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PRIME IDEALS OF THE COORDINATE RING OF QUANTUM SYMPLECTIC SPACE

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INTRODUCTION

In [1], K. R. Goodearl and E. S. Letzter study prime and primitive ideals in certain iterated Ore extensions of an infinite field $\mathbf{k}$ of arbitrary characteristic, which include several quantized algebras at non roots of unity, among them the quantized algebras $\mathcal{O}_q(\mathfrak{sp}\mathbf{k}^{2\times n})$ of symplectic spaces. The general framework to work in is to consider some group H acting as automorphism on a ring R which give the set $H-\mathrm{Spec}(R)$ consisting of all H-prime ideals of R. The *H-stratification* of the prime spectrum Spec(R) is then defined as

$$\mathrm{Spec}(R) = \biguplus_{J\in H-\mathrm{Spec}(R)} \mathrm{Spec}_J(R), \tag{1}$$

where each stratum $\mathrm{Spec}_J(R)$ consists of those prime ideals P of R such that $\bigcap_{h\in H} h(P) = J$.

In the case that H is a torus of rank n acting rationally on a noetherian algebra R over an infinite field $\mathbf{k}$ (see [1] for details), the strata $\mathrm{Spec}_J(R)$

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corresponding to completely prime H-invariant ideals J of R are described in [1, Theorem 6.6] as follows.

(a) For each completely prime H-invariant ideal J of R, there exists an Ore set $\mathcal{E}_J$ in the algebra R/J such that the localization map $R \to R/J \to R_J = (R/J)[\mathcal{E}_J^{-1}]$ induces a homeomorphism of $\mathrm{Spec}_J(R)$ onto $\mathrm{Spec}(R_J)$.
(b) Contraction and extension induce mutually inverse homeomorphisms between $\mathrm{Spec}(R_J)$ and $\mathrm{Spec}(Z(R_J))$, where $Z(R_J)$ is the centre of R_J.
(c) $Z(R_J)$ is a commutative Laurent polynomial ring over an extension of $\mathbf{k}$, in n of fewer indeterminates.

The foregoing description of the H-strata applies to iterated Ore extensions of $\mathbf{k}$ under suitable conditions ([1, Section 4]), which include $\mathcal{O}_q(\mathfrak{sp}\mathbf{k}^{2\times n})$. For such a type of iterated Ore extensions, there are finitely many H-prime ideals which are all completely prime.

The aim of this note is to give an more explicit description of the H-stratification of the spectra of the coordinate algebras of quantum symplectic spaces $\mathcal{O}_q(\mathfrak{sp}\mathbf{k}^{2\times n})$ in the following aspects.

1. We prove that the H-prime ideals are just the ideals generated by the admissible sets in the sense of [2]. More explicitly, consider the finite subset $\wp_n$ of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ as defined later in (7). The map $J \mapsto J \cap \wp_n$ gives a bijection between the H-prime ideals of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ and the admissible subsets of $\wp_n$ (Proposition 2.5). As a consequence, we compute the number of H-prime ideals and, hence, the number of H-strata (Corollary 2.6).

2. For each H-prime ideal J, let $T = J \cap \wp_n$ the corresponding admissible set. We give explicitly a McConnell-Pettit $\mathbf{k}$-algebra $\mathbf{P}(Q_T)$, which is strictly contained in R_J, such that the J-th stratum is described as

$$\mathrm{Spec}_J(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))) = \{P \in \mathrm{Spec}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))) | P \cap \wp_n = T\},$$

and it is homeomorphic to the spectrum of $\mathbf{P}(Q_T)$ (Theorem 3.4).

3. By using [3], we obtain that the each stratum is homeomorphic to the spectrum of the centre $Z(\mathbf{P}(Q_T))$ of $\mathbf{P}(Q_T)$ for a suitable admissible set T. We prove that the number of indeterminates in the Laurent polynomial ring $Z(\mathbf{P}(Q_T))$ over $\mathbf{k}$ is exactly the number of connected components of odd length in the connected decomposition of T (Proposition 3.7).

Our methods allow to give an effective description (modulo Commutative Algebra) of $\mathrm{Spec}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ for a given n (Theorem 3.10). This is possible because each prime ideal in the stratum $\mathrm{Spec}_T(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ is recognized as the inverse image under an explicitly defined algebra homomorphism Φ_T connecting $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ and the McConnell-Pettit algebra $\mathbf{P}(Q_T)$ (Theorem 3.4). It follows from [1, Corollary 6.9) that the primitive ideals of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ are precisely the maximal prime ideals of each stratum, which allows, in conjunction with our results, to deduce a clean description of the primitive spectrum $\mathrm{Prim}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ very close to [2, Theorem 7.1]. As an illustration, we compute $\mathrm{Spec}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ and $\mathrm{Prim}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times 2})))$, in the algebraically closed case (see the Figure 1).

1 DEFINITIONS AND BASIC PROPERTIES

Throughout this note, we will consider different quantum spaces, so we will use some convenient notation. Let $\Lambda = (\lambda_{ij})$ be a $p \times p$ matrix with entries in $\mathbf{k}$, such that $\lambda_{ii} = 1$ and $\lambda_{ji} = \lambda_{ij}^{-1}$. Consider the $\mathbf{k}$-algebra $\mathbf{k}_\Lambda[t_1, \ldots, t_p]$ generated by $t_1, \ldots, t_p$ subject to the relations $t_i t_j = \lambda_{ij} t_j t_i$. This is called the *coordinate algebra of the p-dimensional quantum affine space* associated to Λ and it is the iterated Ore extension

$$k_\Lambda[t_1, \ldots, t_p] = \mathbf{k}[t_1][t_2; \sigma_2] \cdots [t_p; \sigma_p] \tag{2}$$

where $\sigma_i(t_j) = \lambda_{ij} t_j$ for every $1 \leq j < i \leq m$. This $\mathbf{k}$-algebra is a noetherian domain, and its skew field of fractions is denoted by $\mathbf{k}_\Lambda(t_1, \ldots, t_p)$. An useful intermediate algebra is the McConnell–Pettit algebra $\mathbf{P}(Q_\Lambda) = \mathbf{k}_\Lambda[t_1^{\pm 1}, \ldots, t_p^{\pm 1}]$ (see [4]).

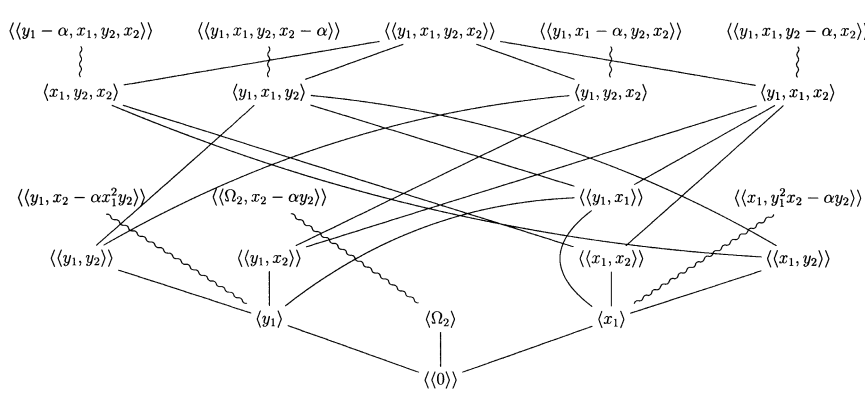

Figure 1. The prime spectrum of $\mathcal{O}_q(\mathfrak{sp}(k^{2\times 2}))$ (k is algebraically closed).

Definition 1.1. *Let q be a non-zero element in $\mathbf{k}$. I. M Musson found* [5, §1.1] *that the coordinate ring $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ of the quantum symplectic space* (*cf.* [6, *Definition* 14] *or* [7, §4]) *is the* $\mathbf{k}$*-algebra generated by $y_1, x_1, \ldots, y_n, x_n$ satisfying the following relations*

$$\begin{aligned} &y_j x_i = q^{-1} x_i y_j, \qquad y_j y_i = q y_i y_j \quad (1 \le i < j \le n) \\ &x_j x_i = q^{-1} x_i x_j, \qquad x_j y_i = q y_i x_j \quad (1 \le i < j \le n) \\ &x_i y_i - q^2 y_i x_i = (q^2 - 1) \sum_{l=1}^{i-1} q^{i-l} y_l x_l \quad (1 \le i \le n) \end{aligned} \tag{3}$$

By [2, Proposition 1.10] or [8, Example 6], $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ can be written as an iterated Ore extension

$$R_0 \subseteq R_1 \subseteq \cdots \subseteq R_n = \mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$$

where $R_0 = \mathbf{k}$ and $R_k = R_{k-1}[y_k; \alpha_k][x_k; \beta_k, \delta_k]$ for $k \ge 1$, with

$$\begin{aligned} &\alpha_k(x_l) = q^{-1} x_l = \beta_k(x_l) && (1 \le l < k \le n) \\ &\alpha_k(y_l) = q y_l = \beta_k(y_l) && (1 \le l < k \le n) \\ &\beta_k(y_k) = q^2 y_k && (1 < k \le n) \\ &\delta_k(R_{k-1}) = 0, \ \ \delta_k(y_k) = (q^2 - 1) \sum_{l-1}^{k-1} q^{k-l} y_l x_l && (1 \le k \le n) \end{aligned}$$

The quantum space *attached* to $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ is $\mathbf{k}_{Q_n}[Y_1, X_1, \ldots, Y_n, X_n]$, where Q_n is the matrix

$$\begin{array}{c} \\ Y_1 \\ X_1 \\ Y_2 \\ X_2 \\ \vdots \\ Y_n \\ X_n \end{array} \begin{array}{c} \begin{array}{ccccccc} Y_1 & X_1 & Y_2 & X_2 & \cdots & Y_n & X_n \end{array} \\ \begin{pmatrix} 1 & q^{-2} & q^{-1} & q^{-1} & \cdots & q^{-1} & q^{-1} \\ q^2 & 1 & q & q & \cdots & q & q \\ q & q^{-1} & 1 & q^{-2} & \cdots & q^{-1} & q^{-1} \\ q & q^{-1} & q^2 & 1 & \cdots & q & q \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q & q^{-1} & q & q^{-1} & \cdots & 1 & q^{-2} \\ q & q^{-1} & q & q^{-1} & \cdots & q^2 & 1 \end{pmatrix} \end{array} \tag{4}$$

In order to classify the prime ideals of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ we shall assume that the parameter q is not a root of unity. Consider the elements $\Omega_i = \sum_{l=1}^{i} q^{i-l} y_l x_l$ $(i \ge 1)$. For $i = 0$, let us write $\Omega_0 = 0$. From [2, Lemma 1.3] we get

$$\begin{aligned} &\Omega_i y_k = q^2 y_k \Omega_i, \quad \Omega_i x_k = q^{-2} x_k \Omega_i \quad (k \le i) \\ &\Omega_i x_k = x_k \Omega_i, \quad \Omega_i y_k = y_k \Omega_i \quad (i < k) \\ &\Omega_i \Omega_k = \Omega_k \Omega_i \quad (\text{for all } i, k) \end{aligned} \tag{5}$$

$$\begin{aligned} \Omega_i &= \sum_{j<l\leq i} q^{i-l} y_l x_l + q^{i-j}\Omega_j, \quad (j \leq i) \\ x_i y_i - q^2 y_i x_i &= (q^2-1) q \Omega_{i-1} \\ x_i y_i - y_i x_i &= (q^2-1)\Omega_i. \end{aligned} \tag{6}$$

Remark 1.2. Since $\delta_k \beta_k = q^2 \beta_k \delta_k$ for every index $k > 1$ and q is not a root of unity, it follows from [9, Theorem 2.3] that every prime ideal of $\mathcal{O}(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ is completely prime.

Following [2], a subset T of

$$\wp_n = \{y_1, x_1, \Omega_1, \ldots, y_n, x_n, \Omega_n\} \tag{7}$$

is said to be *admissible* if it satisfies the following conditions

$$\begin{aligned} &(1)\ x_i \text{ or } y_i \in T \Leftrightarrow \Omega_i \text{ and } \Omega_{i-1} \in T,\ \forall i \geq 2. \\ &(2)\ x_1 \text{ or } y_1 \in T \Leftrightarrow \Omega_1 \in T. \end{aligned} \tag{8}$$

For such a set we denote by $\operatorname{ind}(T) = \{i \in \{1, \ldots, n\} \mid \Omega_i \in T\}$; an index $i \in \operatorname{ind}(T)$ is said to be *removable* if $y_i \in T$ and $x_i \in T$. We say that T is *connected* if for any i, j in $\operatorname{ind}(T)$ such that $i < k < j$, then $k \in \operatorname{ind}(T)$. Let $\mathcal{J}_T = \{j \in \{1, \ldots, n\} \mid y_j \in T\}$, and $\mathcal{I}_T = \{i \in \{1, \ldots, n\} \mid x_i \in T\}$, one can observe that T is an admissible set with no removable indices if and only if $\mathcal{J}_T \cap \mathcal{I}_T = \emptyset$. Let $S = T \cap \{y_1, x_1, \ldots, y_n, x_n\}$, where T is a connected admissible set, the *length* of T is defined by

$$\text{length(T)} = \begin{cases} |S| & \text{if } 1 \in \operatorname{ind}(T) \\ |S|+1 & \text{if } 1 \notin \operatorname{ind}(T) \end{cases}$$

Let $T_1 \cup T_2 \cup \cdots \cup T_r$ be the connected decomposition of T. The length of T is defined by $\operatorname{length}(T) = \sum_{k=1}^{r} \operatorname{length}(T_k)$. The reader is referred to [2] for more properties of admissible sets.

2 THE COMPUTATION OF THE *H*-PRIME IDEALS

Consider the algebraic torus $H = (\mathbf{k}^{\times})^n$ of rank n acting on $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ by $\mathbf{k}$-algebra automorphisms (see [1, 5.2] or [10, 3.5]). In this section we will show that the H-prime ideals of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ are just the ideals generated by admissible sets. We shall need some control on the Gelfand-Kirillov dimension of certain localizations of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$, which is provided by the following proposition.

Proposition 2.1. *Let W be any subset of $\{1, \dots, n\}$ and consider the multiplicative subset $\mathcal{Y}$ of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ generated by all $y_k,\ k \in W$. Then $\mathcal{Y}$ is a right Ore set and the Gelfand–Kirillov dimension of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))\mathcal{Y}^{-1}$ equals $2n$.*

Proof. The multiplicative subset generated by a single y_k is right Ore by [11, Lemma 1.4]. This, in conjunction with [12, Lemma 4.1], gives that $\mathcal{Y}$ is right Ore and, thus, the algebra $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))\mathcal{Y}^{-1}$ makes sense. It is well-known that $\mathrm{GKdim}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))) \leq \mathrm{GKdim}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))\mathcal{Y}^{-1})$, so we will prove the other inequality. Consider the $\mathbf{k}$-algebra S generated by the variables $y_1, x_1, \dots, y_n, x_n$, satisfying the relations (3) and new variables $z_k,\ k \in W$, with the following additional relations for each $k \in W$.

$$\begin{array}{ll} z_j z_k = q z_k z_j & (j \in W,\ j > k) \\ x_j z_k = q^{-1} z_k x_j & (1 \leq j \leq n, j \neq k) \\ x_k z_k = q^{-2} z_k x_k + (1 - q^2)\ \sum_{l=1}^{k-1} q^{k-l-2} y_l x_l z_k^2 & \\ y_j z_k = q z_k y_j & (1 \leq j < k) \\ y_j z_k = q^{-1} z_k y_j & (k < j \leq n) \\ y_k z_k = z_k y_k = 1 & \end{array} \tag{9}$$

There is a surjective homomorphism of algebras $S \to \mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))\mathcal{Y}^{-1}$ sending y_i to y_i, x_i to x_i and z_k to y_k^{-1}. Therefore, $\mathrm{GKdim}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))\mathcal{Y}^{-1}) \leq \mathrm{GKdim}(S)$ and, thus, it is enough to prove that this last dimension equals 2n. To see this, order the variables

$$z_{i_1} < \cdots z_{i_m} < y_1 < x_1 < \cdots < y_n < x_n$$

where $W = \{i_1 < \cdots < i_m\}$. Let $\leq_{\mathbf{w}}$ be the weighted lexicographical ordering on $\mathbb{N}^{2n+m}$ defined by the vector

$$\mathbf{w} = (\underbrace{1, \dots, 1}_{(m)}, 1, 2, 1, 4, \dots, 1, 2n,)$$

By [13, Proposition 3.2], S can be endowed with a finite-dimensional $\mathbb{N}^{2n+m}$-filtration with respect to the order $\leq_{\mathbf{w}}$ such that the associated $\mathbb{N}^{2n+m}$-graded algebra G(S) is semi-commutative, namely, it is generated by finitely many homogeneous elements $z_{i_1}, \dots, z_{i_m}, y_1, x_1, \dots, y_n, x_n$ that commute up to a nonzero scalar and, in addition, $y_k z_k = 0$ for every $k \in W$. Therefore, G(S) is a factor of the coordinate algebra of an *2n+m*-dimensional quantum affine space by the ideal generated by the elements $y_k z_k,\ k \in W$. By [14, Theorem 4.4.7] or [15, Theorem 4.10], it is clear that $\mathrm{GKdim}(G(S)) = 2n + m - m$ and, by [13, Corollary 2.12], we have $\mathrm{GKdim} G(S) = \mathrm{GKdim}(G(S)) = 2n$. □

Fix an admissible set T, and let $T = T_1 \cup T_2 \cup \dots T_r$ be the decomposition of T in connected components, with $i_l = \min(\mathrm{ind}(T_l))$, $j_l = \max(\mathrm{ind}(T_l))$. Notice that the following is always true: $j_l < i_{l+1} - 1$ for all $l \in \{1, \ldots, r-1\}$. Let Q_T be the matrix obtained from Q_n by deleting the rows and columns corresponding to the variables x_k with $k \in \mathcal{I}_T, x_{i_l}$, $l \in \{1, \ldots, r\}$ (we will not delete the row and the column corresponding to x_1 if $i_1 = 1$ and $x_1 \notin T$) and y_k with $k \in \mathcal{J}_T$. If $k \in \cup_{l=1}^{r}\{i_l+1, \ldots, j_l\}$, then v_k will denote x_k if $k \notin \mathcal{I}_T$ and y_k if $k \notin \mathcal{J}_T$.

The image of an element $r \in \mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ in the factor algebra $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))/\langle T\rangle$ will be denoted by $\bar{r}$. This algebra is generated by $\bar{x}_k, \bar{y}_k$, where $k \in \{1, \ldots, i_1\} \cup \{j_1+1, \ldots, i_2\} \cup \cdots \cup \{j_{r-1}+1, \ldots, i_r\} \cup \{j_r, \ldots, n\}$ and $\bar{v}_{i_l+1}, \ldots, \bar{v}_{jl}$ for all $l \in \{1, \ldots, r\}$.

If $k \in \bigcup_{l=1}^{r}\{i_l+1, \ldots, j_l\}$ then the symbol V_k will denote a variable X_k for $k \notin \mathcal{I}_T$, a variable Y_k for $k \notin \mathcal{J}_T$ and the absence of variable when $k \in \mathcal{I}_T \cap \mathcal{J}_T$. With this notation, define the quantum affine space A_T *attached* to $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))/\langle T\rangle$ as follows

$$\begin{aligned} A_T = \mathbf{k}_{Q_T}[&Y_1, X_1, \ldots, X_{i_1-1}, Y_{i_1}, V_{i_1+1}, \ldots, V_{j_1}, Y_{j_1+1}, \ldots, X_{i_r-1}, \\ &Y_{i_r}, V_{i_r+1}, \ldots, V_{j_r}, Y_{j_r+1}, \ldots, Y_n, X_n], \end{aligned} \tag{10}$$

and consider the algebra $B_T = A_T\mathbb{Y}_T^{-1}$, where $\mathbb{Y}_T$ is the multiplicative subset of A_T generated by all Y_k (it is understood that $Y_k = V_k$ for $k \in \{i_1+1, \ldots, j_1\} \cup \cdots \cup \{i_r+1, \ldots, j_r\} \backslash \mathcal{J}_T$). We will denote $R = \mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$, consider the algebra homomorphism $\Psi_T : R/\langle T\rangle \to B_T$ defined by

$$\begin{aligned}
\Psi_T(\bar{v}_k) &= V_k && (k = i_1+1, \ldots, j_1, \\
& && \quad \vdots \\
& && \quad i_r+1, \ldots, j_r, \ \textit{and}\ k \notin \mathcal{I}_T \cap \mathcal{J}_T) \\
\Psi_T(\bar{y}_k) &= Y_k && (k = 1, \ldots, i_1, \\
& && \quad j_1+1, \ldots, i_2 \\
& && \quad \vdots \\
& && \quad j_{r-1}+1 \ldots i_r \\
& && \quad j_r+1, \ldots, n) \\
\Psi_T(\bar{x}_k) &= X_k + qW_{k-1}Y_k^{-1} && (k = 1, \ldots, i_1-1, \\
& && \quad j_1+2, \ldots, i_2-1, \\
& && \quad \vdots \\
& && \quad j_{r-1}+2, \ldots, i_r-1, \\
& && \quad j_r+2, \ldots, n) \\
\Psi_T(\bar{x}_{j_l+1}) &= X_{j_l+1} && (l = 1, \ldots, r) \\
\Psi_T(\bar{x}_{i_l}) &= qW_{i_l-1}Y_{i_l}^{-1} && (l = 1, \ldots, r)
\end{aligned}$$

where $W_k = -Y_kX_k$ for every $k \geq 1$ and $W_0 = 0$.

For each $k \notin \mathcal{J}_T$, let $\bar{\mathcal{Y}}_k$ be the multiplicative subset of $R/\langle T \rangle$ generated by $\bar{y}_k$. By [11, 1.4], these are right Ore multiplicative subsets. This implies, after [12, Lemma 4.1], that the multiplicative subset $\bar{\mathcal{Y}}_T$ generated by the $\bar{y}_k$'s, with $k \notin \mathcal{J}_T$ is a right Ore set. Therefore, it makes sense to extend Ψ_T to $(R/\langle T \rangle)\bar{\mathcal{Y}}_T^{-1}$. Finally, let $\mathcal{Y}_T$ be the multiplicative subset of R generated by those y_k with $k \notin \mathcal{J}_T$. We know that $\mathcal{Y}_T \cap \langle T \rangle = \emptyset$ so by [16, Proposition 3.6.15] we have $\frac{R\mathcal{Y}_T^{-1}}{\langle T \rangle \mathcal{Y}_T^{-1}} \cong (\frac{R}{\langle T \rangle})\bar{\mathcal{Y}}_T^{-1})$. By composing Ψ_T with this isomorphism we get a homomorphism of algebras from $\frac{R\mathcal{Y}_T^{-1}}{\langle T \rangle \mathcal{Y}_T^{-1}}$ to B_T which is also denoted by Ψ_T.

A similar algebra homomorphism was given in [17, Section 3.2] for quantum Weyl algebras. The following is the symplectic version of [17, Proposition 3.2.1].

Proposition 2.2. *The mapping*

$$\Psi_T : \frac{R\mathcal{Y}_T^{-1}}{\langle T \rangle \mathcal{Y}_T^{-1}} \to B_T$$

is a **k**-*algebra isomorphism.*

Proof. It is clear that Ψ_T is surjective, and $\langle T \rangle \subseteq \ker(\Psi_T)$. From [12, Lemma 3.16], we have $\mathrm{GKdim}(\frac{R\mathcal{Y}_T^{-1}}{\langle T \rangle \mathcal{Y}_T^{-1}}) \leq \mathrm{GKdim}(R\mathcal{Y}_T^{-1}) - ht(\langle T \rangle)$ which implies, by Proposition 2.1, that $\mathrm{GKdim}(\frac{R\mathcal{Y}_T^{-1}}{\langle T \rangle \mathcal{Y}_T^{-1}}) \leq 2n - ht(\langle T \rangle)$. By [2, Theorem 3.3], $2n - ht(\langle T \rangle) = 2n - \mathrm{length(T)} = 2n - (\sum_{l=1}^{r}(j_l - i_l + 1) + \mathrm{cardinal}(\mathcal{I}_T \cap > \mathcal{J}_T))$. But this last number equals $\mathrm{GKdim}(B_T)$. Hence, $\mathrm{GKdim}(\frac{R\mathcal{Y}_T^{-1}}{\langle T \rangle \mathcal{Y}_T^{-1}}) \leq \mathrm{GKdim}(B_T)$. Since $\langle T \rangle$ is a completely prime ideal [2, Theorem 2.7], it follows from [12, Proposition 3.15] that Ψ_T is a **k**-algebra isomorphism. □

Let H denote the torus $(\mathbf{k}^{\times})^n$ and consider its action on $R = \mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ as given in [10, 3.5]

$$(h_1, \dots, h_n) \cdot y_i = h_i y_i.$$
$$(h_1, \dots, h_n) \cdot x_i = h_i^{-1} x_i.$$

For any subset $X \subseteq \mathbb{N}_n = \{1, 2, \dots, n\}$, we denote by H_X the torus $\{(h_i)_{i \in X} | h_i \in (\mathbf{k}^x)\}$. Let T be an admissible set of R and A_T the quantum space attached to $R/\langle T \rangle$. We denote by N_T the set of indices of the variables that appear in A_T, this is a subset of $\mathbb{N}_n$. By H_T we denote the torus H_{N_T}. The torus H_T acts on the variables appearing in A_T as follows,

$$(h_i)_{i \in N_T} \cdot Y_k = h_k Y_k, \quad k \in N_T$$
$$(h_i)_{i \in N_T} \cdot X_l = h_l^{-1} X_l, \quad l \in N_T.$$

For example if $n=5$, $T=\{y_1,x_1,\Omega_1\}\cup\{\Omega_3,y_4,\Omega_4\}$ then $A_T=\mathbf{k}_{Q_T}[Y_2,X_2,Y_3,X_4,Y_5,X_5]$ and so $N_T=\{2,3,4,5\}$; the action of H_T on A_T is

$$\begin{aligned} h\cdot Y_2 &= h_2Y_2, \quad h.X_2=h_2^{-1}X_2\\ h\cdot Y_3 &= h_3Y_3\\ h\cdot X_4 &= h_4^{-1}X_4\\ h\cdot Y_5 &= h_5Y_5, \quad h.X_5=h_5^{-1}X_5 \end{aligned}$$

for any $h\in H_T$.

Consider the canonically extended action of H_T to the localization $B_T=A_T\mathbb{Y}^{-1}$. For each $h\in H_T$, we have the following automorphism of $R/\langle T\rangle\bar{\mathcal{Y}}_T^{-1}$

$$\frac{R}{\langle T\rangle}\bar{\mathcal{Y}}_T^{-1}\overset{\Psi_T}{\rightarrow}B_T\overset{h}{\rightarrow}B_T\overset{\Psi_T^{-1}}{\rightarrow}\frac{R}{\langle T\rangle}\bar{\mathcal{Y}}_T^{-1}.$$

where h denote the extension of h to B_T.

Definition 2.3. *We define the action of the torus H_T on $\frac{R}{\langle T\rangle}\bar{\mathcal{Y}}_T^{-1}$ as follows. Given $h\in H_T$, define*

$$h\cdot x=(\Psi_T^{-1}h\Psi_T)(x),$$

for every $x\in\frac{R}{\langle T\rangle}\bar{\mathcal{Y}}_T^{-1}$.

Lemma 2.4. *Consider H_T as a factor group of the torus H. The action of H_T induced on $R/\langle T\rangle$ by that of H coincides with the action given in Definition* 2.3.

Proof. The $\mathbf{k}$-algebra $R/\langle T\rangle$ is generated by the elements

$$\begin{aligned} \bar{y}_l &= y_l+\langle T\rangle, \quad l\notin\mathcal{J}_T,\\ \bar{x}_k &= x_k+\langle T\rangle, \quad k\notin\mathcal{I}_T. \end{aligned}$$

Let $h\in H_T$ and $l\notin\mathcal{J}_T$, then $\Psi_T^{-1}h\Psi_T(\bar{y}_l)=\Psi_T^{-1}h(Y_l)=h_l\Psi_T^{-1}(Y_l)=h_l\bar{y}_l$. Now let $k\notin\mathcal{I}_T$, we know that

$$\Psi_T(\bar{x}_k)=\begin{cases} X_k & \text{if } k-1\in\operatorname{ind}(T)\\ qW_{k-1}Y_k^{-1} & \text{if } k-1\notin\operatorname{ind}(T),\ k\in\operatorname{ind}(T)\\ X_k+qW_{k-1}Y_k^{-1} & \text{if } k-1\notin\operatorname{ind}(T),\ k\notin\operatorname{ind}(T)\end{cases}$$

In the first case, $\Psi_T^{-1}h\Psi_T(\bar{x}_k)=\Psi_T^{-1}h(X_k)=h_k^{-1}\Psi_T^{-1}(X_k)=h_k^{-1}\bar{x}_k$. In the second case, $\Psi_T^{-1}h\Psi_T(\bar{x}_k)=\Psi_T^{-1}h(qW_{k-1}Y_k^{-1})=h_k^{-1}\bar{x}_k$, because $h(W_l)=W_l$ for any $l\notin\operatorname{ind}(T)$.

In the third case, $\Psi_T^{-1} h \Psi_T(\bar{x}_k) = \Psi_T^{-1}(h_k^{-1} X_k + q W_{k-1} h_k^{-1} Y_k^{-1}) = h_k^{-1} \Psi_T^{-1} (X_k + q W_{k-1} Y_k^{-1}) = h_k^{-1} \bar{x}_k$. The lemma now is clear. □

Let us denote by $\mathcal{A}_n(\mathcal{O}_q(\mathfrak{sp}\mathbf{k}^{2\times n}))$ the set of all admissible sets of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$.

Proposition 2.5. *There is a bijection* ζ *between* H-Spec$(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ *and* $\mathcal{A}_n(\mathcal{O}_q(\mathfrak{sp})(\mathbf{k}^{2\times n}))$ *defined by*

$$\begin{array}{llcl} \zeta: & H-\mathrm{Spec}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))) & \to & \mathcal{A}_n(\mathcal{O}_q(\mathfrak{sp}\mathbf{k}^{2\times n})) \\ & J & \mapsto & J \cap \wp_n. \end{array}$$

With the inverse maps

$$\begin{array}{llcl} \zeta^{-1}: & \mathcal{A}_n(\mathcal{O}_q(\mathfrak{sp}\mathbf{k}^{2\times n})) & \to & H-\mathrm{Spec}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))) \\ & T & \mapsto & \langle T \rangle. \end{array}$$

Proof. By [2, Theorem 2.7], each admissible set generates a prime ideal which is clearly H-invariant. Thus ζ^{-1} is well defined. If J is an H-prime ideal of $R = \mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$, then by [1, Proposition 4.2] it is a completely prime ideal. So $J \cap \wp_n = T$ is an admissible set of R. This shows that ζ is well defined. By [2, Theorem 3.3] we have $\zeta\zeta^{-1} = id$. Let us show that $J = \langle T \rangle$ for every H-prime ideal J, when $J \cap \wp_n = T$, which gives $\zeta^{-1}\zeta = id$. Suppose that $\langle T \rangle \subsetneq J$ for a contradiction. By the Lemma 2.4 the ideals $(J/\langle T \rangle)\overline{\mathcal{Y}}_T^{-1}$ and $\mathcal{P} = \Psi_T((J/\langle T \rangle)\overline{\mathcal{Y}}_T^{-1})$ are H_T-prime ideals of $\frac{R}{\langle T \rangle}\overline{\mathcal{Y}}_T^{-1}$ and B_T respectively. Therefore B_T is not H-simple. Let $\overline{\mathcal{J}}_T$ denotes $\mathbb{N}_n \backslash \mathcal{J}_T = \{j_1, \ldots, j_r\}$. It is clear that B_T is an iterated Ore extension of the form

$$B_T = \mathbf{k}_{\overline{Q}}[Y_{j_1}^{\pm 1}, \ldots, Y_{j_r}^{\pm 1}][X_{i_1}, \beta_{i_1}] \cdots [X_{i_t}, \beta_{i_t}].$$

where $\mathbf{k}_{\overline{Q}}[Y_{j_1}^{\pm 1}, \ldots, Y_{j_r}^{\pm 1}]$ is the McConnell-Pettit algebra associated to a suitable matrix $\overline{Q}$. The $\mathbf{k}$-automorphisms β_{i_l}, $l = 1, \ldots, t$, arise by the semi-commutativity of X_{i_l} with $Y_{j_1}, \ldots, Y_{j_t}$ and $X_{i_1}, \ldots, X_{i_{l-1}}$. Let us denote

$$B_T^0 = \mathbf{k}_{\overline{Q}}[Y_{j_1}^{\pm 1}, \ldots, Y_{j_r}^{\pm 1}]$$
$$B_T^l = B_T^0[X_{i_1}, \beta_{i_1}] \cdots [X_{i_l}, \beta_{i_l}], \quad l = 1, \ldots, t.$$

The restriction of the action of H_T on each $\mathbf{k}$-algebra B_T^l, $l = 1, \ldots, t$, satisfies the hypothesis of [1, 3.1]. To see this claim take $l \in \{1, \ldots, t\}$. Suppose that there exists $k_0 \in \{1, \ldots, r\}$ such that $j_{k_0} = i_l$. Define, in this case, an element $h_0 = (h_i)_{i \in N_T} \in H_T$ by

$$h_i = \begin{cases} q^2 & \text{if } i = j_{k_0}, \\ q & \text{if } i \neq j_{k_0}. \end{cases}$$

Otherwise, take $h_0 = (q, \ldots, q) \in H_T$. The restriction of h_0 to B_T^{l-1}, coincides with the $\mathbf{k}$-automorphism β_{i_l}. Since q is not a root of unity, we can apply [1, Lemma 3.3], in each iteration. Denote by $\mathcal{P}^l = \mathcal{P} \cap B_T^l,\ l = 1, \ldots, t-1$. The restriction of the action of H_T to $B_T^0 = \mathbf{k}_{\overline{Q}}[Y_{j_1}^{\pm 1}, \ldots, Y_{j_r}^{\pm 1}]$ is the action of the torus $H_{\bar{\mathcal{J}}_T}$, which is the natural action of the torus $(\mathbf{k}^x)^r$. So by [3, 1.12], B_T^0 is H_T-simple. We use induction on t to show that B_T is H_T-simple, which gives a contradiction. So if $t = 1$ and B_T^1 is not H_T-simple then, by [1, Lemma 3.3], $X_{i_1} \in \mathcal{P}^1$, because B_T^0 is H_T-simple. Thus $x_{i_1} \in J \backslash T$ or $\Omega_{i_1} \in J \backslash T$, which is impossible in view of $J \cap \wp_n = T$. Hence B_T^1 must be H_T-simple. However is we suppose that B_T^t is not H_T-simple, induction hypothesis and [1, Lemma 3.3], implies that $X_{i_t} \in \mathcal{P}$, which is also impossible. In conclusion B_T is H_T-simple. Therefore $J = \langle T \rangle$. □

Corollary 2.6. *Let* $n \in \mathbb{N}$; *if* C_n *denotes the cardinal of* $H - \mathrm{Spec}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$, *then*

$$C_n = \frac{(2+\sqrt{2})^{n+1} - (2-\sqrt{2})^{n+1}}{2\sqrt{2}}.$$

Proof. Using the Proposition 2.5, it suffices to compute the number of the admissible sets. Let $m < n$ and consider T an admissible set of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times m}))$, which contains Ω_m. There are four admissible sets in $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times(m+1)}))$, which contract to T; namely, T, $T \cup \{\Omega_{m+1}, y_{m+1}\}$, $T \cup \{\Omega_{m+1}, x_{m+1}\}$ and $T \cup \{\Omega_{m+1}, y_{m+1}, x_{m+1}\}$. In the case when T does not contain Ω_m, there are only two admissible sets in $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times(m+1)}))$ contracting to T. The number of admissible sets of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times m}))$ that do not contain Ω_m is exactly C_{m-1}. Then we have a linear recursive sequence;

$$C_{m+1} = 4(C_m - C_{m-1}) + 2C_{m-1} = 2(2C_m - C_{m-1}).$$

We know that $C_0 = 1$ (if $n = 0$ we take $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times 0})) = \mathbf{k}$) and $C_1 = 4$, so

$$C_n = \frac{(2+\sqrt{2})^{n+1} - (2-\sqrt{2})^{n+1}}{2\sqrt{2}}.$$

for all $n \in \mathbb{N}$. □

3 THE PRIME AND PRIMITIVE IDEALS

In this section we work out the H-stratification (1) of the prime spectrum of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$. We start with a simpler description of each H-stratum. Let T be an admissible set of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ and let us denote

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$$\mathrm{Spec}_T(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))) = \{P \in \mathrm{Spec}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))|\ P \cap \wp_n = T\}.$$

Lemma 3.1. *Let J be an H-prime ideal of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ and let $T = J \cap \wp_n$ be its correspondent admissible set. Then*

$$\mathrm{Spec}_T(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))) = \mathrm{Spec}_J(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))).$$

Proof. Let $P \in \mathrm{Spec}_T(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ and let J' be an H-prime ideal such that $P \in \mathrm{Spec}_{J'}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$. Put $T' = J' \cap \wp_n$, it is clear that $T' \subseteq T$. Suppose that there exists $u_i \in \{\Omega_i, y_i, x_i\}$, such that $u_i \in T \backslash T'$. The H-invariant ideal $\langle T' \cup \{u_i\}\rangle \subseteq P$ contains strictly J'. This is impossible in a view of the maximality (with respect to the propriety H-invariant) of J', thus $T = T'$. Using the Proposition 2.5, we have $J = J'$. This shows the first inclusion. Let now $P \in \mathrm{Spec}_J(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ and put $P \cap \wp_n = T'$. As J is the maximal H-invariant ideal in P, we have $T = T'$. Hence $P \in \mathrm{Spec}_T(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ which gives the second inclusion. □

Proposition 3.2. *The H-stratification of* $\mathrm{Spec}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ *is given by*

$$\mathrm{Spec}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))) = \bigcup_{T \text{ admissible}} \mathrm{Spec}_T(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))) \tag{11}$$

Proof. This is a consequence of Proposition 2.5 and Lemma 3.1. □

Following [2], let $oc\text{ø}mp(T)$ denote the number of connected components of odd length in the connected decomposition of T. Our aim is to prove that each stratum $\mathrm{Spec}_T(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ is homeomorphic to the prime of the group algebra $\mathbf{k}[\mathbb{Z}^{ocomp(T)}]$, where $\mathbb{Z}^{ocomp(T)}$ denote the free abelian group of rank $ocomp(T)$. Then we give our description of $\mathrm{Spec}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$, Theorem 3.10. When $\mathbf{k}$ is algebraically closed, we determine explicitly the primitive ideals.

The $\mathbf{k}$-algebra obtained by localizing B_T at all X_k (it is understood that if $k \in \{i_1+1,\dots,j_1\} \cup \cdots \cup \{i_r+1,\dots,j_r\} \backslash \mathcal{I}_T$ then $X_k = V_k$) is the McConnell-Pettit $\mathbf{k}$-algebra $\mathbf{P}(Q_T)$. We will denote $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ by R. Consider $\Phi_T : R \to \mathbf{P}(Q_T)$, the composition of the maps

$$R \to \frac{R\mathcal{Y}_T^{-1}}{\langle T\rangle \mathcal{Y}_T^{-1}} \overset{\Psi_T}{\to} B_T \hookrightarrow \mathbf{P}(Q_T).$$

Remark 3.3. Let J be an H-prime ideal of R and $J \cap \wp_n = T$. Let $\mathcal{X}_T$ denote the inverse image in $\frac{R\mathcal{Y}_T^{-1}}{\langle T\rangle \mathcal{Y}_T^{-1}}$ of the multiplicative set of B_T generated by all the X_k's. This is a right Ore set and the corresponding localization R_T satisfies that $R_T \cong \mathbf{P}(Q_T)$. Clearly $R_T \subseteq R_J$, where $J = \langle T\rangle$ and $R_J = (R/J)\mathcal{E}_J^{-1}$, $\mathcal{E}_J$ is the set of all non-zero homogeneous elements, with respect to certain $\mathbb{Z}^n$-grading (see [1, Theorem 6.6]). In the general case one cannot expect $R_T = R_J$. The following is a counter example; take $n = 2$, $T = \{\Omega_2\}$, then the homogeneous element $\bar{1} + \bar{y}_1\bar{x}_1$, of degree $(0, 0) \in \mathbb{Z}^2$, is not invertible in R_T.

Theorem 3.4. *Φ_T induces a homeomorphism Φ_T^{-1} between* $\mathrm{Spec}(\mathbf{P}(Q_T))$ *and* $\mathrm{Spec}_T(R)$ *defined by*:

$$\begin{array}{llll} \Phi_T^{-1}: & \mathrm{Spec}(\mathbf{P}(Q_T)) & \rightarrow & \mathrm{Spec}_T(R) \\ & \mathcal{P} & \mapsto & \Phi_T^{-1}(\mathcal{P}). \end{array}$$

Proof. Notice that $\Phi_T^{-1}(\mathcal{P})$ is prime because every prime ideal in R or $\mathbf{P}(Q_T)$ is completely prime. Next, we have to show that $\Phi_T^{-1}(\mathcal{P}) \in \mathrm{Spec}_T(R)$ for all $\mathcal{P} \in \mathrm{Spec}(\mathbf{P}(Q_T))$. Put $\Phi_T^{-1}(\mathcal{P}) \cap \wp_n = T'$, clearly $T \subseteq T'$. Assume for a contradiction that $T \neq T'$. If there exits an index k such that $\Omega_k \in T' \backslash T$, then $-W_k = \Phi_T(\Omega_k) \in \mathcal{P}$, a contradiction. Otherwise, $Ind(T) = Ind(T')$ and there exists $v_k \in T' \backslash T$. We have in particular that $k \notin \mathcal{I}_T \cap \mathcal{J}_T$. This entails that $0 \neq V_k = \Phi_T(v_k) \in \mathcal{P}$, which is a contradiction. Let us show the injectivity of Φ_T^{-1}. If $\mathcal{P}, \mathcal{P}' \in \mathrm{Spec}(\mathbf{P}(Q_T))$ are such that $\Phi_T^{-1}(\mathcal{P}) = \Phi_T^{-1}(\mathcal{P}')$ then $\frac{\Phi_T^{-1}(\mathcal{P})\mathcal{Y}_T^{-1}}{\langle T\rangle\mathcal{Y}_T^{-1}} = \frac{\Phi_T^{-1}(\mathcal{P}')\mathcal{Y}_T^{-1}}{\langle T\rangle\mathcal{Y}_T^{-1}}$ is a prime ideal of $\frac{R\mathcal{Y}_T^{-1}}{\langle T\rangle\mathcal{Y}_T^{-1}}$, because $\Phi_T^{-1}(\mathcal{P}) \cap \mathcal{Y}_T = \Phi_T^{-1}(\mathcal{P}') \cap \mathcal{Y}_T = \emptyset$. Apply Ψ_T to get $\mathcal{P} = \mathcal{P}'$. To prove the surjectivity, let $P \in \mathrm{Spec}_T(R)$. Thus $P \cap \mathbb{Y}_T = \emptyset$ and $\frac{P\mathcal{Y}_T^{-1}}{\langle T\rangle\mathcal{Y}_T^{-1}}$ is a prime ideal in $\frac{R\mathcal{Y}_T^{-1}}{\langle T\rangle\mathcal{Y}_T^{-1}}$. The ideal $\mathcal{P} = \Psi_T(\frac{P\mathcal{Y}_T^{-1}}{\langle T\rangle\mathcal{Y}_T^{-1}})$ satisfies $\mathcal{P} \cap \mathbb{Y}_T = \emptyset$. We will show that $X_k \notin \mathcal{P}$ for all X_k in $\mathbf{P}(Q_T)$. Suppose $X_k \in \mathcal{P}$ for a contradiction. The two possible values of $\Psi_T^{-1}(X_k)$ are $\overline{x_k}$ and $-\overline{y_k}^{-1}\overline{\Omega_k}$. In the first case we have the contradiction $x_k \in P$, while the second value gives $\Omega_k \in P$, another contradiction. So $\mathcal{P}$ is the inverse image of P. □

Corollary 3.5. *Let T be an admissible set of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$. Then* $\mathrm{Spec}_T(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ *is homeomorphic to* $\mathrm{Spec}(Z(\mathbf{P}(Q_T)))$, *where* $Z(\mathbf{P}(Q_T))$ *is the center of the* $\mathbf{k}$*-algebra* $\mathbf{P}(Q_T)$.

Proof. By [3, Corollary 1.5(b)], the contraction $\mathcal{P} \to \mathcal{P} \cap Z(\mathbf{P}(Q_T))$ gives a homeomorphism between $\mathrm{Spec}(\mathbf{P}(Q_T))$ and $\mathrm{Spec}(Z(\mathbf{P}(Q_T)))$. The result follows from Theorem 3.4. □

By [3, 1.3], the center $Z(\mathbf{P}(Q_T))$ is a Laurent polynomial ring. The variables of this ring are determined by the solutions of the system of equations $\mathcal{M}_T\boldsymbol{m} = 0$, where $\mathcal{M}_T$ is the matrix with integer entries k_{ij}, such that $Q_T = (q^{k_{ij}})$. Our next purpose is to compute the number of independent variables in $Z(\mathbf{P}(Q_T))$.

Lemma 3.6. *Let* $A \in \mathbf{M}_{m\times m}(\mathbb{Z})$, $v \in \mathbf{M}_{m\times 1}(\mathbb{Z})$, $w \in \mathbf{M}_{1\times m}(\mathbb{Z})$ *and* ρ *a nonzero integer. Then*

$$\operatorname{rank}\begin{pmatrix} A & v & v \\ w & 0 & -\rho \\ w & \rho & 0 \end{pmatrix} = 2 + \operatorname{rank} A$$

and

$$\operatorname{rank}\begin{pmatrix} A & v & v & v \\ w & 0 & -2 & -1 \\ w & 2 & 0 & 1 \\ w & 1 & -1 & 0 \end{pmatrix} = 2 + \operatorname{rank} A$$

Proof. Compute the ranks by using minors and suitable row and column elementary operations. □

Proposition 3.7. *Let* T *be an admissible set, and let* $\mathcal{M}_T \in \mathbf{M}_{t\times t}(\mathbb{Z})$ *be its associated matrix. Then rank* $\mathcal{M}_T = t - ocomp(T)$.

Proof. We proceed by induction on n. The cases $n = 1, 2$ are easy. Assume $n > 2$ and let $j = \max(\operatorname{ind}(T))$. If T has some removable index i, let T' be the admissible subset of $\wp_n \backslash \{x_i, y_i\}$ obtained by removing x_i, y_i from T. Notice that $ocomp(T) = ocomp(T')$. Let $\mathcal{M}_{T'}^{n-1}$ the matrix associated to T' with respect to Q_{n-1}. By induction hypothesis, $\operatorname{rank} \mathcal{M}_{T'}^{n-1} = t' - ocomp(T')$. But $t = t'$; in fact, $\mathcal{M}_T = \mathcal{M}_{T'}^{n-1}$ and, thus, $\operatorname{rank} \mathcal{M}_T = t - ocomp(T)$. For T without removable indices, we will consider several cases. Decompose $T = T' \cup T_r$, where T_r is the last connected component of T, and put $i_r = \min(\operatorname{ind}(T_r))$.

Case 1. If $j < n$, then

$$\mathcal{M}_T = \begin{pmatrix} \mathcal{M}_T^{n-1} & v & v \\ w & 0 & -2 \\ w & 2 & 0 \end{pmatrix}$$

where $\mathcal{M}_T^{n-1}$ is the matrix associated to T with respect to Q_{N-1}. By induction hypothesis, $\operatorname{rank} \mathcal{M}_T^{n-1} = t - 2 - ocomp(T)$. By Lemma 3.6, $\operatorname{rank} \mathcal{M}_T = t - ocomp(T)$.

Case 2. Assume $j = n$ and $i_r = j$. In this case, necessarily, $T_r = \{\Omega_n\}$ and we have

$$\mathcal{M}_T = \begin{pmatrix} \mathcal{M}_{T'}^{n-2} & v & v & v \\ w & 0 & -2 & -1 \\ w & 2 & 0 & 1 \\ w & 1 & -1 & 0 \end{pmatrix}$$

By induction hypothesis, $rank\, \mathcal{M}_{T'}^{n-2} = t - 3 - ocomp(T')$. In this case, $ocomp(T') = ocomp(T) - 1$ which, in conjunction with Lemma 3.6, gives our equality $rank\, \mathcal{M}_T = t - ocomp(T)$.

Case 3. Assume $i_r < j = n$ with $j = i_r + 1$. In this case, $T_r = \{\Omega_{n-1}, \Omega_n, x_n\}$ or $T_r = \{\Omega_{n-1}, \Omega_n, y_n\}$. Therefore,

$$\mathcal{M}_T = \begin{pmatrix} \mathcal{M}_{T'}^{n-2} & v & v \\ w & 0 & \varepsilon \\ w & -\varepsilon & 0 \end{pmatrix}$$

In this case $ocomp(T) = ocomp(T')$. Use again induction and Lemma 3.6.

Case 4. This is the last case, where $i_r + 1 < j = n$. Here, $T_r = T_r' \cup \{\Omega_{n-1}, \Omega_n, u_{n-1}, u_n\}$ where $u_{n-1} \in \{y_{n-1}, x_{n-1}\}$, $u_n \in \{y_n, x_n\}$ and $T_r' \neq \emptyset$ is an admissible set with $\mathrm{length}(\mathrm{T}_\mathrm{r}') = \mathrm{length}(\mathrm{T}_\mathrm{r}) - 2$. Now,

$$\mathcal{M}_T = \begin{pmatrix} \mathcal{M}_{T''}^{n-2} & v & v \\ w & 0 & \varepsilon \\ w & -\varepsilon & 0 \end{pmatrix}$$

where $T'' = T' \cup T_r'$ and $\varepsilon \in \{1, -1\}$. By induction, $\mathrm{rank}\, \mathcal{M}_{T''}^{n-2} = t - 2 - ocomp(T'')$. But $ocomp(T'') = ocomp(T)$ and this implies, by Lemma 3.6, the desired equality. □

Definition 3.8. *Let T be an admissible set and $\mathcal{M}_T \in \mathbf{M}_{t\times t}(\mathbb{Z})$, the associated matrix. The linear system of equations over the integers $\mathcal{M}_T \boldsymbol{m} = 0$ where $\boldsymbol{m} \in \mathbb{Z}^t$ will be called the quantum linear system associated to T. We denote by $Null(\mathcal{M}_T)$ the solution free abelian group $\{\boldsymbol{m} \in \mathbb{Z}^t : \mathcal{M}_T \boldsymbol{m} = 0\}$.*

Corollary 3.9. *Let T be an admissible set. Then the rank of the free abelian group* $\mathrm{Null}(\mathcal{M}_T)$ *is $ocomp(T)$.*

Proof. This is the consequence of Proposition 3.7. □

Let T be an admissible set and let

$$\{U^\alpha = U_1^{\alpha_1}, \ldots, U_t^{\alpha_t} : \alpha = (\boldsymbol{\alpha}_1, \ldots, \alpha_t) \in \mathbb{Z}^t\}$$

be the canonical $\mathbf{k}$-basis of $\mathbf{P}(Q_T)$, where the U_l's denote the variables in A_T (see (10)). Let

$$\{\boldsymbol{m}_1^T, \ldots, \boldsymbol{m}_k^T\}$$

be a basis of $Null(\mathcal{M}_T)$. By Corollary 3.9, we have that $k = ocomp(T)$. By [3, 1.3]

$$Z(\mathbf{P}(Q_T)) = \mathbf{k}[(U^{\boldsymbol{m}_1^T})^{\pm 1}, \ldots, (U^{\boldsymbol{m}_k^T})^{\pm 1}]. \tag{12}$$

This is a Laurent polynomial ring in the variables $(U^{\boldsymbol{m}_1^T})^{\pm 1}, \ldots, (U^{\boldsymbol{m}_k^T})^{\pm 1}$ and, thus, it is canonically isomorphic to the group algebra $\mathbf{k}[\mathbb{Z}^{ocomp(T)}]$. Given a prime ideal $\mathfrak{p}$ of $Z(\mathbf{P}(Q_T))$, we denote by $\mathfrak{p}^e$ its extension to $\mathbf{P}(Q_T)$. The set of maximal ideals of $\mathbf{k}[\mathbb{Z}^{ocomp(T)}]$ is denoted by $Max(\mathbf{k}[\mathbb{Z}^{ocomp(T)}])$. We combine our results with [3, 1.3 and Corollary 1.5] to get our main theorem.

Theorem 3.10. *Let*

$$\mathfrak{sp} = \{(T, \mathfrak{p}) \mid T \text{ is an admissible set}, \ \mathfrak{p} \in \operatorname{Spec}(\mathbf{k}[\mathbb{Z}^{ocomp(T)}])\}$$

and

$$\mathcal{P} = \{(T, \mathfrak{p}) \mid T \text{ is an admissible set}, \ \mathfrak{p} \in \operatorname{Max}(\mathbf{k}[\mathbb{Z}^{ocomp(T)}])\}.$$

If q is not a root of unity. Then the map $(T, \mathfrak{p}) \mapsto \Phi_T^{-1}(\mathfrak{p}^e)$ defines a bijection between $\mathfrak{sp}$ and the prime spectrum $\operatorname{Spec}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ whose restriction to $\mathcal{P}$ is a bijection onto the primitive spectrum $\operatorname{Prim}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$.

Proof. The bijection between $\mathfrak{sp}$ and $\operatorname{Spec}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ follows from Theorem 3.4, Corollary 3.5 and (12) in conjunction with the stratification (1). By [13, Example 3.3] the algebra $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ has a $(\mathbb{N}^{2n}, +)$-filtration with a semi commutative associated $\mathbb{N}^{2n}$-graded algebra. Then, using [18, Section 3], $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ satisfies the Nullstellensatz over $\mathbf{k}$. Therefore the bijection between $\mathcal{P}$ and $\operatorname{Prim}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ follows in the same way taking into account [3, Corollary 1.5.(c)]. □

Remark 3.11. Let T be an admissible set. By the Proposition 3.7 and [4, Proposition 1.3] $ocomp(T) = 0$ if and only if $\mathbf{P}(Q_T)$ is a simple algebra. In this case, $\operatorname{Spec}_T(R) = \{\langle T \rangle\}$.

From now on, we suppose that $\mathbf{k}$ is algebraically closed. Let T be an admissible set and let $\{\boldsymbol{m}_1^T, \ldots, \boldsymbol{m}_k^T\}$, $k = ocomp(T)$ be a basis of $Null(\mathcal{M}_T)$. The maximal ideals of $Z(\mathbf{P}(Q_T))$ are of the form

$$\mathfrak{p}(\boldsymbol{\lambda}) = \langle U^{\boldsymbol{m}_1^T} - \lambda_1, \ldots, U^{\boldsymbol{m}_k^T} - \lambda_k \rangle$$

for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \in (\mathbf{k}^*)^{ocomp(T)}$. By Theorem 3.10, the primitive ideals of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ are of the form $\Phi_T^{-1}(\mathfrak{p}(\boldsymbol{\lambda})^e)$, when T runs the set of all admissible sets. We shall exhibit a procedure to compute them from the solutions of the quantum systems defined in 3.8.

For $\boldsymbol{m} = (m_1, \ldots, m_t) \in \mathbb{Z}^t$ we denote,

$$\boldsymbol{m}^+ = \tfrac{1}{2}(m_1 + |m_1|, \ldots, m_t + |m_t|)$$

and

$$\boldsymbol{m}^- = \tfrac{1}{2}(m_1 - |m_1|, \ldots, m_t - |m_t|)$$

where $|m|$ is the absolute value of $m \in \mathbb{Z}$. Then the inverse image of $\mathfrak{p}(\boldsymbol{\lambda})$ in A_T is

$$\langle U^{\boldsymbol{m}_1^{T+}} - \lambda_1 U^{-\boldsymbol{m}_1^{T-}}, \ldots, U^{\boldsymbol{m}_k^{T+}} - \lambda_k U^{-\boldsymbol{m}_k^{T-}} \rangle \tag{13}$$

For each $s = 1, \ldots, k$, let $Y_{\boldsymbol{m}_s^T}(\lambda_s)$ denote an element of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$ such that

$$\Psi_T(Y_{\boldsymbol{m}_s^T}(\lambda_s) + \langle T\rangle) = U^{\boldsymbol{m}_s^{T+}} - \lambda_s U^{-\boldsymbol{m}_s^{T-}}.$$

Then

$$\Phi_T^{-1}(\mathfrak{p}(\boldsymbol{\lambda})^e) = \langle T, Y_{\boldsymbol{m}_1^T}(\lambda_1), \ldots, Y_{\mathbf{m}_k^T}(\lambda_k)\rangle$$

This gives a description of $\mathrm{Prim}(\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n})))$ close to [2, Theorem 7.1].

Corollary 3.12. *The primitive ideals of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times n}))$, when q is not a root of unity, are the maximal elements of each stratum* $\mathrm{Spec}_{\mathrm{T(R)}}$, *where T is an admissible set. So they are of the form*

$$\langle T, Y_{\boldsymbol{m}_1^T}(\lambda_1), \ldots, Y_{\boldsymbol{m}_k^T}(\lambda_k)\rangle$$

where $k = ocomp(T)$ and $\lambda = (\lambda_1, \ldots, \lambda_k) \in (\mathbf{k}^*)^k$.

Remark 3.13. 1) When T is connected, the elements $Y_{\boldsymbol{m}}^{T_1}(\lambda_1)$ are the $a - \lambda_1 b$ of [2, Definition 4.2.(3)]. However, if T is not connected, then the elements $Y_{m_s^T}(\lambda_s)$ can be different from the elements $Y_{T_s}(\lambda_s)$ defined in [2, page 542], as can be easily checked in the case $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times 3}))$ and $T = \{y_1, \Omega_1, \Omega_3\}$.

2) Let T be a connected admissible set. So if T is of even length then $\mathrm{Spec}_T(R) = \{\langle T\rangle\}$, and if T is of odd length then $\mathrm{Spec}_T(R) = \{\langle T\rangle \subset \langle T, Y_{m_1^T}(\lambda)\rangle\}$, $\boldsymbol{m}_1^T$ is a basis of $\mathrm{Null}(\mathcal{M}_T)$.

Example 3.14. We give the prime and primitive spectra of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times 2}))$ when q is not a root of unity. Observe that in this case all the admissible sets are connected, so the prime ideals of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times 2}))$ are of the form 2) in Remark 3.13. The lattice of prime ideals of $\mathcal{O}_q(\mathfrak{sp}(\mathbf{k}^{2\times 2}))$ is drawn in the Figure 1, The primitive ideal generated by a set A is denoted by $\langle\langle A\rangle\rangle$, while prime but not primitive ideals are denoted by $\langle A\rangle$. A line connecting two prime ideals means inclusion. When both ideals belong to the same stratum, we use a wavy line. Lastly, α denotes an arbitrary non-zero element in $\mathbf{k}$.

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