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Infinite Comatrix Corings

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1 Introduction

Among the different aspects in the recent developments of the theory of corings, one of the most intensively studied is the notion of a Galois coring and its relationships with (noncommutative) descent theory for ring extensions and Morita-type equivalence theorems. A coring $\mathfrak{C}$ over a ring A is said to be Galois [2] whenever the canonical map $\mathrm{can} : A \otimes_B A \to \mathfrak{C}$, which sends $a \otimes_B a'$ onto aga', is an isomorphism, where $g \in \mathfrak{C}$ is a group-like element and $B = \{a \in A \mid ag = ga\}$. One of the origins of this notion is the concept of a noncommutative Hopf-Galois extension [8, 18, 21, 24], which can be ultimately traced back to the theory of principal homogeneous spaces for actions of affine groups over affine schemes (see, e.g., [25, Section 18.3]) and the characterization of strongly graded rings [7]. One of the fundamental facts is that a faithfully flat Hopf-Galois extension $B \subseteq A$, for H a Hopf algebra, encodes a canonical equivalence of categories between a category of Hopf modules and the category of right modules over B (see [21]). This has been a model for research in more general frameworks (see [6] for a helpful survey), including coalgebra-Galois extensions for entwining structures [4] and Galois corings with a group-like [2]. In [9] it is shown that in order to formulate a meaningful notion of Galois coring, the group-like can be replaced by a right comodule (a Galois comodule in the terminology of [3, 5]), finitely generated and projective as a module over the ground ring A, whenever the role of Sweedler's canonical coring is played by the associated comatrix coring. This new viewpoint led us to imagine that there was a relation between the notion of Galois coring and the fact that a coalgebra over a field can be reconstructed from its finite-dimensional comodules [16]. This paper contains the mathematical results of our investigations on this idea. A generalization of the notion of comatrix coring (and of

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Galois comodule) will be introduced with this purpose. The role of Galois comodules in noncommutative geometry, as noncommutative principal bundles, is revealed in [3].

We will define our generalized comatrix corings in three ways, being the interplay between these constructions fundamental in our exposition. One of them is inspired in the coalgebra defined in [16, Section 4]. We wish to thank Antonio M. Cegarra for helping us to understand this construction.

2 A specialized introduction: Galois corings with several group-like elements

In this section, we will give our statements without proofs. They are consequences of the results proved in the rest of the paper. The basic notations and notions will be explained later as well.

Let G be a set of group-like elements in a coring $\mathfrak{C}$ over a K-algebra A (K is a commutative ring). For each $g \in G$, we will denote by $[g]A$ the right $\mathfrak{C}$-comodule structure on A, with coaction $A \to A \otimes_A \mathfrak{C}$ sending $a \in A$ onto $1 \otimes_A ga$. The notation $A[g]$ stands for the left $\mathfrak{C}$-comodule structure on A associated to g. For $g, h \in G$, the K-module $\mathrm{Hom}_{\mathfrak{C}}([g]A, [h]A)$ is identified via the map $f \mapsto f(1)$, with $A_{g,h} = \{b \in A \mid hb = bg\}$. Clearly, $A_{g,g}$ is a (unital) subring of A, and $A_{g,h}$ becomes an $(A_{h,h} - A_{g,g})$-bimodule. Consider the ring (not necessarily with unit) $S = M^f_{|G|\times|G|}$ consisting of all $|G| \times |G|$ matrices with coefficients in A and with finitely many nonzero entries. The unit $1 \in A$ at the position (g, g) and zero elsewhere gives an idempotent matrix $1_{g,g} \in S$, and the set of all $1_{g,g}$'s gives a complete set of pairwise orthogonal idempotents for S. The external direct sum $R = \bigoplus_{g,h\in G} A_{g,h}$ may be then considered as a subring of S that contains the idempotents $1_{g,g}$.

On the other hand, for every $g \in G$, we can consider Sweedler's canonical A-coring $A \otimes_{A_{g,g}} A$ [23]. Let $\mathfrak{J}$ be the K-submodule of the coproduct $\bigoplus_{g\in G} A \otimes_{A_{g,g}} A$ generated by all elements of the form $a \otimes_{A_{h,h}} ta' - at \otimes_{A_{g,g}} a'$, where $g, h \in G$, $a, a' \in A$, and $t \in A_{g,h}$. It turns out that $\mathfrak{J}$ is a coideal of the coproduct A-coring $\bigoplus_{g\in G} A \otimes_{A_{g,g}} A$, and we have a new A-coring

$$\mathfrak{R}(G) = \frac{\bigoplus_{g\in G} A \otimes_{A_{g,g}} A}{\mathfrak{J}} = \frac{\bigoplus_{g\in G} A \otimes_{A_{g,g}} A}{\langle a \otimes_{A_{h,h}} ta' - at \otimes_{A_{g,g}} a';\ a, a' \in A,\ t \in A_{g,h},\ g, h \in G\rangle}. \tag{2.1}$$

Moreover, the map $\mathrm{can} : \mathfrak{R}(G) \to \mathfrak{C}$ which sends $a \otimes_{A_{g,g}} a' + \mathfrak{J}$ onto aga' is a homomorphism of A-corings. Obviously, this canonical map is a generalization of the one given in [2] for a single group-like (i.e., $G = \{g\}$). The comodule $\Sigma = \bigoplus_{g \in G}[g]A$ is not finitely generated unless G is finite. Nevertheless, following [2] for the case $G = \{g\}$, it makes sense to say that $\mathfrak{C}$ is a Galois coring (with respect to Σ) whenever can is an isomorphism. Of course, there are examples of situations where can is bijective but Σ is not finitely generated. The key is the following result.

Theorem 2.1. If $\bigoplus_{g \in G}[g]A$ is a generator for $\mathcal{M}^{\mathfrak{C}}$, then $\mathrm{can} : \mathfrak{R}(G) \to \mathfrak{C}$ is an isomorphism of A-corings. □

A relevant example of Galois coring in this new general framework is the following.

Example 2.2. Let A be a G-graded ring, where G is any group. Endow the free left A-module AG with basis G with the right A-module structure given by $ga = agh$, for $a \in A$ homogeneous of degree $h \in G$. Then AG becomes an A-bimodule. Consider the A-coring structure on AG given by the comultiplication defined by $\Delta(ag) = ag \otimes_A g$ and counit defined by $\epsilon(ag) = a$, for every $a \in A$, $g \in G$. The category of right AG-comodules is then isomorphic to the category $gr - A$ of all G-graded right A-modules. This can be proved either by direct computations or by using that AG is the coring built, according to [2, Proposition 2.2] and [1, Example 3.1], from the entwining structure given on $AG \cong A \otimes_K KG$ by the KG-comodule algebra map $a \mapsto \sum_{g \in G} a_g \otimes g$, and the fact that, in this case, the category of Hopf modules $\mathcal{M}_A^{KG}$ is isomorphic to $gr - A$ [17]. Clearly, G is a set of group-like elements, and $\{[g]A : g \in G\}$ is nothing but the set of all shifts of the graded module A. It is known that it is a generating set of small projectives for $gr - A$. By Theorem 2.1, the canonical map is an isomorphism. More generally, for each subgroup H of G such that $\bigoplus_{h \in H}[h]A$ is a (projective) generator for $gr - A$, the canonical map $\mathrm{can} : \mathfrak{R}(H) \to AG$ is an isomorphism of A-corings.

Theorem 2.3. The following statements are equivalent for an A-coring $\mathfrak{C}$ with a set of group-like elements G:

(i) ${}_A\mathfrak{C}$ is flat, $\mathrm{can} : \mathfrak{R}(G) \to \mathfrak{C}$ is an isomorphism, and S is faithfully flat as a left R-module;

(ii) ${}_A\mathfrak{C}$ is flat, $\mathrm{can} : \mathfrak{R}(G) \to \mathfrak{C}$ is an isomorphism, and the category $\mathcal{M}^{\mathfrak{R}(G)}$ of right $\mathfrak{R}(G)$-comodules is equivalent, in a canonical way, to the category $\mathcal{M}_R$ of all unital right R-modules;

(iii) ${}_A\mathfrak{C}$ is flat and $\bigoplus_{g \in G}[g]A$ is a projective generator for $\mathcal{M}^{\mathfrak{C}}$;

(iv) ${}_A\mathfrak{C}$ is flat and the category $\mathcal{M}^{\mathfrak{C}}$ is equivalent, in a canonical way, to the category $\mathcal{M}_R$ of all unital right R-modules. □

We will in fact prove a more general result (Theorem 5.7), characterizing those corings whose category of comodules has a generating set of small projectives. With this purpose, we extend the techniques developed in [9] (in particular, the notion of a comatrix coring) to a larger framework. We thus will obtain an alternative description of the coring $\mathfrak{R}(G)$ as follows: write $\Sigma^{\dagger} = \bigoplus_{g\in G} A[g]$, $\Sigma = \bigoplus_{g\in G}[g]A$. We understand Σ (resp., $\Sigma^{\dagger}$) as the free right (resp., left) A-module with basis $\{[g] : g \in G\}$. Consider the A-coring $\Sigma^{\dagger} \otimes_R \Sigma$ with comultiplication given by

$$\Delta\left(\sum_{g\in G} a_g[g] \otimes_R \sum_{h\in G}[h]a'_h\right) = \sum_{g\in G} a_g[g] \otimes_R [g]1 \otimes_A 1[g] \otimes_R [g]a'_g \tag{2.2}$$

and counit

$$\epsilon\left(\sum_{g\in G} a_g[g] \otimes_R \sum_{h\in G}[h]a'_h\right) = \sum_{g\in G} a_g a'_g. \tag{2.3}$$

Then there exists a canonical isomorphism of A-corings $\mathfrak{R}(G) \cong \Sigma^{\dagger} \otimes_R \Sigma$, and this latter is a sort of generalized comatrix coring. In fact, this setting allows one to extend the methods from [9] to a more general context. As a consequence, we will have that the corings characterized in Theorem 2.3 are those for which ${}_R\Sigma$ is faithfully flat.

Example 2.4. Continuing with Example 2.2, we get that the category $gr - A$ is always equivalent to the category of unital right modules over the ring R. More generally, given a subgroup H of G, Theorem 2.3 gives several conditions which characterize when $gr - A$ is equivalent, in a canonical way, to the category of unital right R-modules, where R is the ring of $|H| \times |H|$ matrices with finitely many nonzero entries, whose coefficients in its (g, h)-entry are the homogeneous elements of A of degree gh^{-1}, where $g, h \in H$.

3 Basic notions

We use the following conventions. We work over a fixed commutative ring K, and all our additive categories are assumed to be K-linear. For instance, all rings in this paper are (not necessarily unitary) K-algebras and all bimodules are assumed to centralize the elements of K. The identity morphism attached to any object X of a category $\mathcal{C}$ is represented by the object itself. Associated to every object C of an additive category $\mathcal{C}$ is its endomorphism ring $\mathrm{End}_{\mathcal{C}}(C)$, whose multiplication is given by the composition of the category. As usual, some special conventions will be understood for the case of endomorphism rings of modules. Thus, if M_R is a (unital) right module over a ring R, then its endomorphism ring in the category $\mathcal{M}_R$ of all unital right R-modules will be denoted by $\mathrm{End}(M_R)$, while if ${}_RN$ is a left R-module, then its endomorphism ring, denoted by $\mathrm{End}({}_RN)$, is, by definition, the opposite of the endomorphism ring of N in the category ${}_R\mathcal{M}$ of all unital left modules over R. We will make use of rings R which need not have a unit, although they will always contain a set of pairwise orthogonal idempotents $\{e_i\}$ which is complete in the sense that $R = \bigoplus_i Re_i = \bigoplus_i e_iR$. In this case, to be unital for a right R-module M means that $M = MR$. The tensor product over R is denoted by $\otimes_R$. We will sometimes replace $\otimes_K$ by $\otimes$.

The notation A is reserved for a K-algebra with unit. We recall from [23] that an A-*coring* is an A-bimodule $\mathfrak{C}$ with two A-bimodule maps

$$\Delta : \mathfrak{C} \longrightarrow \mathfrak{C} \otimes_A \mathfrak{C}, \qquad \epsilon : \mathfrak{C} \longrightarrow A \tag{3.1}$$

such that $(\mathfrak{C} \otimes_A \Delta) \circ \Delta = (\Delta \otimes_A \mathfrak{C}) \circ \Delta$ and $(\epsilon \otimes_A \mathfrak{C}) \circ \Delta = (\mathfrak{C} \otimes_A \epsilon) \circ \Delta = \mathfrak{C}$. A *right* $\mathfrak{C}$*-comodule* is a pair (M, ρ_M) consisting of a right A-module M and an A-linear map $\rho_M : M \to M \otimes_A \mathfrak{C}$ satisfying $(M \otimes_A \Delta) \circ \rho_M = (\rho_M \otimes_A \mathfrak{C}) \circ \rho_M$, $(M \otimes_A \epsilon) \circ \rho_M = M$. A *morphism* of right $\mathfrak{C}$-comodules (M, ρ_M) and (N, ρ_N) is a right A-linear map $f : M \to N$ such that $(f \otimes_A \mathfrak{C}) \circ \rho_M = \rho_N \circ f$; the K-module of all such morphisms will be denoted by $\mathrm{Hom}_{\mathfrak{C}}(M, N)$. The right $\mathfrak{C}$-comodules together with their morphisms form the additive category $\mathcal{M}^{\mathfrak{C}}$. Coproducts and cokernels in $\mathcal{M}^{\mathfrak{C}}$ do exist and can be already computed in $\mathcal{M}_A$. Therefore, $\mathcal{M}^{\mathfrak{C}}$ has arbitrary inductive limits. If ${}_A\mathfrak{C}$ is flat, then $\mathcal{M}^{\mathfrak{C}}$ is an abelian category. The converse is not true, as [10, Example 1.1] shows.

Let $\rho_M : M \to M \otimes_A \mathfrak{C}$ be a comodule structure over an $(A' - A)$-bimodule M and assume that ρ_M is A'-linear (A' denotes a second unitary K-algebra). For any right A'-module X, the right A-linear map $X \otimes_{A'} \rho_M : X \otimes_{A'} M \to X \otimes_{A'} M \otimes_A \mathfrak{C}$ makes $X \otimes_{A'} M$ a right $\mathfrak{C}$-comodule. This leads to an additive functor $- \otimes_{A'} M : \mathcal{M}_{A'} \to \mathcal{M}^{\mathfrak{C}}$. The classical adjointness isomorphism $\mathrm{Hom}_A(Y \otimes_{A'} M, X) \cong \mathrm{Hom}_{A'}(Y, \mathrm{Hom}_A(M, X))$ induces, by restriction, a natural isomorphism $\mathrm{Hom}_{\mathfrak{C}}(Y \otimes_{A'} M, X) \cong \mathrm{Hom}_{A'}(Y, \mathrm{Hom}_{\mathfrak{C}}(M, X))$, for $Y \in \mathcal{M}_{A'}$, $X \in \mathcal{M}^{\mathfrak{C}}$. Therefore, $\mathrm{Hom}_{\mathfrak{C}}(M, -) : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_{A'}$ is right adjoint to $- \otimes_{A'} M : \mathcal{M}_{A'} \to \mathcal{M}^{\mathfrak{C}}$. On the other hand, the functor $- \otimes_A \mathfrak{C}$ is right adjoint to the forgetful functor $U : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ (see [14, Proposition 3.1] and [2, Lemma 3.1]).

Now assume that the $(A' - A)$-bimodule M is also a left comodule over an A'-coring $\mathfrak{C}'$ with structure map $\lambda_M : M \to \mathfrak{C}' \otimes_{A'} M$. It is clear that $\rho_M : M \to M \otimes_A \mathfrak{C}$ is a morphism of left $\mathfrak{C}'$-comodules if and only if $\lambda_M : M \to \mathfrak{C}' \otimes_{A'} M$ is a morphism of right $\mathfrak{C}$-comodules. In this case, we say that M is a $\mathfrak{C}' - \mathfrak{C}$-bicomodule.

A group-like is an element $g \in \mathfrak{C}$ such that $\Delta(g) = g \otimes_A g$ and $\epsilon(g) = 1$. Every group-like defines a right $\mathfrak{C}$-comodule structure $\rho : A \to A \otimes_A \mathfrak{C} \cong \mathfrak{C}$ given by $\rho(a) = ga$.

A left comodule structure is similarly obtained. This process can be reversed: each right $\mathfrak{C}$-comodule structure $\rho : A \to \mathfrak{C}$ gives a group-like $g = \rho(1)$ (see [2, Lemma 5.1]).

A friendly source of information on corings and their comodules is the monograph [5].

4 Reconstruction and infinite comatrix corings

A ring extension $B \subseteq A$ for rings with unit can be understood as a faithful functor between two additive categories with one object. With this idea in mind and with an eye on [16, Section 4], the construction of the canonical A-coring $A \otimes_B A$ from [23] is generalized in this section.

Let K denote a commutative ring. Let $\mathrm{add}(A_A)$ denote the category of all finitely generated projective right modules over an associative K-algebra with unit A. Let $\omega : \mathcal{A} \to \mathrm{add}(A_A)$ be a functor, where $\mathcal{A}$ is a small category. The image of an object P of $\mathcal{A}$ under ω will be denoted by ${}^{\omega}P$, or even by P itself when no confusion may be expected. For each $P \in \mathcal{A}$, there is a canonical homomorphism of K-algebras from $T_P = \mathrm{End}_{\mathcal{A}}(P)$ to $S_P = \mathrm{End}({}^{\omega}P_A)$, which sends t onto $\omega(t)$. The module ${}^{\omega}P$ becomes then a T_P–A-bimodule. If P, Q are objects of $\mathcal{A}$, then the elements of the $(T_Q - T_P)$-bimodule $T_{PQ} = \mathrm{Hom}_{\mathcal{A}}(P, Q)$ act canonically on ${}^{\omega}P$. This action can be thought of as the $(T_Q - A)$-bimodule map

$$\begin{aligned} T_{PQ} \otimes_{T_P} {}^{\omega}P &\longrightarrow {}^{\omega}Q, \\ t \otimes_{T_P} p &\longmapsto tp := \omega(t)(p). \end{aligned} \tag{4.1}$$

The right dual A-modules ${}^{\omega}P^* = \mathrm{Hom}_A({}^{\omega}P, A)$ are then in a natural way $(A - T_P)$-bimodules; the corresponding dual pairings are given by

$$\begin{aligned} {}^{\omega}Q^* \otimes_{T_Q} T_{PQ} &\longrightarrow {}^{\omega}P^*, \\ \varphi \otimes_{T_Q} t &\longmapsto \varphi t := \varphi \circ \omega(t). \end{aligned} \tag{4.2}$$

From now on, we will denote ${}^{\omega}P$ by P. We can associate to every object P of $\mathcal{A}$ its *comatrix A-coring* $P^* \otimes_{T_P} P$ defined as follows [9]. Let $\{e^*_{\alpha_P}, e_{\alpha_P}\}$ be a finite dual basis for the module P_A and define the comultiplication as $\Delta(\varphi \otimes_{T_P} p) = \sum_{\alpha_P} \varphi \otimes_{T_P} e_{\alpha_P} \otimes_A e^*_{\alpha_P} \otimes_{T_P} p$. The counit is given by $\epsilon(\varphi \otimes_{T_P} p) = \varphi(p)$. We can then consider the coproduct of A-corings

$$\mathfrak{P}(\mathcal{A}) = \bigoplus_{P \in \mathcal{A}} P^* \otimes_{T_P} P. \tag{4.3}$$

Every $P \in \mathcal{A}$ is canonically a right $P^* \otimes_{T_P} P$-comodule [9] and, hence, a $\mathfrak{P}(\mathcal{A})$-comodule, with structure map $\rho_P : P \to P \otimes_A \mathfrak{P}(\mathcal{A})$ defined as $\rho_P(p) = \sum_{\alpha_P} p \otimes_A e^*_{\alpha_P} \otimes_{T_P} e_{\alpha_P}$. The assignment $P \mapsto (P, \rho_P)$ is not, at least in the obvious way, a functor from $\mathcal{A}$ to $\mathcal{M}^{\mathfrak{P}(\mathcal{A})}$. In order to remedy this, we will factor out $\mathfrak{P}(\mathcal{A})$ by a coideal.

Lemma 4.1. Let $t \in T_{PQ}$. Then

$$\sum_{\alpha_Q} e_{\alpha_Q} \otimes_A e^*_{\alpha_Q} t = \sum_{\alpha_P} t e_{\alpha_P} \otimes_A e^*_{\alpha_P}. \tag{4.4}$$

□

Proof. The proof follows easily from the dual basis criterion. ■

Lemma 4.2. The K-submodule $\mathfrak{J}$ of $\mathfrak{P}(\mathcal{A})$ generated by the set

$$\left\{\varphi \otimes_{T_Q} tp - \varphi t \otimes_{T_P} p : \varphi \in Q^*,\ p \in P,\ t \in T_{PQ},\ P, Q \in \mathcal{A}\right\} \tag{4.5}$$

is a coideal of $\mathfrak{P}(\mathcal{A})$. □

Proof. It is easily shown that, in fact, $\mathfrak{J}$ is an A-subbimodule of $\mathfrak{P}(\mathcal{A})$. We now prove that $\epsilon(\mathfrak{J}) = 0$. In fact,

$$\epsilon\left(\varphi \otimes_{T_Q} tp - \varphi t \otimes_{T_P} p\right) = \varphi(tp) - \varphi t(p) = \varphi t(p) - \varphi t(p) = 0. \tag{4.6}$$

Now, by using Lemma 4.1, we have

$$\begin{aligned}
&\Delta\left(\varphi \otimes_{T_Q} tp - \varphi t \otimes_{T_P} p\right)\\
&\quad= \sum_{\alpha_Q} \varphi \otimes_{T_Q} e_{\alpha_Q} \otimes_A e^*_{\alpha_Q} \otimes_{T_Q} tp - \sum_{\alpha_P} \varphi t \otimes_{T_P} e_{\alpha_P} \otimes_A e^*_{\alpha_P} \otimes_{T_P} p\\
&\quad= \sum_{\alpha_Q} \varphi \otimes_{T_Q} e_{\alpha_Q} \otimes_A e^*_{\alpha_Q} \otimes_{T_Q} tp - \sum_{\alpha_Q} \varphi \otimes_{T_Q} e_{\alpha_Q} \otimes_A e^*_{\alpha_Q} t \otimes_{T_P} p\\
&\qquad+ \sum_{\alpha_P} \varphi \otimes_{T_Q} t e_{\alpha_P} \otimes_A e^*_{\alpha_P} \otimes_{T_P} p - \sum_{\alpha_P} \varphi t \otimes_{T_P} e_{\alpha_P} \otimes_A e^*_{\alpha_P} \otimes_{T_P} p\\
&\quad= \sum_{\alpha_Q} \varphi \otimes_{T_Q} e_{\alpha_Q} \otimes_A \left(e^*_{\alpha_Q} \otimes_{T_Q} tp - e^*_{\alpha_Q} t \otimes_{T_P} p\right)\\
&\qquad+ \sum_{\alpha_P} \left(\varphi \otimes_{T_Q} t e_{\alpha_P} - \varphi t \otimes_{T_P} e_{\alpha_P}\right) \otimes_A e^*_{\alpha_P} \otimes_{T_P} p.
\end{aligned} \tag{4.7}$$

This proves that $\Delta(\mathfrak{J}) \subseteq \mathrm{Ker}(\pi \otimes_A \pi)$, where $\pi : \mathfrak{P}(\mathcal{A}) \to \mathfrak{P}(\mathcal{A})/\mathfrak{J}$ is the canonical projection. Therefore, $\mathfrak{J}$ is a coideal. ■

Proposition 4.3. Let $\mathfrak{P}(\mathcal{A}) = \bigoplus_{P \in \mathcal{A}} P^* \otimes_{T_P} P$ and define the factor A-coring $\mathfrak{R}(\mathcal{A}) = \mathfrak{P}(\mathcal{A})/\mathfrak{J}$. There is a functor $\mathfrak{R}(\omega_A) : \mathcal{A} \to \mathcal{M}^{\mathfrak{R}(\mathcal{A})}$ making the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\omega_A} & \mathrm{add}\,(A_A) \\ {\scriptstyle \mathfrak{R}(\omega_A)} \downarrow & & \downarrow \\ \mathcal{M}^{\mathfrak{R}(\mathcal{A})} & \xrightarrow[U_A]{} & \mathcal{M}_A \end{array} \tag{4.8}$$

commutative. This functor assigns to every $P \in \mathcal{A}$ the right $\mathfrak{R}(\mathcal{A})$-comodule induced by its canonical $P^* \otimes_{T_P} P$-coaction and the canonical homomorphism of A-corings $\pi : \mathfrak{P}(\mathcal{A}) \to \mathfrak{P}(\mathcal{A})/\mathfrak{J}$. □

Proof. So, the right $\mathfrak{R}(\mathcal{A})$-comodule structure for P is given by

$$\begin{aligned} & P \xrightarrow{\rho_P} P \otimes_A P^* \otimes_{T_P} P \xrightarrow{P \otimes_A \pi} P \otimes_A \mathfrak{R}(\mathcal{A}), \\ & p \longmapsto \sum_{\alpha_P} e_{\alpha_P} \otimes_A \left(e^*_{\alpha_P} \otimes_{T_P} p + \mathfrak{J}\right). \end{aligned} \tag{4.9}$$

Given $\lambda \in \mathrm{Hom}_{\mathcal{A}}(P, Q)$, a straightforward computation shows, with the help of Lemma 4.1, that $\omega(\lambda)$ is $\mathfrak{R}(\mathcal{A})$-colinear. ■

We will now give an alternative description of the A-coring $\mathfrak{R}(\mathcal{A})$. Assume that $\mathcal{A}$ is a subcategory of an additive category $\mathcal{C}$, and that the coproduct $\Sigma = \bigoplus_{P \in \mathcal{A}} P$ exists in the category $\mathcal{C}$. Assume further that the functor $\omega : \mathcal{A} \to \mathrm{add}(A_A)$ is the restriction of a functor $U : \mathcal{C} \to \mathcal{M}_A$ which commutes with the coproduct $\bigoplus_{P \in \mathcal{A}} P$. This is not actually a restriction: such a category $\mathcal{C}$ can be constructed by formally introducing a new object Σ and enlarging the set of morphisms by defining the new hom-sets $\mathrm{Hom}_{\mathcal{C}}(P, Q) = \mathrm{Hom}_{\mathcal{A}}(P, Q)$, $\mathrm{Hom}_{\mathcal{C}}(P, \Sigma) = \bigoplus_{X \in \mathcal{A}} \mathrm{Hom}_{\mathcal{C}}(P, X)$, $\mathrm{Hom}_{\mathcal{C}}(\Sigma, Q) = \prod_{X \in \mathcal{A}} \mathrm{Hom}_{\mathcal{C}}(X, Q)$, and, finally, $\mathrm{Hom}_{\mathcal{C}}(\Sigma, \Sigma) = \prod_{X \in \mathcal{A}} \mathrm{Hom}_{\mathcal{C}}(X, \Sigma)$, for $P, Q \in \mathcal{A}$. The definition of the functor $U : \mathcal{C} \to \mathcal{M}_A$ is then clear.

Consider the endomorphism ring $T = \mathrm{End}_{\mathcal{C}}(\Sigma)$. We then have a canonical structure of a $(T - A)$-bimodule on $U(\Sigma)$, which induces an $(A - T)$-bimodule structure on $U(\Sigma)^* = \mathrm{Hom}_A(\Sigma, A)$. We think there will be no problems when using the notation Σ instead of $U(\Sigma)$. For each object P in $\mathcal{A}$, let $\pi_P : \Sigma \to P$ (resp., $\iota_P : P \to \Sigma$) be the canonical projection (resp., injection).

Proposition 4.4. There is a surjective homomorphism of A-bimodules

$$\Gamma : \bigoplus_{P\in\mathcal{A}} P^* \otimes_{T_P} P \longrightarrow \Sigma^* \otimes_T \Sigma \tag{4.10}$$

whose restriction Γ_P to each $P^* \otimes_{T_P} P$ is given by $\Gamma_P(\varphi \otimes_{T_P} p) = \varphi\pi_P \otimes_T \iota_P(p)$. The kernel of Γ is the coideal $\mathfrak{J}$ and, hence, $\Sigma^* \otimes_T \Sigma$ can be endowed with a structure of A-coring such that Γ is a homomorphism of A-corings. This homomorphism Γ induces an isomorphism of corings $\mathfrak{R}(\mathcal{A}) \cong \Sigma^* \otimes_T \Sigma$. □

Proof. We first check that each Γ_P is well defined. For $\varphi \in P^*$, $p \in P$, and $t \in T_P$, we have

$$\begin{aligned}
\Gamma_P\big(\varphi t \otimes_{T_P} p\big) &= (\varphi t)\pi_P \otimes_T \iota_P(p)\\
&= \varphi\pi_P\iota_P t\pi_P \otimes_T \iota_P(p)\\
&= \varphi\pi_P \otimes_T \iota_P t\pi_P\iota_P(p)\\
&= \varphi\pi_P \otimes_T \iota_P t(p)\\
&= \Gamma_P\big(\varphi \otimes_{T_P} tp\big).
\end{aligned} \tag{4.11}$$

To prove that Γ is surjective, observe that if $\varphi \otimes_T p \in \Sigma^* \otimes_T \Sigma$, then there is a finite set $\mathcal{F}$ of objects of $\mathcal{A}$ such that $p = (\sum_{P\in\mathcal{F}} \iota_P\pi_P)(p)$. Therefore

$$\begin{aligned}
\varphi \otimes_T p &= \varphi \otimes_T \sum_{P\in\mathcal{F}} \iota_P\pi_P(p)\\
&= \sum_{P\in\mathcal{F}} \varphi\iota_P\pi_P \otimes_T \iota_P\pi_P(p)\\
&= \Gamma\Bigg(\sum_{P\in\mathcal{F}} \varphi\iota_P \otimes_{T_P} \pi_P(p)\Bigg).
\end{aligned} \tag{4.12}$$

Next, we check that $\mathfrak{J} \subseteq \operatorname{Ker}\Gamma$. Given a generator $\varphi \otimes_{T_Q} tp - \varphi t \otimes_{T_P} p$ of $\mathfrak{J}$, where $\varphi \in Q^*$, $p \in P$, $t \in T_{PQ}$, and $P, Q \in \mathcal{A}$, apply Γ to obtain

$$\begin{aligned}
\Gamma\big(\varphi \otimes_{T_Q} tp - \varphi t \otimes_{T_P} p\big) &= \varphi\pi_Q \otimes_T \iota_Q(tp) - \varphi t\pi_P \otimes_T \iota_P(p)\\
&= \varphi\pi_Q \otimes_T \iota_Q(tp) - \varphi\pi_Q\iota_Q t\pi_P \otimes_T \iota_P(p)\\
&= \varphi\pi_Q \otimes_T \iota_Q(tp) - \varphi\pi_Q \otimes_T \iota_Q t\pi_P\iota_P(p)\\
&= \varphi\pi_Q \otimes_T \iota_Q(tp) - \varphi\pi_Q \otimes_T \iota_Q(tp) = 0.
\end{aligned} \tag{4.13}$$

We finally check that $\operatorname{Ker}\Gamma \subseteq \mathfrak{J}$. An arbitrary element of $\bigoplus_{P\in\mathcal{A}} P^* \otimes_{T_P} P$ is a finite sum $x = \sum_{a=1}^{n} \sum_{P\in\mathcal{A}} \varphi_{a,P} \otimes_{T_P} p_{a,P}$, for some $\varphi_{a,P} \in P^*$ and $p_{a,P} \in P$ with almost all $p_{a,P} = 0$. Such an element belongs to $\operatorname{Ker}\Gamma$ if and only if $\sum_{a,P} \varphi_{a,P}\pi_P \otimes_T \iota_P(p_{a,P}) = 0$. Since the elements of the form $\iota_P(p)$ generate Σ, there exist [22, Proposition I.8.8] a finite set $\{\nu_k\}_{k\in I} \subseteq \Sigma^*$ and a set $\{g_{a,P,k}\} \subseteq T$ such that

(1) $g_{a,P,k} = 0$ for almost all (a, P, k),
(2) $\sum_{a,P} g_{a,P,k}\iota_P(p_{a,P}) = 0$ for each $k \in I$,
(3) $\varphi_{a,P}\pi_P = \sum_k \nu_k g_{a,P}$ for each $a = 1, \ldots, n$ and each $P \in \mathcal{A}$.

It follows from the third condition that, for every a and P,

$$\varphi_{a,P} = \varphi_{a,P}\pi_P\iota_P = \sum_k \nu_k g_{a,P,k}\iota_P. \tag{4.14}$$

Since each P is finitely generated as a right A-module, it follows that $g_{a,P,k} : P \to \Sigma$ factorizes throughout a finite direct sum $\bigoplus_{Q\in\mathcal{F}} Q$. We can assume that the finite set $\mathcal{F}$ is independent of a and P (recall that only finitely many $g_{a,P,k}$'s are nonzero). Therefore,

$$g_{a,P,k}\iota_P = \left(\sum_{Q\in\mathcal{F}} \iota_Q\pi_Q\right) g_{a,P,k}\iota_P = \sum_{Q\in\mathcal{F}} \iota_Q\pi_Q g_{a,P,k}\iota_P. \tag{4.15}$$

In view of (4.14), we have

$$\varphi_{a,P} = \sum_{k,Q} \nu_{k,Q} t_{Q,a,P,k}, \tag{4.16}$$

where $\nu_{k,Q} = \nu_k\iota_Q \in Q^*$ and $t_{Q,a,P,k} = \pi_Q g_{a,P,k}\iota_P \in T_{PQ}$. On the other hand, condition (2) implies that for each Q, k,

$$\sum_{a,P} t_{Q,a,i,k} p_{a,P} = \sum_{a,P} \pi_Q g_{a,P,k}\iota_{a,P}(p_{a,P}) = \pi_Q\left(\sum_{a,P} g_{a,P,k}\iota_{a,P}(p_{a,P})\right) = 0. \tag{4.17}$$

Finally,

$$\begin{aligned}
&\sum_{a,P} \varphi_{a,P} \otimes_{T_P} p_{a,P} \\
&\quad = \sum_{a,P}\left(\sum_{k,Q} \nu_{k,Q} t_{Q,a,P,k}\right) \otimes_{T_P} p_{a,P} \\
&\quad = \sum_{a,P,k,Q} \nu_{k,Q} t_{Q,a,P,k} \otimes_{T_P} p_{a,P} - \sum_{k,Q} \nu_{k,Q} \otimes_{T_Q} \left(\sum_{a,P} t_{Q,a,P,k} p_{a,P}\right) \\
&\quad = \sum_{a,k}\left(\sum_{P,Q} \nu_{P,Q}\nu_{k,Q} t_{Q,a,P,k} \otimes_{T_P} p_{a,P} - \sum_{P,Q} \nu_{k,Q} \otimes_{T_Q} t_{Q,a,P,k} p_{a,P}\right),
\end{aligned} \tag{4.18}$$

where the first equality follows from (4.16) and the second from (4.17). ■

Definition 4.5. The A-coring $\Sigma^* \otimes_T \Sigma$ will be called the *infinite comatrix coring* associated to the category $\mathcal{A}$ and the functor $\omega : \mathcal{A} \to \mathrm{add}(A_A)$. Its comultiplication Δ is given explicitly as follows: once a (finite) dual basis $\{e^*_{\alpha_P}, e_{\alpha_P}\}$ is chosen for each $P \in \mathcal{A}$ (recall that P_A is finitely generated and projective), we have, from (4.12),

$$\Delta(\varphi \otimes_T p) = \sum_{P \in \mathcal{F}} \sum_{\alpha_P} \varphi \iota_P \pi_P \otimes_T \iota_P(e_{\alpha_P}) \otimes_A e^*_{\alpha_P} \pi_P \otimes_T \iota_P \pi_P(p), \tag{4.19}$$

for $\varphi \otimes_T p \in \Sigma^* \otimes_T \Sigma$, where $\mathcal{F}$ is any finite set of objects of $\mathcal{A}$ such that $p = \sum_{P \in \mathcal{F}} \iota_P \pi_P(p)$. The counit ϵ of $\Sigma^* \otimes_T \Sigma$ is simply the evaluation map $\varphi \otimes_T p \mapsto \varphi(p)$.

Remark 4.6. If ${}_BP_A$ is a $(B-A)$-bimodule, with $P_A \in \mathrm{add}(A_A)$, then the comatrix coring $P^* \otimes_B P$ of [9, Proposition 2.1] is just the infinite comatrix coring associated to the canonical functor from the additive category B to $\mathrm{add}(A_A)$, which sends the unique object of B onto P.

Now assume that $\mathcal{A}$ is a (small) subcategory of the category of right comodules $\mathcal{M}^{\mathfrak{C}}$ over an A-coring $\mathfrak{C}$, and that the functor $\omega : \mathcal{A} \to \mathrm{add}(A_A)$ is the restriction of the underlying functor $U : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ (i.e., we are taking $\mathcal{C} = \mathcal{M}^{\mathfrak{C}}$). Let $\Sigma = \bigoplus_{P \in \mathcal{A}} P$ and $T = \mathrm{End}_{\mathfrak{C}}(\Sigma)$. A straightforward argument shows that the functor $\mathrm{Hom}_{\mathfrak{C}}(\Sigma, -) : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_T$ is right adjoint to $- \otimes_T \Sigma : \mathcal{M}_T \to \mathcal{M}^{\mathfrak{C}}$ (the right comodule structure of $N \otimes_T \Sigma$ is inherited from Σ for each right T-module N). The counit of this adjunction evaluated at $\mathfrak{C}$ gives a homomorphism of A-bimodules

$$\mathrm{Hom}_{\mathfrak{C}}(\Sigma, \mathfrak{C}) \otimes_T \Sigma \longrightarrow \mathfrak{C} \quad \big(f \otimes_T u \longmapsto f(u)\big) \tag{4.20}$$

which, in conjunction with the canonical isomorphism $\Sigma^* \cong \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \mathfrak{C})$ (see, e.g., [5, Subsection 18.10]) gives a homomorphism of A-bimodules

$$\mathrm{can} : \Sigma^* \otimes_T \Sigma \cong \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \mathfrak{C}) \otimes_T \Sigma \longrightarrow \mathfrak{C}. \tag{4.21}$$

Lemma 4.7. The map $\mathrm{can} : \Sigma^* \otimes_T \Sigma \to \mathfrak{C}$, defined explicitly by $\mathrm{can}(\varphi \otimes_T u) = (\varphi \otimes_A \mathfrak{C})\rho_\Sigma(u)$, is a homomorphism of A-corings. □

Proof. By Proposition 4.4, there is a surjective homomorphism of A-corings

$$\Gamma : \bigoplus_{P \in \mathcal{A}} P^* \otimes_{T_P} P \longrightarrow \Sigma^* \otimes_T \Sigma. \tag{4.22}$$

Clearly, it suffices to prove that $\mathrm{can} \circ \Gamma$ is a homomorphism of A-corings and ultimately, that its restriction can_P to $P^* \otimes_{T_P} P$ is a homomorphism of A-corings for every P. This canonical map $\mathrm{can}_P : P^* \otimes_{T_P} P \to \mathfrak{C}$ coincides with the homomorphism of A-corings given in [9, Proposition 3]. ∎

We can now apply Gabriel-Popescu's theorem and state our reconstruction theorem for corings in the following terms.

Theorem 4.8 (reconstruction). Let $\mathfrak{C}$ be an A-coring and assume that the category $\mathcal{M}^{\mathfrak{C}}$ is abelian and that there is a generating set $\mathcal{A}$ of right $\mathfrak{C}$-comodules such that P_A is finitely generated and projective for every $P \in \mathcal{A}$. If $\Sigma = \bigoplus_{P \in \mathcal{A}} P$ and $T = \mathrm{End}_{\mathfrak{C}}(\Sigma)$, then $\mathrm{can} : \Sigma^* \otimes_T \Sigma \to \mathfrak{C}$ is an isomorphism of A-corings. □

Proof. If $\mathcal{M}^{\mathfrak{C}}$ is abelian, then it is a Grothendieck category [10, Proposition 1.2]. Clearly, Σ is then a generator for the category $\mathcal{M}^{\mathfrak{C}}$. By Gabriel-Popescu's theorem [20], the canonical map (4.20) is an isomorphism. We thus get that can is an isomorphism of A-corings. ∎

Remark 4.9. The terminology Galois comodule has been introduced in [3, 5] to refer to a right $\mathfrak{C}$-comodule P such that P_A is finitely generated and projective and $\mathrm{can} : P^* \otimes_T P \to \mathfrak{C}$ is an isomorphism. These comodules played a role in the characterization of corings having a finitely generated projective generator [9, Theorem 3.2] (see also the "Galois comodule structure theorem" [3]), and in the structure theorem for cosemisimple corings [9, Theorem 4.4]. Theorem 4.8 suggests that it makes sense to consider *Galois comodules* without finiteness conditions (the important point here, we think, is that $\Sigma^* \otimes_T \Sigma$ is endowed with a coring structure before assuming that can is an isomorphism). Another possibility would be to say that $\mathcal{A}$ is a *Galois subcategory* of $\mathcal{M}^{\mathfrak{C}}$ whenever the coring homomorphism $\mathrm{can} : \Sigma^* \otimes_T \Sigma \to \mathfrak{C}$ is an isomorphism. Of course, Σ denotes $\bigoplus_{P \in \mathcal{A}} P$.

Corollary 4.10. Let $\mathfrak{C}$ be a coring over a semisimple Artinian ring A. Let $\mathcal{A}$ be a generating set of finitely generated right $\mathfrak{C}$-comodules. If $\Sigma = \bigoplus_{P \in \mathcal{A}} P$ and $T = \mathrm{End}_{\mathfrak{C}}(\Sigma)$, then $\mathrm{can} : \Sigma^* \otimes_T \Sigma \to \mathfrak{C}$ is an isomorphism of A-corings. □

Proof. Since in this case ${}_A\mathfrak{C}$ is obviously flat, we get that $\mathcal{M}^{\mathfrak{C}}$ is abelian. Moreover, every finitely generated comodule is finitely generated as a right A-module. Therefore, our coring is in the hypotheses of Theorem 4.8. ∎

Example 4.11. Every coalgebra C over a field K is isomorphic to $\Sigma^* \otimes_T \Sigma$, where Σ denotes the coproduct of a generating set of finite-dimensional right C-comodules and $T = \mathrm{End}_C(\Sigma)$ (here, $(-)^*$ denotes the K-dual functor). In particular, if B is an algebra over K and Σ is the coproduct of a set of representatives of all finite-dimensional right B-modules, then the finite dual coalgebra B^0, consisting of those $\varphi \in B^*$ such that $\mathrm{Ker}\,\varphi$ contains an ideal I such that B/I is of finite dimension over k, is isomorphic to the coalgebra $\Sigma^* \otimes_T \Sigma$, where $T = \mathrm{End}_B(\Sigma)$.

5 Corings with a generating set of small projectives

We return to our functor $\omega : \mathcal{A} \to \mathrm{add}(A_A)$. Assume that $\mathcal{A}$ embeds in an additive category $\mathcal{C}$ and that the coproduct $\Sigma = \bigoplus_{P \in \mathcal{A}} P$ exists in the category $\mathcal{C}$. Assume further that the functor $\omega : \mathcal{A} \to \mathrm{add}(A_A)$ is the restriction of a functor $U : \mathcal{C} \to \mathcal{M}_A$ which commutes with the coproduct $\bigoplus_{P \in \mathcal{A}} P$. Recall from Section 4 that such a category and functor can be always constructed.

Let R be the two-sided ideal of $T = \mathrm{End}_{\mathcal{C}}(\Sigma)$ given by $R = \bigoplus_{P,Q \in \mathcal{A}} T_{PQ}$, where $T_{PQ} = \mathrm{Hom}_{\mathcal{A}}(P, Q)$. We will consider R as a ring with a complete set of pairwise orthogonal idempotents $\{1_P : P \in \mathcal{A}\}$, although R in general does not have a unit (1_P is the element of R which is the identity of $T_P = \mathrm{End}_{\mathcal{A}}(P)$ at P and zero elsewhere). Then Σ is an $(R - A)$-bimodule in a canonical way, and $\Sigma^\dagger = \bigoplus_{P \in \mathcal{A}} P^*$ becomes an $(A - R)$-bimodule. We thus get an A-bimodule $\Sigma^\dagger \otimes_R \Sigma$.

Lemma 5.1. There is a commutative diagram of surjective homomorphisms of A-bimodules

$$\begin{array}{ccc} \bigoplus_{P \in \mathcal{A}} P^* \otimes_{T_P} P & \xrightarrow{\ \Gamma_1\ } & \Sigma^\dagger \otimes_R \Sigma \\ \downarrow{\scriptstyle \Gamma} & \swarrow{\scriptstyle \Gamma_2} & \\ \Sigma^* \otimes_T \Sigma. & & \end{array} \tag{5.1}$$

□

Proof. For each $P \in \mathcal{A}$, let $\iota_{P^*} : P^* \to \Sigma^\dagger$ and $\iota_P : P \to \Sigma$ denote the canonical inclusions. By $\pi_{P^*} : \Sigma^\dagger \to P^*$ and $\pi_P : \Sigma \to P$ we denote the canonical projections. Some straightforward computations show that the map

$$\gamma_P : P^* \otimes_{T_P} P \longrightarrow \Sigma^\dagger \otimes_R \Sigma \quad (\varphi \otimes_{T_P} p \longmapsto \iota_{P^*}(\varphi) \otimes_R \iota_P(p)) \tag{5.2}$$

is a well-defined homomorphism of A-bimodules. The family $\{\gamma_P : P \in \mathcal{A}\}$ determines the homomorphism of A-bimodules Γ_1 in a unique way. In order to show that Γ_1 is surjective, observe that every element of $\Sigma^\dagger \otimes_R \Sigma$ is a sum of elements of the form $\iota_{P^*}(\varphi) \otimes_R \iota_Q(q)$, for some $\varphi \in P^*$, $q \in Q$. But if $Q \neq P$, then $\iota_{P^*}(\varphi) \otimes_R \iota_Q(q) = \iota_{P^*}(\varphi)\iota_P\pi_P \otimes_R \iota_Q(q) = 0$, which proves that Γ_1 is onto.

Now, we consider the map

$$\Gamma_2 : \Sigma^\dagger \otimes_R \Sigma \longrightarrow \Sigma^* \otimes_T \Sigma \quad \big(\iota_{P^*}(\varphi) \otimes_R \iota_Q(q) \longmapsto \varphi\pi_P \otimes_T \iota_Q(q),\ \varphi \in P^*,\ q \in Q\big). \tag{5.3}$$

To check that Γ_2 is well defined, let $t \in T_{MN} = \mathrm{Hom}_{\mathcal{A}}(M, N)$ and compute

$$\begin{aligned} \varphi\pi_P\iota_N t\pi_M \otimes_T \iota_Q(p) &= \varphi\pi_P\iota_N t\pi_M \otimes_T \iota_Q\pi_Q\iota_Q(q) \\ &= \varphi\pi_P\iota_N t\pi_M\iota_Q\pi_Q \otimes_T \iota_Q(q) \\ &= \begin{cases} \varphi t\pi_P \otimes_T \iota_P(q) & \text{if } P = Q = M = N, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{5.4}$$

$$\begin{aligned} \varphi\pi_P \otimes_T \iota_N t\pi_M\iota_Q(p) &= \varphi\pi_P\iota_P\pi_P \otimes_T \iota_N t\pi_M\iota_Q(q) \\ &= \varphi\pi_P \otimes_T \iota_P\pi_P\iota_N t\pi_M\iota_Q(q) \\ &= \begin{cases} \varphi\pi_P \otimes_T \iota_P t(q) & \text{if } P = Q = M = N, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.5}$$

Finally, in the case $P = Q = M = N$, we have

$$\begin{aligned} \varphi t\pi_P \otimes_T \iota_P(q) &= \varphi\pi_P\iota_P t\pi_P \otimes_T \iota_P(q) \\ &= \varphi\pi_P \otimes_T \iota_P t\pi_P\iota_P(q) \\ &= \varphi\pi_P \otimes_T \iota_P t(q). \end{aligned} \tag{5.6}$$

Diagram (5.1) is clearly commutative. By Proposition 4.4, Γ is surjective and, thus, Γ_2 is so. ■

Recall from Lemma 4.2 that the K-submodule $\mathfrak{J}$ of $\bigoplus_{P\in\mathcal{A}} P^* \otimes_{T_P} P$ generated by the set $\{\varphi \otimes_{T_Q} tp - \varphi t \otimes_{T_P} p : \varphi \in Q^*,\ p \in P,\ t \in T_{PQ},\ P, Q \in \mathcal{A}\}$ is a coideal. The factor A-coring is denoted by $\mathfrak{R}(\mathcal{A})$.

Proposition 5.2. The kernel of Γ_1 is the coideal $\mathfrak{J}$ and, hence, $\Sigma^\dagger \otimes_R \Sigma$ can be endowed with a structure of A-coring such that Γ_1 is a homomorphism of A-corings. In this way, the commutative diagram (5.1) induces a commutative diagram of isomorphisms of A-corings

$$\begin{array}{ccc} \mathfrak{R}(\mathcal{A}) & \xrightarrow{\ \simeq\ } & \Sigma^\dagger \otimes_R \Sigma \\ \downarrow{\scriptstyle \simeq} & \swarrow{\scriptstyle \simeq} & \\ \Sigma^* \otimes_T \Sigma. & & \end{array} \tag{5.7}$$

□

Proof. Since $\mathfrak{J}$ is already the kernel of Γ by Proposition 4.4 and Γ_1 is surjective, it suffices to prove that $\mathfrak{J} \subseteq \operatorname{Ker} \Gamma_1$. But this is a straightforward computation. ■

Proposition 5.3. Every $P \in \mathcal{A}$ is a right $\Sigma^\dagger \otimes_R \Sigma$-comodule with structure map

$$\rho_P : P \longrightarrow P \otimes_A \Sigma^\dagger \otimes_R \Sigma \quad \left(p \longmapsto \sum_{\alpha_P} e_{\alpha_P} \otimes_A \iota_{P^*}\left(e^*_{\alpha_P}\right) \otimes_R \iota_P(p) \right). \tag{5.8}$$

Consider $S = \bigoplus_{P,Q \in \mathcal{A}} \operatorname{Hom}_A(P, Q)$ as a ring (not necessarily with unit). Then

$$\begin{aligned} &\operatorname{Hom}_{\Sigma^\dagger \otimes_R \Sigma}(P, Q) \\ &\quad = \left\{ f \in \operatorname{Hom}_A(P, Q) : f \otimes_R \iota_P(p) = 1_Q \otimes_R \iota_Q\left(f(p)\right) \text{ for every } p \in P \right\}. \end{aligned} \tag{5.9}$$

Therefore, the canonical ring extension $R \to S$ factors through $\bigoplus_{P,Q \in \mathcal{A}} \operatorname{Hom}_{\Sigma^\dagger \otimes_R \Sigma}(P, Q)$.

□

Proof. A homomorphism of right A-modules $f : P \to Q$ belongs to $\operatorname{Hom}_{\Sigma^\dagger \otimes_R \Sigma}(P, Q)$ if and only if

$$\sum_{\alpha_Q} e_{\alpha_Q} \otimes_A \iota_{Q^*}\left(e^*_{\alpha_Q}\right) \otimes_R \iota_Q\left(f(p)\right) = \sum_{\alpha_P} f\left(e_{\alpha_P}\right) \otimes_A \iota_{P^*} \otimes_R \iota_P(p), \tag{5.10}$$

for every $p \in P$. Now,

$$Q \otimes_A \Sigma^\dagger = Q \otimes_A \bigoplus_{P \in \mathcal{A}} P^* \cong \bigoplus_{P \in \mathcal{A}} P \otimes_A Q^* \cong \bigoplus_{P \in \mathcal{A}} \operatorname{Hom}_A(P, Q), \tag{5.11}$$

being a direct summand, as a right ideal, of S. Therefore, $Q \otimes_A \Sigma^\dagger \otimes_R \Sigma$ is identified with a K-submodule of $S \otimes_R \Sigma$ for every $Q \in \mathcal{A}$. With these identifications, equation (5.10) is equivalent to $f \otimes_R \iota_P(p) = 1_Q \otimes_R \iota_Q(f(p))$ for every $p \in P$. ■

We now look at the case where $\mathcal{C} = \mathcal{M}^{\mathfrak{C}}$ is the category of right comodules over an A-coring $\mathfrak{C}$ and $\mathcal{A}$ is a small subcategory of $\mathcal{M}^{\mathfrak{C}}$ whose objects are right $\mathfrak{C}$-comodules which are finitely generated and projective as right modules over A. In this way, $\omega : \mathcal{A} \to \mathrm{add}(A_A)$ is the restriction to $\mathcal{A}$ of the underlying functor $U : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$. Diagram (5.7) can now be completed to

$$\begin{array}{ccc} \mathfrak{R}(\mathcal{A}) & \xrightarrow{\ \simeq\ } & \Sigma^{\dagger} \otimes_R \Sigma \\ \downarrow{\scriptstyle \simeq} & \swarrow{\scriptstyle \simeq} & \downarrow{\scriptstyle \mathrm{can}} \\ \Sigma^{*} \otimes_T \Sigma & \xrightarrow[\mathrm{can}]{} & \mathfrak{C}, \end{array} \tag{5.12}$$

where the horizontal can is defined in (4.21) and the vertical one is just the composite that makes the diagram commute. When Σ is a generator for $\mathcal{M}^{\mathfrak{C}}$, both maps are isomorphisms, and the four A-corings are isomorphic by Theorem 4.8.

The functor $- \otimes_R T : \mathcal{M}_R \to \mathcal{M}_T$ has a right adjoint $\cdot R : \mathcal{M}_T \to \mathcal{M}_R$ which sends X onto $XR = \{xr : r \in R\}$. Consider the diagram of functors

$$\begin{array}{ccccc} \mathcal{M}_A & \underset{U}{\overset{-\otimes_A \mathfrak{C}}{\rightleftarrows}} & \mathcal{M}^{\mathfrak{C}} & \underset{-\otimes_T \Sigma}{\overset{\mathrm{Hom}_{\mathfrak{C}}(\Sigma,-)}{\rightleftarrows}} & \mathcal{M}_T \\ & & & {\scriptstyle \mathcal{F}} \searrow \nwarrow {\scriptstyle -\otimes_R \Sigma} & {\scriptstyle -\otimes_R T} \uparrow \downarrow {\scriptstyle \cdot R} \\ & & & & \mathcal{M}_R, \end{array} \tag{5.13}$$

where $\mathcal{F} = \mathrm{Hom}_{\mathfrak{C}}(\Sigma, -)R$ and $- \otimes_R \Sigma \simeq - \otimes_R T \otimes_T \Sigma$. Thus, $- \otimes_R \Sigma$ is left adjoint to $\mathcal{F}$.

Lemma 5.4. There are natural isomorphisms $\mathrm{Hom}_{\mathfrak{C}}(\Sigma, - \otimes_A \mathfrak{C})R \simeq - \otimes_A \Sigma^{\dagger}$ and $\mathcal{F} \simeq \bigoplus_{P \in \mathcal{A}} \mathrm{Hom}_{\mathfrak{C}}(P, -)$. □

Proof. Since $\mathrm{Hom}_{\mathfrak{C}}(\Sigma, - \otimes_A \mathfrak{C}) \simeq \mathrm{Hom}_A(\Sigma, -)$ naturally, we need just to exhibit a natural isomorphism from $- \otimes_A \Sigma^{\dagger}$ to $\mathrm{Hom}_A(\Sigma, -)R$. Now, observe that $- \otimes_A \Sigma^{\dagger} \simeq \bigoplus_{P \in \mathcal{A}} \mathrm{Hom}_A(P, -)$, and this last functor is easily shown to be naturally isomorphic to $\mathrm{Hom}_A(\Sigma, -)R$ via the isomorphism defined at $X \in \mathcal{M}_A$ by the assignment $f \mapsto f \iota_P \pi_P$ for every $f \in \mathrm{Hom}_A(P, X)$ and every $P \in \mathcal{A}$. This last construction also yields a natural isomorphism $\bigoplus_{P \in \mathcal{A}} \mathrm{Hom}_{\mathfrak{C}}(P, -) \simeq \mathcal{F}$. ■

The left $\mathfrak{C}$-comodule structure of every P^* induces a structure of left $\mathfrak{C}$-comodule on $\Sigma^{\dagger} = \bigoplus_{P \in \mathcal{A}} P^*$, which will be used in Proposition 5.5.

Proposition 5.5. The functor $- \otimes_R \Sigma : \mathcal{M}_R \to \mathcal{M}_A$ is left adjoint to the functor $- \otimes_A \Sigma^\dagger : \mathcal{M}_A \to \mathcal{M}_R$, and this adjunction induces one for right $\mathfrak{C}$-comodules, that is, the functor $- \otimes_R \Sigma : \mathcal{M}_R \to \mathcal{M}^{\mathfrak{C}}$ is left adjoint to the functor $-\square_{\mathfrak{C}} \Sigma^\dagger : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_R$. In particular, $\mathcal{F} \simeq -\square_{\mathfrak{C}} \Sigma^\dagger$. □

Proof. The first adjunction follows from Lemma 5.4 and diagram (5.13). For the second, we first observe that $R = R^2$ is a pure ideal of $T = \mathrm{End}_{\mathfrak{C}}(\Sigma)$. Therefore, R is a T-coring in a canonical way, and the category $\mathcal{M}_R$ of unital right R-modules is isomorphic to the category $\mathcal{M}^R$ of right R-comodules. By [13, Proposition 4.2], $\Sigma^\dagger$ is a quasifinite right R-comodule and $- \otimes_R \Sigma$ is left adjoint to $-\square_{\mathfrak{C}} \Sigma^\dagger$ (in fact, what we have is that $- \otimes_R \Sigma$ is isomorphic to the co-hom functor $h_R(\Sigma^\dagger, -)$). ■

The coring homomorphism $\mathrm{can} : \Sigma^\dagger \otimes_R \Sigma \to \mathfrak{C}$ gives a functor $\mathrm{CAN} : \mathcal{M}^{\Sigma^\dagger \otimes_R \Sigma} \to \mathcal{M}^{\mathfrak{C}}$.

Proposition 5.6. Let $\mathcal{A}$ be a set of right $\mathfrak{C}$-comodules such that every comodule in $\mathcal{A}$ is finitely generated and projective as a right A-module. If $R = \bigoplus_{P,Q \in \mathcal{A}} \mathrm{Hom}_{\mathfrak{C}}(P, Q)$, then $R = \bigoplus_{P,Q \in \mathcal{A}} \mathrm{Hom}_{\Sigma^\dagger \otimes_R \Sigma}(P, Q)$. Furthermore, we have a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{M}^{\Sigma^\dagger \otimes_R \Sigma} & \xrightarrow{\ \mathrm{CAN}\ } & \mathcal{M}^{\mathfrak{C}}. \\ & {\scriptstyle - \otimes_R \Sigma} \nwarrow \quad \nearrow {\scriptstyle - \otimes_R \Sigma} & \\ & \mathcal{M}_R & \end{array} \tag{5.14}$$

□

Proof. It follows from Lemma 4.7 and (5.12) that the right $\mathfrak{C}$-comodule structure map $\rho_P : P \to P \otimes_A \mathfrak{C}$ factorizes as

$$\begin{array}{ccc} P & \xrightarrow{\ \rho_P\ } & P \otimes_A \mathfrak{C}, \\ & {\scriptstyle \boldsymbol{\rho}_P} \searrow \quad \nearrow {\scriptstyle P \otimes_A \mathrm{can}} & \\ & P \otimes_A \Sigma^\dagger \otimes_R \Sigma & \end{array} \tag{5.15}$$

where the right $\Sigma^\dagger \otimes_R \Sigma$-comodule map $\boldsymbol{\rho}_P$ is defined in Proposition 5.3. This gives, on the one hand, that $\mathrm{CAN}(\Sigma_{\Sigma^\dagger \otimes_R \Sigma}) = \Sigma_{\mathfrak{C}}$ and, on the other hand, that $\mathrm{Hom}_{\Sigma^\dagger \otimes_R \Sigma}(P, Q) \subseteq \mathrm{Hom}_{\mathfrak{C}}(P, Q)$ for every $P, Q \in \mathcal{A}$. The converse inclusions $\mathrm{Hom}_{\mathfrak{C}}(P, Q) \subseteq \mathrm{Hom}_{\Sigma^\dagger \otimes_R \Sigma}(P, Q)$ follow from Proposition 5.3. Finally, $\mathrm{CAN}(Y \otimes_R \Sigma_{\Sigma^\dagger \otimes_R \Sigma}) = Y \otimes_R \Sigma_{\mathfrak{C}}$ for every $Y \in \mathcal{M}_R$, whence the commutativity of diagram (5.14). ■

A set $\mathcal{A}$ of objects of a Grothendieck category $\mathcal{C}$ is said to be a *generating set of small projectives* if every object in $\mathcal{A}$ is small and $\bigoplus_{P \in \mathcal{A}} P$ is a projective generator for $\mathcal{C}$.

The following theorem generalizes [21, Theorem 3.7], [2, Theorem 5.6], and [9, Theorem 3.2].

Theorem 5.7. Let $\mathfrak{C}$ be an A-coring and $\mathcal{A}$ a set of right $\mathfrak{C}$-comodules. Let $\Sigma = \bigoplus_{P \in \mathcal{A}} P$ and $\Sigma^\dagger = \bigoplus_{P \in \mathcal{A}} P^*$. Consider the ring extension

$$R = \bigoplus_{P,Q \in \mathcal{A}} \mathrm{Hom}_{\mathfrak{C}}(P,Q) \subseteq \bigoplus_{P,Q \in \mathcal{A}} \mathrm{Hom}_A(P,Q) = S. \tag{5.16}$$

The following statements are equivalent:

(i) ${}_A\mathfrak{C}$ is flat and $\mathcal{A}$ is a generating set of small projectives for $\mathcal{M}^{\mathfrak{C}}$;

(ii) ${}_A\mathfrak{C}$ is flat, every comodule in $\mathcal{A}$ is finitely generated and projective as a right A-module, $\mathrm{can} : \Sigma^\dagger \otimes_R \Sigma \to \mathfrak{C}$ is an isomorphism, and $- \otimes_R \Sigma : \mathcal{M}_R \to \mathcal{M}^{\Sigma^\dagger \otimes_R \Sigma}$ is an equivalence of categories;

(iii) every comodule in $\mathcal{A}$ is finitely generated and projective as a right A-module, $\mathrm{can} : \Sigma^\dagger \otimes_R \Sigma \to \mathfrak{C}$ is an isomorphism, and ${}_R\Sigma$ is faithfully flat;

(iv) ${}_A\mathfrak{C}$ is flat, every comodule in $\mathcal{A}$ is finitely generated and projective as a right A-module, $\mathrm{can} : \Sigma^\dagger \otimes_R \Sigma \to \mathfrak{C}$ is an isomorphism, and ${}_RS$ is faithfully flat;

(v) ${}_A\mathfrak{C}$ is flat and $- \otimes_R \Sigma : \mathcal{M}_R \to \mathcal{M}^{\mathfrak{C}}$ is an equivalence of categories. □

Proof. (i)⇔(v). Since ${}_A\mathfrak{C}$ is flat, it follows from [10, Proposition 1.2] that $\mathcal{M}^{\mathfrak{C}}$ is a Grothendieck category. By Proposition 5.5, $- \otimes_R \Sigma$ is left adjoint to $\mathcal{F} : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_R$. By Lemma 5.4, we can apply Freyd's theorem [11, page 120] in conjunction with Gabriel's theorem [12, Proposition II.2] (see also [15]) to get that $\mathcal{F}$ is an equivalence of categories if and only if $\mathcal{A}$ is a generating set of small projectives for $\mathcal{M}^{\mathfrak{C}}$. Therefore, $- \otimes_R \Sigma : \mathcal{M}_R \to \mathcal{M}^{\mathfrak{C}}$ is itself an equivalence of categories if and only if $\mathcal{A}$ is a generating set of small projectives.

(i)⇒(ii). Since ${}_A\mathfrak{C}$ is flat, $\mathcal{M}^{\mathfrak{C}}$ is a Grothendieck category and the forgetful functor $U_A : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is exact [10, Proposition 1.2]. Moreover, it has an exact right adjoint $- \otimes_A \mathfrak{C}$. If $\mathcal{A}$ is a generating set of small projectives, then every comodule in $\mathcal{A}$ is small and projective as a right A-module. By [19, Section 4.11, Lemma 1], every $P \in \mathcal{A}$ is then finitely generated and projective. On the other hand, Corollary 4.10 and (5.12) imply that $\mathrm{can} : \Sigma^\dagger \otimes_R \Sigma \to \mathfrak{C}$ is an isomorphism of A-corings, and so $\mathrm{CAN} : \mathcal{M}^{\Sigma^\dagger \otimes_R \Sigma} \to \mathcal{M}^{\mathfrak{C}}$ is already an equivalence of categories. By Proposition 5.6, we have that $- \otimes_R \Sigma : \mathcal{M}_R \to \mathcal{M}^{\Sigma^\dagger \otimes_R \Sigma}$ is also an equivalence of categories.

(ii)⇒(iii). The functor $- \otimes_R \Sigma : \mathcal{M}_R \to \Sigma^\dagger \otimes_R \Sigma$ is obviously faithful and exact. Since $\Sigma^\dagger \otimes_R \Sigma \cong \mathfrak{C}$ is flat as a left A-module, we have, by [10, Proposition 1.2], that the forgetful functor $U : \mathcal{M}^{\Sigma^\dagger \otimes_R \Sigma} \to \mathcal{M}_A$ is faithful and exact. Therefore, the functor $- \otimes_R \Sigma : \mathcal{M}_R \to \mathcal{M}_A$ is faithful and exact, that is, ${}_R\Sigma$ is a faithfully flat module.

(iii)⇒(v). Since can is an isomorphism, ${}_A\mathfrak{C}$ is a flat module. By Proposition 5.6, we can apply Proposition 5.5 to the infinite comatrix A-coring $\Sigma^\dagger \otimes_R \Sigma$ to obtain that the cotensor product functor $-\square_{\Sigma^\dagger\otimes_R\Sigma}\Sigma^\dagger : \mathcal{M}^{\Sigma^\dagger\otimes_R\Sigma} \to \mathcal{M}_R$ is right adjoint to the functor $-\otimes_R \Sigma : \mathcal{M}_R \to \mathcal{M}^{\Sigma^\dagger\otimes_R\Sigma}$. Since ${}_R\Sigma$ is flat, we have, by a straightforward generalization of [13, Lemma 2.2] to rings with a complete set of pairwise orthogonal idempotents, the isomorphism $(M\square_{\Sigma^\dagger\otimes_R\Sigma}\Sigma^\dagger)\otimes_R\Sigma \cong M\square_{\Sigma^\dagger\otimes_R\Sigma}(\Sigma^\dagger\otimes_R\Sigma) \cong M$, which turns out to be the inverse of the counity of the adjunction at $M \in \mathcal{M}^{\Sigma^\dagger\otimes_R\Sigma}$. Moreover, if $\eta_X : X \to (X\otimes_R\Sigma)\square_{\Sigma^\dagger\otimes_R\Sigma}\Sigma^\dagger$ is the unity of the adjunction at $X \in \mathcal{M}_R$, then an inverse of $\eta_X \otimes_R \Sigma$ is obtained by the isomorphism $((X\otimes_R\Sigma)\square_{\Sigma^\dagger\otimes_R\Sigma}\Sigma^\dagger)\otimes_R\Sigma \cong (X\otimes_R\Sigma)\square_{\Sigma^\dagger\otimes_R\Sigma}(\Sigma^\dagger\otimes_R\Sigma) \cong X\otimes_R\Sigma$. Since ${}_R\Sigma$ is faithful, we get that η_X is an isomorphism, and hence $-\otimes_R\Sigma : \mathcal{M}_R \to \mathcal{M}^{\Sigma^\dagger\otimes_R\Sigma}$ is an equivalence of categories. It follows from Proposition 5.6 that $-\otimes_R\Sigma : \mathcal{M}_R \to \mathcal{M}^{\mathfrak{C}}$ is an equivalence, as $\mathrm{can} : \Sigma^\dagger\otimes_R\Sigma \to \mathfrak{C}$ is an isomorphism.

The proof of the equivalence (iii)⇔(iv) is that of [9, Theorem 3.2], taking the R-bilinear isomorphism $S \cong \Sigma\otimes_A\Sigma^\dagger$ into account. ■

A consequence of Theorem 5.7 is a version for our functor $\omega : \mathcal{A} \to \mathrm{add}(A_A)$ of the faithfully flat descent theorem for a (noncommutative) ring extension $B \subseteq A$ (in categorical words, this is the case when $\mathcal{A}$ has a single object whose image under ω is A).

So, let $\omega : \mathcal{A} \to \mathrm{add}(A_A)$ be a faithful functor, where $\mathcal{A}$ is a small category. Consider the rings $R = \bigoplus_{P,Q\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(P,Q)$ and $\overline{R} = \bigoplus_{P,Q\in\mathcal{A}} \mathrm{Hom}_{\Sigma^\dagger\otimes_R\Sigma}(P,Q)$ and the ring homomorphism $\lambda : R \to \overline{R}$ defined in Proposition 5.3. Finally, put $\Sigma = \bigoplus_{P\in\mathcal{A}} P$.

Lemma 5.8. The homomorphism of A-corings $\mathrm{can} : \Sigma^\dagger\otimes_{\overline{R}}\Sigma \to \Sigma^\dagger\otimes_R\Sigma$ is an isomorphism. □

Proof. A straightforward computation gives that $\mathrm{can}(\iota_{P^*}(\varphi)\otimes_{\overline{R}}\iota_P(p)) = \iota_{P^*}(\varphi)\otimes_R\iota_P(p)$ for every $\varphi \in P^*$, $p \in P$, and $P \in \mathcal{A}$. So, can is the inverse of the obvious map induced by the ring homomorphism $\lambda : R \to \overline{R}$. ■

Theorem 5.9 (faithfully flat descent)**.** With the previous notations and

$$S = \bigoplus_{P,Q\in\mathcal{A}} \mathrm{Hom}_A(P,Q), \tag{5.17}$$

the following statements are equivalent:

(i) ${}_A(\Sigma^\dagger\otimes_R\Sigma)$ is flat, $\mathcal{A}$ becomes a generating set of small projectives for $\mathcal{M}^{\Sigma^\dagger\otimes_R\Sigma}$, and $\lambda : R \to \overline{R}$ is an isomorphism;

(ii) ${}_A(\Sigma^\dagger\otimes_R\Sigma)$ is flat and $-\otimes_R\Sigma : \mathcal{M}_R \to \mathcal{M}^{\Sigma^\dagger\otimes_R\Sigma}$ is an equivalence of categories;

(iii) ${}_R\Sigma$ is faithfully flat;

(iv) ${}_A(\Sigma^\dagger\otimes_R\Sigma)$ is flat (or ${}_R\Sigma$ is flat) and ${}_RS$ is faithfully flat. □

Proof. (i)⇒(ii). The proof follows from Lemma 5.8 and Theorem 5.7.

(ii)⇒(i) and (iii). If ${}_{\mathcal{A}}(\Sigma^{\dagger} \otimes_R \Sigma)$ is flat and $-\otimes_R \Sigma : \mathcal{M}_R \to \mathcal{M}^{\Sigma^{\dagger}\otimes_R \Sigma}$ is an equivalence, then ${}_R\Sigma$ is faithfully flat and the functor $-\otimes_R \Sigma$ sends any projective generator of $\mathcal{M}_R$ onto a projective generator of the Grothendieck category $\mathcal{M}^{\Sigma^{\dagger}\otimes_R \Sigma}$. Therefore, $\Sigma \cong R \otimes_R \Sigma$ is a projective generator for $\mathcal{M}^{\Sigma^{\dagger}\otimes_R \Sigma}$ since R is a projective generator of $\mathcal{M}_R$ [12]. This means that $\mathcal{A}$ is a generating set of small projectives. By Theorem 5.7, $- \otimes_{\overline{R}} \Sigma : \mathcal{M}_{\overline{R}} \to \mathcal{M}^{\Sigma^{\dagger}\otimes_R \Sigma}$ is an equivalence of categories. That λ is an isomorphism of rings follows now from the commutative diagram of functors

$$\begin{array}{ccc} \mathcal{M}_R & \xrightarrow{-\underset{R}{\otimes}\Sigma} & \mathcal{M}^{\Sigma^{\dagger}\otimes_R \Sigma}, \\ \uparrow{\scriptstyle F} & & \uparrow{\scriptstyle \mathrm{CAN}} \\ \mathcal{M}_{\overline{R}} & \xrightarrow{-\underset{\overline{R}}{\otimes}\Sigma} & \mathcal{M}^{\Sigma^{\dagger}\otimes_R \Sigma} \end{array} \tag{5.18}$$

where F is the restriction scalars functor associated to λ, which turns out to be an equivalence of categories.

(iii)⇒(ii). It follows from Proposition 5.3 that $\lambda \otimes_R \Sigma : R \otimes_R \Sigma \to \overline{R} \otimes_R \Sigma$ is an isomorphism, hence λ is an isomorphism as ${}_R\Sigma$ is faithfully flat. The implication follows from Lemma 5.8 and Theorem 5.7.

(iii)⇔(iv). The proof of the equivalence between (iii) and (iv) in Theorem 5.7 works here. ■

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