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CATEGORIES OF COMODULES AND CHAIN COMPLEXES OF MODULES

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Let $\mathscr{L}(A)$ denote the coendomorphism left R-bialgebroid associated to a left finitely generated and projective extension of rings $R \to A$ with identities. We show that the category of left comodules over an epimorphic image of $\mathscr{L}(A)$ is equivalent to the category of chain complexes of left R-modules. This equivalence is monoidal whenever R is commutative and A is an R-algebra. This is a generalization, using entirely new tools, of results by Pareigis and Tambara for chain complexes of vector spaces over fields. Our approach relies heavily on the noncommutative theory of Tannaka reconstruction, and the generalized faithfully flat descent for small additive categories, or rings with enough orthogonal idempotents.

Keywords: Monoidal categories; chain complexes; ring extension; bialgebroids; Tannakian categories.

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1. Introduction

1.1. *Methodology and motivation overviews*

The starting point of this paper is a result due to Pareigis [25, Theorem 18] which asserts that the category of unbounded complexes of vector spaces is monoidally equivalent to the category of left comodules over a certain Hopf algebra which is neither commutative nor cocommutative. Later on, in [29, Theorem 4.4], Tambara associated to every finite-dimensional algebra A over a field $\Bbbk$, a bialgebra $\mathscr{L}(A)$ (termed *coendomorphism bialgebra*) such that the category of left comodules ${}_{\mathscr{L}(A)}\mathsf{Comod}$ is monoidally equivalent to the category $\mathrm{Ch}_+(\Bbbk)$ of chain complexes of $\Bbbk$-vector spaces. The Hopf algebra considered by Pareigis is recovered by choosing $A = \Bbbk \oplus \Bbbk t$ with $t^2 = 0$, i.e. the trivial extension of $\Bbbk$, and localizing the bialgebra $\mathscr{L}(A)$ using a multiplicative set generated by a single grouplike element. The equivalence of categories established by Tambara relies on the use of a variant of the equivalence between simplicial $\Bbbk$-vector spaces and chain complexes of $\Bbbk$-vector spaces, provided by the normalization functor, due to Dold and Kan, see [11, Theorem 1.9, Corollary 1.12] or [18, Theorem 2.4]. The functor that provides such equivalence is given, in some sense, by tensoring chain complexes with the augmented cochain complex $Q_\bullet$ constructed using the Amitsur cosimplicial vector space attached to the $\Bbbk$-algebra A. Note that $Q_\bullet$ is the universal differential graded $\Bbbk$-algebra of A, given by $Q_0 = \Bbbk$, $Q_1 = A$ and $Q_n = K \otimes_A \cdots \otimes_A K$, $(n-1)$-times for $n \geq 2$, where K is the kernel of the multiplication of A. The construction of this functor will be clarified in Sec. 3, see also the forthcoming paragraphs. A different approach to Pareigis's result, using Tannaka reconstruction for several-objects coalgebras, was also given by McCrudden in [24, Examples 6.6, 6.9], where the same coendomorphism bialgebra $\mathscr{L}(A)$ was constructed for a commutative base ring $\Bbbk$ instate of a field.

A monoidal equivalence between categories of chain complexes of (left) modules and left comodules over bialgebroids, allows one freely to transfer at least the model structure of chain complexes, as was described in [21, Sec. 2.3], to left comodules over bialgebroids. This in fact suggests that the categories of comodules over certain bialgebroids could be endowed within a (monoidal) model structure. This indeed is our main motivation for further investigating the relationship between categories of chain complexes of modules and left comodules over bialgebroids.

Let R be an algebra over a commutative ring $\Bbbk$. The purpose of this paper is to investigate the relationship between the category of left comodules over certain left R-bialgebroids, termed *coendomorphism bialgebroids* coming from the $\times_R$-bialgebra defined in [1] (see also [29, Remark 1.7]), and the category of chain complexes of left R-modules. Tambara's results, and in particular Pareigis's one, are then immediate consequences of the general theory developed here. It is noteworthy that our methods can be seen as new and more conceptual even for the case of vector spaces. Indeed, we will see why concretely the trivial extension of rings, already considered by Pareigis, induces the above equivalence of categories. Our approach makes use of the "noncommutative" Tannakian categories theory following the spirit of [10, 4, 20], as well as of the generalized faithfully flat descent for rings with enough orthogonal idempotents stated in [14]. We mean that all (left) bialgebroids arising here come in fact from the noncommutative version of Tannaka reconstruction process which in our approach involves rings with enough orthogonal idempotents.

In the setting of noncommutative Tannakian categories, one basically starts with a small $\Bbbk$-linear monoidal category $(\mathcal{A}, \otimes, \mathbf{1})$ and a faithful monoidal functor[a] from $\mathcal{A}$ to the category of R-bimodules, $\omega : \mathcal{A} \to {}_R\mathsf{Mod}_R$ (the fiber functor), valued in the category $\mathrm{add}({}_RR)$ of finitely generated and projective left R-modules (i.e. locally free sheaves of finite rank). There are several objects under consideration:

$$\Sigma(\omega) = \bigoplus_{\mathfrak{p} \in \mathcal{A}} \omega(\mathfrak{p}), \quad {}^{\vee}\Sigma(\omega) = \bigoplus_{\mathfrak{p} \in \mathcal{A}} {}^{*}\omega(\mathfrak{p}), \quad \mathscr{G}(\mathcal{A}) = \bigoplus_{\mathfrak{p}, \mathfrak{p}' \in \mathcal{A}} \mathrm{Hom}_{\mathcal{A}^o}(\mathfrak{p}, \mathfrak{p}').$$

Here the second is the right R-module direct sum of the left duals while the third is Gabriel's ring with enough orthogonal idempotents, introduced in [16], attached to the opposite category $\mathcal{A}^o$ of $\mathcal{A}$. Using the canonical actions, we consider $\mathcal{L}(\omega) := \Sigma(\omega) \otimes_{\mathscr{G}(A)} {}^{\vee}\Sigma(\omega)$ as an R^{e}-bimodule, where $R^{\mathsf{e}} := R \otimes_{\Bbbk} R^o$ denotes the enveloping ring. A well-known argument in small additive categories says that the object $\mathcal{L}(\omega)$ solves the following universal problems in R-bimodules

$$\mathrm{Nat}(\omega, - \otimes_R \omega) \cong \mathrm{Hom}_{R-R}(\mathcal{L}(\omega), -),$$

$$\mathrm{Nat}(\omega \otimes_R \omega, - \otimes_R (\omega \otimes_R \omega)) \cong \mathrm{Hom}_{R-R}(\mathcal{L}(\omega) \otimes_{R^{\mathsf{e}}} \mathcal{L}(\omega), -),$$

where the R-bimodule structures of $\mathcal{L}(\omega)$ have been chosen properly. It is indeed this solution which allows us to construct a left R-bialgebroid (or a Hopf bialgebroid if desired). Of course there is an obvious (monoidal) functor connecting left unital $\mathscr{G}(\mathcal{A})$-modules and left $\mathcal{L}(\omega)$-comodule, namely

$$\Sigma(\omega) \otimes_{\mathscr{G}(\mathcal{A})} - : {}_{\mathscr{G}(\mathcal{A})}\mathsf{Mod} \longrightarrow {}_{\mathcal{L}(\omega)}\mathsf{Comod}.$$

In the case when each of the left R-modules $\omega(\mathfrak{p})$ is endowed with a structure of left $\mathfrak{C}$-comodule for some R-coring $\mathfrak{C}$ (or certain left R-bialgebroid), there is a map of R-corings, known as a *canonical map*,

$$\mathsf{can}_{\mathscr{G}(\mathcal{A})} : \mathcal{L}(\omega) \longrightarrow \mathfrak{C}$$

defined by using the left $\mathfrak{C}$-coaction of the $\omega(\mathfrak{p})$'s. This homomorphism of corings is not in general bijective, see [14] for more discussions. The associated coinduction functor of the canonical map leads to the following composition of functors

$$ {}_{\mathscr{G}(\mathcal{A})}\mathsf{Mod} \xrightarrow{\Sigma(\omega) \otimes_{\mathscr{G}(\mathcal{A})} -} {}_{\mathcal{L}(\omega)}\mathsf{Comod} \xrightarrow{(-)\mathsf{can}_{\mathscr{G}(\mathcal{A})}} {}_{\mathfrak{C}}\mathsf{Comod}.$$

Indeed this is a conceptual framework that allows us to compare certain categories of $\Bbbk$-linear functors with the categories of comodules over some corings (or left

[a]Our setting requires an isomorphism only at the level of unit. That is, $R \cong \omega(\mathbf{1})$, while $\omega(- \otimes -) \to \omega(-) \otimes_R \omega(-)$ is not necessarily a natural isomorphism.

bialgebroids). For instance, take $R = \Bbbk$ to be a field and A a finite-dimensional $\Bbbk$-algebra. Consider the associated cochain complex $Q_\bullet$ mentioned above and the monoidal $\Bbbk$-linear category $\Bbbk(\mathbb{N})$ generated by the natural number $\mathbb{N}$. There is a fiber functor $\chi : \Bbbk(\mathbb{N}) \to \mathsf{Mod}_{\Bbbk}$ defined by $\chi(n) = Q_n$ on objects and sending the morphism $n \mapsto n+1$ to the differential $\partial : Q_n \to Q_{n+1}$, for every $n \in \mathbb{N}$. Using the previous arguments and notations, we then arrive to the following composition

$$\mathrm{Ch}_+(\Bbbk) \xrightarrow[\cong]{\mathscr{O}} {}_{\mathscr{G}(\Bbbk(\mathbb{N}))}\mathsf{Mod} \xrightarrow{Q\otimes_{\mathscr{G}(\Bbbk(\mathbb{N}))}-} {}_{\mathcal{L}(\chi)}\mathsf{Comod} \xrightarrow{(-)_{\mathsf{can}_{\mathscr{G}(\Bbbk(\mathbb{N}))}}} {}_{\mathscr{L}(A)}\mathsf{Comod} \tag{1.1}$$

where $\mathscr{O}$ is the canonical equivalence between chain complexes of $\Bbbk$-vector spaces and left unital $\mathscr{G}(\Bbbk(\mathbb{N}))$-modules. This in fact is exactly the functor used by Tambara in the proof of [29, Theorem 4.4]. However, the above process of constructing this functor, is actually entirely different from the one presented in [29]. The detailed construction of the functors involved in (1.1), as well as conditions on the extension $R \to A$ under which this composition gives a monoidal equivalence form a part of the main aim of this paper.

1.2. *A brief description of the main results*

Let $\Bbbk$ be a commutative base ring with 1. Fix a morphism of $\Bbbk$-algebras $R \to A$. Assume that ${}_RA$ is finitely generated and projective left R-module with a finite dual basis $\{e_i, {}^*e_i\}_i$. We consider the monoidal functor $- \times_R A : {}_{R^{\mathsf{e}}}\mathsf{Mod}_{R^{\mathsf{e}}} \to {}_R\mathsf{Mod}_R$, where $(- \times_R -)$ is the Sweedler–Takeuchi's product [27, 28], see Sec. 1.3. We obtain that the restriction of this functor to the category of R^{e}-rings (i.e. the category of monoids in ${}_{R^{\mathsf{e}}}\mathsf{Mod}_{R^{\mathsf{e}}}$) admits a left adjoint which we denoted by $\mathscr{L} : R\text{-}\mathsf{Rings} \to R^{\mathsf{e}}\text{-}\mathsf{Rings}$. Then $\mathscr{L}(A)$ the image of A by the functor $\mathscr{L}$, admits a structure of left R-bialgebroid (termed a *coendomorphism bialgebroid*) such that A is a left $\mathscr{L}(A)$-comodule ring, see Proposition 2.1 and Corollary 2.2. Explicitly, the underlying $\Bbbk$-module $\mathscr{L}(A)$ is given by the following quotient of the tensor R^{e}-ring of $A \otimes_{\Bbbk} {}^*A$:

$$\mathscr{L}(A) := \frac{\mathscr{T}_{R^{\mathsf{e}}}(A \otimes {}^*A)}{\langle \sum_i (a \otimes e_i\varphi) \otimes_{R^{\mathsf{e}}} (a' \otimes {}^*e_i) - (aa' \otimes \varphi), (1 \otimes \varphi) - 1 \otimes \varphi(1)^o \rangle_{\{a,a' \in A, \varphi \in {}^*A\}}}. \tag{1.2}$$

On the other hand, we consider the augmented cochain complex of the universal differential graded ring:

$$Q_\bullet : R \xrightarrow{1} A \xrightarrow{\partial} K \xrightarrow{\partial_2} K \otimes_A K \xrightarrow{\partial_3} K \otimes_A K \otimes_A K \longrightarrow \cdots\cdots \tag{1.3}$$

where K denotes the kernel of $A \otimes_R A \to A$, the multiplication of A. We check that this is in fact a cochain complex of left $\mathscr{L}(A)$-comodules whose components are finitely generated and projective left R-modules. This leads to a fiber functor $\chi : \Bbbk(\mathbb{N}) \to {}_R\mathsf{Mod}_R$ defined in the obvious way, as well as to a canonical map

$\mathsf{can}_B : Q \otimes_B {}^\vee Q \to \mathscr{L}(A)$, where $B = \Bbbk^{(\mathbb{N})} \oplus \Bbbk^{(\mathbb{N})}$ (direct sum of copies of $\Bbbk$) is the ring with enough orthogonal idempotents attached to the small category $\Bbbk(\mathbb{N})$. Using the fiber functor χ, we first endow $Q \otimes_B {}^\vee Q$ with a structure of left R-bialgebroid, and then show that can_B is an isomorphism of left R-bialgebroids. This means, in the sense of [14], that Q is actually a Galois object in the category of left comodules. In this way we arrive to our first main result stated below as Theorem 3.3:

Theorem A. *Let $R \to A$ be a $\Bbbk$-algebra map with A finitely generated and projective as left R-module. Consider the associated left R-bialgebroid $\mathscr{L}(A)$ (see Eq. (1.2) above) and the cochain complex $Q_\bullet$ of Eq. (1.3) with its canonical right unital B-action and left $\mathscr{L}(A)$-coaction, where $B = \Bbbk^{(\mathbb{N})} \oplus \Bbbk^{(\mathbb{N})}$. Then the following statements are equivalent*

(1) *The right R-module ${}_{1 \otimes_\Bbbk R^o}\mathscr{L}(A)$ is flat and the functor $Q \otimes_B - : {}_B\mathsf{Mod} \to {}_{\mathscr{L}(A)}\mathsf{Comod}$ is an equivalence of monoidal categories*;
(2) *Q_B is a faithfully flat unital module.*

Obviously, the category of left unital B-module ${}_B\mathsf{Mod}$ is isomorphic to the category of chain complexes of $\Bbbk$-modules. Thus, Theorem A allows one freely to transfer the monoidal model structure described in [21, Sec. 2.3] for chain complexes of $\Bbbk$-modules, to the categories of left comodules over coendomorphism bialgebroids.

Clearly the unit map $\Bbbk \to R$ can be extended to a morphism of rings with the same set of orthogonal idempotents: $B = \Bbbk^{(\mathbb{N})} \oplus \Bbbk^{(\mathbb{N})} \to R^{(\mathbb{N})} \oplus R^{(\mathbb{N})} = C$. By [12], this enables us to consider the usual adjunction between the scalars-restriction functor and the tensor product functor and, in particular, to define a canonical map can_C with codomain a suitable quotient of $\mathscr{L}(A)$. Thus one can try to extend Theorem A to left unital C-modules. In this way we arrive to our second main theorem which is stated below as Theorem 3.9.

Theorem B. *Let $R \to A$ be a $\Bbbk$-algebra map with A finitely generated and projective as left R-module. Consider $\mathscr{L}(A)$ the associated left R-bialgebroid (see Eq. (1.2) above) and $\mathscr{J}$ the coideal of $\mathscr{L}(A)$ generated by the set of elements $\{1_{\mathscr{L}(A)}(r \otimes 1^o - 1 \otimes r^o)\}_{r \in R}$; denote by $\overline{\mathscr{L}(A)} = \mathscr{L}(A)/\mathscr{J}$ the corresponding quotient R-coring. Consider the cochain complex $Q_\bullet$ of Eq. (1.3) with its structures of right unital C-module and left $\overline{\mathscr{L}(A)}$-comodule. Then the following statements are equivalent*

(1) *The right R-module ${}_{1 \otimes_\Bbbk R^o}\overline{\mathscr{L}(A)}$ is flat and the functor $Q \otimes_C - : {}_C\mathsf{Mod} \to {}_{\overline{\mathscr{L}(A)}}\mathsf{Comod}$ is an equivalence of categories*;
(2) *Q_C is a faithfully flat unital module.*

The problem of obtaining an equivalence of categories as above, is then closely linked to the faithfully flat condition on the right unital module Q. This is in fact not at all easy to check. Our third main result, which is a combination of Theorem 3.10 and Proposition 3.13, gives certain homological conditions under which Q becomes flat (or faithfully flat).

Theorem C. *The notations and assumptions are that of Theorem* B. *Assume further that A_R is finitely generated and projective, and the cochain complex $Q_\bullet$ is exact and splits, in the sense that, for every $m \geq 1$, $Q_m = \partial Q_{m-1} \oplus \overline{Q}_m = \mathrm{Ker}(\partial) \oplus \overline{Q}_m$ as right R-modules, for some right R-module $\overline{Q}_m$. Then Q_C is a flat module. In particular, Q_C is faithfully flat in either one of the following cases.*

(1) $A = R \oplus Rt$, $(t^2 = 0)$, *the trivial extension of R.*
(2) $\Bbbk$ *is a field and R is a division $\Bbbk$-algebra.*

As a consequence of Theorems B and C, we get that for every $\Bbbk$-algebra R, there is a left R-bialgebroid $\mathscr{L}$ such that the category of chain complexes of left R-modules is equivalent to the category of left comodules over an epimorphic image of $\mathscr{L}$. In particular, if R is commutative, then this equivalence is in fact a monoidal equivalence.

1.3. *Basic notions and notations*

Given any Hom-set category $\mathcal{C}$, the notation $X \in \mathcal{C}$ means that X is an object of $\mathcal{C}$. The identity morphism of X will be denoted by X itself. The set of all morphisms $f : X \to X'$ in $\mathcal{C}$ is denoted by $\mathrm{Hom}_{\mathcal{C}}(X, X')$. The identity functor of $\mathcal{C}$ is denoted by $\mathrm{id}_{\mathcal{C}}$. We denote the dual (or opposite) category of $\mathcal{C}$ by $\mathcal{C}^o$. The class of all natural transformations between two functors F and G is denoted by $\mathrm{Nat}(F, G)$.

We work over a ground commutative ring with 1 denoted by $\Bbbk$. Up to Sec. 3, all rings under consideration are $\Bbbk$-algebras, and morphisms of rings are morphisms of $\Bbbk$-algebras. Modules are assumed to be unital modules and bimodules are assumed to be central $\Bbbk$-bimodules. For every ring R, these categories are denoted by ${}_R\mathsf{Mod}$ (left modules), Mod_R (right modules) and ${}_R\mathsf{Mod}_R$ (bimodules) respectively. The tensor product over R, is denoted as usual by $- \otimes_R -$. The unadorned symbol $\otimes$ stands for $\otimes_\Bbbk$ the tensor product over $\Bbbk$.

We denote by $\mathrm{Ch}(R)$ the category of *chain complexes* of left R-modules. That is, complexes of left modules of the form:

$$(M_\bullet, d_\bullet) : \cdots \to M_n \xrightarrow{d_n} \cdots \to M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \to \cdots \to M_{-n} \xrightarrow{d_{-n}} \cdots.$$

Let $\mathrm{Ch}_+(R)$ denote the full subcategory of $\mathrm{Ch}(R)$ consisting of positive chain complexes, i.e. complexes of the form:

$$(M_\bullet, d_\bullet) : \cdots \longrightarrow M_n \xrightarrow{d_n} \cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0.$$

From now on, chain complex of left R-modules will stands for an object of the category $\mathrm{Ch}_+(R)$. When R is commutative (i.e. commutative $\Bbbk$-algebra), we will considered this category in a standard way as a monoidal category with unit object the chain complex $R[0]_\bullet$, where $R[0]_0 = R$, and $R[0]_n = 0$, for $n > 0$.

Given an R-bimodule X, its $\Bbbk$-submodule of R-invariant elements is denoted by

$$X^R := \{x \in X | xr = rx, \forall\, r \in R\}.$$

This in fact defines a functor $(-)^R : {}_R\mathsf{Mod}_R \to \mathsf{Mod}_{\mathcal{Z}(R)}$, where $\mathcal{Z}(R)$ is the centre of R. As usual, we use the symbols $\mathrm{Hom}_{R-}(-,-)$, $\mathrm{Hom}_{-R}(-,-)$ and $\mathrm{Hom}_{R-R}(-,-)$ to denote the Hom-functor of left R-linear maps, right R-linear maps and R-bilinear maps, respectively.

For two bimodules ${}_RP_S$ and ${}_RQ_S$ over rings R and S, we will consider the $\Bbbk$-modules of R-linear maps $\mathrm{Hom}_{R-}(P,Q)$ as an S-bimodule with actions:

$$sf : p \mapsto f(ps), \quad \text{and} \quad fs' : p \mapsto f(p)s', \quad \text{for every}$$

$$f \in \mathrm{Hom}_{R-}(P,Q), s, s' \in S, \quad \text{and} \quad p \in P.$$

Similarly, $\mathrm{Hom}_{-S}(P,Q)$ is considered as an R-bimodule with actions:

$$rg : p \mapsto rg(p), \quad \text{and} \quad gr' : p \mapsto g(r'p), \quad \text{for every}$$

$$g \in \mathrm{Hom}_{-S}(P,Q), r, r' \in R, \quad \text{and} \quad p \in P.$$

Under these considerations, the left dual ${}^*X = \mathrm{Hom}_{R-}(X,R)$ of an R-bimodule X, is an R-bimodule, as well as its right dual $X^* = \mathrm{Hom}_{-R}(X,R)$.

Let R be a ring, for any $r \in R$, we denote by r^o the same element regarded as an element in the opposite ring R^o. Let $R^{\mathsf{e}} := R \otimes R^o$ be the enveloping ring of R. Next, we recall the Sweedler–Takeuchi's [27, 28] product on the category of R^{e}-bimodules, usually denoted by $- \times_R -$. So, given an R^{e}-bimodule M, the underlying $\Bbbk$-module M admits several structures of R-bimodule. Among them, we will select in the forthcoming step the following two ones. The first structure is that of the opposite bimodule ${}_{1\otimes R^o}M_{1\otimes R^o}$ which we denote by M^o. That is, the R-biaction on M^o is given by

$$rm^o = (m(1 \otimes r^o)^o), \quad m^o s = ((1 \otimes s^o)m)^o, \quad m^o \in M^o, r, s \in R. \tag{1.4}$$

The second structure is defined by the left R^{e}-module ${}_{R^{\mathsf{e}}}M$. That is, the R-bimodule $M^l = {}_{R\otimes 1^o}M_R$ whose R-biaction is defined by

$$rm^l = ((r \otimes 1^o)m)^l, \quad m^l s = ((1 \otimes s^o)m)^l, \quad m^l \in M^l, r, s \in R. \tag{1.5}$$

Now, given M and N two R^{e}-bimodules, we set

$$M \times_R N := (\mathbf{M} \otimes_R \mathbf{N})^R,$$

where ${}_R\mathbf{M}_R = M^o$ and ${}_R\mathbf{N}_R = {}_{R\otimes 1^o}N_{R\otimes 1^o}$. The elements of $M\times_R N$ are denoted by $\sum_i m_i \times_R n_i$, for $m_i \in M$ and $n_i \in N$. Henceforth, using these notations and given an element $m\times_R n \in M\times_R N$, we have the following equalities

$$\begin{aligned}(m(1\otimes r^o))\times_R n = m\times_R n(r\otimes 1^o), \quad \text{and}\\ ((1\otimes r^o)m)\times_R n = m\times_R (r\otimes 1^o)n,\end{aligned} \tag{1.6}$$

for every $r, s\in R$. The $\Bbbk$-module $M\times_R N$ is actually an R^{e}-bimodule with actions:

$$\begin{aligned}&(p\otimes q^o).(m\times_R n).(r\otimes s^o)\\ &\quad := ((p\otimes 1^o)m(r\otimes 1^o))\times_R ((1\otimes q^o)n(1\otimes s^o)),\end{aligned} \tag{1.7}$$

for every $r, s, p, q\in R$ and $m\times_R n\in M\times_R N$.

On the other hand, since we have $M_R^o = M_R^l$ for every R^{e}-bimodule M, there is a canonical natural transformation (injective at least as $\Bbbk$-linear map)

$$\Theta_{M,N} : M\times_R N \longrightarrow M^l\otimes_R N^l. \tag{1.8}$$

With this product, the R^{e}-bimodule $S\times_R T$ is an R^{e}-ring whenever S and T are. The multiplication of $S\times_R T$ is defined componentwise, and the identity element is given by $1_S\times_R 1_T$.

An *R-ring* S is a monoid in the monoidal category of R-bimodules, equivalently, a $\Bbbk$-algebra map $R\to S$. Dually, an *R-coring* is a comonoid in ${}_R\mathsf{Mod}_R$, which is by definition a three-tuple $(\mathfrak{C},\Delta,\varepsilon)$ consisting of R-bimodule $\mathfrak{C}$ and two R-bilinear maps $\Delta:\mathfrak{C}\to\mathfrak{C}\otimes_R\mathfrak{C}$ (comultiplication), $\varepsilon:\mathfrak{C}\to R$ (counit) satisfying the usual coassociativity and counitary constraints. In contrast with coalgebras, corings admit several convolution rings. For instance, the *right convolution* of an R-coring $\mathfrak{C}$, is the right dual R-bimodule $\mathfrak{C}^*$ whose multiplication is defined by

$$\sigma\cdot\sigma' = \sigma\circ(\sigma'\otimes_R\mathfrak{C})\circ\Delta,$$

for all $\sigma,\sigma'\in\mathfrak{C}^*$, and its unit is the counit ε of $\mathfrak{C}$. A *morphism of R-corings* is an R-bilinear map $\phi:\mathfrak{C}\to\mathfrak{C}'$ such that $\Delta'\circ\phi = (\phi\otimes_R\phi)\circ\Delta$ and $\varepsilon'\circ\phi=\varepsilon$. A left $\mathfrak{C}$-comodule is pair (N,λ_N) consisting of left R-module N and left R-linear map $\lambda_N : N\to\mathfrak{C}\otimes_R N$ (coaction) compatible in the canonical way with comultiplication and counit. A morphism of left $\mathfrak{C}$-comodules is a left R-linear map which is compatible with coactions. We denote by ${}_{\mathfrak{C}}\mathsf{Comod}$ the category of left $\mathfrak{C}$-comodules. Right comodules are similarly defined. Given any morphism of R-corings $\phi:\mathfrak{C}\to\mathfrak{C}'$ one can define, in the obvious way, a functor $(-)_\phi : {}_{\mathfrak{C}}\mathsf{Comod}\to{}_{\mathfrak{C}'}\mathsf{Comod}$ refereed to as the *coinduction functor*.

For more information on comodules as well as the definitions of bicomodules and cotensor product over corings, the reader is referred to [7]. For the notions of bialgebroids and their basic properties, the reader is referred to [3].

We will also consider here rings with enough orthogonal idempotents. These are central $\Bbbk$-modules B with internal multiplication which admit a decomposition of $\Bbbk$-modules $B=\bigoplus_{p\in\mathcal{P}} B1_p = \bigoplus_{p\in\mathcal{P}} 1_pB$, where $\{1_p\}_{p\in\mathcal{P}}\subsetneq B$ is a set of orthogonal idempotents. Module over a ring with enough orthogonal idempotents stands for $\Bbbk$-central and *unital module*. Recall that M is a left unital B-module provided that M has an associative left B-action which satisfies $M = \bigoplus_{p \in \mathcal{P}} 1_p M$. We denote by ${}_B\mathsf{Mod}$ the category of left unital B-modules.

2. Coendomorphism and Comatrices Bialgebroids

2.1. *Coendomorphism bialgebroid and* $\times_R$*-comodules*

In this subsection we recall from [1] the construction of coendomorphism bialgebroids attached to any finitely generated and projective extension of rings. We also recall from [26] the monoidal structure of the category of comodules over the underlying coring of a given left bialgebroid.

A $\times_R$-coalgebra is an R^{e}-bimodule C together with two R^{e}-bilinear maps $\Delta : \mathsf{C} \to \mathsf{C} \times_R \mathsf{C}$ (comultiplication) and $\varepsilon : \mathsf{C} \to \mathrm{End}_{\Bbbk}(R)$ (counit) which satisfy the coassociativity and counitary properties in the sense of [28, Sec. 4, Definition 4.5], see also [6, 26]. A $\times_R$-coalgebra C is said to be an $\times_R$-*bialgebra* provided that comultiplication and counit are morphisms of R^{e}-rings.

A left $\times_R$-C-*comodule*, is a pair (X, λ_X) consisting of an R-bimodule X and an R-bilinear map $\lambda_X : X \to \mathsf{C} \times_R X$ satisfying, in some sense, the coassociativity and counitary axioms. Morphism between left $\times_R$-C-comodules are R-bilinear maps compatible in the obvious way with the left $\times_R$-C-coactions. This leads to the definition of the category of left $\times_R$-C-comodules. When C is a $\times_R$-bialgebra, this category becomes a monoidal category [26, Proposition 5.6], and the forgetful functor to the category of R-bimodules is a monoidal functor. There is a strong relation which will be clarified in the sequel, between the category of left $\times_R$-comodules over an $\times_R$-bialgebra and the category of left comodules over the underlying R-coring whose structure maps are

$$\mathsf{C} \longrightarrow \mathsf{C} \times_R \mathsf{C} \xrightarrow{\Theta_{C,C}} \mathsf{C}^l \otimes_R \mathsf{C}^l, \qquad \mathsf{C} \xrightarrow{\varepsilon(-)(1_R)} R,$$

where $\Theta_{-,-}$ is the natural transformation of Eq. (1.8).

Let A be an R-ring, that is, a $\Bbbk$-algebra map $R \to A$, and denote by ${}^*A = \mathrm{Hom}_{\Bbbk}(A, \Bbbk)$ the dual of the $\Bbbk$-module $A_{\Bbbk}$. We consider the tensor product $A \otimes {}^*A$ as an R^{e}-bimodule in the following way

$$\begin{gathered}(r \otimes s^o) \cdot (a \otimes \varphi) \cdot (p \otimes q^o) = (rap) \otimes (q\varphi s),\\ p, q, r, s \in R, \quad \text{and} \quad a, \in A, \varphi \in {}^*A,\end{gathered} \tag{2.1}$$

where A and ${}^*A$ are considered as R-bimodules in the usual way.

Assume that ${}_RA$ is finitely generated and projective module and fix a left dual basis $\{e_i, {}^*e_i\}_{1 \le i \le n}$. Define the R^{e}-ring $\mathscr{L}(A)$ by the quotient algebra

$$\mathscr{L}(A) = \mathscr{T}_{R^{\mathsf{e}}}(A \otimes {}^*A)/\mathscr{I} \tag{2.2}$$

where $\mathscr{T}_{R^{\mathsf{e}}}(A \otimes {}^*A) = \bigoplus_{n \in \mathbb{N}} (A \otimes {}^*A)^{\otimes^n_{R^{\mathsf{e}}}}$ is the tensor R^{e}-ring of the R^{e}-bimodule $A \otimes {}^*A$ and where $\mathscr{I}$ is the two-sided ideal generated by the set

$$\left\{ \sum_i ((a \otimes e_i\varphi) \otimes_{R^{\mathsf{e}}} (a' \otimes {}^*e_i)) - (aa' \otimes \varphi); 1_R \otimes \varphi(1_A)^o - (1_A \otimes \varphi) \right\}_{a,a' \in A, \varphi \in {}^*A}. \tag{2.3}$$

We denote by

$$\pi_A : \mathscr{T}_{R^{\mathsf{e}}}(A \otimes {}^*A) \to \mathscr{L}(A) \tag{2.4}$$

the canonical projection. From now on, given a homogeneous elements $(a \otimes \varphi) \in \mathscr{T}_{R^{\mathsf{e}}}(A \otimes {}^*A)$ of degree one, we denote by $\pi_A(a \otimes \varphi)$ its image in $\mathscr{L}(A)$. Next, we recall the structure of $\times_R$-bialgebra of the object $\mathscr{L}(A)$, which is denoted by $a_R(A, A)$ in [29, Remark 1.7]. The underlying structure of an R^{e}-ring, is given by the following composition of algebra maps

$$R^{\mathsf{e}} \xrightarrow{\iota_0} \mathscr{T}_{R^{\mathsf{e}}}(\mathscr{L}(A)) \xrightarrow{\pi_A} \mathscr{L}(A),$$

where ι_m denotes the canonical R^{e}-bilinear injection in degree $m \geq 0$.

Proposition 2.1. *Let A be an R-ring which is finitely generated and projective as left R-module with dual basis $\{({}^*e_i, e_i)\}_i$. Then $\mathscr{L}(A)$ is a $\times_R$-bialgebra with structure maps*

$$\begin{array}{ll} \mathscr{L}(A) \xrightarrow{\quad\Delta\quad} \mathscr{L}(A) \times_R \mathscr{L}(A), & \mathscr{L}(A) \xrightarrow{\varepsilon} \mathrm{End}_{\Bbbk}(R), \\ \pi_A(a \otimes \varphi) \longmapsto \sum_j \pi_A(a \otimes {}^*e_j) \times_R \pi_A(e_j \otimes \varphi), & \pi_A(a \otimes \varphi) \longmapsto [r \mapsto \varphi(ar)]. \end{array}$$

Proof. This is [1, Proposition 1.3.6]. $\square$

The relation between the R-ring structure of A and the $\times_R$-bialgebra structure of $\mathscr{L}(A)$, is expressed as follows.

Corollary 2.2. *Let A be an R-ring such that ${}_RA$ is finitely generated and projective and $\mathscr{L}(A)$ the associated $\times_R$-bialgebra defined in Proposition 2.1. Then A is a left $\times_R$-$\mathscr{L}(A)$-comodule R-ring, that is, A admits a left $\times_R$-$\mathscr{L}(A)$-coaction*

$$\lambda_A : A \longrightarrow \mathscr{L}(A) \times_R A, \quad \left(a \longmapsto \sum_j \pi_A(a \otimes {}^*e_j) \times_R e_j \right)$$

which is also a morphism of R-rings.

Proof. This is [1, Corollary 1.3.7]. $\square$

The $\times_R$-bialgebra $\mathscr{L}(A)$ defined in Proposition 2.1 is refereed to as *coendomorphism R-bialgebroid* since by [6, Theorem 3.1], $\mathscr{L}(A)$ is in fact a (left) bialgebroid whose structure of R^{e}-ring is the map

$$\pi_A \circ \iota_0 : R^{\mathsf{e}} \longrightarrow \mathscr{L}(A),$$

and its structure of R-coring is given as follows. The underlying R-bimodule is $\mathscr{L}(A)^l = {}_{R^{\mathsf{e}}}\mathscr{L}(A)$, the comultiplication and counit are given by

$$\begin{aligned} \Delta : \mathscr{L}(A)^l &\longrightarrow \mathscr{L}(A)^l \otimes_R \mathscr{L}(A)^l, \\ &\left(\pi_A(a \otimes \varphi) \longmapsto \sum_i \pi_A(a \otimes {}^*e_i) \otimes_R \pi_A(e_i \otimes \varphi)\right), \end{aligned} \tag{2.5}$$

$$\varepsilon : \mathscr{L}(A)^l \longrightarrow R, \quad (\pi_A(a \otimes \varphi) \longmapsto \varphi(a)). \tag{2.6}$$

Here is an example of coendomorphism bialgebroid which will be used in the sequel. For more examples of this object, the reader is referred to [1, Sec. 2].

Example 2.3. Let $A = R \oplus Rt$ be the trivial generalized R-ring, i.e. the R-ring which is free as left R-module with basis $1 = (1, 0)$ and $\mathfrak{t} = (0, t)$ such that $\mathfrak{t}^2 = 0$. Using (2.2) and Proposition 2.1, we can easily check that $\mathscr{L}(A)$ is an R-bialgebroid generated by the image of R^{e} and two R^{e}-invariant elements $\{x, y\}$ subject to the relations $xy + yx = 0$, $x^2 = 0$. The comultiplication and counit of it underlying R-coring are given by

$$\Delta(x) = x \otimes_R 1 + y \otimes_R x, \quad \varepsilon(x) = 0,$$

$$\Delta(y) = y \otimes_R y, \quad \varepsilon(y) = 1.$$

The ring A is a left $\mathscr{L}(A)$-comodule ring with coaction: $\lambda : A \to \mathscr{L}(A) \otimes_R A$ sending

$$\lambda(1_A) = 1_{\mathscr{L}(A)} \otimes_R 1_A, \quad \lambda(\mathfrak{t}) = x \otimes_R 1_A + y \otimes_R \mathfrak{t},$$

extended by R-linearity to the whole set of elements of A.

In [26] it was shown that the category of left $\times_R$-comodules over an $\times_R$-bialgebra is a monoidal category such that the forgetful functor to the category of R-bimodules is a monoidal functor. What we will need in the sequel is a monoidal structure on the category of left $\mathscr{L}(A)$-comodules where $\mathscr{L}(A)$ is viewed as an R-coring with structure maps (2.5) and (2.6). The following lemma is a consequence of [26, Proposition 5.6], see also [3, 3.6].

Lemma 2.4. *Let $\mathscr{L}$ be any left R-bialgebroid. Then the category of left $\times_R$-$\mathscr{L}$-comodule is isomorphic to the category of left $\mathscr{L}^l$-comodules over the underlying R-coring $\mathscr{L}^l$. In particular, the category of left $\mathscr{L}^l$-comodules inherits a monoidal structure with unit object $(R, R \to \mathscr{L}^l)$ and the left forgetful functor*

$U : {}_{\mathscr{L}^l}\mathsf{Comod} \to {}_R\mathsf{Mod}$ factors throughout a monoidal functor into the category of R-bimodules. Thus, we have a commutative diagram

$$\begin{tikzcd} {}_{\mathscr{L}^l}\mathsf{Comod} \arrow[rr, "U"] \arrow[dr, dashed] & & {}_R\mathsf{Mod} \\ & {}_R\mathsf{Mod}_R \arrow[ur] & \end{tikzcd}$$

where the dashed arrow is a monoidal functor.

Summing up, given two left $\mathscr{L}^l$-comodules (X, λ_X) and (Y, λ_Y), using Lemma 2.4, we can consider $(X \otimes_R Y, \lambda_{X\otimes_R Y})$ as a left $\mathscr{L}^l$-comodule with coaction

$$\lambda_{X\otimes_R Y} : X \otimes_R Y \to \mathscr{L}^l \otimes_R X \otimes_R Y,$$
$$\left(x \otimes_R y \longmapsto \sum_{(x),(y)} (x_{(-1)}y_{(-1)})^l \otimes_R (x_{(0)} \otimes_R y_{(0)}) \right), \tag{2.7}$$

where we have considered X as R-bimodule with the right R-action given by the action

$$xr = \sum_{(x)} \varepsilon\left(x_{(-1)}(r \otimes 1^o)\right) x_{(0)}, \quad \text{for every } r \in R \text{ and } x \in X. \tag{2.8}$$

2.2. *The complex of left $\mathscr{L}$-comodules $Q_\bullet$*

Keep the assumptions and notations of Sec. 2.1, that is, we are considering an R-ring A over a fixed $\Bbbk$-algebra R. Let us denote by

$$K = \mathrm{Ker}(A \otimes_R A \xrightarrow{\ \mu\ } A\ )$$

the kernel of the multiplication μ of A with canonical derivation

$$\begin{aligned} A &\xrightarrow{\qquad \partial \qquad} K \\ a &\longmapsto (\partial a = 1 \otimes_R a - a \otimes_R 1). \end{aligned}$$

The associated cochain complex is denoted by

$$Q_\bullet : R \xrightarrow{\partial_0 = 1} A \xrightarrow{\partial_1 = \partial} K \xrightarrow{\partial_2} K \otimes_A K \xrightarrow{\partial_3} K \otimes_A K \otimes_A K \longrightarrow \cdots,$$

where $\partial_n : Q_n \to Q_{n+1}$ sends $a_0\partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1}$ to $\partial a_0 \otimes_A \partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1}$, $n \geq 2$.

The following lemma, which will play a key role in Sec. 3.3, characterizes a split ring extension $R \to A$ (in Mod_R) in terms of the cochain complex $Q_\bullet$.

Lemma 2.5. *Let A be any R-ring. Then the following conditions are equivalent.*

(i) *The unit $u : R \to A$ is a split monomorphism in Mod_R.*

(ii) *The cochain complex* $Q_\bullet$ *is exact and splits, in the sense that, for every* $m \geq 1$, *there is a right* R*-module* $\overline{Q}_m$ *such that* $Q_m = \partial Q_{m-1} \oplus \overline{Q}_m = \mathrm{Ker}(\partial) \oplus \overline{Q}_m$, *as right* R*-modules.*

Proof. (ii) $\Rightarrow$ (i). It is trivial.
(i) $\Rightarrow$ (ii). Let us denote by $u^c : A \to \overline{A}$ the cokernel of $u : R \to A$ in ${}_R\mathsf{Mod}_R$. Put $\Omega_0 := R$, $\Omega_1 := A$, and $\Omega_n := A \otimes_R \overline{A} \otimes_R \cdots \otimes_R \overline{A}$, $(n-1)$-fold $\overline{A}$, for $n \geq 2$. Consider now the following split exact sequence of right R-modules

$$0 \longrightarrow \overline{A}^{\otimes_R n} \xrightarrow{\gamma_n} A \otimes_R \overline{A}^{\otimes_R n} \longrightarrow \overline{A}^{\otimes_R n+1} \longrightarrow 0,$$

where $\gamma_n = u \otimes_R \overline{A}^{\otimes_R n}$, for $n \geq 1$. In view of this, we have a split exact cochain complex of right R-modules

$$\Omega_\bullet : \Omega_0 \xrightarrow{d_0} \Omega_1 \xrightarrow{d_1} \Omega_2 \xrightarrow{d_2} \Omega_3 \longrightarrow \cdots,$$

with differential $d_0 = u$, $d_1 = \gamma_1 \circ u^c$, $d_n = \gamma_n \circ (u^c \otimes_R \overline{A}^{\otimes_R n-1})$, for $n \geq 2$. Since Ω_2 is the cokernel of the map $A \otimes_R u$, and the later split by μ the multiplication of A, we obtain the following split exact sequence of R-bimodules

$$0 \longrightarrow A \xrightarrow{A \otimes_R u} A \otimes_R A \xrightarrow{A \otimes_R u^c} \Omega_2 \longrightarrow 0.$$

This gives the split exact sequence

$$0 \longrightarrow \Omega_2 \longrightarrow A \otimes_R A \xrightarrow{\mu} A \longrightarrow 0.$$

Thus we have an R-bilinear isomorphism $\omega_2 : \Omega_2 \to Q_2 = K$. Henceforth, there is an unique A-bimodule structure on Ω_2 which renders ω_2 an A-bilinear isomorphism, namely

$$a \cdot (x \otimes_R \overline{y}) \cdot b = ax \otimes_R \overline{yb} - axy \otimes_R \overline{b}, \quad \text{for every } a, x, y, b \in A,$$

wherein the notation $u^c(z) = \overline{z}$, for every $z \in A$, has been used. Define iteratively $\omega_n : \Omega_n \to Q_n$, for all $n \geq 3$, as the composition

$$\begin{aligned} \Omega_n &= \Omega_{n-1} \otimes_R \overline{A} \cong \Omega_{n-1} \otimes_A (A \otimes_R \overline{A}) \\ &= \Omega_{n-1} \otimes_A \Omega_2 \xrightarrow{\omega_{n-1} \otimes_A \omega_2} Q_{n-1} \otimes_A K = K^{\otimes_A n-1} = Q_n. \end{aligned}$$

By construction, $\omega_\bullet : (\Omega_\bullet, d_\bullet \to (Q_\bullet, \partial_\bullet)$ is a morphism of complexes of R-bimodules. We leave to the reader to check that $\omega_\bullet$ is in fact an isomorphism of cochain complexes. Now, since $(\Omega_\bullet, d_\bullet)$ is split exact in right R-modules, then so is $(Q_\bullet, \partial_\bullet)$. $\square$

Remark 2.6. In the case of finitely generated and projective extension of rings, the left version of condition (i) in Lemma 2.5 implies that ${}_RA$ is in fact faithfully flat module (see, for example [5, Chap. I, Proposition 9, p. 51]). In this case, one can easily show that $Q_\bullet \otimes_R A$ is homotopically trivial which by [17, Théorème 2.4.1] gives condition (ii). In this way, Lemma 2.5 can be seen as a generalization of [2, Propositions 6.1, 6.2].

The convolution product on the left dual chain complex of $Q_\bullet$ is given as follows: For every $\varphi \in {}^*Q_n$ and $\psi \in {}^*Q_m$ with $n, m \geq 1$, we have a left R-linear map

$$\begin{aligned} \varphi \star \psi : Q_{n+m} &\longrightarrow R, \\ x \otimes_A \partial(a) \otimes_A y &\longmapsto \varphi(x\psi(ay)) - \varphi(xa\psi(y)), \end{aligned} \tag{2.9}$$

where $x \in Q_n$, $y \in Q_m$ and $a \in A$. The convolution product with zero degree element is just the left and right R-actions of ${}^*Q_n$, for every $n \geq 1$, namely

$$\begin{aligned} r \star \varphi : Q_n &\longrightarrow R, & \varphi \star s : Q_n &\longrightarrow R, \\ x &\longmapsto \varphi(xr), & x &\longmapsto \varphi(x)s, \end{aligned} \tag{2.10}$$

for every elements $r, s \in R$ and $\varphi \in {}^*Q_n$.

Remark 2.7. The convolution product defined in (2.9) and (2.10) derives from the structure of comonoid of the cochain complex $Q_\bullet$ viewed as an object in the monoidal category of cochain complexes of R-bimodules. Precisely, the identity map $A \otimes_R \cdots \otimes_R A = A^{\otimes_R n} = A^{\otimes_R p} \otimes_R A^{\otimes_R q}$, for $p + q = n$, rereads as a map $Q_n \to Q_p \otimes_R Q_q$ sending $x \otimes_A \partial a \otimes_A y \mapsto x \otimes_R ay - xa \otimes_R y$, for every $x \in Q_p$, $a \in A$ and $y \in Q_q$. Thus $Q = \bigoplus_{n \geq 0} Q_n$ has a structure of differential R-coring in the sense of [9, pp. 6, 7]. Since each Q_n is finitely generated and projective left R-module (see Lemma 2.8 below, of course under the same assumption for the left module ${}_RA$), the comultiplication of Q is transferred to the graded left dual ${}^\vee Q = \bigoplus_{n \geq 0} {}^*Q_n$ which gives a multiplication defined explicitly by (2.9) and (2.10). A comonoidal structure on $Q_\bullet$ could also be obtained by transferring some comonoidal structure of the Amitsur cosimplicial object of R-bimodules induced by A (see [2]), using for this the normalization functor and it structure of comonoidal functor obtained from Eilenberg–Zilber Theorem, see [23, Theorem 8.1, Exercise 4, p. 244] (of course in their dual form). It seems that Tambara's approach [29] runs in this direction. Anyway this approach uses a slightly variant of the category of cosimplicial groups endowed with some monoidal structure which is not the usual one. Since our methods run in a different way, we will not make use of the normalization process here.

In all what follows, we will fix a (left) finitely generated and projective extension $R \to A$ with dual basis $\{e_i, {}^*e_i\}_{1 \leq i \leq n}$. We will denote by $\mathscr{L} := \mathscr{L}(A)$ the corresponding left R-bialgebroid coming from Proposition 2.1, and by π the projection π_A defined in (2.4).

Using this dual basis, one can check that ${}_RQ_2 = {}_RK$ is finitely generated and projective module whose dual basis is given by the set $\{(e_i \partial e_j, {}^*e_i \star {}^*e_j)\}_{i,j}$. Moreover, we have the following.

Lemma 2.8. *Each Q_n, $n \geq 0$, is finitely generated and projective as left R-module. Furthermore, if $\{(\omega_{n,\alpha}, {}^*\omega_{n,\alpha})\}_\alpha$ is a dual basis for Q_n with $n \geq 1$, then $\{(\omega_{n,\alpha} \otimes_A$*

$\partial\omega_{m,\beta}, {}^*\omega_{n,\alpha} \star {}^*\omega_{m,\beta})\}_{\alpha,\beta}$ is a dual basis for Q_{n+m}, while $\{(\omega_{n,\alpha} \otimes_A \omega_{m,\beta}, {}^*\omega_{n,\alpha} \star \partial^*\omega_{m,\beta})\}_{\alpha,\beta}$ is a dual basis for Q_{n+m-1} when $m \geq 2$.

Proof. Straightforward. $\square$

The cochain complex $Q_\bullet$ is actually a complex of left $\mathscr{L}$-comodules.

Proposition 2.9. *The cochain complex $Q_\bullet$ is a complex of left $\mathscr{L}$-comodules. For $n = 0$, the coaction is given by $(R \to \mathscr{L}, r \mapsto \pi(r \otimes 1^o))$ and, for $n \geq 1$, by $\lambda_n : Q_n \to \mathscr{L} \otimes_R Q_n$ sending*

$$a_0\partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1} \longmapsto \sum_{i_0,i_1,\ldots,i_{n-1}} \pi(a_0 \otimes {}^*e_{i_0}) \cdots \pi(a_{n-1} \otimes {}^*e_{i_{n-1}}) \otimes_R (e_{i_0}\partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}}). \tag{2.11}$$

Proof. The statement is trivial for $n = 0$. For $n \geq 1$, the coassociativity of λ_n is deduced using Lemma 2.8 which assert that $\{(e_{i_0}\partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}}, {}^*e_{i_0} \star \cdots \star {}^*e_{i_{n-1}}\}_{i_0,i_1,\cdots,i_{n-1}}$ is a dual basis for Q_n. Here each ${}^*e_{i_0} \star \cdots \star {}^*e_{i_{n-1}}$ is the n-fold convolution product defined in (2.9). The rest of the proof uses the fact that each coaction λ_n, $n \geq 1$, satisfies the equality

$$\lambda_n(\partial b_1 \otimes_A \cdots \otimes_A \partial b_{n-1}) = \sum_{i_1,\ldots,i_{n-1}} \pi(b_1 \otimes {}^*e_{i_1}) \cdots \pi(b_{n-1} \otimes {}^*e_{i_{n-1}}) \otimes_R (\partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}}). \tag{2.12}$$

$\square$

The following lemma will be used in the sequel.

Lemma 2.10. *Given two elements $u_n = a_0\partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1} \in Q_n$ and $u_m = b_0\partial b_1 \otimes_A \cdots \otimes_A \partial b_{m-1} \in Q_m$ with $n, m \geq 1$. Then*

$$\lambda_{n+m-1}(u_n \otimes_A u_m) = \sum_{i_0,\ldots,i_{n-1},j_0,\ldots,j_{m-1}} \otimes(\pi(a_0 \otimes {}^*e_{i_0}) \cdots \pi(a_{n-1} \otimes {}^*e_{i_{n-1}})\pi(b_0 \otimes {}^*e_{j_0}) \cdots \pi(b_{m-1} \otimes {}^*e_{j_{m-1}})) \otimes_R(e_{i_0}\partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}} \otimes_A e_{j_0}\partial e_{j_1} \otimes_A \cdots \otimes_A \partial e_{j_{m-1}}).$$

Furthermore, for every $u \in Q_n, n \geq 1$ and $v \in Q_m$, $m \geq 1$, we have

$$\lambda_{n+m-1}(u \otimes_A v) = \sum u_{(-1)}v_{(-1)} \otimes_R (u_{(0)} \otimes_A v_{(0)}),$$

and

$$\lambda_{n+m}(u \otimes_A \partial v) = \sum u_{(-1)}v_{(-1)} \otimes_R (u_{(0)} \otimes_A \partial v_{(0)}),$$

where Sweedler's notation for coactions is used.

Proof. The proof of the first claim is based upon the observation that the coaction of any $Q_k = K \otimes_A \cdots \otimes_A K$ ($(k-1)$-times), with $k \geq 2$, is induced from that of $A \otimes_R \cdots \otimes_R A$ (k-times). The later is a left $\mathscr{L}$-comodule, by Corollary 2.2 and Lemma 2.4, using the coactions described in (2.7). The last statement is deduced from the first one by left R-linearity. $\square$

2.3. *The infinite comatrix bialgebroid induced by $Q_\bullet$*

Let $Q_\bullet$ be the cochain complex of $\mathscr{L}$-comodules considered in Proposition 2.9. In this subsection we will construct a left bialgebroid associated to $Q_\bullet$ and a canonical map from this left bialgebroid to $\mathscr{L}$. First we recall from [13, 14] the notion of infinite comatrix coring and the canonical map. A different approach to this notion can be found in [30, 8, 19]. We should mention here that this object coincides with the one already constructed in the context of Tannaka–Krein duality over fields or commutative rings, see [10, 4, 22, 20], see also [24]. However, the description given in [14] in terms of tensor product over a ring with enough orthogonal idempotents, seems easier to handle from a computational point of view.

Let $\mathcal{A}$ be a small full sub-category of an additive category. Following [16, p. 346], we can associate to $\mathcal{A}$ the ring with enough orthogonal idempotents $S = \bigoplus_{\mathfrak{p},\mathfrak{p}' \in \mathcal{A}} \mathrm{Hom}_{\mathcal{A}^o}(\mathfrak{p}, \mathfrak{p}')$, where $\mathcal{A}^o$ is the opposite category of $\mathcal{A}$. The category of left unital S-module is denoted by ${}_S\mathsf{Mod}$.

Let us denote by $\mathrm{add}({}_RR)$ the full sub-category of ${}_R\mathsf{Mod}$ consisting of all finitely generated and projective left R-modules. Let $\chi : \mathcal{A} \to \mathrm{add}({}_RR)$ be a faithful functor, refereed to as *fiber functor*. We denote by $\mathfrak{p}^\chi$ the image of $\mathfrak{p} \in \mathcal{A}$ under χ or by $\mathfrak{p}$ itself if no confusion arises. Consider the left R-module direct sum of the $\mathfrak{p}$'s: $\Sigma = \bigoplus_{\mathfrak{p} \in \mathcal{A}} \mathfrak{p}$ (i.e. $\Sigma = \bigoplus_{\mathfrak{p} \in \mathcal{A}} \mathfrak{p}^\chi$) and the right R-module direct sum of their duals: ${}^\vee\Sigma = \bigoplus_{\mathfrak{p} \in \mathcal{A}} {}^*\mathfrak{p}$. It is clear that ${}^\vee\Sigma$ is a left unital S-module while Σ is a right unital S-module. In this way Σ becomes an (R,S)-bimodule and ${}^\vee\Sigma$ an (S,R)-bimodule. Then $\Sigma \otimes_S {}^\vee\Sigma$ is now an R-bimodule whose elements are described as a finite sum of diagonal ones, i.e. of the form $\iota_\mathfrak{p}(u_\mathfrak{p}) \otimes_S \iota_{^*\mathfrak{p}}(\varphi_\mathfrak{p})$ where $(u_\mathfrak{p}, \varphi_\mathfrak{p}) \in \mathfrak{p}^\chi \times ({}^*\mathfrak{p}^\chi)$ and ι_- are the canonical injections in ${}^\vee\Sigma$ and Σ. From now on, we will write $u_\mathfrak{p} \otimes_S \varphi_\mathfrak{p}$ instead of $\iota_\mathfrak{p}(u_\mathfrak{p}) \otimes_S \iota_{^*\mathfrak{p}}(\varphi_\mathfrak{p})$ to denote a generic element of $\Sigma \otimes_S {}^\vee\Sigma$.

This bimodule admits a structure of an R-coring given by the following comultiplication

$$\begin{aligned} \Delta : \Sigma \otimes_S {}^\vee\Sigma &\longrightarrow (\Sigma \otimes_S {}^\vee\Sigma) \otimes_R (\Sigma \otimes_S {}^\vee\Sigma), \\ u_\mathfrak{p} \otimes_S \varphi_\mathfrak{p} &\longmapsto \textstyle\sum_i u_\mathfrak{p} \otimes_S {}^*u_{\mathfrak{p},i} \otimes_R u_{\mathfrak{p},i} \otimes_S \varphi_\mathfrak{p}, \end{aligned} \tag{2.13}$$

where, for a fixed $\mathfrak{p} \in \mathcal{A}$, the finite set $\{(u_{\mathfrak{p},i}, {}^*u_{\mathfrak{p},i})\}_i \subset \mathfrak{p} \times {}^*\mathfrak{p}$ is a left dual basis of the left R-module $\mathfrak{p}$. The counit is just the evaluating map. Note that this comultiplication is independent from the chosen bases. With this structure $\Sigma \otimes_S {}^\vee\Sigma$ is refereed to as the *infinite comatrix coring* associated to the small category $\mathcal{A}$ and the fiber functor χ. On the other hand, each of the left R-modules $\mathfrak{p}^\chi$ is actually a left $(\Sigma \otimes_S {}^{\vee}\Sigma)$-comodule whose coaction, using the above notation, is given by

$$\widetilde{\lambda}_{\mathfrak{p}} : \mathfrak{p} \longrightarrow \Sigma \otimes_S {}^{\vee}\Sigma \otimes_R \mathfrak{p}, \quad \left(u \longmapsto \sum_i u \otimes_S {}^{*}u_{\mathfrak{p},i} \otimes_R u_{\mathfrak{p},i} \right). \tag{2.14}$$

Another description of the infinite comatrices is given in [14, Proposition 5.2] which establishes an isomorphism of R-bimodules

$$\Sigma \otimes_B {}^{\vee}\Sigma \cong \frac{\bigoplus_{\mathfrak{p}\in\mathcal{A}} \mathfrak{p} \otimes_{T_{\mathfrak{p}}} {}^{*}\mathfrak{p}}{\left\langle u\mathfrak{t} \otimes_{T_{\mathfrak{q}}} \varphi - u \otimes_{T_{\mathfrak{p}}} \mathfrak{t}\varphi \right\rangle_{\{u\in\mathfrak{p},\varphi\in{}^{*}\mathfrak{q},\mathfrak{t}\in T_{\mathfrak{q},\mathfrak{p}}\}}}, \tag{2.15}$$

where $T_{\mathfrak{p}} := \mathrm{End}_{\mathcal{A}^o}(\mathfrak{p})$ and $T_{\mathfrak{p},\mathfrak{q}} := \mathrm{Hom}_{\mathcal{A}^o}(\mathfrak{p}, \mathfrak{q})$, for every objects $\mathfrak{p}, \mathfrak{q}$ in $\mathcal{A}$.

Now, let $\mathfrak{C}$ be an R-coring and let $\mathcal{Q}$ be a small full sub-category of the category of comodules ${}_{\mathfrak{C}}\mathsf{Comod}$ whose underlying left R-modules are finitely generated and projective. Denote by $\lambda_{\mathfrak{q}}$ the coaction of $\mathfrak{q} \in \mathcal{Q}$. Then one can directly apply the above constructions, by putting $\chi(\mathfrak{q}) = U(\mathfrak{q})$, where $U : {}_{\mathfrak{C}}\mathsf{Comod} \to {}_R\mathsf{Mod}$ is the left forgetful functor. In this case, the left $\mathfrak{C}$-coaction of $\Sigma = \bigoplus_{\mathfrak{q}\in\mathcal{Q}} \mathfrak{q}$ is right S-linear, while the right $\mathfrak{C}$-coaction of ${}^{\vee}\Sigma$ is left S-linear. Moreover, there is a canonical morphism of R-corings defined by

$$\begin{aligned} \mathsf{can}_S : \Sigma \otimes_S {}^{\vee}\Sigma &\longrightarrow \mathfrak{C}, \\ u_{\mathfrak{q}} \otimes_S \varphi_{\mathfrak{q}} &\longmapsto (\mathfrak{C} \otimes_R \varphi_{\mathfrak{q}}) \circ \lambda_{\mathfrak{q}}(u_{\mathfrak{q}}). \end{aligned} \tag{2.16}$$

Here S is the induced ring from the category $\mathcal{Q}$, that is,

$$S = \bigoplus_{\mathfrak{q},\mathfrak{p}\in\mathcal{Q}} \mathrm{Hom}_{\mathfrak{C}}(\mathfrak{q}, \mathfrak{p}). \tag{2.17}$$

However, the construction of the infinite comatrix coring, as well as the canonical map can, can be also performed for any sub-ring of S with the same set of orthogonal idempotents (i.e. the $\mathfrak{q}$'s identities).

Let us consider the $\Bbbk$-linear category $\Bbbk(\mathbb{N})$ whose objects are the natural numbers $\mathbb{N}$, and homomorphisms sets are defined by

$$\mathrm{Hom}_{\Bbbk(\mathbb{N})}(n, m) = \begin{cases} 0 & \text{if } m \notin \{n, n+1\}, \\ \Bbbk.1_n & \text{if } n = m, \\ \Bbbk.\jmath_n^{n+1} & \text{if } m = n+1, \end{cases}$$

where the last two terms are free $\Bbbk$-modules of rank one. The induced ring with enough orthogonal idempotents is the free $\Bbbk$-module $B = \Bbbk^{(\mathbb{N})} \oplus \Bbbk^{(\mathbb{N})}$ generated by the set $\{\mathfrak{h}_n, \mathfrak{v}_n\}_{n\in\mathbb{N}}$, where $\mathfrak{h}_n$ and $\mathfrak{v}_n$ corresponds to 1_n and j_n^{n+1} respectively, subject to the following relations:

$$\begin{aligned} &\mathfrak{h}_n\mathfrak{h}_m = \delta_{n,m}\mathfrak{h}_n, && \forall\, m, n \in \mathbb{N} \quad \text{(Kronecker delta)}, \\ &\mathfrak{v}_n\mathfrak{v}_m = \mathfrak{v}_m\mathfrak{v}_n = 0, && \forall\, m, n \in \mathbb{N}, \\ &\mathfrak{v}_n\mathfrak{h}_{n+1} = \mathfrak{v}_n = \mathfrak{h}_n\mathfrak{v}_n, && \forall\, m, n \in \mathbb{N}. \end{aligned} \tag{2.18}$$

In other words B is the ring of $(\mathbb{N}\times\mathbb{N})$-matrices over $\Bbbk$ of the form

$$\begin{pmatrix} \Bbbk & \Bbbk & 0 & 0 & & & \\ 0 & \Bbbk & \Bbbk & 0 & & & \\ 0 & 0 & \Bbbk & \Bbbk & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ & & & & \Bbbk & \Bbbk & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.19}$$

consisting of matrices with only possibly two nonzero entries in each row: (i,i) and $(i,i+1)$. It is clear that the category of unital left B-modules is isomorphic to the category $\mathrm{Ch}_+(\Bbbk)$ of chain complexes of $\Bbbk$-modules. Precisely, this isomorphism functor $\mathscr{O}$ sends every chain complex $(V_\bullet,\partial^V)$ to its associated differential graded $\Bbbk$-module $\mathscr{O}(V_\bullet)=\bigoplus_{n\geq 0}V_n$ with the following left B-action

$$\mathfrak{h}_n.\left(\sum_{n\geq 0} v_i\right)=v_n, \quad \text{and} \quad \mathfrak{v}_n.\left(\sum_{n\geq 0} v_i\right)=\partial^V(v_{n+1})$$

and acts in the obvious way on morphisms of chain complexes. The inverse functor is clear.

By Proposition 2.9, we have a faithful functor $\chi:\Bbbk(\mathbb{N})\to {}_{\mathscr{L}}\mathsf{Comod}$ sending $n\to Q_n$, whose composition with the left forgetful functor gives rise to a fiber functor $\chi:\Bbbk(\mathbb{N})\to \mathrm{add}({}_RR)$. Therefore, we can apply the above process to construct an infinite comatrix R-coring $Q\otimes_B {}^\vee Q$ where $Q=\bigoplus_{n\in\mathbb{N}}Q_n$ and ${}^\vee Q=\bigoplus_{n\in\mathbb{N}}{}^*Q_n$ are given by the cochain complex of Sec. 2.2.

Since each of the Q_n's has a structure of R-bimodule for which the differential $\partial_\bullet$ is R-bilinear, we deduce that $Q\otimes_B {}^\vee Q$ is an R^{e}-bimodule with actions

$$(r\otimes s^o)\cdot(u_n\otimes_B\varphi_n)\cdot(p\otimes q^o)=(ru_np)\otimes_B(s\varphi_np), \tag{2.20}$$

for every $p,q,r,s\in R$ and $u_n\in Q_n$ and $\varphi_n\in {}^*Q_n$. In view of this R^{e}-biaction, the infinite comatrix R-coring has ${}_{R^{\mathsf{e}}}(Q\otimes_B {}^\vee Q)$ as its underlying R-bimodule.

Next we will construct an R^{e}-ring structure on the R^{e}-bimodule $(Q\otimes_B {}^\vee Q)$. Part of this construction needs the following general lemma which can be found, under a slightly different form, in [10, 4, 20]. We adopt the following general notations: For any small $\Bbbk$-linear category $\mathcal{C}$, we denote by $\mathrm{Funct}_f(\mathcal{C},\mathrm{add}({}_RR))$ the category of $\Bbbk$-linear faithful functors valued in $\mathrm{add}({}_RR)$, i.e. of fiber functors on $\mathcal{C}$. For any object $\chi:\mathcal{C}\to\mathrm{add}({}_RR)$, we denote by $\mathcal{L}(\chi)$ the associated infinite comatrix R-coring defined by the isomorphism of (2.15). Lastly, we consider $\Sigma:\mathrm{Funct}_f(\mathcal{C},\mathrm{add}({}_RR))\to\mathsf{Mod}_{S(\mathcal{C})}$ the canonical functor to the category of right unital $S(\mathcal{C})$-modules (recall that $S(\mathcal{C})$ is the induced ring of $\mathcal{C}^o$).

That is,

$$\Sigma(\chi) := \bigoplus_{\mathfrak{c}\in\mathcal{C}} \mathfrak{c}^{\chi}, \quad \Sigma(\gamma) := \bigoplus_{\mathfrak{c}\in\mathcal{C}} \gamma_{\mathfrak{c}} \tag{2.21}$$

for every fiber functor χ and natural transformation γ between fiber functors.

Lemma 2.11. *Let $\mathcal{A}$ be a small $\Bbbk$-linear category and let $\chi_1, \chi_2 : \mathcal{A} \to {}_R\mathsf{Mod}_R$ be functors with images in* $\mathrm{add}({}_RR)$*. Define $(\chi_1 \otimes_R \chi_2) : \mathcal{A} \times \mathcal{A} \to {}_R\mathsf{Mod}_R$ by setting*

$$(\chi_1 \otimes_R \chi_2)(\mathfrak{p}, \mathfrak{q}) = \chi_1(\mathfrak{p}) \otimes_R \chi_2(\mathfrak{q}), \quad \textit{for } \mathfrak{p}, \mathfrak{q} \in \mathcal{A}.$$

Then

(i) *There is a left R^{e}-linear isomorphism $\mathcal{L}(\chi_1 \otimes_R \chi_2) \cong \mathcal{L}(\chi_1) \otimes_{R^{\mathsf{e}}} \mathcal{L}(\chi_2)$.*
(ii) *For every R-bimodule M, there is a natural isomorphism*

$$\mathrm{Nat}((\chi_1 \otimes_R \chi_2),\, M \otimes_R (\chi_1 \otimes_R \chi_2)) \longrightarrow \mathrm{Hom}_{R-R}(\mathcal{L}(\chi_1) \otimes_{R^{\mathsf{e}}} \mathcal{L}(\chi_2),\, M)$$
$$\sigma \longmapsto \left[(u \otimes_S \varphi) \otimes_{R^{\mathsf{e}}} (v \otimes_S \psi) \mapsto \textstyle\sum_i m_i \varphi(p_i \psi(q_i))\right],$$

where we set $\sigma_{(\mathfrak{p},\mathfrak{q})}(u \otimes_R v) = \sum_i m_i \otimes_R p_i \otimes_R q_i \in M \otimes_R \mathfrak{p} \otimes_R \mathfrak{q}$, for every $u \in \mathfrak{p}$, $\varphi \in {}^*\mathfrak{p}$, $v \in \mathfrak{q}$, $\psi \in {}^*\mathfrak{q}$ and $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{A} \times \mathcal{A}$.

Proof. Straightforward. $\square$

Let us come back to our situation. We are considering the functor

$$\chi : \Bbbk(\mathbb{N}) \longrightarrow {}_{\mathscr{L}}\mathsf{Comod}, \quad \text{sending } n \longmapsto Q_n. \tag{2.22}$$

On the one hand, we already observed that the composition of χ with the left forgetful functor gives rise to a fiber functor $\Bbbk(\mathbb{N}) \to \mathrm{add}({}_RR)$. On the other hand, we can consider also the fiber functor $\chi : \Bbbk(\mathbb{N}) \to {}_R\mathsf{Mod}_R$ obtained by composing the functor $\chi : \Bbbk(\mathbb{N}) \to {}_{\mathscr{L}}\mathsf{Comod}$ with the functor ${}_{\mathscr{L}}\mathsf{Comod} \to {}_R\mathsf{Mod}_R$ stated in Lemma 2.4. Therefore, it is clear from Lemma 2.11, that there is a bijective correspondence between multiplications on $\mathcal{L}(\chi) = (Q \otimes_B {}^{\vee}Q)$ and natural transformations $(\chi \otimes_R \chi) \to \mathcal{L}(\chi) \otimes_R (\chi \otimes_R \chi)$. One of those natural transformations can be constructed using the left $\mathcal{L}(\chi)$-coaction on the Q_n's, as defined in (2.14). Thus we have the following statement.

Lemma 2.12. *Let $Q_{\bullet}$ be the cochain complex of Sec. 2.2, and $(Q \otimes_B {}^{\vee}Q)$ the associated R-coring. Then there is a natural transformation $(\chi \otimes_R \chi) \to \mathcal{L}(\chi) \otimes_R (\chi \otimes_R \chi)$ given by: $\widetilde{\lambda}_{n,m} : Q_n \otimes_R Q_m \to (Q \otimes_B {}^{\vee}Q) \otimes_R (Q_n \otimes_R Q_m)$*

$$\begin{aligned} &u_n \otimes_R u_m \\ &\quad \longmapsto \sum_{\alpha,\beta} [(u_n \otimes_A u_m) \otimes_B ({}^*\omega_{n,\alpha} \star \partial^* \omega_{m,\beta}) + (u_n \otimes_A \partial u_m) \otimes_B ({}^*\omega_{n,\alpha} \star {}^*\omega_{m,\beta})] \\ &\qquad \otimes_R (\omega_{n,\alpha} \otimes_R \omega_{m\beta}) \end{aligned}$$

for every $n, m \geq 1$, *and by* $\widetilde{\lambda}_{0,n} = \widetilde{\lambda}_{n,0} : Q_n \to (Q \otimes_B {}^{\vee}Q) \otimes_R Q_n$, *sending* $u_n \longmapsto \sum_\alpha (u_n \otimes_B {}^*\omega_{n,\alpha}) \otimes_R \omega_{n,\alpha}$, *where* $\{(\omega_{n,\alpha}, {}^*\omega_{n,\alpha})\}$ *is a dual basis for* ${}_RQ_n$, $n \geq 1$.

Proof. This is a routine computation using definitions and dual bases notions. □

The following lemma will be used in the sequel.

Lemma 2.13. *Let* $\{(\omega_{n,\alpha}, {}^*\omega_{n,\alpha})\}_\alpha$ *be a dual basis for* ${}_RQ_n$ *with* $n > 0$. *Then, for every element* $u_n \in Q_n$, $u_m \in Q_m$, *and* $\varphi_n \in {}^*Q_n$, $\varphi_m \in {}^*Q_m$, *we have*

$$\sum_{\alpha,\beta} [(u_n \otimes_A \partial u_m) \otimes_B ({}^*\omega_{n,\alpha} \star {}^*\omega_{m,\beta})] \times_R [(\omega_{n,\alpha} \otimes_A \omega_{m,\beta}) \otimes_B (\varphi_n \star \partial\varphi_m)] = 0$$

and

$$\sum_{\alpha,\beta} [(u_n \otimes_A u_m) \otimes_B ({}^*\omega_{n,\alpha} \star \partial^*\omega_{m,\beta})] \times_R [(\omega_{n,\alpha} \otimes_A \partial\omega_{m,\beta}) \otimes_B (\varphi_n \star \varphi_m)] = 0$$

as elements in the R^{e}*-bimodule* $(Q \otimes_B {}^{\vee}Q) \times_R (Q \otimes_B {}^{\vee}Q)$.

Proof. Straightforward. □

We then arrive to the R^{e}-ring structure of $(Q \otimes_B {}^{\vee}Q)$.

Proposition 2.14. *There is a structure of* R^{e}*-ring on* $\mathscr{D} := (Q \otimes_B {}^{\vee}Q)$ *given by the extension of rings* $R^{\mathsf{e}} \to \mathscr{D}$ *sending* $r \otimes s^o \mapsto (r \otimes_B s)$ (*i.e.* $\iota_0(r) \otimes_B \iota_0(s)$, ι_0 *is the canonical injection*), *where the multiplication of* $\mathscr{D}$ *is defined by the following rules*: *for every pair of generic elements* $(u_n \otimes_B \varphi_n)$ *and* $(u_m \otimes_B \varphi_m)$ *of* $\mathscr{D}$ *with* $n, m > 0$, *we set*

$$\begin{aligned}(u_n \otimes_B \varphi_n) \cdot (u_m \otimes_B \varphi_m) = {} & ((u_n \otimes_A \partial u_m) \otimes_B (\varphi_n \star \varphi_m)) \\ & + ((u_n \otimes_A u_m) \otimes_B (\varphi_n \star \partial\varphi_m))\end{aligned}$$

and

$$(u_n \otimes_B \varphi_n) \cdot (r \otimes_B s) = (u_n r \otimes_B s\varphi_n),$$

$$(r \otimes_B s) \cdot (u_n \otimes_B \varphi_n) = (r u_n \otimes_B \varphi_n s), \quad \forall\, r, s \in R.$$

Proof. Using Lemmas 2.8 and 2.13, one can show that each of the maps $\widetilde{\lambda}_{n,m}$ given in Lemma 2.12 is coassociative with respect to the comultiplication of $Q \otimes_B {}^{\vee}Q$. Hence, its image by the natural isomorphism of Lemma 2.11 leads to the stated associative multiplication. The unitary property is clear. □

Remark 2.15. As we have seen, the construction of an R^{e}-ring structure on $\mathscr{D}$ is not an immediate task. Part of this difficulty is clearly due to the fact that the natural transformations which lead to the multiplications on $\mathscr{D}$ are not easy to construct. The other part is probably due to the fact that, although the category $\Bbbk(\mathbb{N})$ is a monoidal category, the fiber functor $\chi : \Bbbk(\mathbb{N}) \to {}_R\mathsf{Mod}_R$ given by the complex $Q_\bullet$ is not strong monoidal since the local "comultiplication" maps $Q_{n+m} \to Q_n \otimes_R Q_m$, $m, n \geq 1$, see Remark 2.7, do not necessarily form a natural isomorphism. Thus χ does not satisfy the usual condition of a fiber functor, namely, being a strict monoidal functor. Of course, this has prevented us from directly using general results already existing in the literature, for example [20].

Proposition 2.16. *Set $\mathscr{D} := {}_{R^{\mathsf{e}}}(Q \otimes_B {}^{\vee}Q)_{R^{\mathsf{e}}}$, where $Q_\bullet$ is the cochain complex defined in Sec.* 2.2*. Then $\mathscr{D}$ has a structure of left R-bialgebroid.*

Proof. Is a routine computation which uses Lemmas 2.13 and 2.8, as well as Proposition 2.14. □

2.4. *The isomorphism between comatrices and coendomorphisms bialgebroids*

Now, we come back to the canonical map. As mentioned in the preamble of Sec. 2.3, there is a canonical map given explicitly by (2.16). Thus, using the $\mathscr{L}$-coactions of Proposition 2.9, we have a morphism of R-corings $\mathsf{can}_B : \mathscr{D}^l \longrightarrow \mathscr{L}^l$ sending

$$(u_n \otimes_B \varphi_n) \longmapsto \sum_{i_0, i_1, \ldots, i_{n-1}} \pi(a_0 \otimes {}^*e_{i_0}) \cdots \pi(a_{n-1} \otimes {}^*e_{i_{n-1}}) \varphi_n (e_{i_0} \partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}}), \tag{2.23}$$

where $u_n = a_0 \partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1} \in Q_n$, and $\mathsf{can}_B(r \otimes_B s) = \pi(r \otimes s^o)$, for $r, s \in R$.

Our next goal is to show that can_B is an isomorphism of left R-bialgebroids. To this end, we will need the following proposition.

Proposition 2.17. *For every $n \geq 1$, $u_n = a_0 \partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1} \in Q_n$ and $\varphi_n \in {}^*Q_n$, we have the following equality*

$$(u_n \otimes_B \varphi_n) = \sum\nolimits_{i_0, i_1, \ldots, i_{n-1}} [(a_0 \otimes_B {}^*e_{i_0}) \cdot (a_1 \otimes_B {}^*e_{i_1}) \cdots (a_{n-1} \otimes_B {}^*e_{i_{n-1}})] \, \varphi_n (e_{i_0} \partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}})$$

viewed as elements in the left R^{e}-module $\mathscr{D}^l$. In particular, $\mathscr{D}$ is generated, as an R^{e}-ring, by the image of R^{e} and the set of elements $\{(e_i \otimes_B {}^*e_j)\}_{i,j}$ (recall that $\{(e_i, {}^*e_i)\}_i$ is a dual basis of ${}_RA$).

Proof. It follows by induction, using the dual basis of the Q_n's given in Lemma 2.8. □

Theorem 2.18. *The canonical map $\mathsf{can}_B : \mathscr{D} \to \mathscr{L}$ of* (2.23) *is an isomorphism of left R-bialgebroids.*

Proof. First we will show that can_B is a multiplicative map. By Proposition 2.17 this is equivalent to show that

$$\mathsf{can}_B(a \otimes_B \varphi)\, \mathsf{can}_B(u_n \otimes_B \varphi_n) = \mathsf{can}_B((a \otimes_B \varphi)(u_n \otimes_B \varphi_n)), \tag{2.24}$$

for every $a \in A$, $\varphi \in {}^*A$, $u_n \in Q_n$, $\varphi_n \in {}^*Q_n$ with $n \geq 1$. Equality (2.24), is proved by direct computation. Since can_B preserves the unit, we deduce that can_B is a morphism of R^{e}-rings. The inverse of can_B is constructed as follows. It is clear that the map $\zeta : (A \otimes {}^*A) \to \mathscr{D}$ sending $a \otimes \varphi \mapsto a \otimes_B \varphi$ is an R^{e}-bilinear map. Therefore, it is canonically extended to the tensor algebra $\zeta : \mathscr{T}_{R^{\mathsf{e}}}((A \otimes {}^*A)) \to \mathscr{D}$, as $\mathscr{D}$ is an R^{e}-ring. Now, for every $a, b \in A$ and $\varphi \in {}^*A$, one shows that

$$\zeta\left(\sum_i (a \otimes e_i\varphi) \otimes_{R^{\mathsf{e}}} (b \otimes {}^*e_i)\right) = \zeta(ab \otimes \varphi),$$

where $\{(e_i, {}^*e_i)\}_i$ is the dual basis of ${}_RA$. This means that ζ factors throughout the canonical projection $\pi : \mathscr{T}_{R^{\mathsf{e}}}((A \otimes {}^*A)) \to \mathscr{L}$, and so we have an algebra map $\zeta : \mathscr{L} \to \mathscr{D}$. Given $a \in A$ and $\varphi \in {}^*A$, we have

$$\begin{aligned} \mathsf{can}_B \circ \zeta(\pi(a \otimes \varphi)) = \mathsf{can}_B(a \otimes_B \varphi) &= \sum_i \pi(a \otimes_B {}^*e_i)\varphi(e_i) \\ &= \pi\left(\sum_i a \otimes {}^*e_i\varphi(e_i)\right) = \pi(a \otimes \varphi). \end{aligned}$$

This implies that $\mathsf{can}_B \circ \zeta = id_{\mathscr{L}}$. Now, take $u_n \in Q_n$, $n \geq 1$, of the form $u_n = a_0\partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1}$ and $\varphi_n \in {}^*Q_n$. Then, by Proposition 2.17, we have

$$\zeta \circ \mathsf{can}_B(u_n \otimes_B \varphi_n) = u_n \otimes_B \varphi_n,$$

and this shows that $\zeta \circ \mathsf{can}_B = id_{\mathscr{D}}$. □

Corollary 2.19. *Let $(\mathscr{L}^l)^*$ be the right convolution ring of the R-coring $\mathscr{L}^l$. Then there is an isomorphism of rings $(\mathscr{L}^l)^* \cong \mathrm{End}(Q_B)$.*

Proof. We know that each $\mathfrak{h}_n{}^\vee Q = {}^*Q_n$ is a finitely generated and projective right R-module, where the $\mathfrak{h}_i$'s are defined in (2.18). The same property holds true for each right R-module of the form $\mathfrak{e}_{i_1,i_n}{}^\vee Q$, where $\mathfrak{e}_{i_1,i_n} = \mathfrak{h}_{i_1} + \cdots + \mathfrak{h}_{i_n}$. This means that the unital bimodule ${}_B{}^\vee Q_R$ satisfies the second condition of [12, Proposition 5.1] for each idempotent which belongs to the set of local units of B. Therefore we have, as in the proof of [12, Proposition 5.1], that the functor $- \otimes_B {}^\vee Q$ is left adjoint to $- \otimes_R Q$. Hence

$$\mathrm{Hom}_{-R}(\mathscr{D}, R) = \mathrm{Hom}_{-R}(Q \otimes_B {}^\vee Q, R) \cong \mathrm{Hom}_{-B}(Q, Q).$$

Now, we conclude by Theorem 2.18. □

3. Categories of Comodules and Chain Complexes of Modules

This section contains our main results, namely Theorems 3.3, 3.9 and 3.10. As a consequence of these results, we obtain that the category of chain complexes of left R-modules is always equivalent to the category of left comodules over a quotient R-coring of the left R-bialgebroids $\mathscr{L}(A)$ constructed in Example 2.3. When R is commutative, this quotient inherits a left R-bialgebroid structure from $\mathscr{L}(A)$, and the stated equivalence is actually a monoidal equivalence. This will clarify the equivalence of categories already constructed by Pareigis and Tambara, [25, 29].

3.1. *Monoidal equivalence between chain complexes of* $\Bbbk$*-modules and left* $\mathscr{L}$*-comodules*

In this subsection we will use the isomorphism of bialgebroids stated in Theorem 2.18 to show that the following are equivalent: (1) Q_B is faithfully flat, (2) the underlying module ${}_{R\otimes 1^o}\mathscr{L}$ of $\mathscr{L}$ is flat and the functor $Q\otimes_B - : {}_B\mathsf{Mod} \to {}_{\mathscr{L}}\mathsf{Comod}$ is a monoidal equivalence of categories. This is our first main result, and stated below as Theorem 3.3.

Remark 3.1. Let $B = \Bbbk^{(\mathbb{N})} \oplus \Bbbk^{(\mathbb{N})}$ be the ring with enough orthogonal idempotents associated to the small $\Bbbk$-linear category $\Bbbk(\mathbb{N})$ considered in Sec. 2.3, see (2.19). We have already observed in Sec. 2.3 that the category of unital left B-modules ${}_B\mathsf{Mod}$ is in a canonical way isomorphic to the category $\mathrm{Ch}_+(\Bbbk)$ of chain complexes of $\Bbbk$-modules. Therefore, ${}_B\mathsf{Mod}$ inherits a structure of monoidal category. Recall that B is generated as a free $\Bbbk$-module by the set of elements $\{\mathfrak{h}_n, \mathfrak{v}_n\}_{n\in\mathbb{N}}$ with $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ as a set of orthogonal idempotents given by (2.18). The multiplication of two object $X, Y \in {}_B\mathsf{Mod}$, is then given by

$$X \ominus Y = \bigoplus_{n\in\mathbb{N}} \left(\bigoplus_{i+j=n} \mathfrak{h}_i X \otimes \mathfrak{h}_j X \right).$$

That is, $\mathfrak{h}_n(X \ominus Y) = \bigoplus_{i+j=n} \mathfrak{h}_i X \otimes \mathfrak{h}_j Y$, for every $n \in \mathbb{N}$, and for every $k \geq 1$, $l \geq 1$ with $k + l = m$, we have

$$\mathfrak{v}_{m-1}(\mathfrak{h}_k x \otimes \mathfrak{h}_l y) = \mathfrak{v}_{k-1} x \otimes \mathfrak{h}_l y + (-1)^k \mathfrak{h}_k x \otimes \mathfrak{v}_{l-1} y,$$

(i.e. the Leibniz rule), and

$$\mathfrak{v}_{m-1}(\mathfrak{h}_0 x \otimes \mathfrak{h}_m y) = \mathfrak{h}_0 x \otimes \mathfrak{v}_{m-1} y, \quad \mathfrak{v}_n(\mathfrak{h}_n x \otimes \mathfrak{h}_0 y) = \mathfrak{v}_{n-1} x \otimes \mathfrak{h}_0 y$$

for every $x \in X$, $y \in Y$ and $m, n \geq 1$. The multiplication of B-linear maps is obvious. The unit object is the left unital B-module $\Bbbk_{[0]}$ whose underlying $\Bbbk$-module is $\Bbbk$, and whose B-action is given by

$$\mathfrak{h}_n \Bbbk_{[0]} = \begin{cases} 0 & \text{if } n \neq 0 \\ \Bbbk & \text{if } n = 0. \end{cases}$$

We know that the cochain complex $Q_\bullet$ of Sec. 2.2 induces an $\mathscr{L}$-comodule $Q = \bigoplus_{n\in\mathbb{N}} Q_n$ whose coaction is easily seen to be right B-linear. Thus, $Q \otimes_B - : {}_B\mathsf{Mod} \to {}_{\mathscr{L}}\mathsf{Comod}$, acting in the obvious way, is a well-defined functor. This functor is in fact monoidal.

Lemma 3.2. *Consider the monoidal categories* ${}_B\mathsf{Mod}$ *and* ${}_{\mathscr{L}}\mathsf{Comod}$, *with structure, respectively, given in Remark* 3.1 *and Lemma* 2.4. *Then* $Q\otimes_B - : {}_B\mathsf{Mod} \to {}_{\mathscr{L}}\mathsf{Comod}$ *is a monoidal functor, with structure*

$$\Gamma^2_{X,Y} : (Q \otimes_B X) \otimes_R (Q \otimes_B Y) \longrightarrow Q \otimes_B (X \ominus Y),$$

explicitly given by

$$\Gamma^2_{X,Y}((u_n \otimes_B \mathfrak{h}_n x) \otimes_R (u_m \otimes_B \mathfrak{h}_m y))$$

$$= \begin{cases} (u_n \otimes_A u_m) \otimes_B (\mathfrak{h}_n x \otimes \mathfrak{v}_{m-1} y) + (u_n \otimes_A \partial u_m) \otimes_B (\mathfrak{h}_n x \otimes \mathfrak{h}_m y), & n, m \geq 1, \\ u_n u_m \otimes_B (\mathfrak{h}_n x \otimes \mathfrak{h}_m y), \quad n = 0 \ \text{or}\ m = 0, \end{cases}$$

for every $u_n \in Q_n$, $u_m \in Q_m$, $x \in X$ *and* $y \in Y$, *and* $\Gamma^0 : R \to Q \otimes_B \Bbbk_{[0]}$ *sending* $r \mapsto r \otimes_B \mathfrak{h}_0 1$.

Proof. The fact that $\Gamma^2_{X,Y}$ is a well-defined map comes from the observation that the right R-action of $Q \otimes_B X$ as left $\mathscr{L}$-comodule is given by the right R-action of Q viewed as left $\mathscr{L}$-comodule. That is, the one given by the rule (2.8). Now, it is easily seen that the right R-action of Q given by (2.8) is exactly the right R-action of Q we started with (i.e. that which comes from the inclusion ${}_RK_R \subset A \otimes_R A$). A direct computation, using Lemma 2.10, shows that $\Gamma^2_{X,Y}$ is left $\mathscr{L}$-colinear, for each X, Y. We leave to the reader the proof of the associativity and unitary properties of the pair $(\Gamma^2_{-,-}, \Gamma^0)$. □

Our first main result is the following.

Theorem 3.3. *Let* R *be an algebra over a commutative ground ring* $\Bbbk$, *and* A *an* R-*ring which is finitely generated and projective as left* R-*module. Consider the associated left* R-*bialgebroid given in Proposition* 2.1 *and let* $B = \Bbbk^{(\mathbb{N})} \oplus \Bbbk^{(\mathbb{N})}$ *be the ring with enough orthogonal idempotents of* (2.19). *Consider the cochain complex* $Q_\bullet$ *of Sec.* 2.2 *with its canonical right unital* B-*action and left* $\mathscr{L}$-*coaction. Then the following statements are equivalent*

(1) *The right module* $\mathscr{L}^l_R$ *is flat and the functor* $Q \otimes_B - : {}_B\mathsf{Mod} \longrightarrow {}_{\mathscr{L}}\mathsf{Comod}$ *is an equivalence of monoidal categories*;
(2) Q_B *is a faithfully flat unital module.*

Proof. The monoidal condition is, by Lemma 3.2, always satisfied, so it can be omitted in the proof. Henceforth, we only need to show that $\mathscr{L}^l_R$ is flat and $Q \otimes_B -$ is an equivalence, if and only if Q_B is a faithfully flat module. By the left version of the generalized faithfully flat descent theorem [14, Theorem 5.9], we know that Q_B

is faithfully flat if and only if $\mathscr{D}^l_R = {}_{1\otimes R^o}(Q \otimes_B {}^\vee Q)$ is flat and $Q \otimes_B - : {}_B\mathsf{Mod} \to {}_{\mathscr{D}^l}\mathsf{Comod}$ is an equivalence of category. We then conclude by Theorem 2.18. $\square$

Notice that, when Q_B is faithfully flat, the inverse of the functor of $Q \otimes_B - : {}_B\mathsf{Mod} \to {}_{\mathscr{L}}\mathsf{Comod}$ is given by the cotensor product ${}^\vee Q \square_{\mathscr{L}} - : {}_{\mathscr{L}}\mathsf{Comod} \to {}_B\mathsf{Mod}$. The structure of bicomodule on ${}^\vee Q$ is given as follows. Recall that Q is in fact an $(\mathscr{L}, B)$-bicomodule, that is, the left $\mathscr{L}$-coaction of Q is right B-linear. So, since each of the Q_n, $n \geq 0$, is finitely generated and projective left R-module, each of the left duals ${}^*Q_n$ admits a right $\mathscr{L}$-coaction, for which ${}^\vee Q$ becomes a $(B, \mathscr{L})$-bicomodule.

The condition $\mathscr{L}^l_R$ is flat, stated in item (1) of Theorem 3.3, seems to be redundant. But, although we can deduce form the equivalence of categories that the category of left $\mathscr{L}$-comodule is abelian, we cannot affirm that the forgetful functor ${}_{\mathscr{L}}\mathsf{Comod} \to {}_R\mathsf{Mod}$ is left exact. Thus, $\mathscr{L}^l_R$ is not necessarily a flat module, see [15, Proposition 2.1].

Consider the category $\mathrm{Ch}_+(\Bbbk)$ of chain complexes of $\Bbbk$-modules and denote by $\mathscr{O} : \mathrm{Ch}_+(\Bbbk) \to {}_B\mathsf{Mod}$ the canonical isomorphism of categories explicitly given in Sec. 2.3. In the case when $R = \Bbbk$ is a field, it is known that Q_B is always faithfully flat wherever $\dim_{\Bbbk}(A) < \infty$ (a complete proof for a noncommutative field, that is, a division ring is given in Theorem 3.10). We thus obtain the following corollary.

Corollary 3.4 ([29, Theorem 4.4]). *Let $\Bbbk$ be a field and A an $\Bbbk$-algebra such that $1 < \dim_{\Bbbk}(A) < \infty$. Consider the associated coendomorphism $\Bbbk$-bialgebra $\mathscr{L}$ given in Proposition* 2.1. *Then the category $\mathrm{Ch}_+(\Bbbk)$ of chain complexes of $\Bbbk$-modules is monoidally equivalent, via the functor $(Q \otimes_B -) \circ \mathscr{O} : \mathrm{Ch}_+(\Bbbk) \to {}_{\mathscr{L}}\mathsf{Comod}$, to the category of left $\mathscr{L}$-comodules.*

Proof. By the foregoing observations, this is a direct consequence of Theorem 3.3. $\square$

Explicitly, the composition of the functor given in Corollary 3.4 with the forgetful functor ${}_{\mathscr{L}}\mathsf{Comod} \to {}_{\Bbbk}\mathsf{Mod}$ is given as follows. For any chain complex $V_\bullet$ in $\mathrm{Ch}_+(\Bbbk)$, we have

$$Q \otimes_B \mathscr{O}(V_\bullet) = \frac{\bigoplus_{n\geq 0}(Q_n \otimes V_n)}{\langle \partial u_n \otimes x_{n+1} - u_n \otimes \partial x_{n+1} \rangle_{n\geq 0}}.$$

3.2. *Equivalence between chain complexes of R-modules and $\overline{\mathscr{L}}$-comodules*

Our main aim here is to extend the result of Theorem 3.3 to the category $\mathrm{Ch}_+(R)$ of chain complexes over left R-modules. In other words, we are interested in relating the category of chain complexes of left R-modules and the category of left $\mathscr{L}(A)$-comodules over the left R-bialgebroid of Proposition 2.1. Precisely, we show an analog of Theorem 3.3 where $\mathscr{L}$ is replaced by its quotient R-coring $\overline{\mathscr{L}} := \mathscr{L}(A)/\langle 1_{\mathscr{L}}(r \otimes 1^o - 1 \otimes r^o)\rangle_{r\in R}$ and the ring B by its extension $C = R^{(\mathbb{N})} \oplus R^{(\mathbb{N})}$. This is our second main result, see Theorem 3.9. Of course, in this case, the monoidal equivalence of categories is reduced to an equivalence, unless the base ring R is commutative and the extension A is an R-algebra.

Let A be an R-ring and assume that ${}_RA$ is a finitely generated and projective module. Fix a dual basis $\{(e_i, {}^*e_i)\}_i$ for ${}_RA$, and consider $\mathscr{L} := \mathscr{L}(A)$ the left R-bialgebroid of Proposition 2.1. We denote by $\pi : \mathscr{T}_{R^{\mathsf{e}}}((A \otimes {}^*A)) \to \mathscr{L}$ the canonical projection.

Lemma 3.5. *Let $\mathscr{J}$ be the left ideal of $\mathscr{L}$ generated by the following set of elements*

$$\{\pi(ar \otimes \varphi) - \pi(a \otimes r\varphi)\}_{a\in A,\varphi\in {}^*A,r\in R}\,.$$

Then $\mathscr{J}$ is a coideal of the underlying R-coring $\mathscr{L}^l$.

Proof. An easy computation shows that

$$\pi(ar \otimes \varphi) - \pi(a \otimes r\varphi) = \pi(a \otimes \varphi)(r \otimes 1^o - 1 \otimes r^o),$$

for every elements $a \in A$, $\varphi \in {}^*A$ and $r \in R$. Thus, $\mathscr{J}$ as left R^{e}-bimodule is generated by the set $\{\mathsf{g}_r := 1_{\mathscr{L}}.(r\otimes 1^o - 1\otimes r^o)\}_{r\in R}$. For arbitrary elements $x \in \mathscr{L}$ and $r \in R$, we get

$$\varepsilon(x\mathsf{g}_r) = \varepsilon(x.(1 \otimes \varepsilon(\mathsf{g}_r)^o)) = 0,$$

as $\varepsilon(\mathsf{g}_r) = 0$. Hence, $\varepsilon(\mathscr{J}) = 0$. On the other hand, for every $r \in R$, we have

$$\Delta(\mathsf{g}_r) = (1_{\mathscr{L}} \otimes_R 1_{\mathscr{L}})(r \otimes 1^o) - (1_{\mathscr{L}} \otimes_R 1_{\mathscr{L}})(1 \otimes r^o).$$

Using these equalities we can show that, for every $x \in \mathscr{L}$ and $r \in R$, we have

$$\begin{aligned}\Delta(x\mathsf{g}_r) &= \sum_{(x)} x_{(1)} \otimes_R x_{(2)}(r \otimes 1^o) - \sum_{(x)} x_{(1)} \otimes_R x_{(2)}(1 \otimes r^o)\\ &= \sum_{(x)} x_{(1)} \otimes_R x_{(2)}(r \otimes 1^o - 1 \otimes r^o),\end{aligned}$$

where $\Delta(x) = \sum_{(x)} x_{(1)} \otimes_R x_{(2)}$. Therefore, $(\overline{\pi} \otimes_R \overline{\pi}) \circ \Delta(x\mathsf{g}_r) = 0$, for every $x \in \mathscr{L}$ and $r \in R$, where $\overline{\pi} : \mathscr{L} \to \mathscr{L}/\mathscr{J}$ is the canonical projection. Thus $\mathscr{J}$ is a coideal of $\mathscr{L}$. $\square$

Denote by $\overline{\mathscr{L}} := \mathscr{L}/\mathscr{J}$ the quotient R-coring and by $\overline{\pi} : \mathscr{L} \to \overline{\mathscr{L}}$ the canonical projection. Notice that $\overline{\pi}$ is also left $\mathscr{L}$-colinear. Consider the cochain complex $Q_\bullet$ of Sec. 2.2. We know, by Proposition 2.9, that each Q_n is a left $\mathscr{L}$-comodule. Hence each of them is a left $\overline{\mathscr{L}}$-comodule with coaction

$$\overline{\lambda}_n : Q_n \to \mathscr{L} \otimes_R Q_n \to \overline{\mathscr{L}} \otimes_R Q_n, \quad n \geq 0.$$

Lemma 3.6. *The $\mathscr{L}$-coaction $\overline{\lambda}_n$ is right R-linear. That is, Q_n is an $(\overline{\mathscr{L}}, R)$-bicomodule (here R is considered as a the trivial R-coring).*

Proof. For $n = 0$ the statement is trivial since $\overline{\lambda}_0(r) = (r \otimes 1^o)\overline{\pi}(1_{\mathscr{L}}) = \overline{\pi}(1_{\mathscr{L}})(1 \otimes r^o)$, for every $r \in R$. Take $n \geq 1$ and an element $u_n \in Q_n$ of the form $u_n = a_0 \partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1}$. Then, for every $r \in R$, we have

$$\begin{aligned}
\overline{\lambda}_n(u_n r) &= \sum_{\alpha} \overline{\pi}(\pi(a_0 \otimes {}^*e_{i_0}) \cdots \pi(a_{n-1} r \otimes {}^*e_{i_{n-1}})) \otimes_R \omega_{n,\alpha}, \\
&\qquad \text{where } \alpha = (i_0, \ldots, i_{n-1}), \quad \text{and} \quad \omega_{n,\alpha} = e_{i_0} \partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}} \\
&= \sum_{\alpha} \overline{\pi}(\pi(a_0 \otimes {}^*e_{i_0}) \cdots \pi(a_{n-1} \otimes r^* e_{i_{n-1}})) \otimes_R \omega_{n,\alpha} \\
&= \sum_{\alpha} (\overline{\pi} \otimes_R Q_n)[\pi(a_0 \otimes {}^*e_{i_0}) \cdots \pi(a_{n-1} \otimes {}^*e_{i_{n-1}})(1 \otimes r^o) \otimes_R \omega_{n,\alpha}] \\
&= \sum_{\alpha} (\overline{\pi} \otimes_R Q_n)[\pi(a_0 \otimes {}^*e_{i_0}) \cdots \pi(a_{n-1} \otimes {}^*e_{i_{n-1}}) \otimes_R \omega_{n,\alpha} r] \\
&= \sum_{\alpha} \overline{\pi}(\pi(a_0 \otimes {}^*e_{i_0}) \cdots \pi(a_{n-1} \otimes {}^*e_{i_{n-1}})) \otimes_R \omega_{n,\alpha} r \\
&= \overline{\lambda}_n(u_n) r,
\end{aligned}$$

where in the fourth equality we have used that each Q_n is in fact a left $\times_R$-$\mathscr{L}$-comodule, that is, $\lambda_n(Q_n) \subseteq \mathscr{L} \times_R Q_n$. We then conclude by linearity. □

Remark 3.7. The quotient R-coring $\overline{\mathscr{L}}$ does not admit, in general, a structure of left R-bialgebroid. However, if we assume that R is commutative (i.e. a commutative $\Bbbk$-algebra) and that A is an R-algebra, then the left ideal $\mathscr{J}$ is in fact a two-sided ideal, since in this case we have the following equalities

$$\mathsf{g}_r \pi(a \otimes \varphi) = \pi(a \otimes \varphi)\mathsf{g}_r, \quad \text{for every } r \in R,\ \ a \in A,\ \text{ and } \ \varphi \in A^*.$$

In view of this, a direct verification shows that $\overline{\mathscr{L}}$ is an R-bialgebroid such that the canonical surjection $\overline{\pi} : \mathscr{L} \to \overline{\mathscr{L}}$ is a morphism of R-bialgebroids.

Let us consider the $\Bbbk$-linear category $R(\mathbb{N})$ whose objects are the natural numbers $\mathbb{N}$ and homomorphisms sets are defined by

$$\mathrm{Hom}_{R(\mathbb{N})}(n, m) = \begin{cases} 0 & \text{if } m \notin \{n, n+1\}, \\ R.1_n = 1_n.R & \text{if } n = m, \\ R.\jmath_n^{n+1} = \jmath_n^{n+1}.R & \text{if } m = n+1. \end{cases} \tag{3.1}$$

The last two terms are copies of ${}_R R_R$ viewed as an R-bimodule which is free as left and right R-module of rank one, generated by an invariant element. The composition is defined using the regular R-biactions of ${}_RR_R$. The induced ring with enough orthogonal idempotents is the free left R-module $C = R^{(\mathbb{N})} \oplus R^{(\mathbb{N})}$ generated by elements $\{\mathfrak{h}_n, \mathfrak{u}_n\}_{n\in\mathbb{N}}$ subject to the following relations:

$$\begin{aligned} &\mathfrak{h}_n\mathfrak{h}_m = \delta_{n,m}\mathfrak{h}_n, && \forall\, n,m \in \mathbb{N} \quad \text{(Kronecker delta)},\\ &\mathfrak{u}_n\mathfrak{u}_m = \mathfrak{u}_m\mathfrak{u}_n = 0, && \forall\, n,m \in \mathbb{N},\\ &\mathfrak{u}_n\mathfrak{h}_{n+1} = \mathfrak{u}_n = \mathfrak{h}_n\mathfrak{u}_n, && \forall\, n,m \in \mathbb{N}. \end{aligned} \tag{3.2}$$

In other words C is the ring of $(\mathbb{N}\times\mathbb{N})$-matrices over R of the form

$$C = \begin{pmatrix} R & R & 0 & 0 & & & \\ 0 & R & R & 0 & & & \\ 0 & 0 & R & R & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ & & & 0 & R & R & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{3.3}$$

i.e. with possibly non-zero entries in each row: (i,i) and $(i,i+1)$. C is also free as right R-module, since the generators are invariant. One can easily check that the category of chain complexes of left R-modules $\mathrm{Ch}_+(R)$ is equivalent to the category of unital left C-modules. Let B be the ring with enough orthogonal idempotents of (2.19). There is a morphism of rings $B \to C$ with the same set of orthogonal idempotents. In this way, we have by [12, p. 733] the usual adjunction between left unital B-modules and C-modules using restriction of scalars and the tensor product functor $C \otimes_B -$.

By Lemma 3.6, we have a morphism of rings $R \to \mathrm{End}_{\overline{\mathscr{L}}}(Q_n)$, for every $n \geq 0$. This leads to a faithful functor from the category $R(\mathbb{N})$ to the category of $(\overline{\mathscr{L}}, R)$-bicomodules (here R is considered as a trivial R-coring) $\chi' : R(\mathbb{N}) \to {}_{\overline{\mathscr{L}}}\mathsf{Comod}_R$. The composition of χ' with the forgetful functor gives rise then to a fiber functor $\omega : R(\mathbb{N}) \to {}_R\mathsf{Mod}_R$ whose image is in $\mathrm{add}({}_RR)$. Therefore, we can apply the constructions performed in Sec. 2.3. Thus, we have an infinite comatrix R-coring $Q \otimes_C {}^{\vee}Q$ together with a canonical map $\mathsf{can}_C : Q \otimes_C {}^{\vee}Q \longrightarrow \overline{\mathscr{L}}$ sending

$$u_n \otimes_C \varphi_n \xrightarrow{\ \mathsf{can}_C\ } \sum_{i_0,\ldots,i_{n-1}} \overline{\pi}(\pi(a_0 \otimes {}^*e_{i_0}) \cdots \pi(a_{n-1} \otimes {}^*e_{i_{n-1}}))$$
$$\varphi_n(e_{i_0}\partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}}). \tag{3.4}$$

Clearly we have a surjective map $\vartheta : Q \otimes_B {}^{\vee}Q \to Q \otimes_C {}^{\vee}Q$. Moreover, we have a commutative diagram with exact rows relating the two R-corings morphisms can_B

and can_C (see Eqs. (2.23) and (3.4))

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(\vartheta) \arrow[r] \arrow[d, dashed] & Q \otimes_B {}^{\vee}Q \arrow[r, "\vartheta"] \arrow[d, "\mathsf{can}_B"] & Q \otimes_C {}^{\vee}Q \arrow[r] \arrow[d, "\mathsf{can}_C"] & 0 \\ 0 \arrow[r] & \mathscr{J} \arrow[r] & \mathscr{L} \arrow[r, "\overline{\pi}"] & \overline{\mathscr{L}} \arrow[r] & 0 \end{tikzcd} \tag{3.5}$$

Proposition 3.8. *In diagram* (3.5), *we have the following equality* $\mathsf{can}_B(\mathrm{Ker}(\vartheta)) = \mathscr{J}$. *In particular, the map* can_C *of Eq.* (3.4) *is an isomorphism of R-corings.*

Proof. The inclusion $\mathsf{can}_B(\mathrm{Ker}(\vartheta)) \subseteq \mathscr{J}$ is clear from the commutative diagram (3.5). Conversely, consider arbitrary elements $y \in \mathscr{L}$ and $r \in R$. We need to show that $y\mathsf{g}_r \in \mathsf{can}_B(\mathrm{Ker}(\vartheta))$, where g_r are as in the proof of Lemma 3.5. There is no loss of generality if we assume that $y = x\pi(a \otimes \varphi)$, for some $x \in \mathscr{L}$ and $a \in A$, $\varphi \in {}^*A$. Since can_B is, by Theorem 2.18, bijective, there exists $u \in Q \otimes_B {}^{\vee}Q$ such that $x = \mathsf{can}_B(u)$. In view of this, $y\mathsf{g}_r = \mathsf{can}_B(u(ar \otimes_B \varphi - a \otimes_B r\varphi))$, as can_B is multiplicative. We need to check that $\vartheta(u(ar \otimes_B \varphi - a \otimes_B r\varphi)) = 0$. However, this is directly obtained from the following equality

$$\vartheta((u_n \otimes_B \varphi_n)(ar \otimes_B - a \otimes_B r\varphi)) = 0, \text{ for every } u_n \in Q_n,\ \ \varphi_n \in {}^*Q_n,$$

whose proof follows by induction on n. The last statement to prove is a consequence of the first one, since the diagram (3.5) has exact rows. □

Our second main result is the following theorem.

Theorem 3.9. *Let R be an algebra over a commutative ground ring $\Bbbk$, and A an R-ring which is finitely generated and projective as left R-module. Consider the associated left R-bialgebroid $\mathscr{L}$ stated in Proposition* 2.1 *and $\mathscr{J}$ the coideal of $\mathscr{L}$ generated by the set of elements $\{1_{\mathscr{L}}(r \otimes 1^o - 1 \otimes r^o)\}_{r \in R}$. Denote by $\overline{\mathscr{L}} = \mathscr{L}/\mathscr{J}$ the corresponding quotient R-coring. Let $C = R^{(\mathbb{N})} \oplus R^{(\mathbb{N})}$ be the ring with enough orthogonal idempotents induced from the small $\Bbbk$-linear category $R(\mathbb{N})$ defined by relations* (3.1). *Consider the cochain complex $Q_{\bullet}$ given in Sec.* 2.2 *with its canonical right unital C-action and left $\overline{\mathscr{L}}$-coaction as in Lemma* 3.6. *Then the following statements are equivalent*

(1) *The right module $\overline{\mathscr{L}}^{l}_{R}$ is flat and the functor $Q \otimes_C - : {}_C\mathsf{Mod} \longrightarrow {}^{\overline{\mathscr{L}}}\mathsf{Comod}$ is an equivalence of categories*;
(2) *Q_C is a faithfully flat unital module.*

Proof. By the left version of the generalized faithfully flat descent Theorem [14, Theorem 5.9], we know that $(Q \otimes_C {}^{\vee}Q)_R$ is flat and $Q \otimes_C - : {}_C\mathsf{Mod} \to {}^{Q \otimes_C {}^{\vee}Q}\mathsf{Comod}$ is an equivalence of categories, if and only if Q_C is faithfully flat. We then deduced the stated equivalence by using the isomorphism of R-corings $\mathsf{can}_C : Q \otimes_C {}^{\vee}Q \cong \overline{\mathscr{L}}$ established in Proposition 3.8. $\square$

Notice that, if Q_C is faithfully flat, then the inverse functor of $Q \otimes_C - : {}_C\mathsf{Mod} \to {}_{\overline{\mathscr{L}}}\mathsf{Comod}$ is given by the cotensor product ${}^{\vee}Q \square_{\overline{\mathscr{L}}} - : {}_{\overline{\mathscr{L}}}\mathsf{Comod} \to {}_C\mathsf{Mod}$. Here the structure of $(C, \overline{\mathscr{L}})$-bicomodule of ${}^{\vee}Q$ is deduced, as was observed in Sec. 3.1, from that of Q using the fact that each of the Q_n's is finitely generated and projective left R-module.

3.3. *Conditions under which Q_C is faithfully flat*

As was seen in Theorems 3.3 and 3.9, a sufficient and necessary condition for establishing an equivalence of categories of left comodules and chain complexes, is the faithfully flatness of the unital right module Q. The proof of this fact is actually the most difficult task in this theory. In this subsection we will analyze assumptions under which Q_C is faithfully flat.

The following is our third main result.

Theorem 3.10. *The notations and assumptions are that of Theorem* 3.9. *Assume further that A_R is finitely generated and projective, and the cochain complex $Q_\bullet$ is exact and splits, in the sense that, for every $m \geq 1$, $Q_m = \partial Q_{m-1} \oplus \overline{Q}_m =$ $\mathrm{Ker}(\partial) \oplus \overline{Q}_m$ as right R-modules, for some right R-module $\overline{Q}_m$. Then Q_C is a flat module. Furthermore, if $\Bbbk$ is a field and R is a division $\Bbbk$-algebra, then Q_C is faithfully flat.*

Proof. We first consider the following family of right R-modules

$$Q^{(m)} = \begin{cases} \partial Q_m \oplus \overline{Q}_m & \text{for } m \geq 1, \\ \partial Q_0 \oplus Q_0 & \text{for } m = 0, \end{cases}$$

which we claim to be a family of right unital flat C-modules. Using this claim we can easily deduce that Q_C is a flat module since we know that $Q_C = \bigoplus_{m \geq 0} Q_C^{(m)}$. The structure of unital right C-module of each $Q^{(m)}$ is given as follows: Denote by $\mathsf{i}_m : \partial Q_m \to Q^{(m)}$, $\bar{\mathsf{i}}_m : \overline{Q}_m \to Q^{(m)}$ the canonical injections and by j_m, $\bar{\mathsf{j}}_m$ their canonical projections. For every element $u^{(m)} \in Q^{(m)}$, we set

$$u^{(m)}\mathfrak{h}_n = \begin{cases} 0 & \text{if } n \notin \{m, m+1\}, \\ \bar{\mathsf{i}}_m \bar{\mathsf{j}}_m(u^{(m)}) & \text{if } n = m, \\ \mathsf{i}_m \mathsf{j}_m(u^{(m)}) & \text{if } n = m+1, \end{cases}$$

$$u^{(m)}\mathfrak{u}_n = \begin{cases} 0 & \text{if } n \neq m, \\ \mathsf{i}_m(\gamma_m \bar{\mathsf{j}}_m(u^{(m)})) & \text{if } n = m, \end{cases}$$

where $\gamma_m : \overline{Q}_m \to Q_m \to \partial Q_m$. That is, the obtained cochain complexes have the following form

$$
\begin{tikzcd}
Q^{(m)}_{\bullet} : 0 \arrow[r] & 0 \cdots\cdots 0 \arrow[r] & \overline{Q}_m \arrow[rr, dashed, "\gamma_m"] \arrow[dr] & & \partial Q_m \arrow[r] & 0 \arrow[r] & 0 \cdots\cdots \\
& & & Q_m \arrow[ur] & & &
\end{tikzcd}
\tag{3.6}
$$

Put $\mathfrak{e}_{n,n+1} = \mathfrak{h}_n + \mathfrak{h}_{n+1}$, for every $n \geq 0$. These are idempotents elements in C, and the induced rings, i.e. $\mathfrak{e}_{n,n+1} C \mathfrak{e}_{n,n+1}$ are all isomorphic to the upper-triangular matrices over R. That is, of the form

$$C_{n,n+1} := \mathfrak{e}_{n,n+1} C \mathfrak{e}_{n,n+1} = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}, \quad \text{for every } n \in \mathbb{N}.$$

It is clear that, for every $m \geq 0$, we have $Q^{(m)} \mathfrak{e}_{m,m+1} = Q^{(m)}$. Therefore, there is an isomorphism of right unital C-modules

$$Q^{(m)} \mathfrak{e}_{m,m+1} \bigotimes_{C_{m,m+1}} \mathfrak{e}_{m,m+1} C \cong Q^{(m)}. \tag{3.7}$$

Next we will show that each of the right $C_{m,m+1}$-modules $Q^{(m)} \mathfrak{e}_{m,m+1} = Q^{(m)}$ is finitely generated and projective. This fact, combined with the isomorphisms (3.7), establish the above claim.

For $m = 0$, it is clear that the right $C_{0,1}$-module

$$Q^{(0)} = R \oplus R = \begin{pmatrix} R & R \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix} C_{0,1}$$

is finitely generated and projective. Now take $m \geq 1$, under the hypothesis A_R is finitely generated and projective, we can show, as in Lemma 2.8, that each right R-module Q_m is also finitely generated and projective. Thus, we can consider a dual basis $\{(\overline{q}_{m,k}, \overline{q}^*_{m,k})\}_k$ for each right R-module $\overline{Q}_m$. In this way, we have a right $C_{m,m+1}$-linear map

$$\theta^*_{m,k} : Q^{(m)} \longrightarrow C_{m,m+1}, \quad \left[u^{(m)} \longmapsto \begin{pmatrix} \overline{q}^*_{m,k}(\bar{\mathfrak{j}}_m(u^{(m)}) & \overline{q}^*_{m,k}(\overline{x}_m) \\ 0 & 0 \end{pmatrix} \right],$$

where $\overline{x}_m \in \overline{Q}_m$ is the projection of $x_m \in Q_m = \partial Q_{m-1} \oplus \overline{Q}_m$, defined by $\mathfrak{j}_m(u^{(m)}) = \partial x_m \in \partial Q_m$. We should mention that, under our assumptions, the maps $\theta^*_{m,k}$ are well defined. Effectively, if there is some other element $y_m \in Q_m$ such that $\mathfrak{j}_m(u^{(m)}) = \partial x_m = \partial y_m$, then $x_m - y_m \in \mathrm{Ker}(\partial_m) = \partial Q_{m-1}$ which means that they have equal image $\overline{x}_m = \overline{y}_m$ in $\overline{Q}_m \cong Q_m / \partial Q_{m-1}$. It is convenient to check that $\theta^*_{m,k}$ are right $C_{m,m+1}$-linear. But first we will identify the right module $\overline{Q}_m$ with the quotient of Q_m, $\overline{Q_m} = Q_m / \partial Q_{m-1}$. The right $C_{m,m+1}$-action of $Q^{(m)}$ is given as follows: Take an element $u^{(m)} \in Q^{(m)}$ and write it in the form $u^{(m)} = (\overline{q_m}, \partial p_m)$ for some elements $q_m, p_m \in Q_m$. Here $\mathsf{j}_m(u^{(m)}) = \partial p_m$ and $\bar{\mathsf{j}}_m(u^{(m)}) = \overline{q_m}$. So

$$(\overline{q_m}, \partial p_m) \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} = \Big(\overline{q_m} r_{11}, \partial q_m r_{12} + \partial p_m r_{22}\Big),$$

for every element $\begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$ in $C_{m,m+1}$. Therefore,

$$\begin{aligned}
\theta^*_{m,k}\left((\overline{q_m}, \partial p_m) \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}\right) &= \theta^*_{m,k}\left(\overline{q_m} r_{11}, \partial q_m r_{12} + \partial p_m r_{22}\right) \\
&= \begin{pmatrix} \overline{q}^*_{m,k}(\overline{q_m} r_{11}) & \overline{q}^*_{m,k}\left(\overline{q_m} r_{12} + \overline{p}_m r_{22}\right) \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \overline{q}^*_{m,k}(\overline{q_m} r_{11}) & \overline{q}^*_{m,k}(\overline{q_m} r_{12}) + \overline{q}^*_{m,k}(\overline{p}_m r_{22}) \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \overline{q}^*_{m,k}(\overline{q_m}) r_{11} & \overline{q}^*_{m,k}(\overline{q_m}) r_{12} + \overline{q}^*_{m,k}(\overline{p}_m) r_{22} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \overline{q}^*_{m,k}(\overline{q_m}) & \overline{q}^*_{m,k}(\overline{p}_m) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} \\
&= \theta^*_{m,k}(\overline{q_m}, \partial p_m) \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}.
\end{aligned}$$

Take an arbitrary element $(\overline{q_m}, \partial p_m) \in Q^{(m)}$, we have

$$\begin{aligned}
(\overline{q_m}, \partial p_m) &= (\overline{q_m}, 0) + (0, \partial p_m) \\
&= (\overline{q_m}, 0) + (\overline{p_m}, 0) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \sum_k (\overline{q}_{m,k} \overline{q}^*_{m,k}(\overline{q_m}), 0) + \sum_k (\overline{q}_{m,k} \overline{q}^*_{m,k}(\overline{p_m}), 0) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \sum_k (\overline{q}_{m,k}, 0) \begin{pmatrix} \overline{q}^*_{m,k}(\overline{q_m}) & 0 \\ 0 & 0 \end{pmatrix} + \sum_k (\overline{q}_{m,k}, 0) \begin{pmatrix} \overline{q}^*_{m,k}(\overline{p_m}) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \sum_k (\overline{q}_{m,k}, 0) \begin{pmatrix} \overline{q}^*_{m,k}(\overline{q_m}) & 0 \\ 0 & 0 \end{pmatrix} + \sum_k (\overline{q}_{m,k}, 0) \begin{pmatrix} 0 & \overline{q}^*_{m,k}(\overline{p_m}) \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

$$= \sum_k (\overline{q}_{m,k}, 0) \begin{pmatrix} \overline{q}^*_{m,k}(\overline{q_m}) & \overline{q}^*_{m,k}(\overline{p_m}) \\ 0 & 0 \end{pmatrix}$$

$$= \sum_k (\overline{q}_{m,k}, 0)\theta^*_{m,k}\left(\overline{q_m}, \partial p_m\right),$$

which shows that $\{((\overline{q}_{m,k}, 0), \theta^*_{m,k})\}_k$ is a dual basis for the right $C_{m,m+1}$-module $Q^{(m)}$, and this finishes the proof of the main statement.

If we assume now that $\Bbbk$ is a field and R is a division $\Bbbk$-algebra, then one can show as follows that each $Q^{(m)}\mathfrak{e}_{m,m+1}$ is a progenerator in the category of right $C_{m,m+1}$-modules. This means that $Q^{(m)}\mathfrak{e}_{m,m+1} \otimes_{C_{m,m+1}} - : {}_{C_{m,m+1}}\mathsf{Mod} \to {}_R\mathsf{Mod}$ is a faithful functor. By identifying each ring $C_{m,m+1}$ with the generalized matrix ring $T := \left(\begin{smallmatrix} R & R \\ 0 & R \end{smallmatrix}\right)$, we know that $T = eT \oplus (1-e)T$, where e is the obvious idempotent element. The structure of right T-module of $Q^{(m)}$ is given by the decomposition $Q^{(m)}_T = \partial Q_m \oplus \overline{Q}_m$ with a surjective canonical map $\gamma_m : \overline{Q}_m \to \partial Q_m$ of (3.6). Since R is a division ring and each component of $Q^{(m)}$ is by assumption finite-dimensional with $d = \dim_R(\overline{Q}_m) \geq \dim_R(\partial Q_m) = d'$, we can split $Q^{(m)}$ as

$$Q^{(m)} \cong (eT)^d \oplus ((1-e)T)^{d-d'},$$

and this shows that $Q^{(m)}_T$ is a progenerator. Notice, that if $d = d'$ then we still have the faithfully property. Let $f : X \to Y$ be a morphism of right unital C-modules such that $Q \otimes_C f = 0$. Hence $Q^{(m)} \otimes_C f = 0$, for every $m \geq 0$, as $Q_C = \oplus_{m \geq 0} Q^{(m)}$. Therefore, we have

$$0 = Q^{(m)} \otimes_C f \cong Q^{(m)}\mathfrak{e}_{m,m+1} \bigotimes_{C_{m,m+1}} \mathfrak{e}_{m,m+1}C \otimes_C f,$$

$$\forall\, m \geq 0 \Rightarrow \mathfrak{e}_{m,m+1}C \otimes_C f = 0, \quad \forall\, m \geq 0.$$

This means that $\mathfrak{h}_m C \otimes_C f = 0$, for every $m \geq 0$, and so $f = 0$. This shows that $Q \otimes_C -$ is a faithful functor, which completes the proof. $\square$

Remark 3.11. As one can see, the hypothesis on the complex $Q_\bullet$ in Theorem 3.10, is not easy to check. However, under further conditions on the ring extension $R \to A$, this hypothesis is satisfied. For instance, it is clear from Lemma 2.5 and Remark 2.6 that it is satisfied by assuming that the ring extension $R \to A$ splits either in the category of right or left R-modules. Obviously this includes the case when A is free as right (or left) R-module with 1_A as an element of the canonical basis. In particular, this is the case when R is a division ring.

Corollary 3.12. *Let D be a division $\Bbbk$-algebra over a field $\Bbbk$, and A a D-ring which is finite dimensional as left and right D-vector space with dimension ≥ 2. Consider the associated left D-bialgebroid $\mathscr{L}$ given by Proposition* 2.1 *and its coideal $\mathscr{J}$ of Lemma* 3.5*. Then the category* $\mathrm{Ch}_+(D)$ *of chain complexes of left D-vector spaces is equivalent to the category of left $(\mathscr{L}/\mathscr{J})$-comodules.*

Proof. It follows from Theorems 3.9 and 3.10. □

3.4. *The main example*

Here we will explain why Pareigis's example [25], even in the noncommutative case, always works. Thus, we will check using the first statement of Theorem 3.10 that the cochain complex $Q_\bullet$ associated to the example of the R-ring A considered in Example 2.3, always satisfies condition (2) of Theorem 3.9. In this way the category $\mathrm{Ch}_+(R)$ of chain complexes of left R-modules is always equivalent to the category of left $\overline{\mathscr{L}(A)}$-comodules, where $\mathscr{L}(A)$ is the left R-bialgebroid described in Example 2.3.

Recall from Example 2.3, the R-ring $A = R \oplus Rt$ which is the trivial generalized ring extension of R. Set $1_A = (1, 0)$ and $\mathfrak{t} = (0, t)$, so we have $\mathfrak{t}^2 = 0$. It is easily seen that the kernel of the multiplication of A, i.e. $K = \mathrm{Ker}(A \otimes_R A \to A)$ is free as a left and right R-module with basis $\{\partial\mathfrak{t}, \mathfrak{t}\partial\mathfrak{t}\}$. In fact K is a free A-module with rank one and basis $\partial\mathfrak{t}$. We summarize the properties of the cochain complex $Q_\bullet$, as follows.

Proposition 3.13. *The cochain complex $Q_\bullet$ associated to the trivial generalized ring $A = R \oplus Rt$, fulfils the following properties*:

(i) *For every $m \geq 2$, Q_m is free as a left and right R-module with rank two, and its basis (on both sides) is given by the set $\{\mathfrak{t}\partial\mathfrak{t} \otimes_A \cdots \otimes_A \partial\mathfrak{t}, \partial\mathfrak{t} \otimes_A \cdots \otimes_A \partial\mathfrak{t}\}$.*
(ii) *Q is a homotopically trivial complex.*
(iii) *Q_C is faithfully flat module.*

Proof. (i) This is proved by induction on m.

(ii) The homotopy is given by switching the dual basis. Let $q_m = \partial\mathfrak{t} \otimes_A \cdots \otimes_A \partial\mathfrak{t}$, ($(m-1)$-times) and $q_1 = 1_A$ be the generating element of Q_m. Then we define a homotopy $h_m : Q_{m+1} \to Q_m$ by sending $q_{m+1} \mapsto \mathfrak{t}q_m$ and $\mathfrak{t}q_{m+1} \mapsto q_m$, h_0 is the first projection.

(iii) The fact that Q_C is flat follows from Theorem 3.10, since we know that $Q_\bullet$ is exact and splits either by Lemma 2.5, or by item (ii) and [17, Théorème 2.4.1]. Following the notations of the proof of Theorem 3.10, we can easily see that each right $T = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$-module $Q^{(m)} = \partial Q_{m-1} \oplus \mathfrak{t}q_m R$ is isomorphic to eT, where e is the canonical idempotent of T. Henceforth, the same argument of the last part of the proof of Theorem 3.10 serves to deduce that Q_C is actually a faithfully flat module. □

Corollary 3.14. *Let R be any $\Bbbk$-algebra and $A = R \oplus Rt$ its trivial generalized extension. Consider the left R-bialgebroid $\mathscr{L}(A)$ described in Example 2.3 and its quotient R-coring $\overline{\mathscr{L}(A)}$ by the left ideal $\langle 1_{\mathscr{L}(A)}(r \otimes 1^o - 1 \otimes r^o)\rangle$. Then the functor $Q \otimes_C -$ establishes an equivalence between the categories of chain complexes of left R-modules and the category of left $\overline{\mathscr{L}(A)}$-comodules. In particular, if R is a commutative ring, then $Q \otimes_C -$ establishes in fact a monoidal equivalence.*

Proof. The main claim is an immediate consequence of Proposition 3.13 and Theorem 3.9. In the last statement, the functor in question can be shown to be monoidal using a similar proof of Lemma 3.2. □

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References

[1] A. Ardizzoni, L. El Kaoutit and C. Menini, Coendomorphism left bialgebroids, *J. Algebra Appl.* (In press).

[2] M. Artin, On Azumaya algebras and finite dimensional representations of rings, *J. Algebra* **11** (1969) 532–563.

[3] G. Böhm, Hopf algebroids, in *Handbook of Algebra*, Vol. 6 (Elsevier/North-Holland, Amsterdam, 2009), pp. 173–235.

[4] A. Bruguières, Théorie Tannakienne non commutative, *Comm. Algebra* **22** (1994) 5817–5860.

[5] N. Bourbaki, *Éléments de Mathématique: Algèbre Commutative, Chapitres* 1 *à 4* (Masson, Paris, 1985).

[6] T. Brzeziński and G. Militaru, Bialgebroids, $\times_R$-Bialgebras and duality, *J. Algebra* **251** (2002), 279–294.

[7] T. Brzeziński and R. Wisbauer, *Corings and Comodules*, London Mathematical Society, Vol. 309 (Cambridge University Press, 2003).

[8] S. Caenepeel, E. De Groot and J. Vercruysse, Constructing infinite comatrix corings from colimits, *Appl. Categ. Structures* **14**(5–6) (2006) 539–565.

[9] H. Cartan, Homologie et cohomologie d'une algèbre graduée, *Séminiare Henri Cartan* **11**(2) (1958-1959), 1–20.

[10] P. Deligne, Catégories tannakiennes, in *The Grothendieck Festschrift*, Vol. II, eds. P. Cartier *et al.*, Progress in Mathematics, Vol. 87 (Birkhäuser, Boston, MA, 1990) pp. 111–195.

[11] A. Dold, Homology of symmetric products and other functors of complexes, *Ann. Math.* **68**(1) (1958) 54–80.

[12] L. El Kaoutit, Corings over rings with local units, *Math. Nachr.* **282**(5) (2009) 726–747.

[13] L. El Kaoutit and J. Gómez-Torrecillas, Comatrix corings: Galois coring, Descent theory, and a structure theorem for cosemisimple corings, *Math. Z.* **244** (2003) 887–906.

[14] L. El Kaoutit and J. Gómez-Torrecillas, Infinite comatrix corings, *Int. Math. Res. Not.* **39** (2004) 2017–2037.

[15] L. El Kaoutit, J. Gómez-Torrecillas and F. J. Lobillo, Semisimple corings, *Algebra Colloq.* **11**(4) (2004) 427–442.
[16] P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962) 323–448.
[17] R. Godement, *Topologie Algébrique et Théorie des Faisceaux*, Troisième édition (Hermann, Paris 1973).
[18] P. G. Goerss and J. F. Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics, Vol. 174 (Birkhäuser, Basel, 1999).
[19] J. Gómez-Torrecillas and J. Vercruysse, Comatrix corings and Galois comodules over firm rings, *Algebr. Rep. Theory* **10** (2007) 271–306.
[20] P. H. Hai, Tannaka–Krein duality for Hopf algebroids, *Israel J. Math.* **167** (2008) 193–225.
[21] M. Hovey, *Model Categories*, Mathematical surveys and Monographs, Vol. 63 (American Mathematical Society Providence, RI, 1999).
[22] A. Joyal and R. Street, An introduction to Tannaka duality and quantum groups, in *Category Theory Proceedings, Comom 1990*, Lecture Notes in Mathematics, Vol. 1488 (Springer-Verlag, Berlin, 1991), pp. 411–492.
[23] S. Mac Lane, *Homology*, Grundlehren der Mathematischen Wissenschaften, Vol. 114 (Springer-Verlag, Berlin, 1963).
[24] P. McCrudden, Categories of representations of coalgebroids, *Adv. Math.* **154** (2000) 299–332.
[25] B. Pareigis, A non-commutative non-cocommutative Hopf algebra in "nature", *J. Algebra* **70** (1981) 356–374.
[26] P. Schauenburg, Bialgebras over noncommutative rings and a structure theorem for Hopf bimodules, *Appl. Categ. Structures* **6** (1998) 193–222.
[27] M. Sweedler, Groups of simple algebras, *Publ. Math. Inst. Hautes Etudes Sci.* **44** (1975) 79–189.
[28] M. Takeuchi, Groups of algebras over $A \otimes \overline{A}$, *J. Math. Soc. Japan* **29** (1977) 459–492.
[29] D. Tambara, The coendomorphism bialgebra of an algebra, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **37**(2) (1990) 425–456.
[30] R. Wisbauer, On Galois comodules, *Comm. Algebra* **34** (2006) 2683–2711.