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EXTENDED DISTRIBUTIVE LAW. COWREATH OVER CORINGS

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We introduce and study comodules over cowreath defined over a given coring. If our coring arises from an entwining structure, we then give a procedure to construct a cowreath from a given cowreath over the factor coalgebra. We also include the dual notions, that is, wreaths over ring extension and their modules. In particular, we show that the study of twisted algebras and twisted bimodules already introduced in [A. Čap, H. Schichl and J. Vanžura, On twisted tensor products of algebras, *Commun. Algebra* **23** (1995) 4701–4735] has its origin in the study of wreath and their bimodules.

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1. Introduction

In the series of papers [6, 7, 18, 24, 25], a notion called a twisted tensor product algebra and their twisted bimodules were studied from several point of views (e.g. noncommutative differential geometry, C^*-algebra, Hopf algebra, etc.). Roughly speaking, this is a deformation of the tensor product algebra, in the sense that we substitute the usual flip map by another one (called the *twisting map*) which we require to be compatible with the multiplication and the unit of both factor algebras. To each algebra, one can obviously associate a monad (or triple [8]) on the category of vector spaces. In this way, twisting maps between two algebras are in 1-1 correspondence with distributive laws, in the sense of Beck [1], between the associated monads. The resulting monad from a distributive law leads in fact to the twisted tensor product algebra.

The notion of wreath, or extended distributive law, in a given bicategory is a formal generalizations of distributive law, and was introduced by Lack and Street in [17] (see also [22]). The notion of cowreath is in some sense dual to that of wreath. For instance, consider Bim the bicategory of bimodules (unital rings, unital bimodules, bilinear maps) and $(\mathfrak{C} : A)$ a comonad in Bim (i.e. $\mathfrak{C}$ is an A-coring, see Subsec. 1.3). Attached to $(\mathfrak{C} : A)$ there is a category $\mathscr{R}_{(\mathfrak{C}:A)}$ called the right Eilenberg–Moore category of $(\mathfrak{C} : A)$, see Sec. 2. It turns out that this category is the Hom-category of the 0-cell $(\mathfrak{C} : A)$ in the right Eilenberg–Moore bicategory $\mathsf{REM}(\mathsf{Bim})$ of the bicategory Bim, explicitly given in [3] and generally introduced in [17]. Thus, $\mathscr{R}_{(\mathfrak{C}:A)}$ inherits from $\mathsf{REM}(\mathsf{Bim})$ a structure of monoidal category. In this way, right cowreath is then defined to be a comonoid in the monoidal category $\mathscr{R}_{(\mathfrak{C}:A)}$, see Definition 3.1. Let $(A : T)$ be a monad in Bim, that is, A and T are two unital rings and there is a ring extension $\iota : A \to T$. Then one can dually construct the monoidal category $\mathscr{R}_{(A:T)}$, and thus define a right wreath as monoid in this category, see Definition 5.2.

The purpose of this paper is two-fold. The first aim is to introduce and study comodules over cowreath defined over a given coring. If this coring arises from an entwining structure [4], we then give a procedure of constructing examples of cowreaths with noncommutative base ring. The second aim is to place the constructions of twisted product algebras and twisted bimodules into the frameworks of wreaths and their bimodules defined over a ring extension. We also extend the results of [7] to the case of noncommutative ring extension. It is noteworthy that these two aims are as follows interrelated. We make the statements and proofs for the case of cowreaths over coring, then use freely these results in their dual form for the case of wreaths over ring extension.

We shall proceed as follows. In the current section, we present the basic properties of (co)monads, lifted and continuous functors, which we will use in the sequel. In Sec. 2, we recall the definition of the category $\mathscr{R}_{(\mathfrak{C}:A)}$ for a given coring $(\mathfrak{C} : A)$. Next, we show the useful result of Proposition 2.4 which says that the category $\mathscr{R}_{(\mathfrak{C}:A)}$ is monoidally isomorphic to the category of continuous $(- \otimes_A \mathfrak{C})$-lifted functors (see Subsec. 1.2 for definition). If $\mathfrak{C} = A \otimes_{\mathbb{K}} C$ where $(A, C)_{\mathfrak{a}}$ is an entwining structure over a commutative ring $\mathbb{K}$, then we show using Proposition 2.4 that the functor $A \otimes_{\mathbb{K}} - \otimes_{\mathbb{K}} A : \mathscr{R}_{(C:\mathbb{K})} \to \mathscr{R}_{(\mathfrak{C}:A)}$ is an opmonoidal functor. Thus, the image $A \otimes_{\mathbb{K}} M \otimes_{\mathbb{K}} A$ of any right C-cowreath M is a right $\mathfrak{C}$-cowreath. In Sec. 3, we use Proposition 2.4 in order to give in terms of tensor product over the base ring A, a simplest and equivalent definition of right $\mathfrak{C}$-cowreath. In particular, we give a proof of the fact that an object in $\mathscr{R}_{(\mathfrak{C}:A)}$ is right cowreath (i.e. comonoid) if and only if its associated $(- \otimes_A \mathfrak{C})$-lifted functor admits a comonad structure on the category of right $\mathfrak{C}$-comodules, Proposition 3.2. Another characterization of right $\mathfrak{C}$-cowreath in terms of cowreath product as well as right (or left) comodules over a given right $\mathfrak{C}$-cowreath $(M, \mathfrak{m})$ are discussed in Sec. 4 (here $(M, \mathfrak{m})$ is an object of $\mathscr{R}_{(\mathfrak{C}:A)}$). We show that $(M, \mathfrak{m})$ is a right cowreath if and only if $\mathfrak{C} \otimes_A M$ admits a compatible structure of A-coring (is this coring which we refer to as a *cowreath product* of $\mathfrak{C}$ by M), Proposition 4.1. Here Proposition 2.4 was again the key in providing a definition of right $(M, \mathfrak{m})$-comodule in terms of tensor product over A, Proposition 4.3. In Sec. 5, we will give without proofs results analogs to those stated in previous sections for wreaths over ring extension. This will be used in Sec. 6 to extend the main results of [7] to the noncommutative case, Propositions 6.1 and 6.2.

Notations and basic notions. Given any Hom-set category $\mathscr{C}$, the notation $X \in \mathscr{C}$ means that X is an object of $\mathscr{C}$. The identity morphism of X will be denoted by X itself. The set of all morphisms $f : X \to X'$ in $\mathscr{C}$ is denoted by $\mathrm{Hom}_{\mathscr{C}}(X, X')$. The identity functor of $\mathscr{C}$ is denoted by $\mathbb{1}_{\mathscr{C}}$. A natural transformation between two functors $\mathsf{F}, \mathsf{G} : \mathscr{C} \to \mathscr{B}$, is denoted by $\beta_{-} : \mathsf{F} \to \mathsf{G}$. If $\mathsf{H} : \mathscr{B} \to \mathscr{A}$ and $\mathsf{J} : \mathscr{D} \to \mathscr{C}$ are other functors, then $\beta_{\mathsf{J}(-)}$ (or β_{J}) denotes the natural transformation defined at each object $Z \in \mathscr{D}$ by $\beta_{\mathsf{J}(Z)} : \mathsf{FJ}(Z) \to \mathsf{GJ}(Z)$, while $\mathsf{H}\beta_{-}$ (or $\mathsf{H}\beta$) denotes the natural transformation defined at each object $X \in \mathscr{C}$ by $\mathsf{H}(\beta_X) : \mathsf{HF}(X) \to \mathsf{HG}(X)$.

1.1. *Monads and comonads*

Recall from [8] that a comonad (or cotriple) on a category $\mathscr{C}$ is a functor $\mathsf{C} : \mathscr{C} \to \mathscr{C}$ with two natural transformations $\Theta : \mathsf{C} \to \mathsf{C}^2$ (comultiplication) and $\vartheta : \mathsf{C} \to \mathbb{1}_{\mathscr{C}}$ (counit) such that $\Theta_{\mathsf{C}} \circ \Theta = \mathsf{C}\Theta \circ \Theta$ and $\vartheta_{\mathsf{C}} \circ \Theta = \mathsf{C}\vartheta \circ \Theta = \mathsf{C}$. The objects of the category of C-comodules (or C-coalgebras) $\mathscr{C}^{\mathsf{C}}$ are pairs (X, d_X) where $X \in \mathscr{C}$ and $\mathsf{d}_X : X \to \mathsf{C}(X)$ is a morphism in $\mathscr{C}$ such that $\Theta_X \circ \mathsf{d}_X = \mathsf{C}(\mathsf{d}_X) \circ \mathsf{d}_X$ and $\vartheta_X \circ \mathsf{d}_X = X$. A morphism in $\mathscr{C}^{\mathsf{C}}$ is a morphism $f : X \to X'$ in $\mathscr{C}$ such that $\mathsf{d}_{X'} \circ f = \mathsf{C}(f) \circ \mathsf{d}_X$. Any adjunction $S : \mathscr{D} \rightleftarrows \mathscr{C} : T$ with S is left adjoint to T (notation $S \dashv T$) induces a structure of comonad $(ST, S\eta_T, \epsilon)$, where $\eta : \mathbb{1}_{\mathscr{D}} \to TS$ and $\epsilon : ST \to \mathbb{1}_{\mathscr{C}}$ are, respectively, the unit and counit of this adjunction. In this situation, we say that $S \dashv T$ cogenerates the comonad $(ST, S\eta_T, \epsilon)$. Associated to a comonad C there is an universal cogenerator. That is, an adjunction $S^{\mathsf{C}} : \mathscr{C}^{\mathsf{C}} \rightleftarrows \mathscr{C} : T^{\mathsf{C}}$ with $S^{\mathsf{C}} \dashv T^{\mathsf{C}}$, where S^{C} is the forgetful functor and T^{C} sends any object $Y \in \mathscr{C}$ to the comodule $T^{\mathsf{C}}(Y) = (\mathsf{C}(Y), \Theta_Y)$ and any morphism g to $T^{\mathsf{C}}(g) = \mathsf{C}(g)$. The universal property [8, Theorem 2.2] asserts that, for any other cogenerator $S : \mathscr{D} \rightleftarrows \mathscr{C} : T$ of $(\mathsf{C}, \Theta, \vartheta)$, there exists a unique functor $L : \mathscr{D} \to \mathscr{C}^{\mathsf{C}}$ such that $S^{\mathsf{C}}L = S$ and $L\eta = {\eta^{\mathsf{C}}}_L$, where η^{C} is the unit of the adjunction $S^{\mathsf{C}} \dashv T^{\mathsf{C}}$. The last two equalities imply $LT = T^{\mathsf{C}}$. The functor L is refereed to as *factorization functor* (or *comparison functor*), and sends any object $D \in \mathscr{D}$ to the comodule $L(D) = (S(D), S\eta_D)$ and any morphism h to $L(h) = S(h)$. The comparison functor is not in general an equivalence of categories, see [14, Theorem 2.7] for more details.

The notion of monad is dual to that of comonad. Explicitly, a monad (or triple) on $\mathscr{C}$ is a three-tuple (A, μ, η), where $\mathsf{A} : \mathscr{C} \to \mathscr{C}$ is a functor and $\mu : \mathsf{A}^2 \to \mathsf{A}$, $\eta : \mathbb{1}_{\mathscr{C}} \to \mathsf{A}$ are natural transformations satisfying the associativity and unitary properties. The category $\mathscr{C}_{\mathsf{A}}$ of A-modules (or A-algebras) is dually defined, and there is a universal generator $T_{\mathsf{A}} : \mathscr{C} \rightleftarrows \mathscr{C}_{\mathsf{A}} : S_{\mathsf{A}}$ attached to (A, μ, η), where S_{A} denotes the forgetful functor. In contrast with comonads, here we have an adjunction of the form $T_{\mathsf{A}} \dashv S_{\mathsf{A}}$, i.e. T_{A} is left adjoint functor to S_{A}. Analogously there is a universal property as well as a notion of comparison functor.

1.2. *Continuous and lifted functors*

Let $\mathscr{C}$ and $\mathscr{D}$ be two additive categories with cokernels and arbitrary direct sums. Recall that an additive functor $\mathsf{F} : \mathscr{C} \to \mathscr{D}$ is said to be *continuous* provided that F is right exact and preserves direct sums. Thus F commutes with inductive limits. Assume further that $\mathscr{C}$ has a subgenerator U, that is, every object of $\mathscr{C}$ is a sub-object of an U-generated one. Then the proof of [9, Lemma 5.1] can be adapted to this setting in order to show that natural transformations between two continuous functors form a set (see also [10, Proposition 2.6]). Henceforth, continuous endo-functors of the category $\mathscr{C}$ and their natural transformations form a Hom-set category (or Set-category) which we denote by $\overline{\mathsf{Funct}}(\mathscr{C})$. This situation can be applied to the category of comodules $\mathscr{C}^{\mathsf{C}}$ whenever $\mathscr{C}$ is a Grothendieck category and the underlying functor $\mathsf{C} : \mathscr{C} \to \mathscr{C}$ is continuous, see [10, Lemma 2.5].

Let us recall from [16] (see also [27]) the notion of lifted functors. Given a comonad $(\mathsf{C}, \Theta, \vartheta)$ on category $\mathscr{C}$, a functor $\mathsf{F} : \mathscr{C} \to \mathscr{C}$ is said to be C-*lifted* if there exists a functor $\bar{\mathsf{F}} : \mathscr{C}^{\mathsf{C}} \to \mathscr{C}^{\mathsf{C}}$ rendering commutative the following diagram

$$\begin{tikzcd}
\mathscr{C}^{\mathsf{C}} \arrow[r, "\bar{\mathsf{F}}"] \arrow[d, "S^{\mathsf{C}}"'] & \mathscr{C}^{\mathsf{C}} \arrow[d, "S^{\mathsf{C}}"] \\
\mathscr{C} \arrow[r, "\mathsf{F}"] & \mathscr{C}.
\end{tikzcd}$$

In this way, we say that $\bar{\mathsf{F}}$ is a C-*lifted functor* of F. A C-lifted functor of F when exists is not necessarily unique, see Remark 2.7. In fact the lifting Theorem (dual version of [16, Lemma 1]) asserts that there is an 1-1 correspondence between a C-lifted functors $\bar{\mathsf{F}}$ of F and natural transformations $\mathscr{E}^{\mathsf{F}} : \mathsf{FC} \to \mathsf{CF}$ compatible in the obvious way with the counit and the comultiplication of C. This correspondence is given as follows. Given a C-lifted functor $\bar{\mathsf{F}} : \mathscr{C}^{\mathsf{C}} \to \mathscr{C}^{\mathsf{C}}$ of F, we define for every object $X \in \mathscr{C}$ a morphism

$$\mathscr{E}^{\mathsf{F}}_X : \mathsf{FC}(X) \xrightarrow{\mathsf{d}_{\mathsf{FC}(X)}} \mathsf{CFC}(X) \xrightarrow{\mathsf{CF}\vartheta_X} \mathsf{CF}(X),$$

where $\bar{\mathsf{F}}(\mathsf{C}(X), \Theta_X) = (\mathsf{FC}(X), \mathsf{d}_{\mathsf{FC}(X)}) \in \mathscr{C}^{\mathsf{C}}$ is the image by $\bar{\mathsf{F}}$ of the comodule $(\mathsf{C}(X), \Theta_X)$. This leads to a natural transformation $\mathscr{E}^{\mathsf{F}}_{-} : \mathsf{FC} \to \mathsf{CF}$ which is easily shown to be compatible with Θ and ϑ. Conversely, if a compatible natural transformation $\mathscr{E}^{\mathsf{F}} : \mathsf{FC} \to \mathsf{CF}$ is given, then a C-lifted functor $\bar{\mathsf{F}} : \mathscr{C}^{\mathsf{C}} \to \mathscr{C}^{\mathsf{C}}$ is defined by sending $(X, \mathsf{d}_X) \to \bar{\mathsf{F}}(X, \mathsf{d}_X) = (\mathsf{F}(X), \mathscr{E}^{\mathsf{F}} \circ \mathsf{F}(\mathsf{d}_X))$ and acting by F on the morphisms of $\mathscr{C}^{\mathsf{C}}$. In this way, the composition of C-lifted functor is again a C-lifted functor. Explicitly, if $\bar{\mathsf{F}}_i : \mathscr{C}^{\mathsf{C}} \to \mathscr{C}^{\mathsf{C}}$ are C-lifted functors of F_i, $i = 1, 2$, with natural transformations $\mathscr{E}^{\mathsf{F}_i} : \mathsf{F}_i\mathsf{C} \to \mathsf{CF}_i$, $i = 1, 2$. Then $\bar{\mathsf{F}}_1\bar{\mathsf{F}}_2 : \mathscr{C}^{\mathsf{C}} \to \mathscr{C}^{\mathsf{C}}$ is a C-lifted functor of $\mathsf{F}_1\mathsf{F}_2$ with natural transformation

$$\mathscr{E}^{\mathsf{F}_1\mathsf{F}_2} : \mathsf{F}_1\mathsf{F}_2\mathsf{C} \xrightarrow{\mathsf{F}_1\mathscr{E}^{\mathsf{F}_2}} \mathsf{F}_1\mathsf{CF}_2 \xrightarrow{\mathscr{E}^{\mathsf{F}_1}_{\mathsf{F}_2}} \mathsf{CF}_1\mathsf{F}_2.$$

Given a natural transformation $\sigma : \mathsf{F} \to \mathsf{G}$ where $\mathsf{F}, \mathsf{G} : \mathscr{C} \to \mathscr{C}$ are C-lifted. In general, it is not clear that σ can be lifted to a natural transformation $\bar{\mathsf{F}} \to \bar{\mathsf{G}}$. However, if we assume that

$$\begin{tikzcd}
\mathsf{FC} \arrow[r, "\sigma_{\mathsf{C}}"] \arrow[d, "\mathscr{E}^{\mathsf{F}}"'] & \mathsf{GC} \arrow[d, "\mathscr{E}^{\mathsf{G}}"] \\
\mathsf{CF} \arrow[r, "\mathsf{C}\sigma"] & \mathsf{CG}
\end{tikzcd} \tag{1.1}$$

is a commutative diagram, then clearly σ induces a natural transformation $\bar{\sigma} : \bar{\mathsf{F}} \to \bar{\mathsf{G}}$ such that $S^{\mathsf{C}}\bar{\sigma} = \sigma_{S^{\mathsf{C}}}$.

The following lemma whose proof is omitted shows that the property of being continuous can be transferred to a lifted functors.

Lemma 1.1. *Let $\mathscr{C}$ be a Grothendieck category and $(\mathsf{C}, \Theta, \vartheta)$ a comonad on $\mathscr{C}$ whose underlying functor $\mathsf{C} : \mathscr{C} \to \mathscr{C}$ is continuous. Then any C-lifted functor $\bar{\mathsf{F}} : \mathscr{C}^{\mathsf{C}} \to \mathscr{C}^{\mathsf{C}}$ of a continuous functor $\mathsf{F} : \mathscr{C} \to \mathscr{C}$, is also continuous.*

Given a comonad $\Xi : \mathscr{C} \to \mathscr{C}$ on category $\mathscr{C}$, we denote by $S^{\Xi} \dashv T^{\Xi}$ its universal cogenerator, i.e. the universal adjunction $S^{\Xi} : \mathscr{C}^{\Xi} \rightleftarrows \mathscr{C} : T^{\Xi}$. The unit of this adjunction is denoted by $\mathsf{d}_{-}^{\Xi} : \mathbb{1}_{\mathscr{C}^{\Xi}} \to T^{\Xi}S^{\Xi}$. The following lemma is a companion of [17, Proposition 3.1].

Lemma 1.2. *Let $(\mathsf{C}, \Theta, \vartheta)$ be a comonad on a category $\mathscr{C}$, and $\vartheta : S^{\mathsf{C}}T^{\mathsf{C}} \to \mathbb{1}_{\mathscr{C}}$ the counit of its universal cogenerator $S^{\mathsf{C}} \dashv T^{\mathsf{C}}$. Let $\mathsf{F} : \mathscr{C} \to \mathscr{C}$ be a functor which has a C-lifted functor $\bar{\mathsf{F}} : \mathscr{C}^{\mathsf{C}} \to \mathscr{C}^{\mathsf{C}}$. Assume that $(\bar{\mathsf{F}}, \nabla, \omega)$ is a comonad on $\mathscr{C}^{\mathsf{C}}$, and consider the composed comonad $(S^{\mathsf{C}}\bar{\mathsf{F}}T^{\mathsf{C}},\, S^{\mathsf{C}}\bar{\mathsf{F}}\mathsf{d}^{\mathsf{C}}_{\bar{\mathsf{F}}T^{\mathsf{C}}} \circ S^{\mathsf{C}}\nabla_{T^{\mathsf{C}}},\, \vartheta \circ S^{\mathsf{C}}\omega_{T^{\mathsf{C}}})$. Then there is a unique functor $\mathscr{K} : (\mathscr{C}^{\mathsf{C}})^{\bar{\mathsf{F}}} \to \mathscr{C}^{S^{\mathsf{C}}\bar{\mathsf{F}}T^{\mathsf{C}}}$ such that $S^{S^{\mathsf{C}}\bar{\mathsf{F}}T^{\mathsf{C}}}\mathscr{K} = S^{\mathsf{C}}S^{\bar{\mathsf{F}}}$ and $\mathscr{K}(T^{\bar{\mathsf{F}}}\mathsf{d}^{\mathsf{C}}_{S^{\bar{\mathsf{F}}}} \circ \mathsf{d}^{\bar{\mathsf{F}}}) = \mathsf{d}^{S^{\mathsf{C}}\bar{\mathsf{F}}T^{\mathsf{C}}}_{\mathscr{K}}$. Moreover, $\mathscr{K}$ is an isomorphism of categories.*

Proof. Since the following composition of adjunctions

$$\begin{tikzcd}
(\mathscr{C}^{\mathsf{C}})^{\bar{\mathsf{F}}} \arrow[r, shift left, "S^{\bar{\mathsf{F}}}"] & \mathscr{C}^{\mathsf{C}} \arrow[l, shift left, "T^{\bar{\mathsf{F}}}"] \arrow[r, shift left, "S^{\mathsf{c}}"] & \mathscr{C} \arrow[l, shift left, "T^{\mathsf{C}}"]
\end{tikzcd}$$

cogenerates the comonad $S^{\mathsf{C}}\bar{\mathsf{F}}T^{\mathsf{C}}$, the first statement of the lemma is a direct application of the dual version of [8, Theorem 2.2]. For the second one, we can easily show that the inverse functor of $\mathscr{K}$ is $\mathscr{F} : \mathscr{C}^{S^{\mathsf{C}}\bar{\mathsf{F}}T^{\mathsf{C}}} \to (\mathscr{C}^{\mathsf{C}})^{\bar{\mathsf{F}}}$ defined on objects by

$$\mathscr{F}(Y,\, \mathsf{d}_{Y}^{S^{\mathsf{C}}\bar{\mathsf{F}}T^{\mathsf{C}}}) = ((Y_{\xi}, \mathsf{d}^{\mathsf{C}}_{Y_{\xi}}),\, \mathsf{d}^{\bar{\mathsf{F}}}_{(Y_{\xi}, \mathsf{d}^{\mathsf{C}}_{Y_{\xi}})} = F\vartheta_{Y} \circ \mathsf{d}_{Y}^{S^{\mathsf{C}}\bar{\mathsf{F}}T^{\mathsf{C}}}),$$

where $(Y_{\xi}, \mathsf{d}^{\mathsf{C}}_{Y_{\xi}})$ is the C-comodule induced by the comonad morphism $\xi = S^{\mathsf{C}}\omega_{T^{\mathsf{C}}} : S^{\mathsf{C}}\bar{\mathsf{F}}T^{\mathsf{C}} \to S^{\mathsf{C}}T^{\mathsf{C}} = \mathsf{C}$, and acts by identity on morphisms. □

A lifted functors with respect to a given monad (A, μ, η) on a category $\mathscr{C}$, are similarly defined. The lifting Theorem in this case [16, Lemma 1], says that there is an 1-1 correspondence between an A-lifted functors $\bar{\mathsf{F}}$ of $\mathsf{F} : \mathscr{C} \to \mathscr{C}$ and natural transformations $\mathscr{I}^{\mathsf{F}} : \mathsf{AF} \to \mathsf{FA}$ compatible in the obvious way with μ and η.

1.3. *Corings and comodules*

We work over a commutative ground ring with 1 denoted by $\mathbb{K}$. All rings are assumed to be associative $\mathbb{K}$-algebras. Modules are unital modules, and bimodules are left and right unital modules and are assumed to be central $\mathbb{K}$-modules. Given A and B two rings, the category of (A, B)-bimodules is denoted as usual by ${}_A\mathsf{Mod}_B$. The $\mathbb{K}$-module of morphisms in this category will be denoted by $\mathrm{Hom}_{A-B}(-, -)$. The symbol $- \otimes_A -$ between bimodules and bilinear maps denotes the tensor product over A. Let A be a ring, an *A-coring* [23] is a three-tuple $(\mathfrak{C}, \Delta, \varepsilon)$ (denoted also by $(\mathfrak{C} : A)$) consisting of an A-bimodule $\mathfrak{C}$ and two A-bilinear maps

$$\Delta : \mathfrak{C} \to \mathfrak{C} \otimes_A \mathfrak{C} \quad \text{and} \quad \varepsilon : \mathfrak{C} \to A,$$

known as the comultiplication and the counit of $\mathfrak{C}$, which satisfy

$$(\mathfrak{C} \otimes_A \Delta) \circ \Delta = (\Delta \otimes_A \mathfrak{C}) \circ \Delta, \quad (\mathfrak{C} \otimes_A \varepsilon) \circ \Delta = \mathfrak{C} = (\varepsilon \otimes_A \mathfrak{C}) \circ \Delta.$$

It is clear that any A-coring $\mathfrak{C}$ induces a comonad on the categories of both right and left A-modules. Conversely, if $(\mathsf{C}, \Theta, \vartheta)$ is a comonad, say on the category of right A-modules Mod_A, such that C is a continuous functor, then $\mathsf{C}(A)$ admits a structure of an A-coring whose induced comonad is naturally isomorphic to C, see [10, Proposition 3.5].

A *right $\mathfrak{C}$-comodule* is a pair (M, ρ_M) with M a right A-module and $\rho_M : M \to M \otimes_A \mathfrak{C}$ a right A-linear map (called right $\mathfrak{C}$-coaction) satisfying two equalities: $(\rho_M \otimes_A \mathfrak{C}) \circ \rho_M = (M \otimes_A \Delta) \circ \rho_M$ and $(M \otimes_A \varepsilon) \circ \rho_M = M$. A *morphism* of right $\mathfrak{C}$-comodules $f : (M, \rho_M) \to (M', \rho_{M'})$ is a right A-linear map $f : M \to M'$ which is compatible with coactions: $\rho_{M'} \circ f = (f \otimes_A \mathfrak{C}) \circ \rho_M$ (f is right $\mathfrak{C}$-colinear). The $\mathbb{K}$-module of all colinear maps will be denoted by $\mathrm{Hom}_{\mathfrak{C}}(M, M')$. We denote by $\mathsf{Comod}_{\mathfrak{C}}$ the category of all right $\mathfrak{C}$-comodules, and by $\mathscr{U}_{\mathfrak{C}} : \mathsf{Comod}_{\mathfrak{C}} \rightleftarrows \mathsf{Mod}_A : - \otimes_A \mathfrak{C}$ the universal cogenerator of the comonad $- \otimes_A \mathfrak{C} : \mathsf{Mod}_A \to \mathsf{Mod}_A$. Left $\mathfrak{C}$-comodules are symmetrically defined, we use the Greek letter λ_{-} to denote their coactions. If more than one coring are handled, we then use the notations $\rho^{\mathfrak{C}}_{-}$, $\rho^{\mathfrak{D}}_{-}$ and $\lambda^{\mathfrak{C}}_{-}$, $\lambda^{\mathfrak{D}}_{-}$ to distinguish between $\mathfrak{C}$-coactions and $\mathfrak{D}$-coactions.

An A-bilinear map $\phi : \mathfrak{D} \to \mathfrak{C}$ is a *morphism of A-corings* if it satisfies $\varepsilon_{\mathfrak{C}} \circ \phi = \varepsilon_{\mathfrak{D}}$ and $(\phi \otimes_A \phi) \circ \Delta_{\mathfrak{D}} = \Delta_{\mathfrak{C}} \circ \phi$. Given a morphism of A-corings $\phi : \mathfrak{D} \to \mathfrak{C}$, one can associated to it the so-called *induction functor* $(-)_\phi : \mathsf{Comod}_{\mathfrak{D}} \to \mathsf{Comod}_{\mathfrak{C}}$ sending a comodule $(X, \rho^{\mathfrak{D}}_X)$ to $(X, \rho^{\mathfrak{C}}_X = (X \otimes_A \phi) \circ \rho^{\mathfrak{D}}_X)$ and acting by identity on morphisms, see [13] and [5].

The category of (right) $\mathfrak{C}$-comodules is in general not an abelian category, it has cokernels and arbitrary direct sums which can be already computed in the category of A-modules. However, if ${}_A\mathfrak{C}$ is a flat module, then $\mathsf{Comod}_{\mathfrak{C}}$ becomes a

Grothendieck category, see [11, 5]. Let $\mathfrak{D}$ be a B-coring, a $(\mathfrak{C}, \mathfrak{D})$-*bicomodule* is a three-tuple $(M, \rho_M^{\mathfrak{D}}, \lambda_M^{\mathfrak{C}})$ consisting of an (A, B)-bimodule M and A-B-bilinear maps $\rho_M^{\mathfrak{D}} : M \to M \otimes_B \mathfrak{D}$, $\lambda_M^{\mathfrak{C}} : M \to \mathfrak{C} \otimes_A M$ such that $(M, \rho_M^{\mathfrak{D}})$ is right $\mathfrak{D}$-comodule and $(M, \lambda_M^{\mathfrak{C}})$ is left $\mathfrak{C}$-comodule with compatibility condition: $(\mathfrak{C} \otimes_A \rho_M^{\mathfrak{D}}) \circ \lambda_M^{\mathfrak{C}} = (\lambda_M^{\mathfrak{C}} \otimes_B \mathfrak{D}) \circ \rho_M^{\mathfrak{D}}$. A *morphism of bicomodules* is a left and right colinear map (say *bicolinear map*). We use the notation $\mathrm{Hom}_{\mathfrak{C}-\mathfrak{D}}(M, M')$ for the $\mathbb{K}$-module of all bicolinear maps. The category of all $(\mathfrak{C}, \mathfrak{D})$-bicomodule is denoted by ${}_{\mathfrak{C}}\mathsf{Comod}_{\mathfrak{D}}$. Obviously any ring A is a coring over itself, with comultiplication the isomorphism $A \cong A \otimes_A A$ and counit the identity on A. In this way, an $(A, \mathfrak{D})$-bicomodule is just a right $\mathfrak{D}$-comodule (M, ρ_M) whose underlying module M is an (A, B)-bimodule and whose coaction ρ_M is an A-B-bilinear map. The category of $(A, \mathfrak{D})$-bicomodules is denoted by ${}_A\mathsf{Comod}_{\mathfrak{D}}$. For more details on comodules, definitions and basic properties of bicomodules and the cotensor product, the reader is referred to monograph [5].

Throughout all sections, the symbol $-\otimes-$ stands for $-\otimes_A-$ the tensor product over a fixed base ring A.

2. Eilenberg–Moore Monoidal Category Associated to a Coring, and Lifted Functors

In this section, we give a complete and detailed proof of the fact that the right Eilenberg–Moore monoidal category associated to a given coring $\mathfrak{C}$ is monoidally equivalent to certain monoidal category of $(- \otimes \mathfrak{C})$-lifted functors, Proposition 2.4. If $\mathfrak{C}$ arises from some an entwining structure, we then establish using this proposition an *op*monoidal functor from the right Eilenberg–Moore category of the factor coalgebra to the right Eilenberg–Moore category of $\mathfrak{C}$, Lemma 2.8.

For a coring $(\mathfrak{C} : A)$ with comultiplication Δ and counit ε, we consider as in [3] (see [17] for general notions), the right Eilenberg–Moore additive category $\mathscr{R}_{(\mathfrak{C} : A)}$ defined by the following data:

Objects: Are pairs $(M, \mathfrak{m})$ consisting of an A-bimodule M and A-bilinear map $\mathfrak{m} : \mathfrak{C} \otimes M \to M \otimes \mathfrak{C}$ (the *twisting map*) such that

$$(M \otimes \Delta) \circ \mathfrak{m} = (\mathfrak{m} \otimes \mathfrak{C}) \circ (\mathfrak{C} \otimes \mathfrak{m}) \circ (\Delta \otimes M), \tag{2.1}$$

$$(M \otimes \varepsilon) \circ \mathfrak{m} = \varepsilon \otimes M, \tag{2.2}$$

where in the second equality M was identified with $A \otimes M$ and with $M \otimes A$ via the obvious isomorphisms.

Morphisms: Given any object $(M, \mathfrak{m})$ one can easily check that $\mathfrak{C} \otimes M$ is a $\mathfrak{C}$-bicomodule with left $\mathfrak{C}$-coaction $\lambda_{\mathfrak{C} \otimes M} = \Delta \otimes M$ and right $\mathfrak{C}$-coaction $\rho_{\mathfrak{C} \otimes M} = (\mathfrak{C} \otimes \mathfrak{m}) \circ (\Delta \otimes M)$. By [3, Proposition 2.2], the $\mathbb{K}$-modules of morphisms in $\mathscr{R}_{(\mathfrak{C} : A)}$ are then defined (in unreduced form) by

$$\mathrm{Hom}_{\mathscr{R}_{(\mathfrak{C} : A)}}((M, \mathfrak{m}),\, (M', \mathfrak{m}')) := \mathrm{Hom}_{\mathfrak{C}-\mathfrak{C}}(\mathfrak{C} \otimes M,\, \mathfrak{C} \otimes M').$$

That is, a morphism $\varphi : (M, \mathfrak{m}) \to (M', \mathfrak{m}')$ in $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$ is an A-bilinear map $\varphi : \mathfrak{C} \otimes M \to \mathfrak{C} \otimes M'$ which satisfies

$$(\Delta \otimes M') \circ \varphi = (\mathfrak{C} \otimes \varphi) \circ (\Delta \otimes M), \tag{2.3}$$

$$(\mathfrak{C} \otimes \mathfrak{m}') \circ (\Delta \otimes M') \circ \varphi = (\varphi \otimes \mathfrak{C}) \circ (\mathfrak{C} \otimes \mathfrak{m}) \circ (\Delta \otimes M). \tag{2.4}$$

The category $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$ is in fact a multiplicative additive category (i.e. an additive monoidal category). It multiplication is defined as follows. Given two objects $(M, \mathfrak{m})$ and $(M', \mathfrak{m}')$ of $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$, we define a new object of $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$ by

$$(M, \mathfrak{m}) \underset{(\mathfrak{C}\,:\,A)}{\overset{r}{\otimes}} (M', \mathfrak{m}') := (M \otimes M', (M \otimes \mathfrak{m}') \circ (\mathfrak{m} \otimes M')).$$

If $\varphi : (M, \mathfrak{m}) \to (M', \mathfrak{m}')$ and $\psi : (N, \mathfrak{n}) \to (N', \mathfrak{n}')$ are morphisms in $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$, then their multiplication is defined by the composition

$$\begin{tikzcd}
\mathfrak{C} \otimes M \otimes N \arrow[rrr, dashed, "\varphi \underset{(\mathfrak{C}\,:\,A)}{\overset{r}{\otimes}} \psi"] \arrow[d, "\Delta \otimes M \otimes N"'] & & & \mathfrak{C} \otimes M' \otimes N' \\
\mathfrak{C} \otimes \mathfrak{C} \otimes M \otimes N \arrow[dr, "\mathfrak{C} \otimes \varphi \otimes N"'] & & & \mathfrak{C} \otimes M' \otimes \mathfrak{C} \otimes N' \arrow[u, "\mathfrak{C} \otimes M' \otimes \varepsilon \otimes N'"] \\
& \mathfrak{C} \otimes \mathfrak{C} \otimes M' \otimes N \arrow[r, "\mathfrak{C} \otimes \mathfrak{m}' \otimes N"'] & \mathfrak{C} \otimes M' \otimes \mathfrak{C} \otimes N \arrow[ur, "\mathfrak{C} \otimes M' \otimes \psi"] &
\end{tikzcd} \tag{2.5}$$

Equivalently $\varphi \underset{(\mathfrak{C}\,:\,A)}{\overset{r}{\otimes}} \psi = (\mathfrak{C} \otimes M' \otimes \varepsilon \otimes N') \circ (\varphi \otimes \psi) \circ (\mathfrak{C} \otimes \mathfrak{m} \otimes N) \circ (\Delta \otimes M \otimes N)$. The identity object for this multiplication is proportioned by the pair $(A, \mathfrak{C})$ where the identity of $\mathfrak{C}$ was identified with the isomorphism $\mathfrak{C} \otimes A \cong A \otimes \mathfrak{C}$.

There is, up to a monoidal isomorphism of categories, an alternative description of the category $\mathscr{R}_{(\mathfrak{C}:A)}$ which uses the notion of *lifted functors*. This was mentioned in [17, p. 256] without any indication on the proof. We will give in our case a complete and detailed proof. We start by applying the lifting Theorem (see Subsec. 1.2) to the comonad $- \otimes \mathfrak{C} : \mathsf{Mod}_A \to \mathsf{Mod}_A$. As consequence, we obtain the following two lemmas.

Lemma 2.1. *Let* $(\mathfrak{C} : A)$ *be any coring and* M *an* A*-bimodule. There is an* 1-1 *correspondence between*

- (i) $\mathfrak{C}$*-bicomodule structures on* $\mathfrak{C} \otimes M$*, with underlying left* $\mathfrak{C}$*-coaction* $\lambda_{\mathfrak{C} \otimes M} = \Delta \otimes M$;
- (ii) A*-bilinear maps* $\mathfrak{m} : \mathfrak{C} \otimes M \to M \otimes \mathfrak{C}$ *such that* $(M, \mathfrak{m})$ *is an object of the category* $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$;
- (iii) $(- \otimes \mathfrak{C})$*-Lifted functor* $\overline{(- \otimes M)} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$ *of the functor* $- \otimes M : \mathsf{Mod}_A \to \mathsf{Mod}_A$.

The image of $(U, \rho_U) \in \mathsf{Comod}_{\mathfrak{C}}$ *is given by*

$$\overline{(-\otimes M)}(U, \rho_U) = (U \otimes M,\, (U \otimes \mathfrak{m}) \circ (\rho_U \otimes M)) \in \mathsf{Comod}_{\mathfrak{C}}.$$

Conversely, if $\overline{(-\otimes M)}$ *is a* $(-\otimes \mathfrak{C})$*-lifted functor, then the twisting map* $\mathfrak{m}$ *is given by* $\mathfrak{m} = (\varepsilon \otimes M \otimes \mathfrak{C}) \circ \rho_{\mathfrak{C}\otimes M}$*, where* $\rho_{\mathfrak{C}\otimes M}$ *is the* $\mathfrak{C}$*-coaction of the right comodule* $\overline{(-\otimes M)}(\mathfrak{C}, \Delta)$*.*

Lemma 2.2. *Let* M *and* M' *be two* A*-bimodules satisfying the equivalent conditions of Lemma 2.1. Then*

(a) *For every right module* $U \in \mathsf{Mod}_A$ *and every object* $(V, \rho_V) \in {}_A\mathsf{Comod}_{\mathfrak{C}}$*, we have*

$$\overline{(-\otimes M)}(U \otimes V, U \otimes \rho_V) = (U \otimes V \otimes M,\, U \otimes \rho_{V\otimes M}),$$

where $\rho_{V\otimes M}$ *is the* $\mathfrak{C}$*-coaction of the comodule* $\overline{(-\otimes M)}(V, \rho_V)$*.*

(b) *If* $\Gamma_{-} : \overline{(-\otimes M)} \to \overline{(-\otimes M')}$ *is a natural transformation between two lifted functors, then*

$$\Gamma_{(U\otimes V,\, U\otimes \rho_V)} = U \otimes \Gamma_{(V,\, \rho_V)},$$

for every $U \in \mathsf{Mod}_A$ *and every* $(V, \rho_V) \in {}_A\mathsf{Comod}_{\mathfrak{C}}$*.*

Proof. Use the free representation of right A-modules. □

The following completes Lemma 2.1.

Lemma 2.3. *Let* $(\mathfrak{C} : A)$ *be a coring. Assume a continuous functor* $\mathsf{F} : \mathsf{Mod}_A \to \mathsf{Mod}_A$ *is given. Then there is an* 1-1 *correspondence between*

(i) $(-\otimes \mathfrak{C})$*-lifted functors* $\bar{\mathsf{F}} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$ *of* F*;*
(ii) A*-bilinear maps* $\mathfrak{m} : \mathfrak{C} \otimes \mathsf{F}(A) \to \mathsf{F}(A) \otimes \mathfrak{C}$ *such that* $(\mathsf{F}(A), \mathfrak{m})$ *is an object of the category* $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$*;*
(iii) $(-\otimes \mathfrak{C})$*-lifted functor* $\overline{(-\otimes \mathsf{F}(A))} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$ *of the functor* $-\otimes \mathsf{F}(A) : \mathsf{Mod}_A \to \mathsf{Mod}_A$*.*

Moreover, the canonical natural isomorphism $\mathsf{F} \cong -\otimes \mathsf{F}(A)$ *induces a natural isomorphism* $\bar{\mathsf{F}} \cong \overline{(-\otimes \mathsf{F}(A))}$ *between two corresponding* $(-\otimes \mathfrak{C})$*-lifted functors.*

Proof. By Watts's Theorem [26] each continuous functor $\mathsf{F} : \mathsf{Mod}_A \to \mathsf{Mod}_A$ is naturally isomorphic to the tensor product functor $-\otimes \mathsf{F}(A)$, where $\mathsf{F}(A)$ is in a natural way an A-bimodule. Let us denote by $\Upsilon : \mathsf{F} \to -\otimes \mathsf{F}(A)$ this canonical isomorphism. The stated correspondence is given as follows, wherein the properties of Υ stated in [10, Lemma 3.4] were implicitly used. To each twisting map $\mathfrak{m} : \mathfrak{C} \otimes \mathsf{F}(A) \to \mathsf{F}(A) \otimes \mathfrak{C}$ it corresponds a compatible natural transformation

$$\begin{tikzcd}
\mathsf{F}(X \otimes \mathfrak{C}) \arrow[r, "\Upsilon_{X \otimes \mathfrak{C}}"] \arrow[rrd, dashed, "\mathscr{E}^{\mathsf{F}}_{X}"'] & X \otimes \mathfrak{C} \otimes \mathsf{F}(A) \arrow[r, "X \otimes \mathfrak{m}"] & X \otimes \mathsf{F}(A) \otimes \mathfrak{C} \arrow[d, "\Upsilon_X^{-1} \otimes \mathfrak{C}"] \\
& & \mathsf{F}(X) \otimes \mathfrak{C}
\end{tikzcd}$$

which via the lifting Theorem (see Subsec. 1.2) leads to a $(- \otimes \mathfrak{C})$-lifted func tor $\bar{\mathsf{F}} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$ of F. Conversely, assume a $(- \otimes \mathfrak{C})$-lifted functor $\bar{\mathsf{F}} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$ of F is given and let $\mathscr{E}^{\mathsf{F}} : \mathsf{F} \circ (- \otimes \mathfrak{C}) \to \mathsf{F}(-) \otimes \mathfrak{C}$ be its corresponding compatible natural transformation. We define a twisting map by

$$\mathfrak{m} : \mathfrak{C} \otimes \mathsf{F}(A) \xrightarrow{\Upsilon_{\mathfrak{C}}^{-1}} \mathsf{F}(\mathfrak{C}) \cong \mathsf{F}(A \otimes \mathfrak{C}) \xrightarrow{\mathscr{E}^{\mathsf{F}}_{A}} \mathsf{F}(A) \otimes \mathfrak{C}.$$

This establishes the equivalence (i) $\Leftrightarrow$ (ii), and finishes the proof of the first statement since (ii) $\Leftrightarrow$ (iii) was shown in Lemma 2.1.

Consider a twisting map $\mathfrak{m}$ as in (ii). It is clear that the corresponding natural transformation (via the lifting Theorem and Lemma 2.1) of the $(-\otimes\mathfrak{C})$-lifted functor $\overline{(- \otimes \mathsf{F}(A))}$ is given by $- \otimes \mathfrak{m}$. As above we set $\mathscr{E}^{\mathsf{F}}_{-} := (\Upsilon_{-}^{-1} \otimes \mathfrak{C}) \circ (- \otimes \mathfrak{m}) \circ \Upsilon_{- \otimes \mathfrak{C}}$. Hence we can show using [10, Lemma 3.4(b)], that the following diagram

$$\begin{tikzcd}
\mathsf{F}(X \otimes \mathfrak{C}) \arrow[r, "\Upsilon_{X \otimes \mathfrak{C}}"] \arrow[d, "\mathscr{E}^{\mathsf{F}}_{X}"'] & X \otimes \mathfrak{C} \otimes \mathsf{F}(A) \arrow[d, "X \otimes \mathfrak{m}"] \\
\mathsf{F}(X) \otimes \mathfrak{C} \arrow[r, "\Upsilon_X \otimes \mathfrak{C}"] & X \otimes \mathsf{F}(A) \otimes \mathfrak{C}
\end{tikzcd}$$

is commutative for every right A-module X. Thus the condition of Eq. (1.1) for the natural isomorphism Υ is fulfilled. Therefore, $\bar{\mathsf{F}} \cong \overline{(- \otimes \mathsf{F}(A)}$ via Υ. $\square$

Applying the arguments of the preamble of Subsec. 1.2 to the category of right comodules $\mathsf{Comod}_{\mathfrak{C}}$ over an A-coring $\mathfrak{C}$, we can thus define the category of continuous endo-functor $\overline{\mathsf{Funct}}(\mathsf{Comod}_{\mathfrak{C}})$ (here the right $\mathfrak{C}$-comodule $(\mathfrak{C}, \Delta_{\mathfrak{C}})$ is clearly a subgenerator of the category $\mathsf{Comod}_{\mathfrak{C}}$). Now, we can consider the category of lifted continuous functors with respect to the comonad $- \otimes \mathfrak{C} : \mathsf{Mod}_A \to \mathsf{Mod}_A$ which we denote by $\overline{\mathsf{Funct}_{\mathfrak{C}}}(\mathsf{Mod}_A)$. The objects of $\overline{\mathsf{Funct}_{\mathfrak{C}}}(\mathsf{Mod}_A)$ are then $(- \otimes \mathfrak{C})$-lifted functors $\bar{\mathsf{F}} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$ of a continuous functors $\mathsf{F} : \mathsf{Mod}_A \to \mathsf{Mod}_A$, and morphisms are natural transformations. By Lemma 1.1, $\overline{\mathsf{Funct}_{\mathfrak{C}}}(\mathsf{Mod}_A)$ is a full subcategory of the category $\overline{\mathsf{Funct}}(\mathsf{Comod}_{\mathfrak{C}})$. On the other hand, as we have seen a composition of two $(- \otimes \mathfrak{C})$-lifted functors $\bar{\mathsf{F}} \circ \bar{\mathsf{G}}$ is clearly a $(- \otimes \mathfrak{C})$-lifted functor of the composition $\mathsf{F} \circ \mathsf{G}$. Since the identity functor $\mathbb{1}_{\mathsf{Comod}_{\mathfrak{C}}}$ belongs to $\overline{\mathsf{Funct}_{\mathfrak{C}}}(\mathsf{Mod}_A)$, we then conclude that $\overline{\mathsf{Funct}_{\mathfrak{C}}}(\mathsf{Mod}_A)$ inherits the monoidal structure of the monoidal category $\overline{\mathsf{Funct}}(\mathsf{Comod}_{\mathfrak{C}})$.

In this way, we can also consider the full subcategory $\mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$ of $\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$, defined by the class of objects of the form $\bar{X} := \overline{- \otimes X} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$, for some A-bimodule X. By Lemma 2.3, we know that each object of the category $\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$ is isomorphic to an object of the full subcategory $\mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$.

The category $\mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$ clearly inherits the monoidal structure of lifted continuous functors. We express its multiplication in the opposite way. That is, for two objects $\bar{X}$ and $\bar{Y}$ in $\mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$, we define

$$\bar{X} \bullet \bar{Y} \;:=\; \overline{X \otimes Y},$$

which is in fact the $(- \otimes \mathfrak{C})$-lifted composed functor $\overline{(- \otimes Y)} \circ \overline{(- \otimes X)}$. The multiplication of morphisms in $\mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$ is given by Godement's product

$$\Phi \bullet \Psi \;:=\; \overline{X \otimes Y} \xrightarrow{\;\;\overline{Y}\Phi\;\;} \overline{X' \otimes Y} \xrightarrow{\;\;\Psi_{\bar{X}'}\;\;} \overline{X' \otimes Y'},$$

for every pair of natural transformations $\Phi : \bar{X} \to \bar{X}'$ and $\Psi : \bar{Y} \to \bar{Y}'$. The following Proposition, which is a companion of Lemma 2.1, can be deduced from [17, Sec. 2.1], see also [10, Proposition 2.6]. We give an elementary proof in our case.

Proposition 2.4. *Let* $(\mathfrak{C} : A)$ *be a coring, and consider the above monoidal categories* $\mathscr{R}_{(\mathfrak{C}:\, A)}$ *and* $\mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$. *There is a monoidal isomorphism of categories*:

$$\begin{tikzcd}
\mathscr{R}_{(\mathfrak{C}:\, A)} \arrow[r, "\mathscr{F}"] & \mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A) \\
(X, \mathfrak{x}) \arrow[r] & \Big[\overline{X} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}},\ \Big(\rho_{U \otimes X} = (U \otimes \mathfrak{x}) \circ (\rho_U \otimes X)\Big)\Big] \\
{[\varphi : (X, \mathfrak{x}) \to (X', \mathfrak{x}')]} \arrow[r] & \Big[\Phi : \overline{X} \to \overline{X'},\ \Big(\Phi_{(U,\, \rho_U)} = (U \otimes \varepsilon \otimes X') \circ (U \otimes \varphi) \circ (\rho_U \otimes X)\Big)\Big]
\end{tikzcd}$$

for every comodule $(U, \rho_U) \in \mathsf{Comod}_{\mathfrak{C}}$. *The inverse functor is given by*

$$\begin{tikzcd}
\mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A) \arrow[r, "\mathscr{G}"] & \mathscr{R}_{(\mathfrak{C}:\, A)} \\
\big[\bar{X} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}\big] \arrow[r] & (X,\, \mathfrak{x} = (\varepsilon \otimes X \otimes \mathfrak{C}) \circ \rho_{\mathfrak{C} \otimes X}) \\
\big[\Phi : \bar{X} \to \bar{X}'\big] \arrow[r] & \big[\Phi_{(\mathfrak{C},\, \Delta)} : \mathfrak{C} \otimes X \to \mathfrak{C} \otimes X'\big].
\end{tikzcd}$$

Proof. By Lemma 2.1, the functors $\mathscr{F}$ and $\mathscr{G}$ are well-defined on objects. Let us show that they are also well-defined on morphisms. We start with a morphism $\varphi : \mathfrak{C} \otimes X \to \mathfrak{C} \otimes X'$ in the category $\mathscr{R}_{(\mathfrak{C}:\, A)}$. We need to show that $\Phi_{-} = \mathscr{F}(\varphi)_{-}$ is right $\mathfrak{C}$-colinear at each right $\mathfrak{C}$-comodule. So take a right $\mathfrak{C}$-comodule (U, ρ_U), we have

$$\begin{aligned}
\rho_{U\otimes X'}\circ\Phi_{(U,\rho_U)} &= (U\otimes\mathfrak{x}')\circ(\rho_U\otimes X')\circ(U\otimes\varepsilon\otimes X')\circ(U\otimes\varphi)\circ(\rho_U\otimes X)\\
&= (U\otimes\mathfrak{x}')\circ(U\otimes\mathfrak{C}\otimes\varepsilon\otimes X')\circ(\rho_U\otimes\mathfrak{C}\otimes X')\\
&\quad\circ(U\otimes\varphi)\circ(\rho_U\otimes X)\\
&= (U\otimes\mathfrak{x}')\circ(U\otimes\mathfrak{C}\otimes\varepsilon\otimes X')\circ(U\otimes\mathfrak{C}\otimes\varphi)\\
&\quad\circ(\rho_U\otimes\mathfrak{C}\otimes X)\circ(\rho_U\otimes X)\\
&= (U\otimes\mathfrak{x}')\circ(U\otimes\mathfrak{C}\otimes\varepsilon\otimes X')\circ(U\otimes\mathfrak{C}\otimes\varphi)\\
&\quad\circ(U\otimes\Delta\otimes X)\circ(\rho_U\otimes X)\\
&\overset{(2.3)}{=} (U\otimes\mathfrak{x}')\circ(U\otimes\mathfrak{C}\otimes\varepsilon\otimes X')\circ(U\otimes\Delta\otimes X')\\
&\quad\circ(U\otimes\varphi)\circ(\rho_U\otimes X)\\
&= (U\otimes\mathfrak{x}')\circ(U\otimes\varphi)\circ(\rho_U\otimes X).
\end{aligned}$$

A similar computations lead to the equality $(\Phi_{(U,\rho_U)}\otimes\mathfrak{C})\circ\rho_{U\otimes\mathfrak{C}} = (U\otimes\mathfrak{x}')\circ(U\otimes\varphi)\circ(\rho_U\otimes X)$. Thus $\Phi_{(U,\rho_U)}$ is right $\mathfrak{C}$-colinear, for every right $\mathfrak{C}$-comodule (U,ρ_U). An easy verification shows that $\Phi_{-}:\bar{X}\to\bar{X}'$ is natural. Conversely, let $\Phi_{-}:\bar{X}\to\bar{X}'$ be a morphism in the category $\mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$. We claim that $\mathscr{G}(\Phi)=\Phi_{(\mathfrak{C},\,\Delta)}:\mathfrak{C}\otimes X\to\mathfrak{C}\otimes X'$ is a morphism of $\mathfrak{C}$-bicomodules. The map $\Phi_{(\mathfrak{C},\,\Delta)}$ is A-bilinear, because Δ is A-bilinear and Φ_{-} is natural. By definition $\Phi_{(\mathfrak{C},\,\Delta)}:\mathfrak{C}\otimes X\to\mathfrak{C}\otimes X'$ is right $\mathfrak{C}$-colinear. Since Φ_{-} is natural, we have

$$(\Delta\otimes X')\circ\Phi_{(\mathfrak{C},\,\Delta)}=\Phi_{(\mathfrak{C}\otimes\mathfrak{C},\,\mathfrak{C}\otimes\Delta)}\circ(\Delta\otimes X).$$

Using Lemma 2.2(b), we get

$$(\Delta\otimes X')\circ\Phi_{(\mathfrak{C},\,\Delta)}=(\mathfrak{C}\otimes\Phi_{(\mathfrak{C},\,\Delta)})\circ(\Delta\otimes X).$$

Thus $\mathscr{G}(\Phi)$ is left $\mathfrak{C}$-colinear, and this proves the claim. Next, we show that $\mathscr{F}$ and $\mathscr{G}$ are mutually inverse functors. To this end, let $(X,\mathfrak{x})$ be an object in $\mathscr{R}_{(\mathfrak{C}:\,A)}$, then

$$\begin{aligned}
\mathscr{G}\circ\mathscr{F}(X,\mathfrak{x}) &= \mathscr{G}(\bar{X})\\
&= (X,(\varepsilon\otimes X\otimes\mathfrak{C})\circ\rho_{\mathfrak{C}\otimes X})\\
&= (X,(\varepsilon\otimes X\otimes\mathfrak{C})\circ(\mathfrak{C}\otimes\mathfrak{x})\circ(\Delta\otimes X))\\
&= (X,\mathfrak{x}\circ(\varepsilon\otimes\mathfrak{C}\otimes X)\circ(\Delta\otimes X))\\
&= (X,\mathfrak{x}).
\end{aligned}$$

For every morphism $\varphi:(X,\mathfrak{x})\to(X',\mathfrak{x}')$, we have

$$\begin{aligned}
\mathscr{G}\circ\mathscr{F}(\varphi) &= \mathscr{F}(\varphi)_{(\mathfrak{C},\,\Delta)}\\
&= (\mathfrak{C}\otimes\varepsilon\otimes X')\circ(\mathfrak{C}\otimes\varphi)\circ(\Delta\otimes X)\\
&\overset{(2.3)}{=} (\mathfrak{C}\otimes\varepsilon\otimes X')\circ(\Delta\otimes X')\circ\varphi=\varphi.
\end{aligned}$$

Therefore, $\mathscr{G}\circ\mathscr{F}=\mathbb{1}_{\mathscr{R}_{(\mathfrak{C}:\,A)}}$. Conversely, for every object $\bar{X}$ in $\mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$, we have $\mathscr{F}\circ\mathscr{G}(\bar{X})=\mathscr{F}(X,\mathfrak{x})$, where $\mathfrak{x}=(\varepsilon\otimes X\otimes\mathfrak{C})\circ\rho_{\mathfrak{C}\otimes X}$. Let (U,ρ_U)

be any right $\mathfrak{C}$-comodule, since $\bar{X}(\rho_U) = \rho_U \otimes X$ is right $\mathfrak{C}$-colinear, we have $\rho_{U \otimes \mathfrak{C} \otimes X} \circ (\rho_U \otimes X) = (\rho_U \otimes X \otimes \mathfrak{C}) \circ \rho_{U \otimes X}$. Applying Lemma 2.2(a) to this equality, we get

$$(U \otimes \rho_{\mathfrak{C} \otimes X}) \circ (\rho_U \otimes X) = (\rho_U \otimes X \otimes \mathfrak{C}) \circ \rho_{U \otimes X},$$

which implies that

$$\begin{aligned}(U \otimes \mathfrak{x}) \circ (\rho_U \otimes X) &= (U \otimes \varepsilon \otimes X \otimes \mathfrak{C}) \circ (U \otimes \rho_{\mathfrak{C} \otimes X}) \circ (\rho_U \otimes X) \\ &= (U \otimes \varepsilon \otimes X \otimes \mathfrak{C}) \circ (\rho_U \otimes X \otimes \mathfrak{C}) \circ \rho_{U \otimes X} \\ &= \rho_{U \otimes X}.\end{aligned}$$

This means that $\mathscr{F} \circ \mathscr{G}(\bar{X})(U, \rho_U) = \bar{X}(U, \rho_U)$, for every right $\mathfrak{C}$-comodule (U, ρ_U). Since it is clear that $\mathscr{F} \circ \mathscr{G}(\bar{X})(f) = \bar{X}(f)$, for every right $\mathfrak{C}$-colinear map f, we get $\mathscr{F} \circ \mathscr{G} = \mathbb{1}_{\overline{\mathsf{LFunct}_{\mathfrak{C}}}(\mathsf{Mod}_A)}$. Therefore, $\mathscr{F}$ and $\mathscr{G}$ are mutually inverse. Finally, the functor $\mathscr{F}$ is monoidal, since we have

$$\mathscr{F}\left((X, \mathfrak{x}) \underset{(\mathfrak{C} : A)}{\overset{r}{\otimes}} (X', \mathfrak{x}')\right) = \mathscr{F}(X, \mathfrak{x}) \bullet \mathscr{F}(X', \mathfrak{x}'), \text{ and}$$

$$\mathscr{F}(A, \mathfrak{C}) \cong \bar{A},$$

for every pair of objects $(X, \mathfrak{x})$ and $(X', \mathfrak{x})$ in $\mathscr{R}_{(\mathfrak{C} : A)}$. □

Example 2.5. Let $(\mathfrak{C} : A)$ be any coring. Define a map

$$\mathfrak{c} : \mathfrak{C} \otimes \mathfrak{C} \longrightarrow \mathfrak{C} \otimes \mathfrak{C} \quad (c \otimes c' \longmapsto \; c_{(1)} \otimes c_{(2)}\varepsilon(c') + \varepsilon(c)c'_{(1)} \otimes c'_{(2)} - c \otimes c'),$$

where we have used Sweedler's notation for the comultiplication $\Delta(c) = c_{(1)} \otimes x_{(2)}$, $c \in \mathfrak{C}$ (summation understood). Then $(\mathfrak{C}, \mathfrak{c})$ is an object of $\mathscr{R}_{(\mathfrak{C} : A)}$. By definition $\mathfrak{c}$ is an A-bilinear map. It is clear that $(\mathfrak{C} \otimes \varepsilon) \circ \mathfrak{c} = \varepsilon \otimes \mathfrak{C}$. Now, for every pair $(c, c') \in \mathfrak{C} \times \mathfrak{C}$, we have

$$\begin{aligned}&(\mathfrak{c} \otimes \mathfrak{C}) \circ (\mathfrak{C} \otimes \mathfrak{c}) \circ (\Delta \otimes \mathfrak{C})(c \otimes c') \\ &\quad = (\mathfrak{c} \otimes \mathfrak{C}) \circ (\mathfrak{C} \otimes \mathfrak{c})(c_{(1)} \otimes c_{(2)} \otimes c') \\ &\quad = (\mathfrak{c} \otimes \mathfrak{C})(c_{(1)} \otimes (c_{(2)} \otimes c_{(3)}\varepsilon(c') + \varepsilon(c_{(2)})c'_{(1)} \otimes c'_{(2)} - c_{(2)} \otimes c')) \\ &\quad = (\mathfrak{c} \otimes \mathfrak{C})(c_{(1)} \otimes c_{(2)} \otimes c_{(3)}\varepsilon(c') + c \otimes c'_{(1)} \otimes c'_{(2)} - c_{(1)} \otimes c_{(2)} \otimes c').\end{aligned}$$

Since we know that $\mathfrak{c} \circ \Delta = \Delta$, we have

$$\begin{aligned}&(\mathfrak{c} \otimes \mathfrak{C}) \circ (\mathfrak{C} \otimes \mathfrak{c}) \circ (\Delta \otimes \mathfrak{C})(c \otimes c') \\ &\quad = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}\varepsilon(c') + \varepsilon(c)c'_{(1)} \otimes c'_{(2)} \otimes c'_{(3)} - c \otimes c'_{(1)} \otimes c'_{(2)} \\ &\quad = (\mathfrak{C} \otimes \Delta) \circ \mathfrak{c}(c \otimes c').\end{aligned}$$

Therefore, $\mathfrak{c}$ satisfies equalities (2.1) and (2.2), and so $(\mathfrak{C}, \mathfrak{c})$ is an object of $\mathscr{R}_{(\mathfrak{C} : A)}$ (this is dual to [21, Proposition 1.7]). The lifted functor associated to $(\mathfrak{C}, \mathfrak{c})$ sends any right $\mathfrak{C}$-comodule (U, ρ_U) to the right $\mathfrak{C}$-comodule $(U \otimes \mathfrak{C}, \rho_{U \otimes \mathfrak{C}})$, where

$$\rho_{U \otimes \mathfrak{C}}(u \otimes c) = u_{(0)} \otimes u_{(1)} \otimes u_{(2)}\varepsilon(c) + u \otimes c_{(1)} \otimes c_{(2)} - u_{(0)} \otimes u_{(1)} \otimes c,$$

and $\rho_U(u) = u_{(0)} \otimes u_{(1)}, u \in U$ (summation understood).

Example 2.6. Let $(\mathfrak{D} : A)$ be another coring and $\phi : \mathfrak{D} \to \mathfrak{C}$ a morphism of A-corings. Consider the map

$$\mathfrak{d} : \mathfrak{C} \otimes \mathfrak{D} \longrightarrow \mathfrak{D} \otimes \mathfrak{C} \qquad (c \otimes d \longmapsto \varepsilon(c) d_{(1)} \otimes \phi(d_{(2)})).$$

It is clear that $\mathfrak{d}$ is A-bilinear, and that $(\mathfrak{D} \otimes \varepsilon) \circ \mathfrak{d} = \varepsilon \otimes \mathfrak{D}$. Furthermore, for every pair of elements $(c, d) \in \mathfrak{C} \times \mathfrak{D}$, we have

$$\begin{aligned}(\mathfrak{d} \otimes \mathfrak{C}) \circ (\mathfrak{C} \otimes \mathfrak{d}) \circ (\Delta \otimes \mathfrak{D})(c \otimes d) &= (\mathfrak{d} \otimes \mathfrak{C}) \circ (\mathfrak{C} \otimes \mathfrak{d})(c_{(1)} \otimes c_{(2)} \otimes d) \\ &= (\mathfrak{d} \otimes \mathfrak{C})(c_{(1)} \otimes \varepsilon(c_{(2)}) d_{(1)} \otimes \phi(d_{(2)})) \\ &= \varepsilon(c) d_{(1)} \otimes \phi(d_{(2)}) \otimes \phi(d_{(3)}) \\ &= (\mathfrak{D} \otimes \Delta) \circ \mathfrak{d}(c \otimes d).\end{aligned}$$

That is, $\mathfrak{d}$ satisfies equalities (2.1) and (2.2), and hence $(\mathfrak{D}, \mathfrak{d})$ is an object of $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$. The lifted functor associated to $(\mathfrak{D}, \mathfrak{d})$ is nothing but the tensor product by the right $\mathfrak{C}$-comodule $(\mathfrak{D}, (\mathfrak{D} \otimes \phi) \circ \Delta_{\mathfrak{D}})$, which is the image of $(\mathfrak{D}, \Delta_{\mathfrak{D}})$ by the induction functor $(-)_{\phi}$. This in fact comes from a more general setting. Namely, if we have any (A', A)-coring morphism $(\phi, \varphi) : (\mathfrak{C}' : A') \to (\mathfrak{C} : A)$ in the sense of [13]. That is, $\varphi : A' \to A$ is a morphism of rings and $\phi : \mathfrak{C}' \to \mathfrak{C}$ is by scalar restriction an A'-bilinear map satisfying $\varepsilon_{\mathfrak{C}} \circ \phi = \varphi \circ \varepsilon_{\mathfrak{C}'}$ and $\Delta_{\mathfrak{C}} \circ \phi = \omega_{A',A} \circ (\phi \otimes_{A'} \phi) \circ \Delta_{\mathfrak{C}'}$, where $\omega_{A',A}$ is the obvious map. Then one can prove that $(A \otimes_{A'} \mathfrak{C}' \otimes_{A'} A, \mathfrak{m})$ is an object of $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$, wherein $\mathfrak{m} : \mathfrak{C} \otimes_{A'} \mathfrak{C}' \otimes_{A'} A \to A \otimes_{A'} \mathfrak{C}' \otimes_{A'} \mathfrak{C}$ sends $c \otimes_{A'} c' \otimes_{A'} a \mapsto \varepsilon(c) \otimes_{A'} c'_{(1)} \otimes_{A'} \phi(c'_{(2)}) a$, for every element $a \in A$, $c \in \mathfrak{C}$, and $c' \in \mathfrak{C}'$.

Remark 2.7. Let $\phi : \mathfrak{C} \to \mathfrak{C}$ be any endomorphism of A-corings, we then get an object $(\mathfrak{C}, \mathfrak{d}) \in \mathscr{R}_{(\mathfrak{C}\,:\,A)}$ as in Example 2.6. On the other hand, we can consider the object $(\mathfrak{C}, \mathfrak{c}) \in \mathscr{R}_{(\mathfrak{C}\,:\,A)}$ constructed in Example 2.5. These two different objects clearly induce two different $(- \otimes \mathfrak{C})$-lifted functors of the same functor $- \otimes \mathfrak{C} : \mathsf{Mod}_A \to \mathsf{Mod}_A$.

Recall from [4] that an entwining structure over $\mathbb{K}$ is a data $(A, C)_{\mathfrak{a}}$ consisting of $\mathbb{K}$-algebra A with multiplication μ and unit 1, $\mathbb{K}$-coalgebra C with comultiplication Δ and counit ε, and a $\mathbb{K}$-module map $\mathfrak{a} : C \otimes_{\mathbb{K}} A \to A \otimes_{\mathbb{K}} C$ satisfying

$$\mathfrak{a} \circ (C \otimes_{\mathbb{K}} \mu) = (\mu \otimes_{\mathbb{K}} C) \circ (A \otimes_{\mathbb{K}} \mathfrak{a}) \circ (\mathfrak{a} \otimes_{\mathbb{K}} A), \tag{2.6}$$

$$\mathfrak{a} \circ (C \otimes_{\mathbb{K}} 1) = 1 \otimes_{\mathbb{K}} C; \tag{2.7}$$

$$(A \otimes_{\mathbb{K}} \Delta) \circ \mathfrak{a} = (\mathfrak{a} \otimes_{\mathbb{K}} C) \circ (C \otimes_{\mathbb{K}} \mathfrak{a}) \circ (\Delta \otimes_{\mathbb{K}} A), \tag{2.8}$$

$$(A \otimes_{\mathbb{K}} \varepsilon) \circ \mathfrak{a} = \varepsilon \otimes_{\mathbb{K}} A. \tag{2.9}$$

By [2, Proposition 2.2] the corresponding A-coring is the A-bimodule $\mathfrak{C} = A \otimes_{\mathbb{K}} C$ with obvious left A-action, and the right A-action is given by $(a' \otimes_{\mathbb{K}} c).a = a' \mathfrak{a}(c \otimes_{\mathbb{K}} a)$, for every $a, a' \in A$, $c \in C$. The comultiplication map is $\Delta_{\mathfrak{C}} = A \otimes_{\mathbb{K}} \Delta$, and the counit is $\varepsilon_{\mathfrak{C}} = A \otimes_{\mathbb{K}} \varepsilon$. For instance, assume a $\mathbb{K}$-bialgebra $\mathcal{H}$ is given together with a right $\mathcal{H}$-comodule algebra A and right $\mathcal{H}$-module coalgebra C. That is, the right $\mathcal{H}$-coaction $\rho_A : A \to A \otimes_{\mathbb{K}} \mathcal{H}$, $a \mapsto a_{(0)} \otimes_{\mathbb{K}} a_{(1)}$ (summation understood), is a morphism of $\mathbb{K}$-algebras, while the right action $\centerdot : C \otimes_{\mathbb{K}} \mathcal{H} \to C$ is a morphism of $\mathbb{K}$-coalgebras. It is clear that the map $\mathfrak{a} : C \otimes_{\mathbb{K}} A \to A \otimes_{\mathbb{K}} C$ sending $c \otimes_{\mathbb{K}} a \mapsto a_{(0)} \otimes_{\mathbb{K}} (c \centerdot a_{(1)})$, for every $c \in C$ and $a \in A$, satisfies all equalities (2.6)–(2.9). Thus $(A, C)_{\mathfrak{a}}$ is an entwining structure over $\mathbb{K}$.

Let $(A, C)_{\mathfrak{a}}$ be an entwining structure over $\mathbb{K}$ and consider as above the associated A-coring $\mathfrak{C} = A \otimes_{\mathbb{K}} C$. We have a canonical morphism of corings, in the sense of Example 2.6, $(C : \mathbb{K}) \to (\mathfrak{C} : A)$ which sends $c \mapsto 1 \otimes_{\mathbb{K}} c$ and $\mathbb{K} \to A$ is the unit of A. In this way, we can construct two functors connecting right comodules over C and $\mathfrak{C}$. Thus to each right C-comodule (X, ρ_X^C), we can associate a right $\mathfrak{C}$-comodule $(X \otimes_{\mathbb{K}} A, \rho^{\mathfrak{C}}_{X \otimes_{\mathbb{K}} A} = (X \otimes_{\mathbb{K}} \mathfrak{a}) \circ (\rho_X^C \otimes_{\mathbb{K}} A))$. Conversely, to each right $\mathfrak{C}$-comodule $(Y, \rho_Y^{\mathfrak{C}})$, we associate a right C-comodule $(Y, (\iota_Y \otimes_{\mathbb{K}} C) \circ \rho_Y^{\mathfrak{C}})$, where $\iota_Y : Y \otimes A \cong Y$ is the obvious natural isomorphism. We have in fact an adjunction $- \otimes_{\mathbb{K}} A : \mathsf{Comod}_C \rightleftarrows \mathsf{Comod}_{\mathfrak{C}} : \mathcal{O}$ with $- \otimes_{\mathbb{K}} A \dashv \mathcal{O}$.

Given any entwining structure $(A, C)_{\mathfrak{a}}$ over $\mathbb{K}$, one can immediately check that $(A, \mathfrak{a})$ is an object of $\mathscr{R}_{(C : \mathbb{K})}$. Furthermore, we have the following lemma.

Lemma 2.8. *Let* $(A, C)_{\mathfrak{a}}$ *be an entwining structure over* $\mathbb{K}$ *and* $\mathfrak{C} = A \otimes_{\mathbb{K}} C$ *its associated* A*-coring. The functor* $A \otimes_{\mathbb{K}} - \otimes_{\mathbb{K}} A : \mathscr{R}_{(C : \mathbb{K})} \to \mathscr{R}_{(\mathfrak{C} : A)}$, *defined over objects by*

$$(N, \mathfrak{n}) \to (A \otimes_{\mathbb{K}} N \otimes_{\mathbb{K}} A,\ (A \otimes_{\mathbb{K}} N \otimes_{\mathbb{K}} \mathfrak{a}) \circ (A \otimes_{\mathbb{K}} \mathfrak{n} \otimes_{\mathbb{K}} A))$$

(*up to natural isomorphisms*), *and over morphisms by* $f \to A \otimes_{\mathbb{K}} f \otimes_{\mathbb{K}} A$ (*up to natural isomorphisms*), *is an* op*monoidal functor.*

Proof. Let us denote by $\mathcal{F} : \mathscr{R}_{(C : \mathbb{K})} \to \mathscr{R}_{(\mathfrak{C} : A)}$ the stated functor. Consider $(N, \mathfrak{n})$ an object of $\mathscr{R}_{(C : \mathbb{K})}$, so we have a diagram

$$\begin{tikzcd}
 & \mathsf{Comod}_C \arrow[rr, "\bar{N}"] \arrow[dl, "\mathscr{U}_C"'] \arrow[dd, shift right, "-\otimes_{\mathbb{K}} A"] & & \mathsf{Comod}_C \arrow[dl, "\mathscr{U}_C"] \arrow[dd, shift right, "-\otimes_{\mathbb{K}} A"] \\
\mathsf{Mod}_{\mathbb{K}} \arrow[rr, "-\otimes_{\mathbb{K}} N"] \arrow[dd, shift right, "-\otimes_{\mathbb{K}} A"] & & \mathsf{Mod}_{\mathbb{K}} \arrow[dd, shift right, "-\otimes_{\mathbb{K}} A"] & \\
 & \mathsf{Comod}_{\mathfrak{C}} \arrow[uu, shift right, "\mathcal{O}"] \arrow[rr, dotted] \arrow[dl, "\mathscr{U}_{\mathfrak{C}}"] & & \mathsf{Comod}_{\mathfrak{C}} \arrow[uu, shift right, "\mathcal{O}"] \arrow[dl, "\mathscr{U}_{\mathfrak{C}}"] \\
\mathsf{Mod}_A \arrow[uu, shift right, "\mathcal{U}"] \arrow[rr, "-\otimes(A\otimes_{\mathbb{K}} N\otimes_{\mathbb{K}} A)"'] & & \mathsf{Mod}_A \arrow[uu, shift right, "\mathcal{U}"] &
\end{tikzcd}$$

where the verticals pairwise arrows represent a canonical adjunction. Set $\mathcal{F}(\bar{N}) := (- \otimes_{\mathbb{K}} A)\, \bar{N} \mathcal{O} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$ the functor defined by the dot arrow which completes the bottom square in the above diagram. The functors involving this diagram satisfy a natural isomorphism $(- \otimes_{\mathbb{K}} N \otimes_{\mathbb{K}} A)\mathcal{U} \cong - \otimes \mathcal{F}(N) : \mathsf{Mod}_A \to \mathsf{Mod}_A$ and the following equalities $\mathscr{U}_{\mathfrak{C}}\,(- \otimes_{\mathbb{K}} A) = (- \otimes_{\mathbb{K}} A)\,\mathscr{U}_C\,,\ \mathscr{U}_C\,\mathcal{O} = \mathcal{U}\,\mathscr{U}_{\mathfrak{C}}$. Using these, we obtain an isomorphism $\mathscr{U}_{\mathfrak{C}}\,\overline{\mathcal{F}(N)} \cong (- \otimes \mathcal{F}(N))\,\mathscr{U}_{\mathfrak{C}}$. Thus $\overline{\mathcal{F}(N)}$ is (up to natural isomorphism) a $(- \otimes \mathfrak{C})$-lifted functor of $- \otimes \mathcal{F}(N) : \mathsf{Mod}_A \to \mathsf{Mod}_A$. These in fact define a functor

$$\begin{tikzcd}
\mathsf{L}\overline{\mathsf{Funct}}_C(\mathsf{Mod}_{\mathbb{K}}) \arrow[r, "\bar{\mathcal{F}}"] & \mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A) \\
\bar{N} \arrow[r] & \bar{\mathcal{F}}(\bar{N}) = \overline{\mathcal{F}(N)} = (- \otimes_{\mathbb{K}} A)\,\bar{N}\mathcal{O} \\
\left[\Phi : \bar{N} \to \bar{N}'\right] \arrow[r] & \left[(- \otimes_{\mathbb{K}} A)\Phi_{\mathcal{O}} : \overline{\mathcal{F}(N)} \to \overline{\mathcal{F}(N')}\right].
\end{tikzcd}$$

One can easily check that $\mathcal{F} = \mathscr{G}^{\mathfrak{C}} \circ \bar{\mathcal{F}} \circ \mathscr{F}^C$, where $\mathscr{G}^{\mathfrak{C}}$ and $\mathscr{F}^C$ are the functors defined in Proposition 2.4, respectively, for $(\mathfrak{C} : A)$ and $(C : \mathbb{K})$. Henceforth, it suffices to show that $\bar{\mathcal{F}}$ is an *op*monoidal functor. So, let $\bar{N}$ and $\bar{N}'$ be two objects in the category $\mathsf{L}\overline{\mathsf{Funct}}_C(\mathsf{Mod}_{\mathbb{K}})$, we have

$$\bar{\mathcal{F}}(\bar{N} \bullet \bar{N}') = \bar{\mathcal{F}}(\overline{N \otimes N'}) = (- \otimes_{\mathbb{K}} A)\,\overline{N \otimes N'}\,\mathcal{O}, \quad \text{and}$$
$$\bar{\mathcal{F}}(\bar{N}) \bullet \bar{\mathcal{F}}(\bar{N}') = ((- \otimes_{\mathbb{K}} A)\,\bar{N}'\,\mathcal{O})\,((- \otimes_{\mathbb{K}} A)\,\bar{N}\,\mathcal{O}).$$

Define the natural transformation $\bar{\mathcal{F}}^2_{\bar{N},\,\bar{N}'} : \bar{\mathcal{F}}(\overline{N \otimes N'}) \to \bar{\mathcal{F}}(\bar{N}) \bullet \bar{\mathcal{F}}(\bar{N}')$ by putting $\bar{\mathcal{F}}^2_{\bar{N},\,\bar{N}'} = (- \otimes_{\mathbb{K}} A)\,\bar{N}'\eta_{\bar{N}\,\mathcal{O}}$, where $\eta_- : \mathbb{1}_{\mathsf{Comod}_C} \to \mathcal{O}(- \otimes_{\mathbb{K}} A)$ is the unit of the adjunction $(- \otimes_{\mathbb{K}} A) \dashv \mathcal{O}$. On the other hand, the counit of this adjunction $\theta : (-\otimes_{\mathbb{K}} A)\mathcal{O} \to \mathbb{1}_{\mathsf{Comod}_{\mathfrak{C}}}$, gives a natural morphism $\bar{\mathcal{F}}^0 : \bar{\mathcal{F}}(\overline{\mathbb{1}_{\mathsf{Mod}_{\mathbb{K}}}}) = \bar{\mathcal{F}}(\mathbb{1}_{\mathsf{Comod}_C}) = (-\otimes_{\mathbb{K}} A)\,\mathcal{O} \to \mathbb{1}_{\mathsf{Comod}_{\mathfrak{C}}}$. We need to show the associativity and the unitary properties of $\bar{\mathcal{F}}^2$ and $\bar{\mathcal{F}}^0$. Let $\bar{N}''$ be another object of $\mathsf{L}\overline{\mathsf{Funct}}_C(\mathsf{Mod}_{\mathbb{K}})$, so we have

$$\begin{aligned}
&[\bar{\mathcal{F}}(\bar{N}) \bullet \bar{\mathcal{F}}^2_{\bar{N}',\bar{N}''}] \circ \bar{\mathcal{F}}^2_{\bar{N},\bar{N}' \bullet \bar{N}''} \\
&\quad = \left[\bar{\mathcal{F}}^2_{\bar{N}',\bar{N}''}\right]_{\bar{\mathcal{F}}(\bar{N})} \circ \bar{\mathcal{F}}^2_{\bar{N},\bar{N}' \bullet \bar{N}''} \\
&\quad = [\bar{\mathcal{F}}^2_{\bar{N}',\bar{N}''}]_{\bar{\mathcal{F}}(\bar{N})} \circ \bar{\mathcal{F}}^2_{\bar{N},\overline{N' \otimes N''}} \\
&\quad = [(- \otimes_{\mathbb{K}} A)\bar{N}''\eta_{\bar{N}'\mathcal{O}}]_{(-\otimes_{\mathbb{K}} A)\bar{N}\mathcal{O}} \circ (- \otimes_{\mathbb{K}} A)\overline{N' \otimes N''}\eta_{\bar{N}\mathcal{O}} \\
&\quad = (- \otimes_{\mathbb{K}} A)\bar{N}''[\eta_{\bar{N}'\mathcal{O}(-\otimes_{\mathbb{K}} A)} \circ \bar{N}'\eta]_{\bar{N}\mathcal{O}} \\
&\quad = (- \otimes_{\mathbb{K}} A)\bar{N}''[\mathcal{O}(- \otimes_{\mathbb{K}} A)\bar{N}'\eta \circ \eta_{\bar{N}'}]_{\bar{N}\mathcal{O}}, \quad \text{by naturality of } \eta \\
&\quad = (- \otimes_{\mathbb{K}} A)\bar{N}''[\mathcal{O}(- \otimes_{\mathbb{K}} A)\bar{N}'\eta_{\bar{N}\mathcal{O}} \circ \eta_{\bar{N}'\bar{N}\mathcal{O}}] \\
&\quad = ((- \otimes_{\mathbb{K}} A)\bar{N}''\mathcal{O}(- \otimes_{\mathbb{K}} A)\bar{N}'\eta_{\bar{N}\mathcal{O}}) \circ (- \otimes_{\mathbb{K}} A)\bar{N}''\eta_{\bar{N}'\bar{N}\mathcal{O}}
\end{aligned}$$

$$
\begin{aligned}
&= (\bar{\mathcal{F}}(\bar{N}'')\bar{\mathcal{F}}^2_{\bar{N},\bar{N}'}) \circ \bar{\mathcal{F}}^2_{\bar{N}\bullet\bar{N}',\bar{N}''} \\
&= (\bar{\mathcal{F}}^2_{\bar{N},\bar{N}'} \bullet \bar{\mathcal{F}}(\bar{N}'')) \circ \bar{\mathcal{F}}^2_{\bar{N}\bullet\bar{N}',\bar{N}''},
\end{aligned}
$$

which gives the associativity property. Finally, the unitary property is obtained by comparing the following two computations:

$$
\begin{aligned}
(\bar{\mathcal{F}}(\bar{N}) \bullet \theta) \circ \bar{\mathcal{F}}^2_{\bar{N},\,\overline{\mathbb{1}_{\mathsf{Mod}_{\mathbb{K}}}}} &= \theta_{\bar{\mathcal{F}}(\bar{N})} \circ (- \otimes_{\mathbb{K}} A)\eta_{\bar{N}\mathcal{O}} \\
&= \theta_{(-\otimes_{\mathbb{K}} A)\,\bar{N}\,\mathcal{O}} \circ (- \otimes_{\mathbb{K}} A)\eta_{\bar{N}\mathcal{O}} \\
&= [\theta_{(-\otimes_{\mathbb{K}} A)} \circ (- \otimes_{\mathbb{K}} A)\eta]_{\bar{N}\,\mathcal{O}} = (- \otimes_{\mathbb{K}} A)\,\bar{N}\,\mathcal{O} = \bar{\mathcal{F}}(\bar{N})
\end{aligned}
$$

and

$$
\begin{aligned}
(\theta \bullet \bar{\mathcal{F}}(\bar{N})) \circ \bar{\mathcal{F}}^2_{\overline{\mathbb{1}_{\mathsf{Mod}_{\mathbb{K}}}},\,\bar{N}} &= (- \otimes_{\mathbb{K}} A)\,\bar{N}\,\mathcal{O}\,\theta \circ [(- \otimes_{\mathbb{K}} A)\bar{N}\eta_{\mathcal{O}}] \\
&= (- \otimes_{\mathbb{K}} A)\bar{N}[\mathcal{O}\theta \circ \eta_{\mathcal{O}}] \\
&= (- \otimes_{\mathbb{K}} A)\bar{N}\mathcal{O} = \bar{\mathcal{F}}(\bar{N}). \qquad \square
\end{aligned}
$$

Remark 2.9. Note that Lemma 2.8 is in fact a special case of more general statement. Precisely, we can state the following: *Let* $(\mathsf{C}, \Theta, \vartheta)$ *and* (A, μ, η) *be, respectively, a comonad and monad on a category* $\mathscr{C}$. *Assume that* $\mathfrak{a} : \mathsf{AC} \to \mathsf{CA}$ *is a distributive law (i.e. entwining structure) between* A *and* C, *that is,* $\mathfrak{a}$ *is a natural transformation which is compatible with both structures of* A *and* C, *see* [1]. *Denote by* $(\bar{\mathsf{C}}, \bar{\Theta}, \bar{\vartheta})$ *the comonad structure of the* A*-lifted functor* $\bar{\mathsf{C}} : \mathscr{C}_{\mathsf{A}} \to \mathscr{C}_{\mathsf{A}}$ *attached to* $\mathfrak{a}$, *see Subsec. 1.2. Then, for every* C*-lifted functor* $\bar{\mathsf{F}} : \mathscr{C}^{\mathsf{C}} \to \mathscr{C}^{\mathsf{C}}$ *of a functor* $\mathsf{F} : \mathscr{C} \to \mathscr{C}$ *it corresponds a* $\bar{\mathsf{C}}$*-lifted functor* $\overline{T_{\mathsf{A}}\mathsf{F}S_{\mathsf{A}}} : (\mathscr{C}_{\mathsf{A}})^{\bar{\mathsf{C}}} \to (\mathscr{C}_{\mathsf{A}})^{\bar{\mathsf{C}}}$ *of the functor* $T_{\mathsf{A}}\mathsf{F}S_{\mathsf{A}} : \mathscr{C}_{\mathsf{A}} \to \mathscr{C}_{\mathsf{A}}$, *which is defined by* $\overline{T_{\mathsf{A}}\mathsf{F}S_{\mathsf{A}}} = \underline{\mathsf{A}}\bar{\mathsf{F}}\mathcal{O}$, *where* $\underline{\mathsf{A}} : \mathscr{C}^{\mathsf{C}} \rightleftarrows (\mathscr{C}_{\mathsf{A}})^{\bar{\mathsf{C}}} : \mathcal{O}$ *is the canonical adjunction.*

The functor $\mathcal{O} : (\mathscr{C}_{\mathsf{A}})^{\bar{\mathsf{C}}} \to \mathscr{C}^{\mathsf{C}}$ sends any $\bar{\mathsf{C}}$-comodule $((Y, \mathsf{l}_Y), \mathsf{d}^{\bar{\mathsf{C}}}_{(Y,\mathsf{l}_Y)})$ to the C-comodule $(S_{\mathsf{A}}(Y, \mathsf{l}_Y), S_{\mathsf{A}}\mathsf{d}^{\bar{\mathsf{C}}}_{(Y,\mathsf{l}_Y)})$, and acts by identity on morphisms (here $S_{\mathsf{A}} : \mathscr{C}_{\mathsf{A}} \to \mathscr{C}$ is the forgetfull functor of Subsec. 1.1). While $\underline{\mathsf{A}}$ sends any comodule $(X, \mathsf{d}^{\mathsf{C}}_X) \in \mathscr{C}^{\mathsf{C}}$ to $((\mathsf{A}(X), \mu_X), \mathsf{d}^{\bar{\mathsf{C}}}_{(A(X),\,\mu_X)} = \mathfrak{a}_X \circ \mathsf{A}(\mathsf{d}^{\mathsf{C}}_X)) \in (\mathscr{C}_{\mathsf{A}})^{\bar{\mathsf{C}}}$ and any morphism $f \in \mathscr{C}^{\mathsf{C}}$ to $\mathsf{A}(f)$. The comonad structure $(\bar{\mathsf{C}}, \bar{\Theta}, \bar{\vartheta})$ is well-defined, since both natural transformations Θ and ϑ satisfy the condition of Eq. (1.1). Thus, $\bar{\Theta}$ and $\bar{\vartheta}$ are defined such that $S_{\mathsf{A}}\bar{\Theta} = \Theta_{S_{\mathsf{A}}}$ and $S_{\mathsf{A}}\bar{\vartheta} = \vartheta_{S_{\mathsf{A}}}$.

Remark 2.10. Reversing the twisting maps, one can construct, for any coring $(\mathfrak{C} : A)$, another monoidal category denoted by $\mathscr{L}_{(\mathfrak{C}:\,A)}$. The objects of $\mathscr{L}_{(\mathfrak{C}:\,A)}$ are pairs $(\mathfrak{l}, L)$ consisting of an A-bimodule L and A-bilinear map $\mathfrak{l} : L \otimes \mathfrak{C} \to \mathfrak{C} \otimes L$

compatible with the comultiplication and the counit, i.e. satisfying the equalities

$$(\varepsilon \otimes L) \circ \mathfrak{l} = L \otimes \varepsilon, \tag{2.10}$$

$$(\Delta \otimes L) \circ \mathfrak{l} = (\mathfrak{C} \otimes \mathfrak{l}) \circ (\mathfrak{l} \otimes \mathfrak{C}) \circ (L \otimes \Delta). \tag{2.11}$$

Here the $\mathfrak{C}$-bicomodule structure of $L \otimes \mathfrak{C}$ is given by $\rho_{L\otimes\mathfrak{C}} = L \otimes \Delta$ and $\lambda_{L\otimes\mathfrak{C}} = (\mathfrak{l} \otimes \mathfrak{C}) \circ (L \otimes \Delta)$. The $\mathbb{K}$-modules of morphisms in this category are defined by

$$\mathrm{Hom}_{\mathscr{L}_{(\mathfrak{C}:\,A)}}((\mathfrak{l}, L),\, (\mathfrak{l}', L')) \;:=\; \mathrm{Hom}_{\mathfrak{C}-\mathfrak{C}}(L \otimes \mathfrak{C},\, L' \otimes \mathfrak{C}).$$

The multiplications of this monoidal category are defined as follows: Given $\gamma\colon (\mathfrak{l}, L) \to (\mathfrak{l}', L')$ and $\sigma : (\mathfrak{k}, K) \to (\mathfrak{k}', K')$ two morphisms in $\mathscr{L}_{(\mathfrak{C}:\,A)}$, the multiplication of objects is defined by

$$(\mathfrak{l}, L) \underset{(\mathfrak{C}\,:\,A)}{\overset{l}{\otimes}} (\mathfrak{k}, K) = ((\mathfrak{l} \otimes K) \circ (L \otimes \mathfrak{k}),\, L \otimes K),$$

and that of morphisms is given by

$$\begin{aligned}\gamma \underset{(\mathfrak{C}\,:\,A)}{\overset{l}{\otimes}} \sigma = {} & (L' \otimes \varepsilon \otimes K' \otimes \mathfrak{C}) \circ (\gamma \otimes K' \otimes \mathfrak{C}) \circ (L \otimes \mathfrak{k}' \otimes \mathfrak{C}) \\ & \circ (L \otimes \sigma \otimes \mathfrak{C}) \circ (L \otimes K \otimes \Delta).\end{aligned} \tag{2.12}$$

The left version of Proposition 2.4 is expressed as follows.

Proposition 2.11. *There is a monoidal isomorphism between* $\mathscr{L}_{(\mathfrak{C}:\,A)}$ *and the category of* $(\mathfrak{C} \otimes -)$*-lifted functors* $\overline{\mathsf{LFunct}}_{\mathfrak{C}}({}_A\mathsf{Mod})$ *whose objects are* $(\mathfrak{C} \otimes -)$*-lifted functors* $\bar{L} : {}_{\mathfrak{C}}\mathsf{Comod} \to {}_{\mathfrak{C}}\mathsf{Comod}$ *of* $L \otimes - : {}_A\mathsf{Mod} \to {}_A\mathsf{Mod}$, *for some* A*-bimodule* L, *and morphisms are natural transformations.*

3. Cowreath Over Corings and Examples

In this section we recall, in terms of the tensor product over the base ring A, the definition of cowreath over a given coring $(\mathfrak{C} : A)$. If our coring arises from entwining structure $(A, C)_{\mathfrak{a}}$, then we prove in Proposition 3.5 a procedure to construct a new cowreath over the coring $A \otimes C$ from a given cowreath over the coalgebra C.

Definition 3.1. Let $(\mathfrak{C} : A)$ be a coring. A *cowreath over* $(\mathfrak{C} : A)$ (or $\mathfrak{C}$-*cowreath*) is a comonoid in the additive monoidal category $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$ defined in Sec. 2. A *wreath over* $\mathfrak{C}$ (or $\mathfrak{C}$-*wreath*) is a monoid in $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$. Notice, that here in fact we are defining a *right wreath* and a *right cowreath*. The left notions are defined in the monoidal category $\mathscr{L}_{(\mathfrak{C}\,:A)}$ of Remark 2.10.

The following Proposition can be deduced from [17, Sec. 3]. For sake of completeness, we include a detailed proof.

Proposition 3.2. *Let* $(\mathfrak{C} : A)$ *be a coring, and* $(M, \mathfrak{m})$ *an object of the category* $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$. *The following statements are equivalent*

- (i) $(M, \mathfrak{m})$ *is a* $\mathfrak{C}$*-coureath.*
- (ii) *There are* $\mathfrak{C}$*-bicolinear maps* $\xi : \mathfrak{C} \otimes M \to \mathfrak{C}$ *and* $\delta : \mathfrak{C} \otimes M \to \mathfrak{C} \otimes M \otimes M$ *satisfying the following equalities*:

$$(\xi \otimes M) \circ \delta = \mathfrak{C} \otimes M, \tag{3.1}$$

$$(M \otimes \xi) \circ (\mathfrak{m} \otimes M) \circ \delta = \mathfrak{m}, \tag{3.2}$$

$$(M \otimes \delta) \circ (\mathfrak{m} \otimes M) \circ \delta = (\mathfrak{m} \otimes M \otimes M) \circ (\delta \otimes M) \circ \delta. \tag{3.3}$$

- (iii) *The* $(- \otimes \mathfrak{C})$*-lifted functor* $\bar{M} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$ *of the functor* $- \otimes M : \mathsf{Mod}_A \to \mathsf{Mod}_A$ *has a structure of comonad.*

Proof. (i) $\Rightarrow$ (iii) Is a consequence of Proposition 2.4.
(iii) $\Rightarrow$ (ii) Assume that $(\bar{M}, \Theta, \vartheta)$ is a comonad on $\mathsf{Comod}_{\mathfrak{C}}$, and put $\delta = \Theta_{(\mathfrak{C},\,\Delta)} : \mathfrak{C} \otimes M \to \mathfrak{C} \otimes M \otimes M$, $\xi = \vartheta_{(\mathfrak{C},\,\Delta)} : \mathfrak{C} \otimes M \to \mathfrak{C}$. Since Θ and ϑ are natural, δ and ξ are morphisms of $\mathfrak{C}$-bicomodules. We claim that the maps δ and ξ satisfy equalities (3.1)–(3.3). By Eq. (2.1), $\mathfrak{m}$ is a morphism of right $\mathfrak{C}$-comodules, so we have $\bar{M}^2(\mathfrak{m}) \circ \Theta_{(\mathfrak{C} \otimes M,\, \rho_{\mathfrak{C} \otimes M})} = \Theta_{(M \otimes \mathfrak{C},\, M \otimes \Delta)} \circ \bar{M}(\mathfrak{m})$, where $\rho_{\mathfrak{C} \otimes M} = (\mathfrak{C} \otimes \mathfrak{m}) \circ (\Delta \otimes M)$. Using Lemma 2.2(b), we obtain $\bar{M}^2(\mathfrak{m}) \circ \Theta_{(\mathfrak{C} \otimes M,\, \rho_{\mathfrak{C} \otimes M})} = (M \otimes \Theta_{(\mathfrak{C},\,\Delta)})\text{\c{i}}rc\bar{M}(\mathfrak{m})$. Applying the coassociativity of Θ evaluated at the right $\mathfrak{C}$-comodule $(\mathfrak{C}, \Delta)$, we deduce that

$$(\mathfrak{m} \otimes M \otimes M) \circ (\Theta_{(\mathfrak{C},\,\Delta)} \otimes M) \circ \Theta_{(\mathfrak{C},\,\Delta)} = (M \otimes \Theta_{(\mathfrak{C},\,\Delta)}) \circ (\mathfrak{m} \otimes M) \circ \Theta_{(\mathfrak{C},\,\Delta)},$$

which means equality (3.3) for $\delta = \Theta_{(\mathfrak{C},\,\Delta)}$. Now, the right counitary property $\bar{M}\vartheta \circ \Theta = \bar{M}$ evaluated at the right $\mathfrak{C}$-comodule $(\mathfrak{C},\ \Delta)$ clearly gives equality (3.1). Finally, using again that $\mathfrak{m}$ is right $\mathfrak{C}$-colinear, we get

$$\begin{aligned} \mathfrak{m} \circ \vartheta_{\bar{M}(\mathfrak{C},\,\Delta)} \circ \Theta_{(\mathfrak{C},\,\Delta)} &= \mathfrak{m} \\ &= \vartheta_{(M \otimes \mathfrak{C},\, M \otimes \Delta)} \circ \bar{M}(\mathfrak{m}) \circ \Theta_{(\mathfrak{C},\,\Delta)} \\ &= (M \otimes \vartheta_{(\mathfrak{C},\,\Delta)}) \circ (\mathfrak{m} \otimes M) \circ \Theta_{(\mathfrak{C},\,\Delta)}, \end{aligned}$$

where Lemma 2.2(b) was used in the third equality. This gives equality (3.2) for the above δ and ξ, and finishes the proof of the claim.
(ii) $\Rightarrow$ (i) Using the definition of the multiplication $- \overset{r}{\underset{(\mathfrak{C}\,:\,A)}{\otimes}} -$ given in Eq. (2.5), we get that $(M, \mathfrak{m})$ is a comonoid in $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$ if and only if there exist $\xi : \mathfrak{C} \otimes M \to \mathfrak{C}$ and $\delta : \mathfrak{C} \otimes M \to \mathfrak{C} \otimes M \otimes M$ morphisms in $\mathscr{R}_{(\mathfrak{C}:\,A)}$ satisfying the following equalities

$$(\mathfrak{C} \otimes M \otimes \varepsilon) \circ (\mathfrak{C} \otimes M \otimes \xi) \circ (\mathfrak{C} \otimes \mathfrak{m} \otimes M) \circ (\Delta \otimes M \otimes M) \circ \delta = \mathfrak{C} \otimes M, \tag{3.4}$$

$$(\mathfrak{C} \otimes \varepsilon \otimes M) \circ (\mathfrak{C} \otimes \xi \otimes M) \circ (\Delta \otimes M \otimes M) \circ \delta = \mathfrak{C} \otimes M, \tag{3.5}$$

$$\begin{aligned} &(\mathfrak{C} \otimes M \otimes M \otimes \varepsilon \otimes M) \circ (\mathfrak{C} \otimes M \otimes \mathfrak{m} \otimes M) \circ (\mathfrak{C} \otimes \mathfrak{m} \otimes M \otimes M) \\ &\quad \circ (\mathfrak{C} \otimes \delta \otimes M) \circ (\Delta \otimes M \otimes M) \circ \delta \\ &= (\mathfrak{C} \otimes M \otimes \varepsilon \otimes M \otimes M) \circ (\mathfrak{C} \otimes M \otimes \delta) \\ &\quad \circ (\mathfrak{C} \otimes \mathfrak{m} \otimes M) \circ (\Delta \otimes M \otimes M) \circ \delta. \end{aligned} \tag{3.6}$$

By hypothesis, we then need to show that equalities (3.1)–(3.3) imply (3.4)–(3.6). To this end, we have

$$\begin{aligned}
&(\mathfrak{C}\otimes M\otimes\varepsilon)\circ(\mathfrak{C}\otimes M\otimes\xi)\circ(\mathfrak{C}\otimes\mathfrak{m}\otimes M)\circ(\Delta\otimes M\otimes M)\circ\delta\\
&\overset{(2.3)}{=}(\mathfrak{C}\otimes M\otimes\varepsilon)\circ(\mathfrak{C}\otimes M\otimes\xi)\circ(\mathfrak{C}\otimes\mathfrak{m}\otimes M)\circ(\mathfrak{C}\otimes\delta)\circ(\Delta\otimes M)\\
&=(\mathfrak{C}\otimes M\otimes\varepsilon)\circ(\mathfrak{C}\otimes((M\otimes\xi)\circ(\mathfrak{m}\otimes M)\circ\delta))\circ(\Delta\otimes M)\\
&\overset{(3.2)}{=}(\mathfrak{C}\otimes M\otimes\varepsilon)\circ(\mathfrak{C}\otimes\mathfrak{m})\circ(\Delta\otimes M)\overset{(2.2)}{=}(\mathfrak{C}\otimes\varepsilon\otimes M)\circ(\Delta\otimes M)=\mathfrak{C}\otimes M,
\end{aligned}$$

which gives equality (3.4). Equality (3.5) is obtained as follows:

$$\begin{aligned}
&(\mathfrak{C}\otimes\varepsilon\otimes M)\circ(\mathfrak{C}\otimes\xi\otimes M)\circ(\Delta\otimes M\otimes M)\circ\delta\\
&\overset{(2.3)}{=}(\mathfrak{C}\otimes\varepsilon\otimes M)\circ(\mathfrak{C}\otimes\xi\otimes M)\circ(\mathfrak{C}\otimes\delta)\circ(\Delta\otimes M)\\
&=(\mathfrak{C}\otimes\varepsilon\otimes M)\circ(\mathfrak{C}\otimes((\xi\otimes M)\circ\delta))\circ(\Delta\otimes M)\\
&\overset{(3.1)}{=}(\mathfrak{C}\otimes\varepsilon\otimes M)\circ(\Delta\otimes M)=\mathfrak{C}\otimes M.
\end{aligned}$$

Finally, the coassociativity, that is, equality (3.6) is derived from the following computation:

$$\begin{aligned}
&(\mathfrak{C}\otimes M\otimes M\otimes\varepsilon\otimes M)\circ(\mathfrak{C}\otimes M\otimes\mathfrak{m}\otimes M)\circ(\mathfrak{C}\otimes\mathfrak{m}\otimes M\otimes M)\circ(\mathfrak{C}\otimes\delta\otimes M)\\
&\qquad\circ(\Delta\otimes M\otimes M)\circ\delta\\
&\overset{(2.3)}{=}(\mathfrak{C}\otimes M\otimes M\otimes\varepsilon\otimes M)\circ(\mathfrak{C}\otimes M\otimes\mathfrak{m}\otimes M)\circ(\mathfrak{C}\otimes\mathfrak{m}\otimes M\otimes M)\\
&\qquad\circ(\mathfrak{C}\otimes\delta\otimes M)\circ(\mathfrak{C}\otimes\delta)\circ(\Delta\otimes M)\\
&=(\mathfrak{C}\otimes M\otimes M\otimes\varepsilon\otimes M)\circ(\mathfrak{C}\otimes M\otimes\mathfrak{m}\otimes M)\circ(\mathfrak{C}\otimes((\mathfrak{m}\otimes M\otimes M)\\
&\qquad\circ(\delta\otimes M)\circ\delta))\circ(\Delta\otimes M)\\
&\overset{(3.3)}{=}(\mathfrak{C}\otimes M\otimes M\otimes\varepsilon\otimes M)\circ(\mathfrak{C}\otimes M\otimes\mathfrak{m}\otimes M)\circ(\mathfrak{C}\otimes M\otimes\delta)\circ(\mathfrak{C}\otimes\mathfrak{m}\otimes M)\\
&\qquad\circ(\mathfrak{C}\otimes\delta)\circ(\Delta\otimes M)\\
&\overset{(2.3)}{=}(\mathfrak{C}\otimes M\otimes M\otimes\varepsilon\otimes M)\circ(\mathfrak{C}\otimes M\otimes\mathfrak{m}\otimes M)\circ(\mathfrak{C}\otimes M\otimes\delta)\circ(\mathfrak{C}\otimes\mathfrak{m}\otimes M)\\
&\qquad\circ(\Delta\otimes M\otimes M)\circ\delta\\
&=(\mathfrak{C}\otimes M\otimes((M\otimes\varepsilon)\circ\mathfrak{m})\otimes M)\circ(\mathfrak{C}\otimes M\otimes\delta)\circ(\mathfrak{C}\otimes\mathfrak{m}\otimes M)\\
&\qquad\circ(\Delta\otimes M\otimes M)\circ\delta\\
&\overset{(2.2)}{=}(\mathfrak{C}\otimes M\otimes\varepsilon\otimes M\otimes M)\circ(\mathfrak{C}\otimes M\otimes\delta)\circ(\mathfrak{C}\otimes\mathfrak{m}\otimes M)\circ(\Delta\otimes M\otimes M)\circ\delta.
\end{aligned}$$

□

Example 3.3. Of course any A-coring $\mathfrak{C}$ can be seen as a cowreath over the trivial coring $(A:A)$.

Let C and D be two $\mathbb{K}$-coalgebras. It is clear that (D,τ) belongs to $\mathscr{R}_{(C:\mathbb{K})}$, where $\tau : C\otimes_{\mathbb{K}} D\to D\otimes_{\mathbb{K}} C$ is the usual flip. Consider the maps $\xi = C\otimes_{\mathbb{K}}\varepsilon_D : C\otimes_{\mathbb{K}} D\to C$ and $\delta = C\otimes_{\mathbb{K}}\Delta_D : C\otimes_{\mathbb{K}} D\to C\otimes_{\mathbb{K}} D\otimes_{\mathbb{K}} D$. One can easily prove that those maps define in fact a morphism $\xi : (D,\tau)\to(\mathbb{K},C)$ and

$\delta : (D,\tau) \to (D,\tau) \overset{r}{\underset{(C:\mathbb{K})}{\otimes}} (D,\tau)$ in the monoidal category $\mathscr{R}_{(C:\mathbb{K})}$. Moreover, ξ and δ satisfy Eqs. (3.1)–(3.3) with $(\mathfrak{C} : A) = (C : \mathbb{K})$. Therefore, (D,τ) is in our terminology a C-cowreath.

Example 3.4. Let $(\mathfrak{C} : A)$ and $(\mathfrak{D} : A)$ be two corings. Assume that there is an A-bilinear map $\mathfrak{d} : \mathfrak{C} \otimes \mathfrak{D} \to \mathfrak{D} \otimes \mathfrak{C}$ satisfying

$$(\mathfrak{D} \otimes \varepsilon_{\mathfrak{C}}) \circ \mathfrak{d} = \varepsilon_{\mathfrak{C}} \otimes \mathfrak{D}, \tag{3.7}$$

$$(\mathfrak{D} \otimes \Delta_{\mathfrak{C}}) \circ \mathfrak{d} = (\mathfrak{d} \otimes \mathfrak{C}) \circ (\mathfrak{C} \otimes \mathfrak{d}) \circ (\Delta_{\mathfrak{C}} \otimes \mathfrak{D}), \tag{3.8}$$

$$(\varepsilon_{\mathfrak{D}} \otimes \mathfrak{C}) \circ \mathfrak{d} = \mathfrak{C} \otimes \varepsilon_{\mathfrak{D}}, \tag{3.9}$$

$$(\Delta_{\mathfrak{D}} \otimes \mathfrak{C}) \circ \mathfrak{d} = (\mathfrak{D} \otimes \mathfrak{d}) \circ (\mathfrak{d} \otimes \mathfrak{D}) \circ (\mathfrak{C} \otimes \Delta_{\mathfrak{D}}). \tag{3.10}$$

Equations (3.7) and (3.8) say that $(\mathfrak{D}, \mathfrak{d})$ is an object of the monoidal category $\mathscr{R}_{(\mathfrak{C}:A)}$. While Eqs. (3.9) and (3.10) say that $(\mathfrak{d}, \mathfrak{C})$ is an object of the monoidal category $\mathscr{L}_{(\mathfrak{D}:A)}$ of Remark 2.10. One can check that $(\mathfrak{D}, \mathfrak{d})$ is a right $\mathfrak{C}$-cowreath with structure maps $\mathfrak{C} \otimes \varepsilon_{\mathfrak{D}}$ and $\mathfrak{C} \otimes \Delta_{\mathfrak{D}}$, and similarly $(\mathfrak{d}, \mathfrak{C})$ is a left $\mathfrak{D}$-cowreath with structure maps $\varepsilon_{\mathfrak{C}} \otimes \mathfrak{D}$ and $\Delta_{\mathfrak{C}} \otimes \mathfrak{D}$.

The following proposition gives, using entwining structures, a method to construct from the commutative case a cowreath with a noncommutative base ring.

Proposition 3.5. *Let* $(A, C)_{\mathfrak{a}}$ *be an entwining structure over* $\mathbb{K}$ *with twisting map* $\mathfrak{a} : C \otimes_{\mathbb{K}} A \to A \otimes_{\mathbb{K}} C$. *Consider its associated coring* $(A \otimes_{\mathbb{K}} C : A)$. *If* $(N, \mathfrak{n})$ *is a* C*-cowreath, then*

$$(A \otimes_{\mathbb{K}} N \otimes_{\mathbb{K}} A, (A \otimes_{\mathbb{K}} N \otimes_{\mathbb{K}} \mathfrak{a}) \circ (A \otimes_{\mathbb{K}} \mathfrak{n} \otimes_{\mathbb{K}} A))$$

is an $(A \otimes_{\mathbb{K}} C)$*-cowreath.*

Proof. Is consequence of Lemma 2.8. ☐

Example 3.6. Let C and D be two $\mathbb{K}$-coalgebras. As in Example 3.3, we consider (D, τ) the obvious C-cowreath, where τ is the flip map. Assume now that we are given an entwining structure $(A, C)_{\mathfrak{a}}$, and let $\mathfrak{C} = A \otimes_{\mathbb{K}} C$ be its associated A-coring. Then, by Proposition 3.5, $(A \otimes_{\mathbb{K}} D \otimes_{\mathbb{K}} A, (A \otimes_{\mathbb{K}} N \otimes_{\mathbb{K}} \mathfrak{a}) \circ A \otimes_{\mathbb{K}} \tau \otimes_{\mathbb{K}} A)$ is a $\mathfrak{C}$-cowreath.

Example 3.7. Let H be a $\mathbb{K}$-bialgebra and C a left H-comodule coalgebra [19, p. 26]. That is, C is a left H-comodule with coaction $\lambda : C \to H \otimes_{\mathbb{K}} C$ which satisfies the following equalities:

$$(H \otimes_{\mathbb{K}} \varepsilon_C) \circ \lambda = 1_H \cdot \varepsilon_C,$$

$$(H \otimes_{\mathbb{K}} \Delta_C) \circ \lambda = (\mu \otimes_{\mathbb{K}} C \otimes_{\mathbb{K}} C) \circ (H \otimes_{\mathbb{K}} \tau \otimes_{\mathbb{K}} C) \circ (\lambda \otimes_{\mathbb{K}} \lambda) \circ \Delta_C,$$

where τ is the usual flip, and μ is the multiplication map of H. Considering the $\mathbb{K}$-linear map

$$\mathfrak{h} := (\mu \otimes_{\mathbb{K}} C) \circ (H \otimes_{\mathbb{K}} \tau) \circ (\lambda \otimes_{\mathbb{K}} H) : C \otimes_{\mathbb{K}} H \to H \otimes_{\mathbb{K}} C,$$

we can directly show that $(H, \mathfrak{h})$ is actually an object of the category $\mathscr{R}_{(C:\mathbb{K})}$. Moreover, we have that $(H, \mathfrak{h})$ is a C-cowreath with structure maps $\delta := C \otimes_{\mathbb{K}} \Delta_H$ and $\xi := C \otimes_{\mathbb{K}} \varepsilon_H$. An alternative proof of this fact can be deduced from Proposition 4.1 and [19, Proposition 1.6.18] which says that $C \otimes_{\mathbb{K}} H$ is a $\mathbb{K}$-coalgebra with structure maps:

$$\begin{aligned}\Delta' = (C \otimes_{\mathbb{K}} \mu \otimes_{\mathbb{K}} C \otimes_{\mathbb{K}} H) \circ (C \otimes_{\mathbb{K}} H \otimes_{\mathbb{K}} \tau \otimes_{\mathbb{K}} H)\\ \circ (C \otimes_{\mathbb{K}} \lambda \otimes_{\mathbb{K}} H \otimes_{\mathbb{K}} H) \circ (\Delta_C \otimes_{\mathbb{K}} \Delta_H),\\ \varepsilon' = \varepsilon_C \otimes_{\mathbb{K}} \varepsilon_H.\end{aligned}$$

Now, assume that we are given an entwining structure $(A, C)_{\mathfrak{a}}$. Then, by Proposition 3.5

$$(A \otimes_{\mathbb{K}} H \otimes_{\mathbb{K}} A,\, (A \otimes_{\mathbb{K}} H \otimes_{\mathbb{K}} \mathfrak{a}) \circ (A \otimes_{\mathbb{K}} \mathfrak{h} \otimes_{\mathbb{K}} A))$$

is a $\mathfrak{C}$-cowreath, where $\mathfrak{C} = A \otimes_{\mathbb{K}} C$ is the canonical A-coring induced by the entwining $\mathfrak{a}$.

Example 3.8. Entwining structures give also an example of a wreath over coalgebras. Explicitly, given any entwining structure $(A, C)_{\mathfrak{a}}$ over $\mathbb{K}$ with $\mathfrak{a} : C \otimes_{\mathbb{K}} A \to A \otimes_{\mathbb{K}} C$. As we have already observe, $(A, \mathfrak{a})$ is an object of the monoidal category $\mathscr{R}_{(C:\mathbb{K})}$. Taking $\boldsymbol{\eta} = C \otimes_{\mathbb{K}} 1 : C \to C \otimes_{\mathbb{K}} A$ and $\boldsymbol{\mu} = C \otimes_{\mathbb{K}} \mu : C \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A \to C \otimes_{\mathbb{K}} A$. One can easily check that $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ are in fact morphisms of C-bicomodules. That is, $\boldsymbol{\eta} : (\mathbb{K}, C) \to (A, \mathfrak{a})$ and $\boldsymbol{\mu} : (A, \mathfrak{a}) \underset{(C:\mathbb{K})}{\overset{r}{\otimes}} (A, \mathfrak{a}) \to (A, \mathfrak{a})$ are morphisms in $\mathscr{R}_{(C:\mathbb{K})}$. Moreover, $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ endow $(A, \mathfrak{a})$ with a structure of monoid in the monoidal category $\mathscr{R}_{(C:\mathbb{K})}$. Thus, $(A, \mathfrak{a})$ is in our terminology a right C-wreath. Conversely, let $(A, \mu, 1_A)$ be a $\mathbb{K}$-algebra such that $(A, \mathfrak{a})$ is a right C-wreath for some $\mathbb{K}$-coalgebra C with multiplication and unit, respectively, $C \otimes_{\mathbb{K}} \mu$ and $C \otimes_{\mathbb{K}} 1_A$. Then $(A, C)_{\mathfrak{a}}$ is an entwining structure over $\mathbb{K}$. Namely, since $C \otimes_{\mathbb{K}} \mu$ and $C \otimes_{\mathbb{K}} 1_A$ are C-bicolinear maps, we have

$$\begin{aligned}(C \otimes_{\mathbb{K}} \mathfrak{a}) \circ (\Delta \otimes_{\mathbb{K}} A) \circ (C \otimes_{\mathbb{K}} \mu) &= (C \otimes_{\mathbb{K}} \mu \otimes_{\mathbb{K}} C) \circ (C \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} \mathfrak{a})\\ &\quad \circ (C \otimes_{\mathbb{K}} \mathfrak{a} \otimes_{\mathbb{K}} A) \circ (\Delta \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A),\\ (C \otimes_{\mathbb{K}} \mathfrak{a}) \circ (\Delta \otimes_{\mathbb{K}} A) \circ (C \otimes_{\mathbb{K}} 1_A) &= (C \otimes_{\mathbb{K}} 1_A \otimes_{\mathbb{K}} C) \circ \Delta.\end{aligned}$$

These equalities clearly imply (2.6) and (2.7). An example of the above situation is the following one. Let G be any group and $A = \oplus_{x \in G} A_x$ any G-graded algebra. There is an entwining structure $(A, \mathbb{K}[G])_{\mathfrak{a}}$, where $\mathbb{K}[G]$ is the usual coalgebra of grouplike elements, and $\mathfrak{a} : \mathbb{K}[G] \otimes_{\mathbb{K}} A \to A \otimes_{\mathbb{K}} \mathbb{K}[G]$ sends $x \otimes_{\mathbb{K}} a_y \mapsto a_y \otimes_{\mathbb{K}} xy$, for every homogeneous element $a_y \in A_y$ and $x, y \in G$. By previous arguments, every G-graded algebra $A = \oplus_{x \in G} A_x$ is then a right $\mathbb{K}[G]$-wreath.

4. Cowreath Products and Comodules Over Cowreath

In this section, we first give a detailed proof of the fact that an object $(M, \mathfrak{m}) \in \mathscr{R}_{(\mathfrak{C}:A)}$ is a cowreath if and only if the A-bimodule $\mathfrak{C} \otimes M$ admits a compatible structure of A-coring (this coring is known as *the cowreath product* of $\mathfrak{C}$ by M). We then show that the category of $(A, \mathfrak{C} \otimes M)$-bicomodules is isomorphic via a comparison functor to the category of right $\bar{M}$-comodule, where $\bar{M}$ is viewed as a comonad on the category of $(A, \mathfrak{C})$-bicomodules. We also give a simplest and equivalent definitions of the objects and morphisms of the category of (right) comodules over a given cowreath.

The following proposition gives, using cowreath product introduced in [17], an elementary characterization of cowreath. We include a detailed proof in our case.

Proposition 4.1. *Let* $(\mathfrak{C} : A)$ *be any coring and* M *an* A*-bimodule. The following statements are equivalent*

(i) $\mathfrak{C} \otimes M$ *is an* A*-coring with a left* $\mathfrak{C}$*-colinear comultiplication* Δ'*, and there exists a morphism of* A*-corings which is a left* $\mathfrak{C}$*-colinear map* $\xi : \mathfrak{C} \otimes M \to \mathfrak{C}$*, such that*

$$(\xi \otimes \varepsilon \otimes M) \circ \Delta' = \mathfrak{C} \otimes M. \tag{4.1}$$

(ii) *There exists an* A*-bilinear map* $\mathfrak{m} : \mathfrak{C} \otimes M \to M \otimes \mathfrak{C}$ *such that* $(M, \mathfrak{m}) \in \mathscr{R}_{(\mathfrak{C}:\,A)}$ *and admits a structure of* $\mathfrak{C}$*-cowreath.*

If one of the above condition is satisfied, then we refer to $\mathfrak{C} \otimes M$ *as the* cowreath product *of* $\mathfrak{C}$ *by* M*.*

Proof. (ii) $\Rightarrow$ (i) Let us denote by $\xi : \mathfrak{C} \otimes M \to \mathfrak{C}$ and $\delta : \mathfrak{C} \otimes M \to \mathfrak{C} \otimes M \otimes M$ the $\mathfrak{C}$-bicolinear structure maps of the cowreath $(M, \mathfrak{m})$. By Proposition 3.2, the functor $\bar{M} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$ has a comonad structure. Since $\mathscr{U}_{\mathfrak{C}} : \mathsf{Comod}_{\mathfrak{C}} \rightleftarrows \mathsf{Mod}_A : (- \otimes \mathfrak{C})$ is an adjunction with $\mathscr{U}_{\mathfrak{C}} \dashv (- \otimes \mathfrak{C})$, we deduce from [15, Theorem 4.2] (or the dual of [8, Proposition 2.3]), that $\mathscr{U}_{\mathfrak{C}} \bar{M}(- \otimes \mathfrak{C}) : \mathsf{Mod}_A \to \mathsf{Mod}_A$ is a comonad with continuous underlying functor. Therefore, $(\mathscr{U}_{\mathfrak{C}} \bar{M}(- \otimes \mathfrak{C}))(A) \cong \mathfrak{C} \otimes M$ admits by [10, Lemma 2.1] a structure of an A-coring with the following comultiplication and counit

$$\Delta' = (\mathfrak{C} \otimes \mathfrak{m} \otimes M) \circ (\mathfrak{C} \otimes \delta) \circ (\Delta \otimes M), \quad \varepsilon' = \varepsilon \circ \xi. \tag{4.2}$$

Let us show that ξ is morphism of corings. By definition ξ is an A-bilinear map compatible with both counits ε and ε'. The compatibility of ξ with comultiplications Δ and Δ' is deduced as follows:

$$\begin{aligned}
(\xi \otimes \xi) \circ \Delta' &= (\xi \otimes \xi) \circ (\mathfrak{C} \otimes \mathfrak{m} \otimes M) \circ (\mathfrak{C} \otimes \delta) \circ (\Delta \otimes M) \\
&= (\mathfrak{C} \otimes \xi) \circ (\xi \otimes \mathfrak{C} \otimes M) \circ (\mathfrak{C} \otimes \mathfrak{m} \otimes M) \circ (\mathfrak{C} \otimes \delta) \circ (\Delta \otimes M)
\end{aligned}$$

$$\begin{aligned} &\overset{(2.3)}{=} (\mathfrak{C}\otimes\xi)\circ(\xi\otimes\mathfrak{C}\otimes M)\circ(\mathfrak{C}\otimes\mathfrak{m}\otimes M)\circ(\Delta\otimes M\otimes M)\circ\delta \\ &\overset{(2.4)}{=} (\mathfrak{C}\otimes\xi)\circ(\Delta\otimes M)\circ(\xi\otimes M)\circ\delta \overset{(3.1)}{=} (\mathfrak{C}\otimes\xi)\circ(\Delta\otimes M) \\ &\overset{(2.3)}{=} \Delta\circ\xi. \end{aligned}$$

Equation (4.1) follows from the following computation:

$$\begin{aligned} (\xi\otimes\varepsilon\otimes M)\circ\Delta' &= (\xi\otimes\varepsilon\otimes M)\circ(\mathfrak{C}\otimes\mathfrak{m}\otimes M)\circ(\mathfrak{C}\otimes\delta)\circ(\Delta\otimes M) \\ &= (\xi\otimes M)\circ(\mathfrak{C}\otimes M\otimes\varepsilon\otimes M)\circ(\mathfrak{C}\otimes\mathfrak{m}\otimes M)\circ(\mathfrak{C}\otimes\delta) \\ &\quad\circ(\Delta\otimes M) \\ &= (\xi\otimes M)\circ(\mathfrak{C}\otimes\varepsilon\otimes M\otimes M)\circ(\mathfrak{C}\otimes\delta)\circ(\Delta\otimes M) \\ &\overset{(2.3)}{=} (\xi\otimes M)\circ(\mathfrak{C}\otimes\varepsilon\otimes M\otimes M)\circ(\Delta\otimes M\otimes M)\circ\delta \\ &= (\xi\otimes M)\circ\delta \overset{(3.1)}{=} \mathfrak{C}\otimes M. \end{aligned}$$

(i) ⇒ (ii) Set $\rho^{\mathfrak{C}}_{\mathfrak{C}\otimes M} = (\mathfrak{C}\otimes M\otimes\xi)\circ\Delta'$, where Δ' and ξ are the stated maps. An easy verification shows that $(\mathfrak{C}\otimes M, \rho^{\mathfrak{C}}_{\mathfrak{C}\otimes M}, \Delta\otimes M)$ is a $\mathfrak{C}$-bicomodule. Hence $(M, (\varepsilon\otimes M\otimes\mathfrak{C})\circ\rho^{\mathfrak{C}}_{\mathfrak{C}\otimes M}) \in \mathscr{R}_{(\mathfrak{C}:\,A)}$ by Lemma 2.1. Now, using Eq. (4.1), one can check that ξ and $\delta = (\mathfrak{C}\otimes M\otimes\varepsilon\otimes M)\circ\Delta'$ are a $\mathfrak{C}$-bicolinear maps which satisfy equalities (3.1)–(3.3) with respect to the twisting map $\mathfrak{m} = (\varepsilon\otimes M\otimes\mathfrak{C})\circ\rho^{\mathfrak{C}}_{\mathfrak{C}\otimes M}$. Thus, $(M,\mathfrak{m})$ is actually a right $\mathfrak{C}$-cowreath. □

Our next aim is to establish an isomorphism of categories between the category of $\bar{M}$-comodule, where the functor $\bar{M} : {}_A\mathsf{Comod}_{\mathfrak{C}} \to {}_A\mathsf{Comod}_{\mathfrak{C}}$ is viewed as a comonad with the same structure of Proposition 3.2(iii), and the category ${}_A\mathsf{Comod}_{\mathfrak{C}\otimes M}$ of $(A, \mathfrak{C}\otimes M)$-bicomodules, where $\mathfrak{C}\otimes M$ is the cowearth product of $\mathfrak{C}$ by M.

Let $(M,\mathfrak{m})$ be a right $\mathfrak{C}$-cowreath, and consider its associated comonad $\bar{M} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$. It is easily checked that $\bar{M}$ induces also a comonad on the category ${}_A\mathsf{Comod}_{\mathfrak{C}}$ with the same comultiplication and counit. We denote this comonad also by $\bar{M} : {}_A\mathsf{Comod}_{\mathfrak{C}} \to {}_A\mathsf{Comod}_{\mathfrak{C}}$. Let us denote by $(-\otimes\mathfrak{C}) : {}_A\mathsf{Mod}_A \rightleftarrows {}_A\mathsf{Comod}_{\mathfrak{C}} : {}_A\mathscr{U}_{\mathfrak{C}}$ the universal cogenerator of the comonad $-\otimes\mathfrak{C} : {}_A\mathsf{Mod}_A \to {}_A\mathsf{Mod}_A$. Clearly, we have ${}_A\mathscr{U}_{\mathfrak{C}}\,\bar{M} = (-\otimes M)\;{}_A\mathscr{U}_{\mathfrak{C}}$. That is, $\bar{M} : {}_A\mathsf{Comod}_{\mathfrak{C}} \to {}_A\mathsf{Comod}_{\mathfrak{C}}$ still is a lifted functor of the functor $-\otimes M : {}_A\mathsf{Mod}_A \to {}_A\mathsf{Mod}_A$. On the other hand, the composed comonad ${}_A\mathscr{U}_{\mathfrak{C}}\,\bar{M}\,(-\otimes\mathfrak{C})$ coincides by Proposition 4.1 with the comonad $(-\otimes\mathfrak{C}\otimes M) : {}_A\mathsf{Mod}_A \to {}_A\mathsf{Mod}_A$. Therefore, we can apply Lemma 1.2 by taking for $\mathscr{C}$ the category of bimodules ${}_A\mathsf{Mod}_A$, for C the comonad $(-\otimes\mathfrak{C})$, and for F the functor $(-\otimes M)$. So, we have a functor

$$\begin{tikzcd} ({}_A\mathsf{Comod}_{\mathfrak{C}})^{\bar{M}} \arrow[r, "\mathscr{K}"] & {}_A\mathsf{Comod}_{\mathfrak{C}\otimes M} \\ ((X,\rho^{\mathfrak{C}}_X),\, \mathsf{d}^{\bar{M}}_{(X,\,\rho^{\mathfrak{C}}_X)}) \arrow[r] & (X,\, \rho^{\mathfrak{C}\otimes M}_X = (\rho^{\mathfrak{C}}_X\otimes M)\circ \mathsf{d}^{\bar{M}}_{(X,\,\rho^{\mathfrak{C}}_X)}) \\ f \arrow[r] & f \end{tikzcd} \tag{4.3}$$

such that the following diagram is commutative

$$\begin{tikzcd}
({}_A\mathsf{Comod}_{\mathfrak{C}})^{\bar{M}} \arrow[r, "\mathscr{K}"] \arrow[d, shift left, "S^{\bar{M}}"] & {}_A\mathsf{Comod}_{\mathfrak{C}\otimes M} \arrow[d, shift left, "{}_A\mathscr{U}_{\mathfrak{C}\otimes M}"] \\
{}_A\mathsf{Comod}_{\mathfrak{C}} \arrow[u, shift left, "T^{\bar{M}}"] \arrow[r, shift left, "{}_A\mathscr{U}_{\mathfrak{C}}"] & {}_A\mathsf{Mod}_A. \arrow[l, shift left, "-\otimes\mathfrak{C}"] \arrow[u, shift left, "-\otimes\mathfrak{C}\otimes M"]
\end{tikzcd} \tag{4.4}$$

That is, $(-\otimes\mathfrak{C}\otimes M)=\mathscr{K}\circ T^{\bar{M}}\circ(-\otimes\mathfrak{C})$, and ${}_A\mathscr{U}_{\mathfrak{C}\otimes M}\circ\mathscr{K}={}_A\mathscr{U}_{\mathfrak{C}}\circ S^{\bar{M}}$. The inverse of $\mathscr{K}$ is defined by:

$$\begin{tikzcd}
{}_A\mathsf{Comod}_{\mathfrak{C}\otimes M} \arrow[r, "\mathscr{K}^{-1}"] & ({}_A\mathsf{Comod}_{\mathfrak{C}})^{\bar{M}} \\
(Y,\rho_Y^{\mathfrak{C}\otimes M}) \arrow[r] & \left((Y_\xi,\,\rho^{\mathfrak{C}}_{Y_\xi}),\,,\,\mathsf{d}^{\bar{M}}_{(Y_\xi,\,\rho^{\mathfrak{C}}_{Y_\xi})}=(Y\otimes\varepsilon\otimes M)\circ\rho_Y^{\mathfrak{C}\otimes M}\right) \\
f \arrow[r] & f,
\end{tikzcd} \tag{4.5}$$

where $(Y_\xi,\rho^{\mathfrak{C}}_{Y_\xi})$ is the image of $(Y,\rho_Y^{\mathfrak{C}\otimes M})$ under the induction functor $(-)_\xi$: ${}_A\mathsf{Comod}_{\mathfrak{C}\otimes M}\to{}_A\mathsf{Comod}_{\mathfrak{C}}$ associated to the morphism of A-corings $\xi:\mathfrak{C}\otimes M\to\mathfrak{C}$ of Proposition 4.1. In conclusion we have shown the following.

Corollary 4.2. *Let $(\mathfrak{C}:A)$ be a coring and M an A-bimodule. Assume that there is a cowearth product of $\mathfrak{C}$ by M, Proposition 4.1. Then the functor $\mathscr{K}$ defined in Eq. (4.3) establishes an isomorphism of categories $({}_A\mathsf{Comod}_{\mathfrak{C}})^{\bar{M}}$ and ${}_A\mathsf{Comod}_{\mathfrak{C}\otimes M}$.*

Fix a coring $(\mathfrak{C}:A)$, and let $(M,\mathfrak{m})$ be a $\mathfrak{C}$-cowreath with structure maps $\xi:\mathfrak{C}\otimes M\to\mathfrak{C}$ and $\delta:\mathfrak{C}\otimes M\to\mathfrak{C}\otimes M\otimes M$. Since $(M,\mathfrak{m})$ is a comonoid in the monoidal category $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$, it is natural to ask for the category of (right) $(M,\mathfrak{m})$-comodules. Thus, an object $(X,\mathfrak{x})$ of $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$ is said to be a *right* $(M,\mathfrak{m})$*-comodule* if there exists a morphism $\varrho_{(X,\mathfrak{x})}:(X,\mathfrak{x})\to(X,\mathfrak{x})\overset{r}{\underset{(\mathfrak{C}\,:\,A)}{\otimes}}(M,\mathfrak{m})$ in $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$ which satisfies

$$\begin{gathered}
\left((X,\mathfrak{x})\overset{r}{\underset{(\mathfrak{C}\,:\,A)}{\otimes}}\xi\right)\circ\varrho_{(X,\mathfrak{x})}=(X,\mathfrak{x}),\\
\left((X,\mathfrak{x})\overset{r}{\underset{(\mathfrak{C}\,:\,A)}{\otimes}}\delta\right)\circ\varrho_{(X,\mathfrak{x})}=\left(\varrho_{(X,\mathfrak{x})}\overset{r}{\underset{(\mathfrak{C}\,:\,A)}{\otimes}}(M,\mathfrak{m})\right)\circ\varrho_{(X,\mathfrak{x})}.
\end{gathered} \tag{4.6}$$

Proposition 4.3. *Let $(M,\mathfrak{m})$ be a $\mathfrak{C}$-cowreath with structure maps $\xi:\mathfrak{C}\otimes M\to\mathfrak{C}$ and $\delta:\mathfrak{C}\otimes M\to\mathfrak{C}\otimes M\otimes M$, and consider its associated comonad $\bar{M}:\mathsf{Comod}_{\mathfrak{C}}\to\mathsf{Comod}_{\mathfrak{C}}\in\mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$.*

(a) *Consider $(X,\mathfrak{x})$ an object of $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$. The following conditions are equivalent*

  (i) *$(X,\mathfrak{x})$ is right $(M,\mathfrak{m})$-comodule;*

(ii) *There is a* $\mathfrak{C}$*-bicolinear map* $\varrho_{(X,\mathfrak{x})} : \mathfrak{C} \otimes X \to \mathfrak{C} \otimes X \otimes M$ *which satisfy the following equalities*:

$$(X \otimes \xi) \circ (\mathfrak{x} \otimes M) \circ \varrho_{(X,\mathfrak{x})} = \mathfrak{x}, \tag{4.7}$$

$$(X \otimes \delta) \circ (\mathfrak{x} \otimes M) \circ \varrho_{(X,\mathfrak{x})} = (\mathfrak{x} \otimes M \otimes M) \circ (\varrho_{(X,\mathfrak{x})} \otimes M) \circ \varrho_{(X,\mathfrak{x})}. \tag{4.8}$$

(iii) *The* $(- \otimes \mathfrak{C})$*-lifted functor* $\bar{X} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$ *is a right* $\bar{M}$*-comodule, where* $\bar{M}$ *is viewed as a comonoid in the strict monoidal category* $\mathsf{L\overline{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$.

(b) *Given two right* $(M,\mathfrak{m})$*-comodules* $(X,\mathfrak{x})$ *and* $(X',\mathfrak{x}')$*, a morphism* $\mathsf{f} : (X,\mathfrak{x}) \to (X',\mathfrak{x}')$ *in* $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$ *is a morphism of right* $(M,\mathfrak{m})$*-comodules if and only if*

$$\varrho_{(X',\mathfrak{x}')} \circ \mathsf{f} = (\mathsf{f} \otimes M) \circ \varrho_{(X,\mathfrak{x})}.$$

Proof. (a) The proof of Proposition 3.2 can be adapted to this item, taking into account the equalities of Eq. (4.6).
(b) The map f is a morphism of right $(M,\mathfrak{m})$-comodules if and only if

$$\begin{aligned}
\varrho_{(X,\mathfrak{x})} \circ \mathsf{f} &= \left(\mathsf{f} \overset{r}{\underset{(\mathfrak{C}\,:\,A)}{\otimes}} (M,\mathfrak{m})\right) \circ \varrho_{(X',\mathfrak{x}')} \\
&= (\mathfrak{C} \otimes X' \otimes \varepsilon \otimes M) \circ (\mathfrak{C} \otimes \mathfrak{x}' \otimes M) \circ (\mathfrak{C} \otimes \mathsf{f} \otimes M) \\
&\quad \circ (\Delta \otimes X \otimes M) \circ \varrho_{(X',\mathfrak{x}')} \\
&\overset{(2.2)}{=} (\mathfrak{C} \otimes \varepsilon \otimes X' \otimes M) \circ (\mathfrak{C} \otimes \mathsf{f} \otimes M) \circ (\Delta \otimes X \otimes M) \circ \varrho_{(X',\mathfrak{x}')} \\
&= (\mathfrak{C} \otimes \varepsilon \otimes X' \otimes M) \circ (((\mathfrak{C} \otimes \mathsf{f}) \circ (\Delta \otimes X)) \otimes M) \circ \varrho_{(X',\mathfrak{x}')} \\
&\overset{(2.3)}{=} (\mathfrak{C} \otimes \varepsilon \otimes X' \otimes M) \circ (\Delta \otimes X' \otimes M) \circ (\mathsf{f} \otimes M) \circ \varrho_{(X',\mathfrak{x}')} \\
&= (\mathsf{f} \otimes M) \circ \varrho_{(X',\mathfrak{x}')}.
\end{aligned}$$

□

Clearly $(M,\mathfrak{m})$ is right $(M,\mathfrak{m})$-comodule with coaction $\varrho_{(M,\mathfrak{m})} = \delta$. For any object $(X,\mathfrak{x}) \in \mathscr{R}_{(\mathfrak{C}:\,A)}$, we have $(X \otimes M, (X \otimes \mathfrak{m}) \circ (\mathfrak{x} \otimes M))$ is right $(M,\mathfrak{m})$-comodule with coaction

$$\varrho_{(X\otimes M,\,(X\otimes\mathfrak{m})\circ(\mathfrak{x}\otimes M))} = (\mathfrak{C}\otimes X\otimes\varepsilon\otimes M\otimes M)\circ(\mathfrak{C}\otimes X\otimes\delta)\circ(\mathfrak{C}\otimes\mathfrak{x}\otimes M)\circ(\Delta\otimes X\otimes M).$$

The description of objects and morphisms in the category of left $(M,\mathfrak{m})$-comodules is given by the following proposition.

Proposition 4.4. *Let* $(M,\mathfrak{m})$, ξ, δ *and* $\bar{M}$ *as in Proposition 4.3.*

(a) *Consider* $(X,\mathfrak{x})$ *an object of* $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$*. The following conditions are equivalent*

(i) $(X,\mathfrak{x})$ *is left* $(M,\mathfrak{m})$*-comodule*;

(ii) *There is a* $\mathfrak{C}$*-bicolinear map* $\lambda_{(X,\mathfrak{x})} : \mathfrak{C} \otimes X \to \mathfrak{C} \otimes M \otimes X$ *such that*

$$(\xi \otimes X) \circ \lambda_{(X,\mathfrak{x})} = \mathfrak{C} \otimes X,$$

$$(\mathfrak{m} \otimes M \otimes X) \circ (\delta \otimes X) \circ \lambda_{(X,\mathfrak{x})} = (M \otimes \lambda_{(X,\,\mathfrak{x})}) \circ (\mathfrak{m} \otimes X) \circ \lambda_{(X,\mathfrak{x})}.$$

(iii) *The* $(- \otimes \mathfrak{C})$*-lifted functor* $\bar{X} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$ *is a left* $\bar{M}$*-comodule, where* $\bar{M}$ *is viewed as a comonoid in the strict monoidal category* $\mathsf{L}\overline{\mathsf{Funct}}_{\mathfrak{C}}(\mathsf{Mod}_A)$.

(b) *Given two left* $(M, \mathfrak{m})$*-comodules* $(X, \mathfrak{x})$ *and* $(X', \mathfrak{x}')$*, a morphism* $\mathsf{f} : (X, \mathfrak{x}) \to (X', \mathfrak{x}')$ *in* $\mathscr{R}_{(\mathfrak{C}\,:\,A)}$ *is a morphism of left* $(M, \mathfrak{m})$*-comodules if and only if*

$$(\mathfrak{m} \otimes X') \circ \lambda_{(X', \mathfrak{x}')} \circ \mathsf{f} = (M \otimes \mathsf{f}) \circ (\mathfrak{m} \otimes X) \circ \lambda_{(X, \mathfrak{x})}.$$

Remark 4.5. Using the monoidal isomorphism established in Proposition 2.4, we can easily prove that the category of right $(M, \mathfrak{m})$-comodule over a right $\mathfrak{C}$-cowreath $(M, \mathfrak{m})$ is isomorphic to the category of right $\bar{M}$-comodule over the comonoid $\bar{M} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Comod}_{\mathfrak{C}}$ of Proposition 3.2(iii). The same argument applies to the category of left $(M, \mathfrak{m})$-comodules.

5. The Dual Notions: Wreath Over Ring Extension

In this section, we give without proofs the "dual" version of the most results stated in previous sections. Notice that the notion "dual" is not at all perfect since there are several duals in the present context. This is due probably to the fact that any bicategory admits three kind of dualization: by reversing 1-cells, by reversing 2-cells, or by reversing both of them.

The notion of coring is dual to that of ring. That is, given any ground base ring A, consider its category of A-bimodules ${}_A\mathsf{Mod}_A$ as monoidal category with multiplication the tensor product over A. An A-coring is then a comonoid in the monoidal category ${}_A\mathsf{Mod}_A$, while an A-ring is a monoid in ${}_A\mathsf{Mod}_A$. In this way, an A-ring is just an unital ring extension $\iota : A \to T$ (i.e. unital morphism of rings).

Throughout this section, we fix a ring extension $\iota : A \to T$, which we express by $(A : T)$. The multiplication of T will be denoted by μ (or μ_T) and its unit by 1 (or 1_T). Associated to $(A : T)$ and as in Sec. 2, there is an additive monoidal category $\mathscr{R}_{(A:T)}$, defined by the following data

Objects: Are pairs $(P, \mathfrak{p})$ consisting of an A-bimodule P and an A-bilinear map $\mathfrak{p} : T \otimes P \to P \otimes T$ satisfying

$$(P \otimes \mu) \circ (\mathfrak{p} \otimes T) \circ (T \otimes \mathfrak{p}) = \mathfrak{p} \circ (\mu \otimes P), \tag{5.1}$$

$$\mathfrak{p} \circ (1 \otimes P) = P \otimes 1. \tag{5.2}$$

Given any object $(P, \mathfrak{p})$ and any left T-module X with action $\mathsf{l}_X : T \otimes X \to X$. Then one can easily check that $P \otimes X$ inherits a structure of left T-module given by the action $\mathsf{l}_{P \otimes X} = (P \otimes \mathsf{l}_X) \circ (\mathfrak{p} \otimes T)$. Of course if X is assumed to be a T-bimodule, then $P \otimes X$ becomes also a T-bimodule. In this way, for each object $(P, \mathfrak{p})$, the A-bimodule $P \otimes T$ will be always considered as a T-bimodule.

Morphisms: For any two objects $(P, \mathfrak{p})$ and $(P', \mathfrak{p}')$, the $\mathbb{K}$-module of morphisms is defined by

$$\mathrm{Hom}_{\mathscr{R}_{(A:\,T)}}((P, \mathfrak{p}),\, (P', \mathfrak{p}')) := \mathrm{Hom}_{T-T}(P \otimes T,\, P' \otimes T).$$

The category $\mathscr{R}_{(A:\,T)}$ is monoidal with the multiplication of objects is given by

$$(P, \mathfrak{p}) \overset{r}{\underset{(A:\,T)}{\otimes}} (P', \mathfrak{p}') = (P \otimes P',\, (P \otimes \mathfrak{p}') \circ (\mathfrak{p} \otimes P)).$$

Now, for any pair of morphisms $\alpha : (P, \mathfrak{p}) \to (Q, \mathfrak{q})$ and $\beta : (P', \mathfrak{p}') \to (Q', \mathfrak{q}')$, the morphism $\alpha \overset{r}{\underset{(A:\,T)}{\otimes}} \beta$ is defined by the following composition

$$\begin{tikzcd}
P \otimes P' \otimes T \arrow[rrr, dashed, "\alpha \overset{r}{\underset{(A:\,T)}{\otimes}} \beta"] \arrow[d, "P\otimes 1\otimes P'\otimes T"'] & & & Q \otimes Q' \otimes T \\
P \otimes T \otimes P' \otimes T \arrow[dr, "\alpha\otimes P'\otimes T"'] & & & Q \otimes Q' \otimes T \otimes T \arrow[u, "Q\otimes Q'\otimes \mu"'] \\
& Q \otimes T \otimes P' \otimes T \arrow[r, "Q\otimes \mathfrak{p}'\otimes T"'] & Q \otimes P' \otimes T \otimes T \arrow[ur, "Q\otimes \beta\otimes T"'] &
\end{tikzcd}$$

or equivalently, $\alpha \overset{r}{\underset{(A:\,T)}{\otimes}} \beta = (Q \otimes Q' \otimes \mu) \circ (Q \otimes \mathfrak{q}' \otimes T) \circ (\alpha \otimes \beta) \circ (P \otimes 1 \otimes P' \otimes T)$. The identity object of this multiplication is proportioned by the pair $(A, T \otimes A \cong A \otimes T)$.

Lemma 5.1. *Let* $(P, \mathfrak{p})$ *and* $(Q, \mathfrak{q})$ *be two objects of the category* $\mathscr{R}_{(A:\,T)}$, *and* $f : P \to Q$ *an* A*-bilinear map. The following conditions are equivalent*

(a) $f \otimes T : (P, \mathfrak{p}) \to (Q, \mathfrak{q})$ *is a morphism in* $\mathscr{R}_{(A:\,T)}$;
(b) f *satisfies the equality* $\mathfrak{q} \circ (T \otimes f) = (f \otimes T) \circ \mathfrak{p}$.

Proof. Straightforward. □

Let $(A : T)$ be any ring extension and P an A-bimodule. One can easily check that there is an 1-1 correspondence between

(i) T-bimodule structures on $P \otimes T$ with underlying right T-action $\mathsf{r}_{P\otimes T} = P \otimes \mu$;
(ii) A-bilinear maps $\mathfrak{p} : T \otimes P \to P \otimes T$ such that $(P, \mathfrak{p})$ is an object of the category $\mathscr{R}_{(A:\,T)}$;
(iii) $(T \otimes -)$-Lifted functor $\bar{P} : {}_T\mathsf{Mod} \to {}_T\mathsf{Mod}$ of the functor $P \otimes - : {}_A\mathsf{Mod} \to {}_A\mathsf{Mod}$.

The image by $\bar{P}$ of left T-module (X, l_X) is given by the left module $\bar{P}(X, \mathsf{l}_X) = (P \otimes X, (P \otimes \mathsf{l}_X) \circ (\mathfrak{p} \otimes X))$. Conversely, if $\bar{P}$ is a $(T \otimes -)$-lifted functor, then the twisting map is given by $\mathfrak{p} = \mathsf{l}_{P\otimes T} \circ (T \otimes P \otimes 1)$, where $\mathsf{l}_{P\otimes T}$ is the left T-action of the module $\bar{P}(T,\mu)$. As in the case of corings, we consider the category of lifted continuous functors $\mathsf{L}\overline{\mathsf{Funct}}_T({}_A\mathsf{Mod})$ whose objects are of the form $\bar{P} := \overline{(P\otimes -)} : {}_T\mathsf{Mod} \to {}_T\mathsf{Mod}$ for some A-bimodule P. There is a monoidal isomorphism of categories given by the following mutually inverse functors:

$$\begin{tikzcd}[column sep=large]
\mathscr{R}_{(A:\,T)} \arrow[r, "\mathscr{F}'"] & \mathsf{L}\overline{\mathsf{Funct}}_T({}_A\mathsf{Mod}) \\
(P,\mathfrak{p}) \arrow[r] & \Big[\bar{P} : {}_T\mathsf{Mod} \to {}_T\mathsf{Mod},\ \Big(\mathsf{l}_{P\otimes U} = (P\otimes \mathsf{l}_U)\circ(\mathfrak{p}\otimes U)\Big)\Big] \\
{[\alpha : (P,\mathfrak{p}) \to (P',\mathfrak{p}')]} \arrow[r] & \Big[\Phi : \bar{P} \to \bar{P}',\ \Big(\Phi_{(U,\,\mathsf{l}_U)} = (P'\otimes \mathsf{l}_U)\circ(\alpha\otimes U)\circ(P\otimes 1\otimes U)\Big)\Big]
\end{tikzcd}$$

for every left module $(U,\mathsf{l}_U) \in {}_T\mathsf{Mod}$. The inverse functor of $\mathscr{F}'$ is

$$\begin{tikzcd}[column sep=large]
\mathsf{L}\overline{\mathsf{Funct}}_T({}_A\mathsf{Mod}) \arrow[r, "\mathscr{G}'"] & \mathscr{R}_{(A:\,T)} \\
\big[\bar{Q} : {}_T\mathsf{Mod} \to {}_T\mathsf{Mod}\big] \arrow[r] & (Q,\ \mathfrak{q} = \mathsf{l}_{Q\otimes T}\circ(T\otimes Q\otimes 1)) \\
\big[\Phi : \bar{Q} \to \bar{Q}'\big] \arrow[r] & \big[\Phi_{(T,\,\mu)} : Q\otimes T \to Q'\otimes T\big].
\end{tikzcd}$$

Definition 5.2. Let $(A : T)$ be a ring extension, and consider its associated monoidal category $\mathscr{R}_{(A:\,T)}$. A *wreath* over T (or T-*wreath*) is a monoid in the monoidal category $\mathscr{R}_{(A:T)}$, and *cowreath* (or T-*cowreath*) is a comonoid in $\mathscr{R}_{(A:T)}$.

Remark 5.3. As in the case of corings, in Definition 5.2 we are in fact defining a right wreath and right cowreath. The left versions of those definitions are given in the left monoidal category $\mathscr{L}_{(A:T)}$ whose objects are pairs $(\mathfrak{u}, U)$ consisting of an A-bimodule U and A-bilinear map $\mathfrak{u} : U\otimes T \to T\otimes U$ satisfying the equalities

$$\mathfrak{u}\circ(U\otimes 1_T) = 1_T\otimes U, \tag{5.3}$$

$$\mathfrak{u}\circ(U\otimes\mu) = (\mu\otimes U)\circ(T\otimes\mathfrak{u})\circ(\mathfrak{u}\otimes T). \tag{5.4}$$

The $\mathbb{K}$-modules of morphisms are

$$\mathrm{Hom}_{\mathscr{L}_{(A:\,T)}}((\mathfrak{u},U),\,(\mathfrak{u}',U')) := \mathrm{Hom}_{T-T}(T\otimes U,\,T\otimes U'),$$

where $T\otimes U$ and $T\otimes U'$ are T-bimodules with right T-action given, respectively, by $\mathsf{r}_{T\otimes U} = (\mu\otimes U)\circ(T\otimes\mathfrak{u})$ and $\mathsf{r}_{T\otimes U'} = (\mu\otimes U')\circ(T\otimes\mathfrak{u}')$. The multiplications in this monoidal category are given as follows: Given $\alpha : (\mathfrak{u},U) \to (\mathfrak{v},V)$ and $\beta : (\mathfrak{u}',U') \to (\mathfrak{v}',V')$ two morphisms in $\mathscr{L}_{(A:\,T)}$. The multiplication of objects is defined by

$$(\mathfrak{u},U) \overset{l}{\underset{(A:\,T)}{\otimes}} (\mathfrak{u}',U') = ((\mathfrak{u}\otimes U')\circ(U\otimes\mathfrak{u}'),\,U\otimes U'),$$

and that of morphisms is given by

$$\alpha \underset{(A:T)}{\overset{l}{\otimes}} \beta = (\mu \otimes V \otimes V') \circ (T \otimes \alpha \otimes V') \circ (T \otimes \mathfrak{u} \otimes V') \circ (T \otimes U \otimes \beta) \circ (T \otimes U \otimes 1 \otimes U').$$

This monoidal category is monoidally isomorphic to the category of $(- \otimes T)$-lifted functor $\bar{U} : \mathsf{Mod}_T \to \mathsf{Mod}_T$ of the functor $- \otimes U : \mathsf{Mod}_A \to \mathsf{Mod}_A$, for some A-bimodule U.

The dual version of Proposition 3.2 is the following proposition.

Proposition 5.4. *Let $(A : T)$ be a ring extension, and $(R, \mathfrak{r})$ an object of the category $\mathscr{R}_{(A:T)}$. The following statements are equivalent*

(i) *$(R, \mathfrak{r})$ is a T-wreath.*

(ii) *There are T-bilinear maps $\boldsymbol{\eta} : T \to R \otimes T$ and $\boldsymbol{\mu} : R \otimes R \otimes T \to R \otimes T$ satisfying*

$$\boldsymbol{\mu} \circ (R \otimes \boldsymbol{\eta}) = R \otimes T, \tag{5.5}$$

$$\boldsymbol{\mu} \circ (R \otimes \mathfrak{r}) \circ (\boldsymbol{\eta} \otimes R) = \mathfrak{r}, \tag{5.6}$$

$$\boldsymbol{\mu} \circ (R \otimes \mathfrak{r}) \circ (\boldsymbol{\mu} \otimes R) = \boldsymbol{\mu} \circ (R \otimes \boldsymbol{\mu}) \circ (R \otimes R \otimes \mathfrak{r}). \tag{5.7}$$

(iii) *The $(T \otimes -)$-lifted functor $\bar{R} : {}_T\mathsf{Mod} \to {}_T\mathsf{Mod}$ of the functor $R \otimes - : {}_A\mathsf{Mod} \to {}_A\mathsf{Mod}$ has a structure of monad.*

The wreath product is described in the following proposition (i.e. the dual of Proposition 4.1).

Proposition 5.5. *Let $(A : T)$ be any ring extension and R an A-bimodule. The following statements are equivalent*

(i) *$R \otimes T$ is an A-ring with a right T-linear multiplication $\boldsymbol{\mu}'$, there exists a morphism of A-rings $\boldsymbol{\eta} : T \to R \otimes T$ which is a right T-linear map such that*

$$\boldsymbol{\mu}' \circ (R \otimes 1 \otimes \boldsymbol{\eta}) = R \otimes T.$$

(ii) *There exists an A-linear map $\mathfrak{r} : T \otimes R \to R \otimes T$ such that $(R, \mathfrak{r}) \in \mathscr{R}_{(A:T)}$ and admits a structure of T-wreath.*

If one of the above condition is satisfied, then we refer to $R \otimes T$ as the wreath product *of R by T.*

Notice, that if $\boldsymbol{\eta} : T \to R \otimes T$ and $\boldsymbol{\mu} : R \otimes R \otimes T \to R \otimes T$ are the structure maps of the T-wreath $(R, \mathfrak{r})$, then the multiplication and the unit of the *wreath product* $R \otimes T$ are given by

$$\boldsymbol{\mu}' = (R \otimes \mu) \circ (\boldsymbol{\mu} \otimes T) \circ (R \otimes \mathfrak{r} \otimes T), \quad \boldsymbol{\eta}' = \boldsymbol{\eta} \circ 1.$$

As in the case of cowreath, we have the following corollary.

Corollary 5.6. *Let T be an A-ring and R an A-bimodule. Assume that there is a wreath product of R by T, Proposition 5.5. Then there is an isomorphism of* *categories* $({}_T\mathsf{Mod}_A)_{\bar{R}} \cong {}_{R\otimes T}\mathsf{Mod}_A$*, where the first one is the category of* $\bar{R}$*-module when* $\bar{R}$ *is viewed as a monad on the category of bimodules* ${}_T\mathsf{Mod}_A$*.*

The objects and morphisms in the category of right modules over a given wreath are expressed as follows.

Proposition 5.7. *Let* $(R, \mathfrak{r})$ *be a* T*-wreath with structure maps* $\boldsymbol{\eta} : T \to R \otimes T$ *and* $\boldsymbol{\mu} : R \otimes R \otimes T \to R \otimes T$*.*

(a) *Consider* $(Y, \mathfrak{y})$ *an object of* $\mathscr{R}_{(A:\,T)}$*. The following conditions are equivalent*

- (i) $(Y, \mathfrak{y})$ *is right* $(R, \mathfrak{r})$*-module;*
- (ii) *There is a* T*-bilinear map* $\mathsf{r}_{(Y,\,\mathfrak{y})} : Y \otimes R \otimes T \to Y \otimes T$ *such that*

$$\mathsf{r}_{(Y,\,\mathfrak{y})} \circ (Y \otimes \boldsymbol{\eta}) = Y \otimes T, \tag{5.8}$$

$$\mathsf{r}_{(Y,\,\mathfrak{y})} \circ (Y \otimes \boldsymbol{\mu}) \circ (Y \otimes R \otimes \mathfrak{r}) = \mathsf{r}_{(Y,\,\mathfrak{y})}, \circ (Y \otimes \mathfrak{r}) \circ (\mathsf{r}_{(Y,\,\mathfrak{y})} \otimes R). \tag{5.9}$$

- (iii) *The* $(T \otimes -)$*-lifted functor* $\bar{Y} : {}_T\mathsf{Mod} \to {}_T\mathsf{Mod}$ *is a right* $\bar{R}$*-module, where* $\bar{R} : {}_T\mathsf{Mod} \to {}_T\mathsf{Mod}$ *is viewed as a monoid in the monoidal category* $\mathsf{L}\overline{\mathsf{Funct}}_T({}_A\mathsf{Mod})$*.*

(b) *Given two right* $(R, \mathfrak{r})$*-modules* $(Y, \mathfrak{y})$ *and* $(Y', \mathfrak{y}')$*, a morphism* $\mathsf{g} : (Y, \mathfrak{y}) \to (Y', \mathfrak{y}')$ *in* $\mathscr{R}_{(A:\,T)}$ *is a morphism of right* $(R, \mathfrak{r})$*-modules if and only if*

$$\mathsf{r}_{(Y',\,\mathfrak{y}')} \circ (Y' \otimes \mathfrak{r}) \circ (\mathsf{g} \otimes R) = \mathsf{g} \circ \mathsf{r}_{(Y,\,\mathfrak{y})} \circ (Y \otimes \mathfrak{r}). \tag{5.10}$$

Analogously we describe the objects and morphisms in the category of left modules over a wreath.

Proposition 5.8. *Let* $(R, \mathfrak{r})$*,* $\boldsymbol{\eta} : T \to R \otimes T$ *and* $\boldsymbol{\mu} : R \otimes R \otimes T \to R \otimes T$ *as in Proposition 5.7.*

(a) *Consider* $(Y, \mathfrak{y})$ *an object of* $\mathscr{R}_{(A:\,T)}$*. The following conditions are equivalent*

- (i) $(Y, \mathfrak{y})$ *is left* $(R, \mathfrak{r})$*-module;*
- (ii) *There is a* T*-bilinear map* $\mathsf{l}_{(Y,\mathfrak{y})} : R \otimes Y \otimes T \to Y \otimes T$ *such that*

$$\mathsf{l}_{(Y,\,\mathfrak{y})} \circ (R \otimes \mathfrak{y}) \circ (\boldsymbol{\eta} \otimes Y) = \mathfrak{y}, \tag{5.11}$$

$$\mathsf{l}_{(Y,\,\mathfrak{y})} \circ (R \otimes \mathfrak{y}) \circ (\boldsymbol{\mu} \otimes Y) = \mathsf{l}_{(Y,\,\mathfrak{y})} \circ (R \otimes \mathsf{l}_{(Y,\,\mathfrak{y})}) \circ (R \otimes R \otimes \mathfrak{y}). \tag{5.12}$$

- (iii) *The* $(T \otimes -)$*-lifted functor* $\bar{Y} : {}_T\mathsf{Mod} \to {}_T\mathsf{Mod}$ *is a left* $\bar{R}$*-module, where* $\bar{R} : {}_T\mathsf{Mod} \to {}_T\mathsf{Mod}$ *is viewed as a monoid in the monoidal category* $\mathsf{L}\overline{\mathsf{Funct}}_T({}_A\mathsf{Mod})$*.*

(b) *Given two left* $(R, \mathfrak{r})$*-modules* $(Y, \mathfrak{y})$ *and* $(Y', \mathfrak{y}')$*, a morphism* $\mathsf{g} : (Y, \mathfrak{y}) \to (Y', \mathfrak{y}')$ *in* $\mathscr{R}_{(A:\,T)}$ *is a morphism of left* $(R, \mathfrak{r})$*-modules if and only if*

$$\mathsf{l}_{(Y',\,\mathfrak{y}')} \circ (R \otimes \mathsf{g}) = \mathsf{g} \circ \mathsf{l}_{(Y,\,\mathfrak{y})}. \tag{5.13}$$

Let $(R, \mathfrak{r})$ be a T-wreath and $(Y, \mathfrak{y})$ an object of $\mathscr{R}_{(A:T)}$. Assume that $(Y, \mathfrak{y})$ is a right $(R, \mathfrak{r})$-module and a left $(R, \mathfrak{r})$-module with actions maps $\mathsf{l}_{(Y,\,\mathfrak{y})} : R \otimes Y \otimes T \to Y \otimes T$ and $\mathsf{r}_{(Y,\,\mathfrak{y})} : Y \otimes R \otimes T \to Y \otimes T$. Then one can easily check that $(Y, \mathfrak{y})$ is an $(R, \mathfrak{r})$-*bimodule* if and only if

$$\begin{aligned} &\mathsf{l}_{(Y,\,\mathfrak{y})} \circ (R \otimes Y \otimes \mu) \circ (R \otimes \mathsf{r}_{(Y,\,\mathfrak{y})} \otimes T) \circ (R \otimes Y \otimes R \otimes 1 \otimes T) \\ &\quad = \mathsf{r}_{(Y,\,\mathfrak{y})} \circ (Y \otimes R \otimes \mu) \circ (Y \otimes \mathfrak{r} \otimes T) \circ (\mathsf{l}_{(Y,\,\mathfrak{y})} \otimes R \otimes T) \circ (R \otimes Y \otimes 1 \otimes R \otimes T). \end{aligned} \tag{5.14}$$

6. Examples: Twisted Modules Over Twisted Algebras

In this section, we review Čap *et al.* [7, Sec. 3] constructions of what they call a twisted modules over a twisted tensor algebra, and extend their results to the noncommutative case. Precisely, we use the notion of modules over a wreath to reformulate the problem of constructing twisted modules, and give an analog solution in the noncommutative case.

6.1. *Twisted tensor product algebras are wreath products*

Let T and R be two A-rings with multiplications and units μ_T, 1_T and μ_R, 1_R. Assume that there is an A-bilinear map $\mathfrak{r} : T \otimes R \to R \otimes T$ satisfying

$$\mathfrak{r} \circ (1_T \otimes R) = R \otimes 1_T, \tag{6.1}$$

$$\mathfrak{r} \circ (\mu_T \otimes R) = (R \otimes \mu_T) \circ (\mathfrak{r} \otimes T) \circ (T \otimes \mathfrak{r}), \tag{6.2}$$

$$\mathfrak{r} \circ (T \otimes 1_R) = 1_R \otimes T, \tag{6.3}$$

$$\mathfrak{r} \circ (T \otimes \mu_R) = (\mu_R \otimes T) \circ (R \otimes \mathfrak{r}) \circ (\mathfrak{r} \otimes R). \tag{6.4}$$

Equations (6.1) and (6.2) say that $(R, \mathfrak{r})$ is an object of the category $\mathscr{R}_{(A:\,T)}$, while Eqs. (6.3) and (6.4) say that $(\mathfrak{r}, T)$ is an object of the category $\mathscr{L}_{(A:\,R)}$. Taking the maps $\boldsymbol{\eta} := 1_R \otimes T : T \to R \otimes T$ and $\boldsymbol{\mu} := \mu_R \otimes T : R \otimes R \otimes T \to R \otimes T$, we can easily check, using associativity and unitary properties of μ_R and 1_R, that $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ satisfy Eqs. (5.5)–(5.7) of Proposition 5.4(ii). Equations (6.3) and (6.4) show that both $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ are T-bilinear maps. That is, $(R, \mathfrak{r})$ is a T-wreath with structure maps $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$. The wreath product $R \otimes T$ is by Proposition 5.5 an A-ring extension of T with multiplication and unit

$$\boldsymbol{\mu}' = (\mu_R \otimes \mu_T) \circ (R \otimes \mathfrak{r} \otimes T), \quad 1_{R \otimes T} = 1_R \otimes 1_T.$$

Of course $(\mathfrak{r}, T) \in \mathscr{L}_{(A:\,R)}$ can be also considered as an R-wreath with structure maps $R \otimes 1_T$ and $R \otimes \mu_T$. This in fact will lead to the same wreath product $R \otimes T$, but in this case an A-ring extension of R.

In the commutative case (i.e. $A = \mathbb{K}$), these algebras were refereed to in the literature as *twisted tensor product algebras*, and were intensively studied by several Mathematicians, see [24, 25, 7, 6, 18] and references therein.

6.2. *Example: Ore extensions are wreath products* ([6, *Example 2.11*])

In this subsection, we prove that the classical Ore extension [12, 20] constructed by using left skew derivations are in fact a wreath product (Proposition 5.5) defined over a commutative polynomials rings with one variable. To this end, consider such polynomials ring $T = \mathbb{K}[\mathsf{x}]$, and let B be any ring (i.e. $\mathbb{K}$-algebra). Let ∂ be a left σ-derivation of B, where σ is an endomorphism of rings of B. That is, ∂ is a $\mathbb{K}$-linear map obeying the rule $\partial(bb') = \partial(b)b' + \sigma(b)\partial(b')$, for every $b, b' \in B$. We define by induction the following map $\mathfrak{b} : T \otimes_{\mathbb{K}} B \to B \otimes_{\mathbb{K}} T$,

$$\begin{aligned} \mathfrak{b}(1 \otimes_{\mathbb{K}} b) &= b \otimes_{\mathbb{K}} 1 \\ \mathfrak{b}(\mathsf{x} \otimes_{\mathbb{K}} b) &= \sigma(b) \otimes_{\mathbb{K}} \mathsf{x} + \partial(b) \otimes_{\mathbb{K}} 1, \end{aligned}$$

and if $\mathfrak{b}(\mathsf{x}^n \otimes_{\mathbb{K}} b) = \sum_{1 \leq i \leq n} b_i \otimes_{\mathbb{K}} \mathsf{x}^i$, for some $n \geq 1$, then

$$\mathfrak{b}(\mathsf{x}^{n+1} \otimes_{\mathbb{K}} b) = \sum_{i=1}^{n} (\sigma(b_i) \otimes_{\mathbb{K}} \mathsf{x}^{i+1} + \partial(b_i) \otimes_{\mathbb{K}} \mathsf{x}^i).$$

Following the proof of [12, Proposition 1.10], consider the $\mathbb{K}$-linear endomorphisms ring $E = \mathrm{End}_{\mathbb{K}}(B[t])$ of a commutative polynomial ring over B. Clearly the ring B is identified with its image in E by using left multiplications. The maps σ and ∂ are extended to E as follows: $\sigma(bt^i) = \sigma(b)t^i$ and $\partial(bt^i) = \partial(b)t^i$, for all $b \in B$ and $i = 0, 1, \ldots$. Denote by Y the element of E defined by $Y(f) = \sigma(f)t + \partial(f)$, for all $f \in B[t]$, and construct a map

$$\begin{tikzcd} B \otimes_{\mathbb{K}} \mathbb{K}[\mathsf{x}] \arrow[r, "\tau"] & \mathrm{End}_{\mathbb{K}}(B[t]) \\ b \otimes_{\mathbb{K}} \mathsf{x}^n \arrow[r, mapsto] & bY^n. \end{tikzcd}$$

τ is in fact injective since the left B-submodule generated by the set $\{Y^n\}_{n=0,1,\ldots}$ is a free module. By the associativity of E, we have $\tau \circ \mathfrak{b} \circ (\mu \otimes_{\mathbb{K}} B) = \tau \circ (B \otimes_{\mathbb{K}} \mu) \circ (\mathfrak{b} \otimes_{\mathbb{K}} T) \circ (T \otimes_{\mathbb{K}} \mathfrak{b})$. Whence $\mathfrak{b} \circ (\mu \otimes_{\mathbb{K}} B) = (B \otimes_{\mathbb{K}} \mu) \circ (\mathfrak{b} \otimes_{\mathbb{K}} T) \circ (T \otimes_{\mathbb{K}} \mathfrak{b})$, thus $(\mathfrak{b}, B)$ is an object of the category $\mathscr{R}_{(\mathbb{K}:\,T)}$. Using again the injectivity of the map τ, we can prove that $(\mathfrak{b}, B)$ is a T-wreath with structure maps $\boldsymbol{\eta} : T \to B \otimes_{\mathbb{K}} T$ sending $\mathsf{x}^n \mapsto 1 \otimes_{\mathbb{K}} \mathsf{x}^n$ and $\boldsymbol{\mu} : B \otimes_{\mathbb{K}} B \otimes_{\mathbb{K}} T \to B \otimes_{\mathbb{K}} T$ sending $b \otimes_{\mathbb{K}} b' \otimes_{\mathbb{K}} \mathsf{x}^n \mapsto bb' \otimes_{\mathbb{K}} \mathsf{x}^n$. In this way, the wreath product of B by T associated to $\mathfrak{b}$, is a $\mathbb{K}$-algebra with underlying $\mathbb{K}$-module $B \otimes_{\mathbb{K}} \mathbb{K}[\mathsf{x}]$ and multiplication

$$(b \otimes_{\mathbb{K}} \mathsf{x}^n)(b' \otimes_{\mathbb{K}} \mathsf{x}^m) = \sum_{i=1}^{n} bb_{i'} \otimes_{\mathbb{K}} \mathsf{x}^{i+m},$$

where $\mathfrak{b}(\mathsf{x}^n \otimes_{\mathbb{K}} b') = \sum_{1 \leq i \leq n} b_{i'} \otimes_{\mathbb{K}} \mathsf{x}^i$. This algebra is in fact an extension of B via the map $- \otimes_{\mathbb{K}} 1 : B \to B \otimes_{\mathbb{K}} \mathbb{K}[\mathsf{x}]$. Now, given any ring extension $\phi : B \to S$ assume that there exists an element $Z \in S$ such that $Zb = \sigma(b)Z + \partial(b)$, for all $b \in B$. This condition leads to construct a ring extension $\bar{\phi} : B \otimes_{\mathbb{K}} \mathbb{K}[\mathsf{x}] \to S$ sending $b \otimes_{\mathbb{K}} \mathsf{x}^n \mapsto bZ^n$. It is clear now that $\phi = \bar{\phi} \circ (- \otimes_{\mathbb{K}} 1)$. In conclusion the

$\mathbb{K}$-algebra $B \otimes_{\mathbb{K}} \mathbb{K}[\mathsf{x}]$ satisfies the universal condition of Ore extension, and thus $B \otimes_{\mathbb{K}} \mathbb{K}[\mathsf{x}] = B[Y; \sigma; \partial]$. Notice that $B \otimes_{\mathbb{K}} \mathbb{K}[\mathsf{x}]$ is isomorphic to the subalgebra $\sum_{i=0,1,\dots} BY^i$ of E.

Here in fact we have constructed a wreath with a commutative base ring. An example of wreath with noncommutative base ring can be constructed as above using an iterated Ore extensions over a noncommutative ring A. That is, one can prove that certain iterated Ore extension of type $A[\mathsf{x}_1; \sigma_1; \partial_1][\mathsf{x}_2; \sigma_2; \partial_2]$ is a wreath with base ring A.

6.3. *Čap et al. construction of twisted modules*

The problem concerned in [7, Sec. 3] can be rephrased in the noncommutative setting as follows. Let A be a noncommutative base ring. Assume that T, R and $\mathfrak{r} : T \otimes R \to R \otimes T$ are given as in Subsec. 6.1. Given a (T, A)-bimodule X and (R, A)-bimodule Y, can we make $X \otimes Y$ into a left $(R \otimes_{\mathfrak{r}} T)$-module in a way which is compatible with the inclusion of R, i.e. such that $(r \otimes 1_T).(x \otimes y) = (rx) \otimes y$, for every $r \in R$, $x \in X$, and $y \in Y$? Here $R \otimes_{\mathfrak{r}} T := R \otimes T$ denotes the wreath product of Subsec. 6.1. The left action which the authors of [7] proposed is the following one

$$\mathsf{l}_{X \otimes Y} : R \otimes T \otimes X \otimes Y \xrightarrow{R \otimes \mathfrak{x} \otimes Y} R \otimes X \otimes T \otimes Y \xrightarrow{\mathsf{l}_X \otimes \mathsf{l}_Y} X \otimes Y$$

where $\mathsf{l}_X : T \otimes X \to X$ and $\mathsf{l}_Y : R \otimes Y \to Y$ are, respectively, the A-bilinear left action map of X and Y, and $\mathfrak{x} : T \otimes X \to X \otimes T$ is some A-bilinear map. A sufficient condition which $\mathfrak{x}$ should satisfies in order to answer positively to the above question using the map $\mathsf{l}_{X \otimes Y}$, was given in the commutative case in [7, 3.6 and 3.7] and says the following: $\mathfrak{x}$ is said to be a *left module twisting map* if and only if

$$\mathfrak{x} \circ (1_T \otimes X) = X \otimes 1_R, \tag{6.5}$$

$$\mathfrak{x} \circ (\mu_T \otimes X) = (X \otimes \mu_T) \circ (\mathfrak{x} \otimes T) \circ (T \otimes \mathfrak{x}), \tag{6.6}$$

$$\mathfrak{x} \circ (T \otimes \mathsf{l}_X) = (\mathsf{l}_X \otimes T) \circ (R \otimes \mathfrak{x}) \circ (\mathfrak{r} \otimes X). \tag{6.7}$$

The main result [7, Theorem 3.8] says: If $\mathfrak{x}$ is a left module twisting map, then $\mathsf{l}_{X \otimes Y}$ gives the answer to the above question. A reciprocate implication was also given in that Theorem: If X is projective and for one left faithful module Y the map $\mathsf{l}_{X \otimes Y}$ defines a left action which compatible with the inclusion of R, then $\mathfrak{x}$ is a left module twisting map.

Let us traduce the previous constructions in our context. First of all, it is obvious that Eqs. (6.5) and (6.6) say that $(X, \mathfrak{x})$ is actually an object of the category $\mathscr{R}_{(A:T)}$. Take the A-bilinear map $\mathsf{l}_{(X,\mathfrak{x})} = \mathsf{l}_X \otimes T : R \otimes X \otimes T \to X \otimes T$. It is easily seen that this map satisfies Eqs. (5.11) and (5.12) of Proposition 5.8(ii). By Lemma 5.1, $\mathsf{l}_{(X,\mathfrak{x})}$ is a T-bilinear map if and only if Eq. (6.7) is fulfilled. Therefore, $\mathfrak{x}$ is a left module twisting map if and only if $(X, \mathfrak{x})$ is a left $(R, \mathfrak{r})$-module with action $\mathsf{l}_{(X,\mathfrak{x})} = \mathsf{l}_X \otimes T$.

Thus what Čap *et al.* were constructing is just an induced left module over the wreath $(R, \mathfrak{r})$. Next, we formulate the noncommutative version of [7, Theorem 3.8].

Proposition 6.1. *Let $(R, \mathfrak{r})$ be the T-wreath of Subsec.* 6.1 *and X an (R, A)-bimodule with left R-action l_X. Assume that $(X, \mathfrak{x})$ is also a left $(R, \mathfrak{r})$-module with action $\mathsf{l}_{(X,\,\mathfrak{x})} : R \otimes X \otimes T \to X \otimes T$ and that*

$$\mathsf{l}_{(X,\,\mathfrak{x})} \circ (R \otimes X \otimes 1_T) = (X \otimes 1_T) \circ \mathsf{l}_X.$$

Then, for every (T, A)-bimodule Y, the A-bilinear map

$$\begin{tikzcd}
R \otimes T \otimes X \otimes Y \arrow[r, "R\otimes\mathfrak{x}\otimes T"] \arrow[drr, dashed, "\mathsf{l}_{X\otimes Y}"'] & R \otimes X \otimes T \otimes Y \arrow[r, "\mathsf{l}_{(X,\,\mathfrak{x})}\otimes Y"] & X \otimes T \otimes Y \arrow[d, "X\otimes\mathsf{l}_Y"] \\
 & & X \otimes Y
\end{tikzcd}$$

define a left $(R \otimes_{\mathfrak{r}} T)$-action which is compatible with the inclusion of R.

Conversely, if $X \otimes \mu_T$ preserves equalizers, $- \otimes T$ is a faithful functor, and $\mathsf{l}_{X\otimes T} := (\mathsf{l}_X \otimes \mu_T) \circ (R \otimes \mathfrak{x} \otimes T)$ define a left $(R \otimes_{\mathfrak{r}} T)$-action on $X \otimes T$ which is compatible with the inclusion of R, for some A-bilinear map $\mathfrak{x} : T \otimes X \to X \otimes T$. Then $(X, \mathfrak{x})$ is a left $(R, \mathfrak{r})$-module with action $\mathsf{l}_{(X,\,\mathfrak{x})} = \mathsf{l}_X \otimes T$. In particular $\mathfrak{x}$ is a left module twisting map (i.e. satisfies (6.5)*–*(6.7)*).*

Proof. By definition the map $\mathsf{l}_{(X,\,\mathfrak{x})}$ is unital and associative, that is, it satisfies Eqs. (5.11) and (5.12) in Proposition 5.8(ii). The left action $\mathsf{l}_{X\otimes Y}$ is unital since l_Y and $\mathsf{l}_{(X,\,\mathfrak{x})}$ they are so. The associativity of l_Y and $\mathsf{l}_{(X,\,\mathfrak{x})}$ lead to that of $\mathsf{l}_{X\otimes Y}$, taking into the account that the structure maps of the wreath $(R, \mathfrak{r})$ are $1_R \otimes T$ and $\mu_R \otimes T$. The proof of the reciprocate implication is left to the reader. □

The right version of Proposition 6.1 is expressed in the monoidal category $\mathscr{L}_{(A:\,R)}$ using the R-wreath $(\mathfrak{r}, T)$ of Subsec. 6.1. If we combine both versions, then we get a criterion on $(R \otimes_{\mathfrak{r}} T)$-bimodules of the form $X \otimes Y$ as in [7, Proposition 3.13].

Proposition 6.2. *Let $(R, \mathfrak{r})$ and $(\mathfrak{r}, T)$, respectively, the T-wreath and R-wreath of Subsec. 6.1. Consider X an R-bimodule, and V a T-bimodule with actions l_X, r_X and l_V, r_V. Assume that $(X, \mathfrak{x})$ is also a left $(R, \mathfrak{r})$-module and that $(\mathfrak{v}, V)$ is a right $(\mathfrak{r}, T)$-module with actions, respectively, $\mathsf{l}_{(X,\mathfrak{x})}$ and $\mathsf{r}_{(\mathfrak{v},V)}$, and consider the maps*

$$\mathsf{l}_{X\otimes V} = (X \otimes \mathsf{l}_V) \circ (\mathsf{l}_{(X,\,\mathfrak{x})} \otimes V) \circ (R \otimes \mathfrak{x} \otimes V),$$
$$\mathsf{r}_{X\otimes V} = (\mathsf{r}_X \otimes V) \circ (X \otimes \mathsf{r}_{(\mathfrak{v},V)}) \circ (X \otimes \mathfrak{v} \otimes T).$$

If $X \otimes V$ is an $(R \otimes_{\mathfrak{r}} T)$-bimodule with action $\mathsf{l}_{X\otimes V}$ and $\mathsf{r}_{X\otimes V}$, then we have

$$\begin{aligned}
&(X \otimes \mathsf{l}_V) \circ (\mathfrak{x} \otimes V) \circ (T \otimes \mathsf{r}_X \otimes V) \circ (T \otimes X \otimes \mathfrak{v}) \\
&\quad = (\mathsf{r}_X \otimes V) \circ (X \otimes \mathfrak{v}) \circ (X \otimes \mathsf{l}_V \otimes R) \circ (\mathfrak{x} \otimes V \otimes R).
\end{aligned} \tag{6.8}$$

This $(R \otimes_{\mathfrak{r}} T)$*-biaction on* $X \otimes V$ *is left compatible with the inclusion of* R *and right compatible with the inclusion of* T *whenever* $\mathsf{l}_{(X,\mathfrak{x})}$ *and* $\mathsf{r}_{(\mathfrak{v},V)}$ *satisfy the equalities*

$$\mathsf{l}_{(X,\mathfrak{x})} \circ (R \otimes X \otimes 1_T) = (X \otimes 1_T) \circ \mathsf{l}_X, \quad \mathsf{r}_{(\mathfrak{v},V)} \circ (1_R \otimes X \otimes T) = (1_R \otimes V) \circ \mathsf{r}_Y.$$

Conversely, if we take $\mathsf{l}_{(X,\mathfrak{x})} = \mathsf{l}_X \otimes T$, $\mathsf{r}_{(\mathfrak{v},V)} = R \otimes \mathsf{r}_V$, *and assume that Eq.* (6.8) *is satisfied, then* $X \otimes V$ *is an* $(R \otimes_{\mathfrak{r}} T)$*-bimodule.*

Proof. Analog to that of [7, Proposition 3.13], taking into account Eq. (5.14). □

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