$ for all $i = 1, \dots, n$. Define $R_n^{(C,\Lambda)}(\Bbbk)$ to be the finitely generated $\Bbbk$-algebra with generators $y_1, x_1, \dots, y_n, x_n$, satisfying the following relations

$$\begin{aligned}
&y_j y_i = \lambda_{ji} y_i y_j, && y_j x_i = \lambda_{ji}^{-1} d x_i y_j \quad (j > i)\\
&x_j x_i = \lambda_{ji} c_i^{-1} d^{-1} x_i x_j, && x_j y_i = \lambda_{ji}^{-1} c_i y_i x_j \quad (j > i)\\
&x_i y_i = c_i y_i x_i + \lambda \sum_{l=1}^{i-1} (\lambda d)^{i-1-l} (c_l d - 1) y_l x_l + (d\lambda)^{i-1} u && (i \geq 1).
\end{aligned} \tag{3}$$

This algebra was defined by Oh in [23]. By [23, p. 39], $R_n^{(c,\Lambda)}(\Bbbk)$ is an iterated Ore extension

$$R_0 \subseteq R_1 \subseteq \cdots \subseteq R_n^{(c,\Lambda)}(\Bbbk) = R_n,$$

where $R_0 = \Bbbk$ and $R_k = R_{k/2}[x_k, \beta_k, \delta_k]$, $R_{k/2} = R_{k-1}[y_k, \alpha_k]$ for all $k \geq 1$, and α_i, β_i are algebra automorphisms defined by

$$\begin{aligned}
&\alpha_j(y_i) = \lambda_{ji} y_i, && \alpha_j(x_i) = \lambda_{ji}^{-1} d x_i, && i < j\\
&\beta_j(y_i) = \lambda_{ji}^{-1} c_i y_i, && \beta_j(x_i) = \lambda_{ji} c_i^{-1} d^{-1} x_i, && i < j\\
&\beta_i(y_i) = c_i y_i,
\end{aligned} \tag{4}$$

and each δ_i is a left β_i-derivation defined by

$$\delta_i(y_i) = \lambda \sum_{l=1}^{i-1} (\lambda d)^{i-1-l} (c_l d - l) y_l x_l + (\lambda d)^{i-1} u, \; i > 1$$

$$\delta_i(R_{i-1}) = 0, \; i \geq 1, \qquad \text{and} \qquad \delta_1(y_1) = u.$$

By $\sum_k^n$ we denote the set $\{\alpha_k, \beta_k, \delta_k, \dots, \alpha_n, \beta_n, \delta_n\}$ for each $k = 1, \dots, n$.

This class of algebras includes the quantum Weyl algebras $A_n^{(\mathbf{q},\Lambda)}(\Bbbk)(C = (\mathbf{q}, 1, 1, 1)$ with $\mathbf{q} = (c_i)_{i=1}^n)$, the coordinate rings of quantum symplectic spaces $\mathscr{O}_q(\mathfrak{sp}\Bbbk^{2\times n})(C = (q^2, \dots, q^2, 1, q, 0)$, $\lambda_{ji} = q)$, and the coordinate rings of quantum Euclidean spaces $\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times n})(C = (1, \dots, 1, q^{-2}, q, 0)$, $\lambda_{ji} = q^{-1})$.

Following [23, p. 39], we have

LEMMA 1.2. *Set $z_i = dx_iy_i - y_ix_i$ for $i = 1, \dots, n$, and $z_0 = du$. Then*

$$\begin{array}{ll}
z_jy_i = c_iy_iz_j, & z_jx_i = c_i^{-1}x_iz_j \quad (i \le j) \\
z_jy_i = d^{-1}y_iz_j, & z_jx_i = dx_iz_j \quad (i > j) \\
z_jz_i = z_iz_j & (\textit{all } i, j) \\
x_iy_i = c_iy_ix_i + \lambda z_{i-1} & (i = 2, \dots, n),\ x_1y_1 = c_1y_1x_1 + d^{-1}z_0 \\
z_i = (c_id - 1)\,y_ix_i + d\lambda z_{i-1} & (i = 2, \dots, n),\ z_1 = (c_1d - 1)\,y_1x_1 + z_0.
\end{array}$$

Observe that $\delta_i(y_i) = \lambda z_{i-1}$ for all $i > 1$.

The quantum space *attached* to $R_n^{(C,\Lambda)}(\Bbbk)$ is $\Bbbk_{Q_n}[Y_1, X_1, \dots, Y_n, X_n]$, where Q_n is the matrix defined by

(5)

$$\begin{array}{c|c}
 & \begin{array}{cccccccc} Y_1 & X_1 & Y_2 & X_2 & \cdots & \cdots & Y_n & X_n \end{array} \\
\begin{array}{c} Y_1 \\ X_1 \\ Y_2 \\ X_2 \\ \vdots \\ Y_n \\ X_n \end{array} &
\begin{pmatrix}
1 & c_1^{-1} & \lambda_{21}^{-1} & \lambda_{21}c_1^{-1} & \cdots & \lambda_{n1}^{-1} & \lambda_{n1}c_1^{-1} \\
c_1 & 1 & \lambda_{21}d^{-1} & \lambda_{21}^{-1}c_1d & \cdots & \lambda_{n1}d^{-1} & \lambda_{n1}^{-1}c_1d \\
\lambda_{21} & \lambda_{21}^{-1}d & 1 & c_2^{-1} & \cdots & \lambda_{n2}^{-1} & \lambda_{n2}c_2^{-1} \\
\lambda_{21}^{-1}c_1 & \lambda_{21}c_1^{-1}d^{-1} & c_2 & 1 & \cdots & \lambda_{n2}d^{-1} & \lambda_{n2}^{-1}c_2d \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_{n1} & \lambda_{n1}^{-1}d & \lambda_{n2} & \lambda_{n2}^{-1}d & \cdots & 1 & c_n^{-1} \\
\lambda_{n1}^{-1}c_1 & \lambda_{n1}c_1^{-1}d^{-1} & \lambda_{n2}^{-1}c_2 & \lambda_{n2}c_2^{-1}d^{-1} & \cdots & c_n & 1
\end{pmatrix}.
\end{array}$$

Notice that $Y_1u = duY_1$ and $X_1u = d^{-1}uX_1$.

Remark 1.3. We have $\delta_i\beta_i = c_id\beta_i\delta_i$ for all $i \ge 1$. So if c_id is not a root of unity for every $i = 1, \dots, n$, then [10, Theorem 2.3] each prime ideal of $R_n^{(C,\Lambda)}(\Bbbk)$ is completely prime.

In order to classify the prime and the primitive ideals of $R_n^{(C,\Lambda)}(\Bbbk)$ we will suppose that for each $i = 1, \dots, n$ the scalar c_id is not a root of unity.

Denote by $\wp_n$ the following subset of R:

$$\wp_n = \begin{cases} \{z_1, y_1, x_1, \dots, z_n, y_n, x_n\}, & \text{if } z_0 = 0, \\ \{z_1, z_2, y_2, x_2, \dots, z_n, y_n, x_n\}, & \text{if not.} \end{cases} \tag{6}$$

DEFINITION 1.4 [22]. A subset T of $\wp_n$ is said to be *admissible* if it satisfies the conditions:

(1) y_i or $x_i \in T \Leftrightarrow z_i$ and $z_{i-1} \in T$, for all $i \geq 2$
(2) x_i or $y_i \in T \Leftrightarrow z_1 \in T$, if $z_0 = 0$.

For an admissible set T, let us denote by $\operatorname{ind}(T) = \{i \in \{1, \dots, n\} | z_i \in T\}$. An index $i \in \operatorname{ind}(T)$ is said to be *removable* if T contains x_i and y_i; the set of removable indices is denoted by $\operatorname{Remv}(T)$. If we denote $\mathcal{I}_T = \{i \in \{1, \dots, n\} | y_i \in T\}$ and $\mathcal{I}_T = \{j \in \{1, \dots, n\} | x_j \in T\}$, then $\operatorname{Remv}(T) = \emptyset$ if and only if $\mathcal{I}_T \cap \mathcal{I}_T = \emptyset$. We say that T is *connected* if for any $i, j \in \operatorname{ind}(T)$ such that $i < k < j$, then $k \in \operatorname{ind}(T)$. A connected component of T is a connected admissible subset U of T such that for all connected admissible subsets V of T with $U \subseteq V$ then $U = V$. Each admissible set T decomposes uniquely as $T = T_1 \cup T_2 \cup \cdots \cup T_r$, where T_i are the connected components of T. This decomposition is called a *connected decomposition* of T. Put $i_k = \min(\operatorname{ind}(T_k))$, $j_k = \max(\operatorname{ind}(T_k))$; we will always suppose that $j_{k-1} < i_k - 1$, $k = 2, \dots, r$. An example of such decomposition is to put $n = 3$ and take $T = \{x_1, z_1\} \cup \{z_3\}$ in $R_3^{(C,\Lambda)}(\Bbbk)$ with $z_0 = 0$; so $\{x_1, z_1\}$, $\{z_3\}$ are the connected components of T.

If T is a connected admissible set we define the length of T, denoted by $\operatorname{length}(T)$, as

$$\operatorname{length}(T) = \operatorname{card}(\operatorname{ind}(T)) + \operatorname{card}(\operatorname{Remv}(T)),$$

where $\operatorname{card}(\operatorname{ind}(T))$(resp. $\operatorname{card}(\operatorname{Remv}(T))$) is the cardinal of $\operatorname{ind}(T)$(resp. the cardinal of $\operatorname{Remv}(T)$). The length of a not necessarily connected admissible set T is

$$\operatorname{length}(T) = \sum_{k=1}^{r} \operatorname{length}(T_k),$$

where $T = T_1 \cup T_2 \cup \cdots \cup T_r$ is the connected decomposition of T.

Our next aim is to prove that $\langle T \rangle$ is a prime ideal for every admissible set T.

PROPOSITION 1.5. *For $k \geq 2$, let Q be an (α_k, β_k)-stable prime ideal of R_{k-1} such that $z_{k-1} \notin Q$. Then the ideals $P = QR_k + z_k R_k$ and QR_k are prime extensions of Q to R_k.*

Proof. By [11, Theorem 10.3(ii)], applied to Q in the iterations $R_{k/2}$ and R_k, we have QR_k is a prime extension of Q. We apply [11, Theorem 10.3(iv)] to get the proposition. Consider $\operatorname{frac}(R_{k/2}/QR_{k/2}) = A_{k/2}$, the Goldie quotient ring of $R_{k/2}/QR_{k/2}$. We need to show that the extension of δ_k, denoted also by δ_k, to $A_{k/2}$ is an inner β_k-derivation, where β_k is the extension of β_k to $A_{k/2}$. It is sufficient to show this for the extension of δ_k to the skew Laurent polynomial ring $F_{k/2} = (R_{k-1}/Q)[Y_k^{\pm 1}, \alpha_k]$. Put $u_k = (1 - c_k d)^{-1}\lambda \bar{z}_{k-1} y_k^{-1} \in F_{k/2}$, where $\bar{z}_{k-1}$ is the image of z_{k-1} in $F_{k/2}$. Then $\delta_k(y_k) = u_k y_k - c_k y_k u_k$ and $\delta_k(y_k^{-1}) = -\beta_k(y_k^{-1})\delta_k(y_k)y_k^{-1} = u_k y_k^{-1} - c_k^{-1} y_k^{-1} u_k$. Therefore δ_k is an inner β_k-derivation of $F_{k/2}$, hence $\delta_k(a) = u_k a - \beta_k(a)u_k$, for all $a \in A_{k/2}$. So, by [9, Lemma 1.4], we have the following isomorphisms of $\Bbbk$-algebras

$$\frac{R_k}{QR_k}[C_{k/2}]^{-1} \cong A_{k/2}[x_k, \beta_k, \delta_k] \cong A_{k/2}[x_k - u_k, \beta_k],$$

where $C_{k/2} = (R_{k/2}/QR_{k/2})\backslash\{0\}$. Now, by [11, Theorem 10.3(iv)], the inverse image of the prime ideal $\langle x_k - u_k\rangle$ in R_k is a prime extension of $QR_{k/2}$. Check that $x_k - u_k = -c_k(1 - c_k d)^{-1}\bar{z}_k y_k^{-1}$, where $\bar{z}_k$ is the image of z_k in R_k/QR_k. So the inverse image of $\langle \bar{z}_k y_k^{-1}\rangle$ in R_k is exactly the ideal $P = QR_k + z_k R_k$. ∎

COROLLARY 1.6. *The ideal $\langle z_i\rangle$ is prime for all $i \geq 2$. If $z_0 \neq 0$, then $\langle z_1\rangle$ is a prime ideal too.*

Proof. It is clear that $\langle z_i\rangle \cap R_i = z_i R_i = R_i z_i$ is $\sum_{i+1}^n$-stable ideal for all $i \geq 1$. So it suffices to show that $R_i z_i$ is a prime ideal of R_i. But this follows from Proposition 1.5 with $Q = 0$ in R_{i-1} and $i \geq 2$. For the case $z_0 \neq 0$ and $i = 1$ consider $F_0 = \Bbbk[y_1^{\pm 1}]$ and the element $v_1 = (1 - c_1 d)^{-1} u y_1^{-1} \in F_0$. We denote by δ_1, β_1 the extensions of δ_1, β_1 to F_0, respectively. So $\delta_1(y_1) = v_1 y_1 - c_1 y_1 v_1$ and $\delta_1(y_1^{-1}) = -c_1^{-1} d^{-1} u y_1^{-2}$; thus $\delta_1(y_1^{-1}) = v_1 y_1^{-1} - c_1^{-1} y_1^{-1} v_1$. Therefore δ_1 is an inner β_1-derivation of F_0, and so $F_0[x_1, \beta_1, \delta_1] \cong F_0[x_1 - v_1, \beta_1]$. By the equality $x_1 - v_1 = -(1 - c_1 d)^{-1} c_1 z_1 y_1^{-1}$, the inverse image of the prime ideal $\langle z_1 y_1^{-1}\rangle$ in R_1 is the prime ideal $z_1 R_1$. ∎

PROPOSITION 1.7. *Let I be a $\sum_{j+1}^n$-stable prime ideal of R_j for $j < n - 2$, and T be a connected admissible set of R_n such that $i = \min(\mathrm{ind}(T)) > j + 1$. Then $J = IR_n + \langle T\rangle$ is a prime ideal of R_n.*

Proof. Let k denote $\max(\mathrm{ind}(T))$; we use induction on k. If $k = j + 2$, then $i = k$, and $T = \{z_k\}$. As $z_{j+1} \notin IR_{j+1}$, by Proposition 1.5, we have that $IR_{j+2} + z_{j+2}R_{j+2}$ is a prime $\sum_{j+3}^n$-stable ideal of R_{j+2}. Then $J = (IR_{j+2} + z_{j+2}R_{j+2})R_n$ is a prime extension of $IR_{j+2} + z_{j+2}R_{j+2}$. Now suppose that the proposition is true for any connected admissible set T', with $\max(\mathrm{ind}(T')) < k$. Let $T' = T \cap R_{k-1}$, so T' is a connected admissible set of R_{k-1} and

$$T = \begin{cases} T' \cup \{y_k, z_k\}, & \text{if } k \in \mathcal{I}_T, k \notin \mathcal{L}_T \\ T' \cup \{x_k, z_k\}, & \text{if } k \in \mathcal{L}_T, k \notin \mathcal{I}_T \\ T' \cup \{y_k, x_k, z_k\}, & \text{if } k \in \mathrm{Remv}(T). \end{cases}$$

Put $J_{k-1} = IR_{k-1} + T'R_{k-1}$. So by induction hypothesis $J_{k-1}R_n$ is a prime ideal of R_n. Recall that in each iteration every prime ideal is completely prime; hence J_{k-1} is a $\sum_k^n$-stable prime ideal of R_{k-1}. We claim that J is a prime extension to R_n of the prime ideal J_{k-1}. To show this we will consider the three cases listed below

(1) If $T = T' \cup \{y_k, z_k\}$ then, by the isomorphism

$$\frac{R_k}{J_{k-1}R_k} \cong \frac{R_{k-1}}{J_{k-1}}[y_k, \alpha_k][x_k, \beta_k],$$

the ideal $J_k = J_{k-1}R_k + y_kR_k$ is a prime ideal of R_k. Check that $J_k = IR_k + TR_k$, which is $\sum_{k+1}^n$-stable. Thus $J = J_kR_n$ is a prime extension of J_{k-1} to R_n.

(2) The case $T = T' \cup \{x_k, z_k\}$ is similar to the first one by taking $J_k = J_{k-1}R_k + x_kR_k$.

(3) If $T = T' \cup \{y_k, x_k, z_k\}$, then consider $J_k = J_{k-1}R_k + y_kR_k + x_kR_k$ and $J_{k/2} = J_{k-1}R_{k/2} + y_kR_{k/2}$. It is clear that $J_k = J_{k/2}R_k + x_kR_k$ (observe that $z_{k-1} \in J_{k-1}$) and $J_{k/2}$ is a prime extension of J_{k-1} to $R_{k/2}$, which is β_k-stable. Therefore $J_{k/2}R_k$ is a prime ideal of R_k and we have the following algebra isomorphism:

$$\frac{R_k}{J_{k/2}R_k} \cong \frac{R_{k/2}}{J_{k/2}}[x_k, \beta_k].$$

Hence J_k is a prime ideal of R_K and $J_k = IR_k + TR_k$. This is a $\sum_{k+1}^n$-stable ideal, so $J_kR_n = J$ is a prime extension of J_k to R_n. Thus J is a prime extension of J_{k-1} to R_n. ∎

A *polynormal sequence* $\{a_1, \ldots, a_s\}$ in a ring R is a sequence of elements of R such that a_1 is normal in R, and each a_k is normal modulo the ideal $\langle a_1, \ldots, a_{k-1}\rangle$ for all $k \geq 2$. An ideal I of R generated by a polynormal sequence is called a *polynormal* ideal.

THEOREM 1.8. *Let T be an admissible set of $R_n^{(C,\Lambda)}(\Bbbk)$. Then $\langle T\rangle$ is a polynormal prime ideal.*

Proof. Clearly, T is a polynormal sequence. We prove that T is prime by induction on the number of connected components of T. If T is a connected admissible set with $\min(\mathrm{ind}(T)) > 1$, then $\langle T\rangle$ is prime by Proposition 1.7. Otherwise it is easy to see that $\langle T\rangle$ is a prime extension of $(T \cap R_1)R_1$. Let now $T = T_1 \cup T_2 \cdots \cup T_r$, $r > 1$, $i_k = \min(\mathrm{ind}(T_k))$, $j_k = \max(\mathrm{ind}(T_k))$, $k = 1, \ldots, r$. Consider $I = (T_1 \cup \cdots \cup T_{r-1})R_{j_{r-1}}$ as an ideal of $R_{j_{r-1}}$; by induction hypothesis IR_n is a prime ideal of R_n. As in each Ore iteration $R_k, R_{k/2}, k = 1, \ldots, n$, every prime ideal is completely prime, we have I is a prime ideal of $R_{j_{r-1}}$. Check that $\langle T\rangle = IR_n + T_rR_n$; applying Proposition 1.7 we have $\langle T\rangle$ is a prime ideal of R_n. ∎

Now we need to compute the Gelfand–Kirillov dimension of the factor algebra of $R_n^{(C,\Lambda)}(\mathbb{k})$ by an ideal generated by an admissible set. As $R_n^{(C,\Lambda)}(\mathbb{k})$ is a PBW $\mathbb{k}$-algebra with respect to the graded lexicographical order $\leq_{\text{deglex}}$, we can use [2, Sect. 3; 3, Sect. 4].

Lemma 1.9. *Let T be an admissible set of $R_n^{(C,\Lambda)}(\mathbb{k})$. Then T is a two-sided Gröbner basis.*

Proof. In this proof we use the notation of [2, 3]. Let $T = T_1 \cup \cdots \cup T_r$ be a connected decomposition of an admissible set T. It is clear that ${}^{T}\overline{S^l(u,v)} = 0$ for any $u \in T_s$ and $v \in T_t$ with $s \neq t \in \{1,\ldots,r\}$. Fix $k \in \{1,\ldots,r\}$ and consider T_k, $i = \min(\operatorname{ind}(T_k))$, $j = \max(\operatorname{ind}(T_k))$. By Lemma 1.2, z_s is a semi-commuting element for every $s = 1,\ldots,n$. So ${}^{T_k}\overline{S^l(z_s, v_t)} = 0$ for every $s \in \{i,\ldots,j\}$ and $t \in \{i+1,\ldots,j\}$, where $v_t \in \{y_t, x_t\}$. By (3), we have also ${}^{T_k}\overline{S^l(v_s, v_t)} = 0$ for all $t \neq s \in \{i+1,\ldots,j\}$. It remains to show that ${}^{T_k}\overline{S^l(y_s, x_s)} = 0$ for all $s \in \{i+1,\ldots,j\} \cap \operatorname{Remv}(T_k)$. But $S^l(y_s, x_s) = c_s^{-1}x_sy_s - y_sx_s = \lambda c_s^{-1} z_{s-1} \in T_k$. So from [3, Theorem 3.2], T is a left Gröbner basis of RT. On the other hand, if $t \in \mathcal{I}_T$ then ${}^{T}\overline{y_tx_t} = {}^{T}\overline{c_t^{-1}x_ty_t - \lambda c_t^{-1} z_{t-1}} = 0$, and if $t \in \mathcal{I}_T$ then ${}^{T}\overline{x_ty_t} = {}^{T}\overline{c_ty_tx_t + \lambda z_{t-1}} = 0$. So by [3, Remark 3.13] T is a two-sided Gröbner basis of $\langle T\rangle$. ∎

Using the notations of [2, Sect. 3], the following lemma is easy to prove:

Lemma 1.10. (1) *Let E be a monoideal of $(\mathbb{N}^p, +)$ generated by a minimal set $\{\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^k\}$ such that $\operatorname{Supp}(\boldsymbol{\alpha}^i) \cap \operatorname{Supp}(\boldsymbol{\alpha}^j) = \varnothing$ for $i \neq j$. Then $\dim(E) = p - k$.*

(2) *Let $E = B + \mathbb{N}^p$, with $B = \uplus_{k=1}^r B_k$ be a disjoint union of subsets of $\mathbb{N}^p$. Consider $E_k = B_k + \mathbb{N}^p$ for all $k = 1,\ldots,r$, and suppose that E has a set of generators of disjoint support (as in (1)). Then*

$$\dim(E) = p(1-r) + \sum_{k=1}^{r} \dim(E_i).$$

Let $\epsilon_i = \exp(y_i)$ and $\epsilon_i' = \exp(x_i)$ for every $i = 1,\ldots,n$ (see [2, Sect. 1]).

Proposition 1.11. *Let T be an admissible set of $R_n^{(C,\Lambda)}(\mathbb{k})$. Then*

$$\operatorname{GK}\dim\left(\frac{R_n^{(C,\Lambda)}(\mathbb{k})}{\langle T\rangle}\right) = 2n - \operatorname{length}(T).$$

Proof. Using Lemma 1.9 and [2, Theorem 3.7; 3, Theorem 4.10] we have

$$\operatorname{GK}\dim(R_n^{(C,\Lambda)}(\mathbb{k})/\langle T\rangle) = \dim(\exp(\langle T\rangle)).$$

So it suffices to compute $\dim(\exp(\langle T\rangle))$. Let T be a connected admissible set with $i = \min(\operatorname{ind}(T))$, $j = \max(\operatorname{ind}(T))$. By Lemma 1.9, $\exp(\langle T\rangle)$ is generated by the elements $\epsilon_i + \epsilon'_i, \epsilon_k$ if $k \in \mathcal{J}_T$, and ϵ'_l if $l \in \mathcal{L}_T$, where $k, l = i+1, \ldots, j$. Apply Lemma 1.10(1) to get $\dim(\exp(\langle T \rangle)) = 2n -$ length(T). If T is not connected, consider $T = T_1 \cup \cdots \cup T_r$ the connected decomposition of T. Then, by Lemma 1.9,

$$\exp(T) = \biguplus_{k=1}^{r} \exp(T_k)$$

and $\exp(\langle T \rangle)$ has a set of generators of disjoint support. So by Lemma 1.10(2) we have

$$\begin{aligned} \dim(\exp(\langle T \rangle)) &= 2n(1-r) + \sum_{k=1}^{r} \dim(\exp(\langle T \rangle)) \\ &= 2n(1-r) + \sum_{k=1}^{r} (2n - \mathrm{length}(T_k)) \\ &= 2n - \mathrm{length}(T). \end{aligned}$$

∎

2. THE $\mathcal{H}$-ACTION AND THE $\mathcal{H}$-PRIME IDEALS

Let $\Bbbk$ be an infinite field. We will define a rational action of an algebraic $\Bbbk$-torus on $R_n^{(C,\Lambda)}(\Bbbk)$, which will be shown to satisfy [13, 4.1]. Thus [13, Theorem 6.6, Corollary 6.9] apply to the algebra $R_n^{(C,\Lambda)}(\Bbbk)$. As in [8], we will show that the $\mathcal{H}$-prime ideals are precisely the ideals generated by admissible sets. We need to establish a $\Bbbk$-algebra isomorphism between a localization of a factor algebra $R_n^{(C,\Lambda)}(\Bbbk)/\langle T \rangle$, for a fixed admissible set T, and a localization of a quantum space attached to T.

Remark 2.1. Let G denote a group acting on $R_n^{(C,\Lambda)}(\Bbbk)$ as $\Bbbk$-algebra automorphisms. Assume that y_i, x_i, $i = 1, \ldots, n$ are G-eigenvectors. If $h \in G$ then, by (3), the action of h has one of the following forms

$$\begin{cases} h.y_i = \eta_i y_i & \text{and} \quad h.x_i = \eta_i^{-1} x_i, \quad \text{if } u \neq 0 \\ & \text{or} \\ h.y_i = \eta_i y_i & \text{and} \quad h.x_i = \eta_i^{-1} \theta x_i, \quad \text{if } u = 0; \end{cases} \tag{7}$$

where η_i is the h-eigenvalue of y_i, $i = 1, \ldots, n$, and θ is the common h-eigenvalue of $z_1, \ldots, z_n$. In conclusion, the group G can be replaced by a subgroup of the algebraic torus $(\Bbbk^\times)^n$ or of the torus $(\Bbbk^\times)^n \times \Bbbk^\times$.

In what follows $\mathcal{H}$ will denote the torus $(\Bbbk^\times)^n$ or $(\Bbbk^\times)^{n+1}$, depending on the value of u.

DEFINITION 2.2. We define the following rational action of $\mathcal{H}$ on $R_n^{(C,\Lambda)}(\Bbbk)$:

(1) If $u \neq 0$, then $\mathcal{H} = (\Bbbk^\times)^n$ and for any $h = (h_1 \dots, h_n) \in \mathcal{H}$ we take

$$h.y_i = h_i y_i$$
$$h.x_i = h_i^{-1} x_i.$$

(2) If $u = 0$, then $\mathcal{H} = (\Bbbk^\times)^{n+1}$ and for any $h = (h_1, \dots, h_n, h_{n+1}) \in \mathcal{H}$ we take

$$h.y_i = h_i y_i$$
$$h.x_i = h_i^{-1} h_{n+1} x_i.$$

Remark 2.3. The actions given in Definition 2.2 satisfy the hypothesis of [13, 4.1(c)]. Let us show this claim; recall that each $c_i d$ is not a root of unity. In the case $u \neq 0$ we have $d = 1$, so for any $j \in \{1, \dots, n\}$ the restriction to R_{j-1} and $R_{j/2}$ of the action of the following elements of $\mathcal{H}$

$$h^j = (\lambda_{j1}, \dots, \lambda_{jj-1}, c_j, 1, \dots, 1),$$
$$g^j = (\lambda_{j1}^{-1} c_1, \dots, \lambda_{jj-1}^{-1} c_{j-1}, c_j, 1, \dots, 1)$$

gives the automorphisms α_j, β_j, respectively. If $u = 0$, then for any $j \in \{1, \dots, n\}$ we take

$$h^j = (\lambda_{j1}, \dots, \lambda_{jj-1}, c_j d, 1, \dots, 1, d),$$
$$g^j = (\lambda_{j1}^{-1} c_1, \dots, \lambda_{jj-1}^{-1} c_{j-1}, c_j, 1, \dots, d^{-1}).$$

It is clear that α_j, β_j are the restriction of h^j, g^j, respectively. Hence both actions on $R_n^{(C,\Lambda)}(\Bbbk)$ satisfy [13, 4.1]. Notice that if $d = 1$ and $u = 0$, then we can use the torus $(\Bbbk^\times)^n$ instead of $(\Bbbk^\times)^{n+1}$ with the obvious action. But if $d \neq 1$, we have to enlarge the size of the acting torus in order to "place" the parameter d. The action defined in [13, 5.4] for the quantum Euclidean space $\mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times n})$ does not satisfy [13, 4.1(c)]; see the following example for $n = 2$.

EXAMPLE 2.4. Let $R_2^{(C,\Lambda)}(\Bbbk) = \mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times 2})$ with q not a root of unity. If we suppose that $\mathcal{H} = (\Bbbk^\times)^2$ and the $\mathcal{H}$-action satisfies [13, 4.1(c)], then there exists $h = (h_1, h_2) \in \mathcal{H}$ such that the restriction of h to $\Bbbk[y_1, x_1]$ coincides with the $\Bbbk$-algebra automorphism α_2. Thus

$$\alpha_2(y_1) = q^{-1} y_1 = h.y_1 = h_1 y_1$$
$$\alpha_2(x_1) = q^{-1} x_1 = h.x_1 = h_1^{-1} x_1;$$

this means that $q^2 = 1$, in spite of the assumption that q is not a root of unity.

In the rest of this section the algebra $R_n^{(C,\Lambda)}(\Bbbk)$ will be denoted by R. By Remark 2.3 and [13, Proposition 4.2], each $\mathcal{H}$-prime ideal of R is completely prime and there exist at most 2^{2n}.

Recall that R satisfies the Nullstellensatz, so, by [13, Corollary 6.9], R satisfies the Dixmier–Moeglin Equivalence. The primitive ideals of R are precisely those maximal within their $\mathcal{H}$-strata.

Our next goal is to describe the $\mathcal{H}$-prime ideals in terms of admissible sets. We need to control the Gelfand–Kirillov dimension of certain localizations of R.

PROPOSITION 2.5. *Let W be any subset of $\{1,\dots,n\}$. Consider the multiplicative subset $\mathcal{Y}$ of R generated by y_k, $k \in W$. Then $\mathcal{Y}$ is a right Ore set and the Gelfand–Kirillov dimension of $R\mathcal{Y}^{-1}$ equals $2n$.*

Proof. Compare with [8, Proposition 2.1]. By [9, Lemma 1.4; 18, Lemma 4.1], $\mathcal{Y}$ is a right Ore set of R and so $\mathrm{GKdim}(R) \le \mathrm{GKdim}(R\mathcal{Y}^{-1})$. So we will prove the converse inequality. Consider the $\Bbbk$-algebra S generated by $y_1, x_1, \dots, y_n, x_n$ satisfying the relations (3) and new variables Ω_k, $k \in W$, with the following additional relations

$$
\begin{array}{llll}
 & \Omega_i\Omega_j = \lambda_{ji}^{-1}\Omega_j\Omega_i, & x_i\Omega_j = \lambda_{ji}^{-1}d\Omega_j x_i & (j>i)\\
(8) & \Omega_i x_j = \lambda_{ji}^{-1}c_i x_j\Omega_i, & y_i\Omega_j = \lambda_{ji}\Omega_j y_i & (j>i)\\
 & \Omega_i y_j = \lambda_{ji} y_j\Omega_i & & (j>i)\\
 & \Omega_i y_j = y_i\Omega_i = 1 & &
\end{array}
$$

$$\Omega_k x_k = c_k x_k\Omega_k + \lambda d^{-1}\sum_{l=1}^{k-1}(d\lambda)^{k-1-l}(c_l d-1)y_1x_1\Omega_k^2 + d^{k-3}\lambda^{k-1}z_0\Omega_k^2.$$

There is a surjective homomorphism of algebras $S \to R\mathcal{Y}^{-1}$ sending y_i to y_i, x_i to x_i, and Ω_k to y_k^{-1}. Then $\mathrm{GKdim}(R\mathcal{Y}^{-1}) \le \mathrm{Gkdim}(S)$. We claim that $\mathrm{GKdim}(S) = 2n$. Order the variables

$$\Omega_{k_1} < \cdots < \Omega_{k_m} < y_1 < x_1 < \cdots y_n < x_n,$$

where $W = \{k_1, \dots, k_m\}$. Let $\le_{\mathbf{w}}$ be the weighted lexicographical ordering on $\mathbb{N}^{2n+m}$ defined by the vector

$$\mathbf{w} = (\underbrace{1,\dots,1}_{(m)}, 1, 2, 1, 4, \dots, 1, 2n,).$$

By [7, Proposition 3.2], S can be endowed with an $(\mathbb{N}^{2n+m}, \le_{\mathbf{w}})$-filtration such that the $\mathbb{N}^{2n+m}$-graded algebra $G(S)$ is semi-commutative, namely, it is generated by finitely many homogeneous elements

$$\Omega_{k_1}, \dots, \Omega_{k_m}, y_1, x_1, \dots, y_n, x_n,$$

in addition $y_k\Omega_k = 0$ for every $k \in W$. Therefore $G(S)$ is a factor of the coordinate algebra of an $2n+m$-dimensional quantum affine space by the ideal generated by the elements $y_k\Omega_k$, $k \in W$. By [2, Theorem 4.4.7; 3, Theorem 4.10], it is clear that $\text{GKdim}(G(S)) = 2n + m - m$, and by [7, Corollary 2.12] we have $\text{GKdim}(S) = \text{GKdim}(G(S)) = 2n$. ∎

Fix an admissible set $T = T_1 \cup T_2 \cup \cdots \cup T_r$ with $i_k = \min(\text{ind}(T_k))$, $j_k = \max(\text{ind}(T_k))$. For a prime ideal P of R we denote by $\text{ht}(P)$ the *height* of P. Let us now compute the height of an ideal generated by the admissible set T.

PROPOSITION 2.6. *Let T be an admissible set of R. Then*

$$\text{ht}(\langle T\rangle) = \text{length}(T) = 2n - \text{GK}\dim\left(\frac{R}{\langle T\rangle}\right).$$

Proof. Let us first prove the proposition for a connected admissible set T. Consider $j = \max(\text{ind}(T))$ and $i = \min(\text{ind}(T))$; we use induction on j. If $j = i = 1$ then the possible admissible sets are $\{z_1\}$ if $z_0 \neq 0$ or $\{z_1, x_1\}, \{z_1, y_1\}$ and $\{z_1, y_1, x_1\}$ if $z_0 = 0$. Using Corollary 1.6 and the definition of the length in conjunction with [21, Theorem 4.1.11] we get the result in this case. Suppose that the proposition holds for all connected admissible T' such that $\max(\text{ind}(T')) < j$. We can decompose T as a disjoint union of two sets

$$T = \begin{cases} T' \cup \{z_j, y_j\}, & \text{if } j \in \mathcal{I}_T, j \notin \text{Remv}(T) \\ T' \cup \{z_j, x_j\}, & \text{if } j \in \mathcal{L}_T, j \notin \text{Remv}(T) \\ T' \cup \{z_j, y_j, x_j\}, & \text{if } j \in \text{Remv}(T), \end{cases}$$

where T' is an admissible set of R. So we have a chain of prime ideals

$$\langle 0\rangle \subsetneq \langle T'\rangle \subsetneq \langle T', v_j\rangle = T, \qquad v_j \in \{y_j, x_j\}$$

or

$$\langle 0\rangle \subsetneq \langle T'\rangle \subsetneq \langle T', y_j\rangle \subsetneq \langle T', y_j, x_j\rangle = T.$$

By Theorem 1.8 we know that T is generated by a polynormal sequence. So using [21, Theorem 4.1.11] the chains above are maximal. Therefore $\text{ht}(\langle T\rangle) = \text{ht}(\langle T'\rangle) + \epsilon$, where $\epsilon \in \{1, 2\}$. Now, the result is clear by induction hypothesis. Let T be an admissible set and $T_1 \cup \cdots \cup T_r$ its connected decomposition. We show the proposition in this case by induction on r. For $r = 1$ the proposition has been already proved. Suppose that $r > 1$, so $T = T' \cup T_r$ where $T' = T_1 \cup \cdots \cup T_{r-1}$. We denote $i_r = \min(\text{ind}(T_r))$. So, by Proposition 1.5, we have a chain of prime ideals

$$\langle 0\rangle \subseteq \langle T'\rangle \subsetneq \langle T', z_{i_r}\rangle \subsetneq \cdots \subsetneq \langle T', T_r\rangle = \langle T\rangle.$$

This chain is maximal because $\langle T\rangle$ is a polynormal ideal. The number of prime ideals between $\langle T', z_{i_r}\rangle$ and $\langle T\rangle$ is exactly the number of the variables $y_l, x_l, l \in \mathrm{ind}(T_r)$. So we have $\mathrm{ht}(\langle T\rangle) = \mathrm{ht}(\langle T'\rangle) + \mathrm{length}(T_r)$; hence by induction hypothesis and the definition of the length we have $\mathrm{ht}(\langle T\rangle) = \mathrm{length}(T)$. The second equality follows from Proposition 1.11. ∎

Let us denote by $\mathcal{Y}_T$ the multiplicative set of R generated by $y_j, j \notin \mathcal{I}_T$; by Proposition 2.5 this is a right Ore set of R. Let Q_T be a submatrix of the matrix Q_n defined by deleting the rows and the columns corresponding to the variables $x_i, y_i \in T$ and $x_{i_k}, k = 1, \ldots, r$. If $z_0 = 0$ and $x_1 \notin T$ we will not delete the row and the column corresponding to the variable x_1. Consider A_T the quantum space associated to the matrix Q_T, considered as a subalgebra of the quantum space $A_\varnothing = \Bbbk_{Q_n}[Y_1, X_1, \ldots, Y_n, X_n]$ attached to R. The multiplicative set $\mathbb{Y}_T$ of A_T generated by all the Y_k's is a right Ore set, so consider $B_T = A_T \mathbb{Y}_T^{-1}$. We will denote by $\bar{a}$ the image of $a \in R$ in $\frac{R}{\langle T\rangle}$. Consider the $\Bbbk$-algebra homomorphism $\Psi_T : \frac{R}{\langle T\rangle}\overline{\mathcal{Y}_T}^{-1} \to B_T$ given by

$$\Psi_T(\bar{y}_k) = Y_k \quad (\text{for all } k)$$

$$\Psi_T(\bar{x}_k) = X_k \quad (\text{if } k-1 \in \mathrm{ind}(T), k \geq 2)$$

$$\Psi_T(\bar{x}_k) = X_k + (1 - c_k d)^{-1}\lambda Z_{k-1} Y_k^{-1} \quad (\text{if } 2 \leq k \text{ and } k, k-1 \notin \mathrm{ind}(T))$$

$$\Psi_T(\bar{x}_1) = X_1 + (1 - c_1 d)^{-1} Z_0 Y_1^{-1}$$

$$\Psi_T(\bar{x}_k) = (1 - c_k d)^{-1}\lambda Z_{k-1} Y_k^{-1} \quad (\text{if } k \in \mathrm{ind}(T) \text{ and } k-1 \notin \mathrm{ind}(T)),$$

where $Z_0 = d^{-1}z_0 = u$, $Z_k = (c_k d - 1)Y_k X_k$. It is clear that $\mathcal{Y}_T \cap \langle T\rangle = \varnothing$ so, by [6, Proposition 3.6.15], $R\mathcal{Y}_T^{-1}/\langle T\rangle\mathcal{Y}_T^{-1} \cong (R/\langle T\rangle)\overline{\mathcal{Y}_T}^{-1}$. Composing Ψ_T with this last isomorphism we get a new map which we also denote by Ψ_T. A similar algebra homomorphism was given in [27, Sect. 3.2] in the case of quantized Weyl algebras.

PROPOSITION 2.7. *The mapping*

$$\Psi_T : \frac{R\mathcal{Y}_T^{-1}}{\langle T\rangle\mathcal{Y}_T^{-1}} \to B_T$$

is a $\Bbbk$-algebra isomorphism.

Proof. It is clear that Ψ_T is surjective, and $\langle T\rangle \subseteq \ker(\Psi_T)$. So $\mathrm{GK}\dim(B_T) \leq \mathrm{GK}\dim(R\mathcal{Y}_T^{-1}/\langle T\rangle\mathcal{Y}_T^{-1})$. We claim that $\mathrm{GK}\dim(R\mathcal{Y}_T^{-1}/\langle T\rangle\mathcal{Y}_T^{-1}) \leq \mathrm{GK}\dim(B_T)$. From [18, Lemma 3.16], we have $\mathrm{GK}\dim(R\mathcal{Y}_T^{-1}/\langle T\rangle\mathcal{Y}_T^{-1}) \leq \mathrm{GK}\dim(R\mathcal{Y}_T^{-1}) - \mathrm{ht}(\langle T\rangle)$. Using Proposition 2.5 and Proposition 2.6 we have $\mathrm{GK}\dim(R\mathcal{Y}_T^{-1}/\langle T\rangle\mathcal{Y}_T^{-1}) \leq 2n - \mathrm{length}(T)$. We know that $\mathrm{GK}\dim(A_T) = 2n - \mathrm{length}(T) = \mathrm{GK}\dim(B_T)$. So $\mathrm{GK}\dim(R\mathcal{Y}_T^{-1}/\langle T\rangle\mathcal{Y}_T^{-1}) \leq \mathrm{GK}\dim(B_T)$. Since $\langle T\rangle$ is a completely prime ideal, Theorem 1.8, it follows from [18, Proposition 3.15] that Ψ_T is a $\Bbbk$-algebra isomorphism. ∎

Consider the algebraic torus $\mathcal{H}$ and its action R as in Definition 2.2. For any subset $X \subseteq \{1, \ldots, n\} = \mathbb{N}_n$, we denote by $\mathcal{H}_X$ the torus

$$\mathcal{H}_X = \begin{cases} \{(h_i)_{i\in X\cup\{n+1\}} | h_i \in \mathbb{k}^\times\}, & \text{if } \mathcal{H} = (\mathbb{k}^\times)^{n+1} \\ \{(h_i)_{i\in X} | h_i \in \mathbb{k}^\times\}, & \text{if } \mathcal{H} = (\mathbb{k}^\times)^n. \end{cases}$$

Let T be an admissible set of R; we denote by $I_n(T)$ the set of indices of the variables that appear in A_T; $I_n(T) \subseteq \mathbb{N}_n$. Define the following action of the torus $\mathcal{H}_{I_n(T)} = \mathcal{H}_T$ on A_T. If $\mathcal{H} = (\mathbb{k}^\times)^{n+1}$, then for any $(h_i)_{i\in I_n(T)\cup\{n+1\}} \in \mathcal{H}_T$,

$$(h_i)_{i\in I_n(T)\cup\{n+1\}}.Y_l = h_l Y_l$$

$$(h_i)_{i\in I_n(T)\cup\{n+1\}}.X_k = h_k^{-1} h_{n+1} X_k.$$

If $\mathcal{H} = (\mathbb{k}^\times)^n$, then for any $(h_i)_{i\in I_n(T)} \in \mathcal{H}_T$,

$$(h_i)_{i\in I_n(T)}.Y_l = h_l Y_l$$

$$(h_i)_{i\in I_n(T)}.X_k = h_k^{-1} X_k.$$

Consider the canonically extended action of $\mathcal{H}_T$ to the localization $B_T = A_T \mathbb{Y}^{-1}$. For each $h \in \mathcal{H}_T$, we have the following composite map

$$\frac{R}{\langle T\rangle}\overline{\mathcal{Y}}_T^{-1} \xrightarrow{\Psi_T} B_T \xrightarrow{h} B_T \xrightarrow{\Psi_T^{-1}} \frac{R}{\langle T\rangle}\overline{\mathcal{Y}}_T^{-1},$$

where h denotes the extension of h to B_T.

DEFINITION 2.8. We define the action of the torus $\mathcal{H}_T$ on $\frac{R}{\langle T\rangle}\overline{\mathcal{Y}}_T^{-1}$ as follows. Given $h \in \mathcal{H}_T$, define

$$h.x = (\Psi_T^{-1} h \Psi_T)(x)$$

for every $x \in \frac{R}{\langle T\rangle}\overline{\mathcal{Y}}_T^{-1}$.

The following lemma is clear.

LEMMA 2.9. *Consider $\mathcal{H}_T$ as a factor group of the torus $\mathcal{H}$. The action of $\mathcal{H}_T$ induced on $R/\langle T\rangle$ by that of $\mathcal{H}$ coincides with the restriction of the action defined in Definition* 2.8.

PROPOSITION 2.10. *There is a bijection ζ between $\mathcal{H} - \mathrm{Spec}(R)$ and $\mathcal{A}_n(R)$, the set of all the admissible sets of R, defined by*

$$\begin{aligned} \zeta : \mathcal{H} - \mathrm{Spec}(R) &\longrightarrow \mathcal{A}_n(R) \\ J &\longmapsto J \cap \wp_n. \end{aligned}$$

With the inverse map

$$\begin{aligned} \zeta^{-1} : \mathcal{A}_n(R) &\longrightarrow \mathcal{H} - \mathrm{Spec}(R) \\ T &\longmapsto \langle T\rangle. \end{aligned}$$

Proof. To show that ζ and ζ^{-1} are well defined, we use Theorem 1.8, Remark 2.3, and [13, Proposition 4.2]. Clearly $\zeta\zeta^{-1} = id$; let us show that $\zeta^{-1}\zeta = id$. Consider $J \in \mathscr{H} - \mathrm{Spec}(R)$ such that $J \cap \wp_n = T$ and suppose that $\langle T \rangle \subsetneq J$. Let $\overline{\mathscr{J}_T} = \mathbb{N}_n \backslash \mathscr{J}_T = \{j_1, \ldots, j_r\}$; B_T is an iterated Ore extension of the form

$$B_T = \Bbbk_{\overline{Q}}[Y_{j_1}^{\pm 1}, \ldots, Y_{j_r}^{\pm 1}][X_{i_1}, \beta_{i_1}] \cdots [X_{i_t}, \beta_{i_t}],$$

where $\Bbbk_{\overline{Q}}[Y_{j_1}^{\pm 1}, \ldots, Y_{j_r}^{\pm 1}]$ is the McConnell–Pettit algebra associated to a suitable matrix $\overline{Q}$. To complete the proof we use the same arguments as in the proof of [8, Proposition 2.5] with the following element $h \in \mathscr{H}_T$. Fix $l \in \{1, \ldots, t-1\}$; if $\mathscr{H} = (\Bbbk^{\times})^{n+1}$, then we take

$$h_i = \begin{cases} \lambda_{i_l j_k}^{-1} c_{j_k}, & i = j_k < i_l \\ \lambda_{j_k i_l} d^{-1}, & i = j_k > i_l \\ \lambda_{i_l i_k}^{-1} c_{i_k}, & i = i_k < i_l \\ c_i, & i = i_l \\ d^{-1}, & i = n+1 \\ 1, & \text{otherwise.} \end{cases}$$

If $\mathscr{H} = (\Bbbk^{\times})^n$, then we take

$$h_i = \begin{cases} \lambda_{i_l j_k}^{-1} c_{j_k}, & i = j_k < i_l \\ \lambda_{j_k i_l} d^{-1}, & i = j_k > i_l \\ \lambda_{i_l i_k}^{-1} c_{i_k}, & i = i_k < i_l \\ c_i, & i = i_l \\ 1, & \text{otherwise.} \end{cases}$$

∎

COROLLARY 2.11. *The number of $\mathscr{H}$-prime ideals is*

$$\frac{1}{2}\left[(2+\sqrt{2})^n + (2-\sqrt{2})^n\right] \qquad (u \neq 0)$$

or

$$\frac{1}{2\sqrt{2}}\left[(2+\sqrt{2})^{n+1} - (2-\sqrt{2})^{n+1}\right] \qquad (u = 0).$$

Proof. Similar to that [8, Corollary 2.6; 27, Proposition 3.1.16]. ∎

3. THE $\mathscr{H}$-STRATIFICATION

In this section, we work out the $\mathscr{H}$-stratification (1) of $\mathrm{Spec}(R_n^{(C,\Lambda)}(\Bbbk))$. First, we give a simpler description of each $\mathscr{H}$-stratum. Let T be an admissible set of $R_n^{(C,\Lambda)}(\Bbbk)$ and let us denote

$$\mathrm{Spec}_T\big(R_n^{(C,\Lambda)}(\Bbbk)\big) = \big\{P \in \mathrm{Spec}(R_n^{(C,\Lambda)}(\Bbbk)) \mid P \cap \wp_n = T\big\}.$$

LEMMA 3.1. *Let J be an $\mathcal{H}$-prime ideal of $R_n^{(C,\Lambda)}(\mathbb{k})$ and let T be the admissible set such that $J = \langle T \rangle$. Then*

$$\mathrm{Spec}_T(R_n^{(C,\Lambda)}(\mathbb{k})) = \mathrm{Spec}_J(R_n^{(C,\Lambda)}(\mathbb{k})).$$

Proof. Analogous to [8, Lemma 3.1], using Proposition 2.10. ∎

PROPOSITION 3.2. *The $\mathcal{H}$-stratification of* $\mathrm{Spec}(R_n^{(C,\Lambda)}(\mathbb{k}))$ *is given by*

$$\mathrm{Spec}(R_n^{(C,\Lambda)}(\mathbb{k})) = \biguplus_{T \text{ admissible}} \mathrm{Spec}_T(R_n^{(C,\Lambda)}(k)). \tag{9}$$

Proof. This is a consequence of Proposition 2.10 and Lemma 3.1. ∎

The $\mathbb{k}$-algebra obtained by localizing B_T at all X_k that appear in A_T is the McConnell–Pettit $\mathbb{k}$-algebra $\mathbf{P}(Q_T)$. We will denote $R_n^{(C,\Lambda)}(\mathbb{k})$ by R. As in [8] consider Φ_T the composite map

$$R \to \frac{R\mathcal{Y}_T^{-1}}{\langle T \rangle \mathcal{Y}_T^{-1}} \overset{\Psi_T}{\to} B_T \hookrightarrow \mathbf{P}(Q_T).$$

Remark 3.3 Let J be an $\mathcal{H}$-prime ideal of R and $J \cap \wp_n = T$. Let $\mathcal{X}_T$ denote the inverse image in $R\mathcal{Y}_T^{-1}/\langle T \rangle \mathcal{Y}_T^{-1}$ of the multiplicative set of B_T generated by all the X_k's. This is a right Ore set and the corresponding localization R_T satisfies that $R_T \cong \mathbf{P}(Q_T)$. Clearly $R_T \subseteq R_J$, where $J = \langle T \rangle$ and $R_J = (R/J)\mathcal{E}_J^{-1}$; $\mathcal{E}_J$ is the set of all non-zero homogeneous elements with respect to a certain $\mathbb{Z}^n$-grading (see [13, Theorem 6.6]). In the general case one cannot expect $R_T = R_J$. The following is a counterexample: take $n = 2$, $R_2^{(C,\Lambda)}(\mathbb{k}) = \mathcal{O}_q(\mathfrak{o}\mathbb{k}^{2\times 2})$, $T = \{z_2\}$; then the homogeneous element $\bar{y}_1 + \bar{y}_2$ of degree $(1, 0, 0) \in \mathbb{Z}^3$ is not invertible in R_T.

THEOREM 3.4. *Let T be an admissible set. Then Φ_T induces a homeomorphism between* $\mathrm{Spec}(\mathbf{P}(Q_T))$ *and* $\mathrm{Spec}_T(R)$ *defined by*

$$\begin{aligned} \Phi_T^{-1} : \mathrm{Spec}(\mathbf{P}(Q_T)) &\to \mathrm{Spec}_T(R) \\ \mathcal{P} \qquad &\mapsto \Phi_T^{-1}(\mathcal{P}). \end{aligned}$$

Proof. Notice that $\Phi_T^{-1}(\mathcal{P})$ is prime because every prime ideal in R or $\mathbf{P}(Q_T)$ is completely prime. So it suffices to show that $\Phi_T^{-1}(\mathcal{P}) \in \mathrm{Spec}_T(R)$ for all $\mathcal{P} \in \mathrm{Spec}(\mathbf{P}(Q_T))$. Put $\Phi_T^{-1}(\mathcal{P}) \cap \wp_n = T'$; it is clear that $T \subseteq T'$. We show the other inclusion by contradiction. So suppose that there exists $k \in \mathrm{ind}(T')$ and $k \notin \mathrm{ind}(T)$. Hence, modulo $\langle T \rangle$, we have $\bar{z}_k \neq 0$. So $\Psi_T(\bar{z}_k) = (c_k d - 1)Y_k X_k \in \mathcal{P}$, which is a contradiction with $\mathcal{P} \in \mathrm{Spec}(\mathbf{P}(Q_T))$. Now, if $\mathrm{ind}(T) = \mathrm{ind}(T')$ then there exists $x_k \in T'$ such that $\Psi_T(\bar{x}_k) = X_k \in \mathcal{P}$, also a contradiction. We need to show the injectivity. So let $\mathcal{P}$ and $\mathcal{P}'$ be two elements of $\mathrm{Spec}(\mathbf{P}(Q_T))$ such that $\Phi_T^{-1}(\mathcal{P}) = \Phi_T^{-1}(\mathcal{P}')$. It is clear that

$\Phi_T^{-1}(\mathscr{P}) \cap \mathscr{Y}_T = \Phi_T^{-1}(\mathscr{P}') \cap \mathscr{Y}_T = \varnothing$. Apply Ψ_T to $\Phi_T^{-1}(\mathscr{P})\mathscr{Y}_T^{-1}/\langle T\rangle\mathscr{Y}_T^{-1} = \Phi_T^{-1}(\mathscr{P}')\mathscr{Y}_T^{-1}/\langle T\rangle\mathscr{Y}_T^{-1}$ to get $\mathscr{P} = \mathscr{P}'$. For the surjectivity we take $P \in \mathrm{Spec}_T(R)$, and we put $\mathscr{P} = \Psi_T(P\mathscr{Y}_T^{-1}/\langle T\rangle\mathscr{Y}_T^{-1})$; so $\mathscr{P} \cap \mathbb{Y}_T = \varnothing$. Suppose that there exists an indeterminate X_k of $\mathbf{P}(Q_T)$ such that $X_k \in \mathscr{P}$. Therefore if $\Psi_T(\bar{x}_k) = X_k$ then $x_k \in P$ with $k \notin \mathscr{I}_T$; this is a contradiction with $P \in \mathrm{Spec}_T(R)$. If $\Psi_T(\bar{x}_k) = X_k + (1 - c_k d)^{-1}\lambda Z_{k-1}Y_k^{-1}$, then $\Psi_T^{-1}(X_k) = -(1 - c_k d)^{-1}\bar{y}_k^{-1}\bar{z}_k$. So $k \notin \mathrm{ind}(T)$ and $z_k \in P$; this is also a contradiction with $P \in \mathrm{Spec}_T(R)$. We have shown that the extension of $\mathscr{P}$ to $\mathbf{P}(Q_T)$ is a prime ideal, which is the inverse image of P by Φ_T^{-1}. ∎

COROLLARY 3.5. *Let T be an admissible set. Then* $\mathrm{Spec}_T(R)$ *is homeomorphic to* $\mathrm{Spec}(Z(\mathbf{P}(Q_T)))$, *where* $Z(\mathbf{P}(Q_T))$ *is the center of* $\mathbf{P}(Q_T)$.

Proof. By [12, Corollary 1.5(b)], the contraction $\mathscr{P} \mapsto \mathscr{P} \cap Z(\mathbf{P}(Q_T))$ gives a homeomorphism between $\mathrm{Spec}(\mathbf{P}(Q_T))$ and $\mathrm{Spec}(Z(\mathbf{P}(Q_T)))$. The corollary is consequence of Theorem 3.4. ∎

Let $\mathfrak{p}$ be a prime ideal of $Z(\mathbf{P}(Q_T))$; we denote by $\mathfrak{p}^e$ its extension to $\mathbf{P}(Q_T)$. By $\max(Z(\mathbf{P}(Q_T)))$ we will denote the set of maximal ideals of $Z(\mathbf{P}(Q_T))$. Now we give the analogue of [8, Theorem 3.10].

THEOREM 3.6. *Let*

$$\mathscr{SP} = \{(T, \mathfrak{p}) |\ T \text{ is an admissible set}, \mathfrak{p} \in \mathrm{Spec}(Z(\mathbf{P}(Q_T)))\}$$

and

$$\mathscr{P} = \{(T, \mathfrak{p}) |\ T \text{ is an admissible set}, \mathfrak{p} \in \max(Z(\mathbf{P}(Q_T)))\}.$$

If the parameters $c_i d$, $i = 1, \ldots, n$ are not roots of unity, then the map $(T, \mathfrak{p}) \mapsto \Phi_T^{-1}(\mathfrak{p}^e)$ defines a bijection between $\mathscr{SP}$ and the prime spectrum $\mathrm{Spec}(R_n^{(C,\Lambda)}(\Bbbk))$ *whose restriction to $\mathscr{P}$ is a bijection onto the primitive spectrum* $\mathrm{Prim}(R_n^{(C,\Lambda)}(\Bbbk))$.

Proof. The bijection between $\mathscr{SP}$ and $\mathrm{Spec}(R_n^{(C,\Lambda)}(\Bbbk))$ follows from Theorem 3.4, Corollary 3.5, and the stratification (9). The bijection between $\mathscr{P}$ and $\mathrm{Prim}(R_n^{(C,\Lambda)}(\Bbbk))$ follows from [12, Corollary 1.5(c)], taking into account that $R_n^{(C,\Lambda)}(\Bbbk)$ and $\mathbf{P}(Q_T)$ are Jacobson algebras.

Remark 3.7. From Theorem 3.6 we deduce that the determination of the prime and primitive spectra of $R_n^{(C,\Lambda)}(\Bbbk)$ depends on the computation of the basis of the $\Bbbk$-algebras $Z(\mathbf{P}(Q_T))$ where T runs the set of all admissible sets. Some assumptions on the matrix Q_n allow such a determination by using [12]. Let us see what happens in case $R_n^{(C,\Lambda)}(\Bbbk)$ is one of the following $\Bbbk$-algebras: $\mathscr{O}_q(\mathfrak{sp}\Bbbk^{2\times n})$, $A_n^{(q,\Lambda)}(\Bbbk)$, and $\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times n})$.

(1) Consider $R_n^{(C,\Lambda)}(\Bbbk) = \mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n})$ where q is not a root of unity. Discussing the quantum linear system attached to an admissible set T (see Definition 4.3), it was shown, in [8, Corollary 3.9], that $Z(\mathbf{P}(Q_T)) \cong \Bbbk[\mathbb{Z}^k]$, where $k = \mathrm{ocomp}(T)$ is the number of connected components of odd length in the connected decomposition of T. An effective computation of the primitive ideals in the algebraically closed case was given in [8, Corollary 3.12].

(2) The second example is $R_n^{(C,\Lambda)}(\Bbbk) = A_n^{(\mathbf{q},\Lambda)}(\Bbbk)$, which was studied by Goodearl [9] with $n = 1$, after by Rigal [27, Proposition 3.2.3] and, independently, by Akhazavidegan and Jordan [1, Proposition 4.12] for any n. In this case the McConnell–Pettit algebra $\mathbf{P}(Q_T)$, where $T \subsetneq \wp_n$, is simple. See [1, Proposition 4.9; 27, Proposition 2.3.8] for a characterization of this simplicity in terms of the matrix Q_T. Then $Z(\mathbf{P}(Q_T)) = \Bbbk$ for all $T \subsetneq \wp_n$ (see Proposition 4.1.) By Corollary 3.5, this means that $\mathrm{Spec}_T(A_n^{(\mathbf{q},\Lambda)}(\Bbbk)) = \{\langle T \rangle\}$ for any $T \subsetneq \wp_n$. In the following section we study the case when $T = \wp_n$ and we give the primitive ideals in the algebraically closed case.

(3) The third example is $R_n^{(C,\Lambda)}(\Bbbk) = \mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times n})$ with q not a root of unity. In Section 4 we will show, by solving the quantum linear system attached to T, that $Z(\mathbf{P}(Q_T)) \cong \Bbbk[\mathbb{Z}^k]$, where k is computed from T by easy combinatorial arguments (see Corollary 4.8). For the primitive ideals we give, in the algebraically closed case, an effective method to compute them (see Corollary 4.10).

4. APPLICATION TO $A_n^{(\mathbf{q},\Lambda)}(\Bbbk)$ AND $\mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times n})$

Let $R_n^{(C,\Lambda)}(\Bbbk) = A_n^{(\mathbf{q},\Lambda)}(\Bbbk)$, $C = (q_1, \ldots, q_n, 1, 1, 1)$, so $z_0 = 1$. As in [27], we denote by Γ_n the subgroup of $\Bbbk^\times$ generated by λ_{ij} and q_i for $(i, j) \in \{1, \ldots, n\}^2$ and $i < j$, ($\Gamma_0 = \langle 1 \rangle$). By [27, Proposition 2.3.8] we have

Proposition 4.1. *Let T be an admissible set of $A_n^{(\mathbf{q},\Lambda)}(\Bbbk)$ such that length $(T) < 2n - 1$. If Γ_n is a free group of* rank $\frac{1}{2}n(n+1)$, *then $\mathbf{P}(Q_T)$ is a simple $\Bbbk$-algebra.*

Observe that if length$(T) = 2n - 1$ then

$$T = \wp_n = \{z_1, y_2, x_2, z_2, \ldots, y_n, x_n, z_n\}.$$

We have $\Psi_{\wp_n}(\bar{y}_1) = Y_1$ and $\Psi_{\wp_n}(\bar{x}_1) = (1 - q_1)^{-1} Y_1^{-1}$ which implies that $Z(\mathbf{P}(Q_{\wp_n})) = \Bbbk[Y_1^{\pm 1}]$. Let us denote by $\mathfrak{P}_{\wp_n}$ the set of the ideals of $A_n^{(\mathbf{q},\Lambda)}(\Bbbk)$ containing strictly $\langle \wp_n \rangle$ that are the inverse images by $\Phi_{\wp_n}$ of the non-zero prime ideals of $\Bbbk[Y_1^{\pm 1}]$. By Corollary 3.5, we have

$$\mathrm{Spec}_{\wp_n}(A_n^{(\mathbf{q},\Lambda)}(\Bbbk)) = \{\langle \wp_n \rangle\} \cup \mathfrak{P}_{\wp_n}. \tag{10}$$

COROLLARY 4.2. *If Γ_n is free of* rank $\frac{1}{2}n(n+1)$, *then*

$$\operatorname{Spec}(A_n^{(\mathbf{q},\Lambda)}(\Bbbk)) = \Bigg(\biguplus_{T \text{ admissible}} \{\langle T\rangle\}\Bigg) \cup \mathfrak{P}_{\wp_n}$$

and

$$\operatorname{Prim}(A_n^{(\mathbf{q},\Lambda)}(\Bbbk)) = \Bigg(\biguplus_{T\neq\wp_n, T \text{ admissible}} \{\langle T\rangle\}\Bigg) \cup \mathfrak{P}_{\wp_n}.$$

In particular $A_n^{(\mathbf{q},\Lambda)}(\Bbbk)$ is a primitive $\Bbbk$-algebra. If $\Bbbk$ is algebraically closed, then

$$\operatorname{Spec}(A_n^{(\mathbf{q},\Lambda)}(\Bbbk)) = \Bigg(\biguplus_{T \text{ admissible}} \{\langle T\rangle\}\Bigg) \cup \{\langle \wp_n, x_1-\alpha\rangle\}$$

and

$$\operatorname{Prim}(A_n^{(\mathbf{q},\Lambda)}(\Bbbk)) = \Bigg(\biguplus_{T\neq\wp_n, T \text{ admissible}} \{\langle T\rangle\}\Bigg) \cup \{\langle \wp_n, x_1-\alpha\rangle\},$$

where $\alpha\in\Bbbk^\times$.

Proof. In the general case this is a consequence of Theorem 3.6 and Proposition 4.1 in conjunction with (9) and (10). If $\Bbbk$ is algebraically closed, let $\alpha\in\Bbbk^\times$. So

$$\Phi_{\wp_n}^{-1}(\langle Y_1-\alpha\rangle) = \langle \wp_n, x_1-\alpha^{-1}(1-q_1)^{-1}\rangle,$$

thus $\mathfrak{P}_{\wp_n} = \{\langle \wp_n, x_1-\alpha^{-1}(1-q_1)^{-1}\rangle \mid \alpha\in\Bbbk^\times\}$. Therefore

$$\operatorname{Spec}_{\wp_n}(A_n^{(\mathbf{q},\Lambda)}(\Bbbk)) = \{\langle\wp_n\rangle \subsetneq \langle \wp_n, x_1-\alpha^{-1}(1-q_1)^{-1}\rangle\}.$$

∎

The $\Bbbk$-algebra $\mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times n})$ is obtained as $R_n^{(C,\Lambda)}(\Bbbk)$ with

$$C = (1,\ldots,1,q^{-2},q,0), \qquad \lambda_{ji} = q^{-1},\ 1\le i<j\le n.$$

From now on, $\mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times n})$ will be denoted by R. The defining relations for R are

$$\begin{aligned} &y_jy_i = q^{-1}y_iy_j, && y_jx_i = q^{-1}x_iy_j \quad (j>i)\\ &x_jx_i = qx_ix_j, && x_jy_i = qy_ix_j \quad (j>i)\\ &x_iy_i = y_ix_i + (1-q^2)\sum_{l=1}^{i-1} q^{l-i}y_lx_l. \end{aligned} \tag{11}$$

The normal elements are $z_i = q^{-2}x_iy_i - y_ix_i$, $i = 1, \dots, n$, $z_0 = 0$, and

$$\begin{array}{lll}
z_jy_i = y_iz_j, & z_jx_i = x_iz_j & (i \le j) \\
z_jy_i = q^2y_iz_j, & z_jx_i = q^{-2}x_iz_j & (i > j) \\
z_jz_i = z_iz_j & (\text{all } i, j) & \\
x_iy_i = y_ix_i + qz_{i-1} & \quad (i = 1, \dots, n), & \\
z_i = (q^{-2} - 1)y_ix_i + q^{-1}z_{i-1} & (i = 1, \dots, n), & z_0 = 0.
\end{array}$$

Finally the matrix Q_n is

$$\begin{array}{c}
\begin{array}{ccccccc} Y_1 & X_1 & Y_2 & X_2 & \cdots & Y_n & X_n \end{array} \\
\begin{array}{c} Y_1 \\ X_1 \\ Y_2 \\ X_2 \\ \vdots \\ Y_n \\ X_n \end{array}
\begin{pmatrix}
1 & 1 & q & q^{-1} & \cdots & q & q^{-1} \\
1 & 1 & q & q^{-1} & \cdots & q & q^{-1} \\
q^{-1} & q^{-1} & 1 & 1 & \cdots & q & q^{-1} \\
q & q & 1 & 1 & \cdots & q & q^{-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
q^{-1} & q^{-1} & q^{-1} & q^{-1} & \cdots & 1 & 1 \\
q & q & q & q & \cdots & 1 & 1
\end{pmatrix}.
\end{array}$$

Let T be an admissible set of $R_n = R$ and $T = T_1 \cup \dots \cup T_r$, $i_k = \min(\mathrm{ind}(T_k))$, $j_k = \max(\mathrm{ind}(T))$, $k = 1, \dots, r$ its connected decomposition. We denote by A_T^n the quantum space attached to $R_n/\langle T\rangle$ with the associated matrix $Q_T = (q^{k_{ij}})_{1\le i,j\le t}$, where $k_{ij} \in \{0, 1-1\}$. If $1 \le l \le t$ then the symbol V_l will denote a variable X_l for $l \in \{i_k+1, \dots, j_k\}\backslash\mathcal{J}_T$, a variable Y_l for $l \in \{i_k+1, \dots, j_k\}\backslash\mathcal{J}_T$, $k = 1, \dots, r$, and the absence of a variable when $l \in \mathrm{Remv}(T)$. Let us denote $\mathcal{M}_T^n = (k_{ij})_{1\le i,j\le t} \in \mathbf{M}_{t\times t}(\mathbb{Z})$.

Definition 4.3. Let T be an admissible set of $\mathcal{O}_q(\mathfrak{o}\mathbb{k}^{2\times n})$ and consider the associated matrix $\mathcal{M}_T^n = (k_{ij})_{1\le i,j\le t}$. The *quantum linear system* associated to T is the linear system of equations over the integers $\mathcal{M}_T^n m = 0$, where $m \in \mathbb{Z}^t$.

We denote by $\mathrm{Null}(\mathcal{M}_T^n)$ the torsion free abelian group $\{m \in \mathbb{Z}^t \mid \mathcal{M}_T^n m = 0\}$. Order the variables $Y_1 < X_1 < \cdots < Y_n < X_n$ and consider this ordering inherited by the subset of variables that appear in A_T^n. Let us denote by $C_n(T)$ the number of ordered pairs (W_i, W_j) satisfying the following conditions:

(a) each W_k represents either the variable X_k or Y_k of A_T^n

(b) no variable appears in two different pairs,

(c) $W_i < W_j$ are consecutive and $i < j$.

This $C_n(T)$ is the number of consecutive disjoint submatrices of the form $\left(\begin{smallmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{smallmatrix}\right)$ of the tridiagonal of $\mathcal{M}_T^n$, where $\epsilon \in \{1, -1\}$. For example, if $n = 3$ and $T = \{x_1, z_1\} \cup \{z_3\}$ then $A_T^3 = \mathbb{k}_{Q_T}[Y_1, Y_2, X_2, Y_3]$ hence $C_3(T) = 2$;

the couples are $(Y_1, Y_2), (X_2, Y_3)$. If $n = 4$, $T = \{x_1, z_1, y_2, z_2\} \cup \{z_4\}$ then $A^4_T = \Bbbk_{Q_T}[Y_1, X_2, Y_3, X_3, Y_4]$ and the couples are $(Y_1, X_2), (X_3, Y_4)$, thus $C_4(T) = 2$.

Remark 4.4. (1) Let T be an admissible set such that $\mathrm{Remv}(T) = \varnothing$, and $T = T_1 \cup T_2 \cup \cdots \cup T_r$ the connected decomposition of T with $i_k = \min(\mathrm{ind}(T_k))$, $j_k = \max(\mathrm{ind}(T_k))$, $k = 1, \ldots, r$. Suppose that $j_r = n-1$, so $A^n_T = A^m_T[Y_{i_r}, V_{i_r+1}, \ldots, V_{n-2}, V_{n-1}, Y_n, X_n]$, $m = i_r - 1$. Then

$$C_n(T) = \begin{cases} C_{n-1}(T) + 1 & \text{if } n - i_r - 1 \text{ is odd} \\ C_{n-1}(T) & \text{if } n - i_r - 1 \text{ is even.} \end{cases}$$

If we put $T' = R_{n-2} \cap T$ then $C_n(T) = C_{n-2}(T') + 1$.

(2) Let T be an admissible set with $j = \max(\mathrm{ind}(T)) \leq m - 1$, $m < n$. So $A^n_T = A^{m-1}_T[Y_m, X_m, \ldots, Y_n, X_n]$; hence $C_n(T) = C_m(T) + (n - m)$.

For an integer z we write $[z] = p$ when z is equal to $2p$ or $2p + 1$.

LEMMA 4.5. *Let T be an admissible set such that* $\mathrm{Remv}(T) = \varnothing$ *and consider $T = T_1 \cup T_2 \cup \cdots \cup T_r$ the connected decomposition of T with $i_k =$* $\min(\mathrm{ind}(T_k))$, $j_k = \max(\mathrm{ind}(T_k))$, $k = 1, \ldots, r$.

(1) *Suppose that $j_r < n - 1$ and put*

$$T' = \begin{cases} T_1 \cup \cdots \cup T_r \cup \{z_{n-1}\}, & \textit{if } j_r < n-2 \\ T_1 \cup \cdots \cup T_{r-1} \cup (T_r \cup \{z_{n-2}, x_{n-1}, z_{n-1}\}), & \textit{if } j_r = n-2. \end{cases}$$

Then $C_n(T) = C_{n-1}(T') + 1$.

(2) *Suppose that $j_r = n, r > 1$ and put $T' = T_1 \cup \cdots \cup T_{r-1}$. Then $C_n(T) = C_m(T') + [n - i_r] + 1$ where $m = i_r - 1$.*

Proof. (1) If $j_r < n - 2$ then

$$A^n_{T'} = A^{n-3}_T[Y_{n-2}, X_{n-2}, Y_{n-1}, Y_n, X_n]$$
$$A^n_T = A^{n-3}_T[Y_{n-2}, X_{n-2}, Y_{n-1}, X_{n-1}, Y_n, X_n].$$

This implies that $C_n(T') = C_{n-2}(T) + 1$ and $C_n(T) = C_{n-2}(T) + 2$ whence $C_n(T) = C_n(T') + 1$. Apply the Remark 4.4(1) to T' to get $C_n(T) = C_{n-1}(T') + 1$.

If $j_r = n - 2$ then

$$A^n_{T'} = A^{n-3}_T[V_{n-2}, Y_{n-1}, Y_n, X_n]$$
$$A^n_T = A^{n-3}_T[V_{n-2}, Y_{n-1}, X_{n-1}, Y_n, X_n].$$

Here we distinguish two cases. The first case is that $n - i_r - 1$ is even; by Remark 4.4(1) (with $n - 1$) applied to T, we have $C_{n-1}(T) = C_{n-2}(T) + 1$. Clearly $C_n(T) = C_{n-2}(T) + 2$, whence $C_n(T) = C_{n-1}(T) + 1$. Check that

$C_n(T') = C_{n-1}(T)$. Hence $C_n(T) = C_n(T') + 1$. Applying Remark 4.4(1) to T', we get $C_n(T) = C_{n-1}(T') + 1$. The second case is that $n - i_r - 1$ is odd, so $C_n(T) = C_{n-2}(T) + 1$, $C_n(T') = C_{n-2}(T) + 1$ and thus, $C_n(T) = C_n(T') = C_{n-1}(T') + 1$ by Remark 4.4(1) applied to T'.

(2) If $j_r = n$, then

$$A_T^n = A_T^m[Y_{i_r}, V_{i_r+1}, \ldots, V_n], \qquad m = i_r - 1.$$

So if $n - i_r$ is even then $C_n(T) = C_m(T') + (p+1)$ with $n - i_r = 2p$. The same is true if $n - i_r = 2p + 1$. ∎

LEMMA 4.6. *Let $A \in \mathbf{M}_{m\times m}(\mathbb{Z})$, $v \in \mathbf{M}_{m\times 1}(\mathbb{Z})$, $v^t \in \mathbf{M}_{1\times m}(\mathbb{Z})$ be the transpose of v and $\epsilon, \epsilon' \in \{1, -1\}$. Then*

$$\operatorname{rank}\begin{pmatrix} A & \epsilon' v & \epsilon v \\ -\epsilon' v^t & 0 & \epsilon \\ -\epsilon v^t & -\epsilon & 0 \end{pmatrix} = \operatorname{rank} A + 2$$

and

$$\operatorname{rank}\begin{pmatrix} A & v & -v & v \\ -v^t & 0 & 0 & 1 \\ v^t & 0 & 0 & 1 \\ -v^t & -1 & -1 & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A & v \\ -v^t & 0 \end{pmatrix} + 2.$$

Proof. Compute the ranks by using the minors and suitable row and column elementary operations. ∎

PROPOSITION 4.7. *Let T be an admissible set of R and $\mathcal{M}_T^n \in \mathbf{M}_{t\times t}(\mathbb{Z})$ the associated matrix. Then* $\operatorname{rank} \mathcal{M}_T^n = 2 \times C_n(T)$.

Proof. We use induction on n. The cases $n = 1, 2$ are easy. Suppose that the result holds for all admissible sets in R_m, $m < n$. If there is $i \in \operatorname{Remv}(T)$, then let T' be the admissible subset of $\wp_n \backslash \{y_i, x_i\}$ obtained by removing y_i, x_i from T. Notice that $\mathcal{M}_T^n = \mathcal{M}_{T'}^{n-1}$ so $C_n(T) = C_{n-1}(T')$. The result in this case follows by the induction hypothesis. Suppose now that $\operatorname{Remv}(T) = \emptyset$ and let $T = T_1 \cup T_2 \cup \cdots \cup T_r$ be a connected decomposition with $i_k = \min(\operatorname{ind}(T_k))$, $j_k = \max(\operatorname{ind}(T_k))$, $k = 1, \ldots, r$. We will consider the different possible cases. The notation v stands for a column vector for all its entries equal to 1, and v^t is its transpose.

Case 1. If $j_r < n - 1$ then

$$\mathcal{M}_T^n = \begin{pmatrix} \mathcal{M}_T^{n-2} & v & -v & v & -v \\ -v^t & 0 & 0 & 1 & -1 \\ v^t & 0 & 0 & 1 & -1 \\ -v^t & -1 & -1 & 0 & 0 \\ v^t & 1 & 1 & 0 & 0 \end{pmatrix}.$$

By Lemma 4.6 we have

$$\operatorname{rank} \mathscr{M}_T^n = \operatorname{rank} \begin{pmatrix} \mathscr{M}_T^{n-2} & v \\ -v^t & 0 \end{pmatrix} + 2. \tag{12}$$

Put

$$T' = \begin{cases} T \cup \{z_{n-1}\}, & \text{if } j_r < n-2 \\ T \cup \{z_{n-2}, x_{n-1}, z_{n-1}\}, & \text{if } j_r = n-2, \end{cases}$$

and consider T' as an admissible set of R_{n-1}. We have

$$\mathscr{M}_{T'}^{n-1} = \begin{pmatrix} \mathscr{M}_T^{n-2} & v \\ -v^t & 0 \end{pmatrix}.$$

By the induction hypothesis we have $\operatorname{rank} \mathscr{M}_{T'}^{n-1} = 2 \times C_{n-1}(T')$. Use Lemma 4.5(1) and (12) to get $\operatorname{rank} \mathscr{M}_T^n = 2 \times C_n(T)$.

Case 2. If $j_r = n-1$ then

$$\mathscr{M}_T^n = \begin{pmatrix} \mathscr{M}_{T'}^{n-2} & \epsilon v & v & -v \\ \epsilon v^t & 0 & 1 & -1 \\ -v^t & -1 & 0 & 0 \\ v^t & 1 & 0 & 0 \end{pmatrix},$$

where $\epsilon \in \{1, -1\}$ and $T' = T \cap R_{n-2}$. So Lemma 4.6 implies that

$$\operatorname{rank} \mathscr{M}_T^n = \operatorname{rank} \mathscr{M}_{T'}^{n-2} + 2. \tag{13}$$

By the induction hypothesis we have $\operatorname{rank} \mathscr{M}_{T'}^{n-2} = 2 \times C_{n-2}(T')$. Then $\operatorname{rank} \mathscr{M}_T^n = 2 \times C_n(T)$ by Remark 4.4(1) and (13).

Case 3. If $j_r = n$ put $T' = T_1 \cup \cdots \cup T_{r-1}$, $m = i_r - 1$. Then

$$\mathscr{M}_T^n = \begin{pmatrix} \mathscr{M}_{T'}^m & v & \epsilon_{i_r+1} v & \cdots & \epsilon_{n-1} v & \epsilon_n v \\ -v^t & 0 & \epsilon_{i_r+1} & \cdots & \epsilon_{n-1} & \epsilon_n \\ -\epsilon_{i_r+1} v^t & -\epsilon_{i_r+1} & 0 & \cdots & \epsilon_{n-1} & \epsilon_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\epsilon_{n-1} v^t & -\epsilon_{n-1} & -\epsilon_{n-1} & \cdots & 0 & \epsilon_n \\ -\epsilon_n v^t & -\epsilon_n & -\epsilon_n & \cdots & -\epsilon_n & 0 \end{pmatrix},$$

where $\epsilon_k \in \{1, -1\}$, $k = i_r + 1, \ldots, n$. Apply Lemma 4.6 several times to get

$$\operatorname{rank} \mathscr{M}_T^n = \begin{cases} \operatorname{rank} \begin{pmatrix} \mathscr{M}_{T'}^m & v \\ -v^t & 0 \end{pmatrix} + (n - i_r) & \text{if } (n - i_r) \text{ is even} \\ \operatorname{rank} \mathscr{M}_{T'}^m + (n - i_r + 1) & \text{if } (n - i_r) \text{ is odd.} \end{cases}$$

So we have

$$\operatorname{rank} \mathcal{M}_T^n = \begin{cases} \operatorname{rank} \begin{pmatrix} \mathcal{M}_{T'}^{m-1} & v & -v & v \\ -v^t & 0 & 0 & 1 \\ v^t & 0 & 0 & 1 \\ -v^t & -1 & -1 & 0 \end{pmatrix} + (n - i_r) & \text{if } (n - i_r) \text{ is even} \\ \\ \operatorname{rank} \begin{pmatrix} \mathcal{M}_{T'}^{m-1} & v & -v \\ -v^t & 0 & 0 \\ v^t & 0 & 0 \end{pmatrix} + (n - i_r + 1) & \text{if } (n - i_r) \text{ is odd.} \end{cases}$$

Apply Lemma 4.6 again to get

$$\operatorname{rank} \mathcal{M}_T^n = \operatorname{rank} \begin{pmatrix} \mathcal{M}_{T'}^{m-1} & v \\ -v^t & 0 \end{pmatrix} + 2(p+1), \tag{14}$$

where $p = [n - i_r]$. Put

$$T'' = \begin{cases} T' \cup \{z_m\} & \text{if } j_{r-1} < m - 1 \\ \\ T_1 \cup \cdots \cup T_{r-2} \cup (T_{r-1} \cup \{z_{m-1}, x_m, z_m\}) & \text{if } j_{r-1} = m - 1. \end{cases}$$

Considered as an admissible set of R_m, the matrix associated to T'' is of the form

$$\mathcal{M}_{T''}^m = \begin{pmatrix} \mathcal{M}_{T''}^{m-1} & v \\ -v^t & 0 \end{pmatrix}.$$

Hence by the induction hypothesis we have $\operatorname{rank} \mathcal{M}_{T''}^m = 2 \times C_m(T'')$. If we apply Lemma 4.5(1) to T'' with $n = m + 1$, then $C_{m+1}(T') = C_m(T'') + 1$; hence

$$\operatorname{rank} \mathcal{M}_{T''}^m = 2(C_{m+1}(T') - 1). \tag{15}$$

By Lemma 4.5(2) we have $C_n(T) = C_m(T') + (p + 1)$, and by Remark 4.4(2) we have $C_{m+1}(T') = C_m(T') + 1$, because $j_{r-1} < m$. This implies that $C_n(T) = C_{m+1}(T') - 1 + (p + 1)$. Combining this last equality with (14) and (15) we get $\operatorname{rank} \mathcal{M}_T^n = 2 \times C_n(T)$. ∎

COROLLARY 4.8. *Let T be an admissible set of R and $\mathcal{M}_T^n \in \mathbf{M}_{t\times t}(\mathbb{Z})$ the associated matrix. Then the rank of the free abelian group* $\operatorname{Null}(\mathcal{M}_T^n)$ *is $N_n(T) = t - 2 \times C_n(T)$.*

Proof. This is a consequence of Proposition 4.7. ∎

Let T be an admissible set of R and $\mathcal{M}_T^n \in \mathbf{M}_{t\times t}(\mathbb{Z})$ and let

$$\{U^\alpha = U_1^{\alpha_1} \cdots U_t^{\alpha_t} | \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t) \in \mathbb{Z}^t\}$$

be the $\Bbbk$-basis of $\mathbf{P}(Q_T)$, where the U_l's denote the variables in A_T^n. Let

$$\{\boldsymbol{m}_1^T, \ldots, \boldsymbol{m}_k^T\}$$

be the basis of $\operatorname{Null}(\mathscr{M}_T^n)$. By Corollary 4.8, we have $k = N_n(T)$. Using [12, 1.3] we get

$$Z(\mathbf{P}(Q_T)) = \Bbbk[(U^{\boldsymbol{m}_1^T})^{\pm 1}, \ldots, (U^{\boldsymbol{m}_k^T})^{\pm 1}].$$

This is a Laurent polynomial ring in the variables $(U^{\boldsymbol{m}_1^T})^{\pm 1}, \ldots, (U^{\boldsymbol{m}_k^T})^{\pm 1}$; thus it is canonically isomorphic to the group algebra $\Bbbk[\mathbb{Z}^{N_n(T)}]$.

COROLLARY 4.9. *Consider $\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times n})$ where q is not a root of unity. Let*

$$\mathscr{S}\mathscr{P} = \{(T, \mathfrak{p}) | T \textit{ is an admissible set}, \mathfrak{p} \in \operatorname{Spec}(\Bbbk[\mathbb{Z}^{N_n(T)}])\}$$

and

$$\mathscr{P} = \{(T, \mathfrak{p}) | T \textit{ is an admissible set}, \mathfrak{p} \in \max(\Bbbk[\mathbb{Z}^{N_n(T)}])\}.$$

Then the map $(T, \mathfrak{p}) \mapsto \Phi_T^{-1}(\mathfrak{p}^e)$ defines a bijection between $\mathscr{S}\mathscr{P}$ and $\operatorname{Spec}(\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times n}))$ *whose restriction to $\mathscr{P}$ is a bijection onto* $\operatorname{Prim}(\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times n}))$.

Proof. Apply Theorem 3.6 and Corollary 4.8. ∎

From now on, we suppose that $\Bbbk$ is algebraically closed. Let T be an admissible set and let $\{\boldsymbol{m}_1^T, \ldots, \boldsymbol{m}_k^T\}$, $k = N_n(T)$, be a basis of $\operatorname{Null}(\mathscr{M}_T)$. The maximal ideals of $Z(\mathbf{P}(Q_T))$ are of the form

$$\mathfrak{p}(\boldsymbol{\lambda}) = \langle U^{\boldsymbol{m}_1^T} - \lambda_1, \ldots, U^{\boldsymbol{m}_k^T} - \lambda_k \rangle,$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \in (\Bbbk^\times)^k$. By Corollary 4.9, the primitive ideals of $\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times n})$ are of the form $\Phi_T^{-1}(\mathfrak{p}(\boldsymbol{\lambda})^e)$, when T ranges over the set of all admissible sets. We shall exhibit a procedure to compute them from the solutions of the quantum systems defined in Definition 4.3.

For $\boldsymbol{m} = (m_1, \ldots, m_t) \in \mathbb{Z}^t$ we denote

$$\boldsymbol{m}^+ = \frac{1}{2}(m_1 + |m_1|, \ldots, m_t + |m_t|)$$

and

$$\boldsymbol{m}^- = \frac{1}{2}(m_1 - |m_1|, \ldots, m_t - |m_t|),$$

where $|m|$ is the absolute value of $m \in \mathbb{Z}$. Then the inverse image of $\mathfrak{p}(\boldsymbol{\lambda})$ in A_T is

$$\langle U^{\boldsymbol{m}_1^{T+}} - \lambda_1 U^{-\boldsymbol{m}_1^{T-}}, \ldots, U^{\boldsymbol{m}_k^{T+}} - \lambda_k U^{-\boldsymbol{m}_k^{T-}} \rangle. \tag{16}$$

For each $s = 1, \dots, k$, let $Y_{\boldsymbol{m}_s^T}(\lambda_s)$ denote an element of $\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times n})$ such that

$$\Psi_T(Y_{\boldsymbol{m}_s^T}(\lambda_s) + \langle T\rangle) = U^{\boldsymbol{m}_s^{T+}} - \lambda_s U^{-\boldsymbol{m}_s^{T-}}.$$

Then

$$\Phi_T^{-1}(\mathfrak{p}(\boldsymbol{\lambda})^e) = \langle T, Y_{\boldsymbol{m}_1^T}(\lambda_1), \dots, Y_{\boldsymbol{m}_k^T}(\lambda_k)\rangle.$$

This gives a description of $\mathrm{Prim}(\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times n}))$.

COROLLARY 4.10. *The primitive ideals of $\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times n})$, when q is not a root of unity, are the maximal elements of each stratum* $\mathrm{Spec}_T(\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times n}))$, *where T is an admissible set. If $\Bbbk$ is algebraically closed, then they are of the form*

$$\langle T, Y_{\boldsymbol{m}_1^T}(\lambda_1), \dots, Y_{\boldsymbol{m}_k^T}(\lambda_k)\rangle,$$

where $k = N_n(T)$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) \in (\Bbbk^\times)^k$.

EXAMPLE 4.11. In this example we compute the prime and primitive spectra of $\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times 2})$, where $q \in \Bbbk^\times$ is not a root of unity and $\Bbbk$ is algebraically closed. Each of the 14 strata can be explicitly described. As an illustration, we compute here two of them. First, consider the stratum corresponding to $\varnothing$ and $\{z_1, y_1, x_1\}$. We know that $A_\varnothing = \Bbbk_{Q_2}[Y_1, X_1, Y_2, X_2]$; solving the attached quantum linear system, we get the basis of $\mathrm{Null}(\mathscr{M}_\varnothing)$ which is $\{(-1, 1, 0, 0), (0, 0, 1, 1)\}$. Thus $Z(\mathbf{P}(Q_2)) = \Bbbk[(Y_1^{-1}X_1)^{\pm 1}, (Y_2X_2)^{\pm 1}]$, hence the maximal ideals corresponding to the $\varnothing$-stratum are $\langle z_2 - \gamma, x_1 - \alpha y_1\rangle$, where $\alpha, \gamma \in \Bbbk^\times$. Let us denote by $\mathscr{I}$ the set of the prime ideals of $\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times 2})$ that are the inverse images by $\Phi_\varnothing$ of the non-zero prime but not maximal ideals of $\Bbbk[(Y_1^{-1}X_1)^{\pm 1}, (Y_2X_2)^{\pm 1}]$. So

$$\mathrm{Spec}_\varnothing(\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times 2})) = \{\langle 0\rangle\} \cup \{I | I \in \mathscr{I}\} \cup \{\langle z_2 - \gamma, x_1 - \alpha y_1\rangle\}.$$

Analogously, for $T = \{z_1, y_1, x_1\}$, we get

$$\mathrm{Spec}_T(\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times 2})) = \{\langle y_1, x_1\rangle\} \cup \{J | J \in \mathscr{J}\} \cup \{\langle y_1, x_1, y_2 - \gamma, x_2 - \alpha\rangle\},$$

where $\mathscr{J}$ is the set of the ideals of $\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times 2})$ strictly containing $\langle y_1, x_1\rangle$ that are the inverse images by Φ_T of the non-zero prime but not maximal ideals of $Z(\mathbf{P}(Q_T)) = \Bbbk[y_2^{\pm 1}, x_2^{\pm 1}]$ and $\alpha, \gamma \in \Bbbk^\times$. For any other T, the algebra $Z(\mathbf{P}(Q_T))$ is one-dimensional, and the computations are straightforward. The lattice of prime ideals of $\mathscr{O}_q(\mathfrak{o}\Bbbk^{2\times 2})$ is drawn in Fig. 1. The primitive ideal generated by a set A is denoted by $\langle\langle A\rangle\rangle$, while prime but not primitive ideals are denoted by $\langle A\rangle$. A line connecting two prime ideal means inclusion. When both ideals belong to the same stratum, we use a wavy line.

Following [23, Example 5], the coordinate ring of quantum Euclidean space $\mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times n+1})$, when q has a square root in $\Bbbk$, is the $\Bbbk$-algebra generated by $2n+1$ variables $\omega, y_1, x_1, \dots, y_n, x_n$ satisfying the following relations

$$
\begin{aligned}
&y_j y_i = q^{-1} y_i y_j, \qquad y_j x_i = q^{-1} x_i y_j \quad (j > i)\\
&x_j x_i = q x_i x_j, \qquad x_j y_i = q y_i x_j \qquad (j > i)\\
&y_i \omega = q^{-1} \omega y_i, \qquad x_i \omega = q \omega x_i \qquad (\text{all } i)\\
&x_i y_i = y_i x_i + (1-q^2)\sum_{l=1}^{i-1} q^{l-i} y_l x_l + q^{1-i}(q^{-1/2} - q^{1/2})\omega^2.
\end{aligned} \tag{17}
$$

This $\Bbbk$-algebra is an iterated skew polynomial ring

$$\mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times n+1}) = \Bbbk[\omega][y_1, \alpha_1][x_1, \beta_1, \delta_1] \cdots [y_n, \alpha_n][x_n, \beta_n, \delta_n],$$

where α_i, β_i are algebra automorphisms and δ_i are left β_i-derivations, deduced from the relations (17). Observe that $\beta_i \delta_i = q^{-2} \delta_i \beta_i$ for all $i \geq 1$. Consider $R_{n+1}^{(C,\Lambda)}(\Bbbk)$ with $C = (1, \dots, 1, q^{-2}, q, 0)$, $\lambda_{ij} = q^{-1}$ for $1 \leq i < j \leq n+1$. We have $R_{n+1}^{(C,\Lambda)}(\Bbbk) = \mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times(n+1)})$ and, by [23, Example 5], there is an epimorphism

$$\phi : \mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times(n+1)}) \longrightarrow \mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times n+1})$$

sending $y_1 \mapsto q^{1/2}(1+q)^{-1}\omega$, $x_1 \mapsto \omega$, $y_i \mapsto y_{i-1}$, $x_i \mapsto x_{i-1}$, $(i \geq 2)$, and $\ker(\phi) = \langle y_1 - q^{1/2}(1+q)^{-1}x_1 \rangle$. Denote by

$$\mathrm{Spec}_0(\mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times(n+1)})) = \{P \in \mathrm{Spec}(\mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times(n+1)})) \mid \ker(\phi) \subseteq P\}.$$

Clearly, $\mathrm{Spec}(\mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times n+1}))$ is homeomorphic to $\mathrm{Spec}_0(\mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times(n+1)}))$.

EXAMPLE 4.12. Here we apply the foregoing homeomorphism to compute the prime spectrum of $\mathcal{O}_q(\mathfrak{o}\Bbbk^3)$ when q has a square root in $\Bbbk$ and it is not a root of unity. Let $\beta \in \Bbbk^\times$ and denote by η_β the automorphism of $\mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times 2})$ sending $y_1 \mapsto \beta y_1$, $y_2 \mapsto y_2$ and $x_i \mapsto x_i$, $i = 1, 2$. Consider the epimorphism

$$\phi_\beta : \mathcal{O}_q(\mathfrak{o}\Bbbk^{2\times 2}) \longrightarrow \mathcal{O}_q(\mathfrak{o}\Bbbk^3),$$

sending $y_2 \mapsto \beta y_1$, $x_2 \mapsto x_1$, $y_1 \mapsto q^{1/2}(1+q)^{-1}\omega$, and $x_1 \mapsto \omega$. It is clear that $\ker(\phi_\beta \eta_\beta) = \langle x_1 - \beta^{-1} q^{-1/2}(1+q) y_1 \rangle$. Now fix $\alpha \in \Bbbk^\times$ and put $\beta = \alpha^{-1} q^{-1/2}(1+q)$ so $\ker(\phi_\beta \eta_\beta) = \langle x_1 - \alpha y_1 \rangle$. Using Fig. 1 we get the lattice of prime ideals of $\mathcal{O}_q(\mathfrak{o}\Bbbk^3)$; see Fig. 2. There, $\mathcal{S}$ is the set of the prime ideals of $\mathcal{O}_q(\mathfrak{o}\Bbbk^3)$ which are the image under $\phi_\beta \eta_\beta$ of elements of $\mathcal{J}$, $\alpha, \gamma \in \Bbbk^\times$, and $\check{z}_1 = (q^{-2} - 1)\beta y_1 x_1 + q^{-2}(q^{-1/2} - q^{1/2})\omega^2$.

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