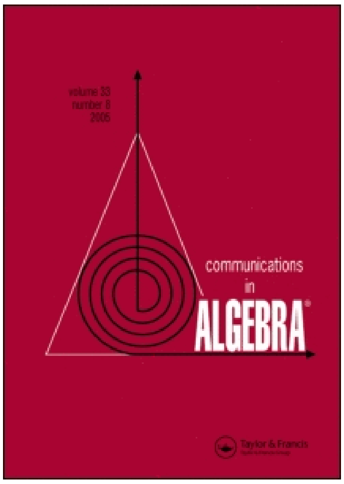

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Compatibility Conditions Between Rings and Corings

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COMPATIBILITY CONDITIONS BETWEEN RINGS AND CORINGS

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We introduce the notion of "bi-monoid" in general monoidal category, generalizing by this the notion of "bialgebra". In the case of bimodules over a noncommutative algebra, we obtain compatibility conditions between rings and corings whenever both structures admit the same underlying bimodule. Several examples are expounded in this case. We also show that there is a class of right modules over a bi-monoid which is a monoidal category and the forgetful functor to the ground category is a strict monoidal functor.

Key Words: Bi-(Co)modules; (Co)Monoides; (Co)Rings; (Co)Wreath; Double distributive law; Monoidal categories.

2000 Mathematics Subject Classification: 16W30; 16D20; 16D90.

INTRODUCTION

Let R be an associative algebra over a commutative ring with identity $\Bbbk$, and consider its category of unital and $\Bbbk$-central bimodules ${}_R\mathscr{M}_R$ as a monoidal category with multiplication given by the tensor product $-\otimes_R-$ and with identity object the regular bimodule ${}_RR_R$. The search of a compatibility conditions between a ring structure (monoid in ${}_R\mathscr{M}_R$, see below) and coring structure (comonoid in ${}_R\mathscr{M}_R$, see below) is a well known problem in noncommutative algebra. Even though the object is the same (i.e., the underlying R-bimodule structures coincide), there is no obvious way in which a tensor product of bimodules can be equipped with a monoid (or comonoid) structure. In this direction, Sweedler (1975b, §5) introduced the $\times_R$-product, and few years later Takeuchi (1977) gave the notion of $\times_A$-bialgebra with noncommutative base ring. This notion can be seen as an approach to a compatibility conditions problem by taking, in an appropriate way, a comonoid in ${}_R\mathscr{M}_R$ and a monoid in ${}_{R\otimes_{\Bbbk}R^o}\mathscr{M}_{R\otimes_{\Bbbk}R^o}$ (R^o is the opposite ring of R).

In this note, we give another approach to the compatibility conditions problem by considering a comonoid and monoid in the same monoidal category. The basic ideas behind our approach are the notions of wreath and cowreath recently introduced in Lack and Street (2002), which are a generalization of distributive

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law due to Beck (1969). In the category of bimodules, wreath and cowreath lead in a formal way to endow certain tensor product with the structure of a ring and of a coring. A double distributive law is then a wreath and a cowreath induced, respectively, by a ring and coring taking the same underlying bimodule. In this way, we arrive to the compatibility conditions by assuming the existence of a double distributive law for a bimodule which admits a structures of both ring and coring.

In Section 1, we review the Eilenberg–Moore categories attached to a monoid (resp., comonoid) in a strict monoidal category. We give, as in the case of the category of bimodules (El Kaoutit, 2006), a simplest and equivalent definition for a wreath (resp., cowreath) over monoid (resp., comonoid), see Proposition 1.11 (resp., Proposition 1.4). In particular, we show that the wreath (resp., cowreath) product satisfies a universal property, Proposition 1.14 (resp., Proposition 1.7). In Section 2, we use the notion of double distributive law (Definition 2.1) in order to give equivalent compatibility conditions for an object with structures of both monoid and comonoid, Proposition 2.2. An object satisfying the equivalent conditions of such proposition is called a *bimonoid*. In Section 3, we introduce the category of right twisted module over a bimonoid. It turns out that this class of right modules is a monoidal category, and the forgetful functor is a strict monoidal (Proposition 3.2). Section 4 presents several applications; we show that the tensor algebra of any algebra with a group algebra is a bimonoid in the category of bimodules over a non-necessary commutative ring. In the case of commutative base rings, we give an example which shows that the class of bialgebras is *strictly contained* in the class of bimonoids in the monoidal category of $\Bbbk$-modules. The theory of bialgebras in a braided monoidal category is recovered as well as the notion of braided bialgebras over commutative rings (Takeuchi, 2002).

Recently, an article by Mesablishvili and Wisbauer was posted in arXiv.math (Mesablishvili and Wisbauer, 2007), where the definition of bimonoid in monoidal category (in the sense of Definition 2.3, see below) was transferred to the case of bimonad on a category. If we consider the 2-category of functors (Categories, Functors, Natural transformations), then the category of endo-functors $\mathsf{Funct}(\mathscr{A}, \mathscr{A})$ on a category $\mathscr{A}$ is in fact a strict monoidal category. Thus a bimonoid in this monoidal category (Definition 2.3) is just a bimonad on $\mathscr{A}$ in the sense of Mesablishvili and Wisbauer (2007, Definition 4.1). Another interesting and different approach was given for a general bicategory in López-López (1976, (1.3.1), p. 7, and (3.1.1), p. 31).

Notations and Basic Notions. Given any Hom-set category $\mathscr{C}$, the notation $X \in \mathscr{C}$ means that X is an object of $\mathscr{C}$. The identity morphism of X will be denoted by X itself. The set of all morphisms $f : X \to X'$ in $\mathscr{C}$ is denoted by $\mathrm{Hom}_{\mathscr{C}}(X, X')$. Let $\mathscr{M}$ be a strict monoidal category with multiplication $- \otimes -$ and identity object $\mathbb{I}$. Recall from Mac Lane (1998, §VII) that a comonoid in $\mathscr{M}$ is a three-tuple $(\mathbb{C}, \Delta, \varepsilon)$ consisting of an object $\mathbb{C}$ and two morphisms $\Delta : \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$ (comultiplication), $\varepsilon : \mathbb{C} \to \mathbb{I}$ (counit) such that $(\varepsilon \otimes \mathbb{C}) \circ \Delta = \mathbb{C} = (\mathbb{C} \otimes \varepsilon) \circ \Delta$ and $(\Delta \otimes \mathbb{C}) \circ \Delta = (\mathbb{C} \otimes \Delta) \circ \Delta$. A morphism of comonoids $\phi : (\mathbb{C}, \Delta, \varepsilon) \to (\mathbb{D}, \Delta', \varepsilon')$ is a morphism $\phi : \mathbb{C} \to \mathbb{D}$ in $\mathscr{M}$ such that $\varepsilon' \circ \phi = \varepsilon$ (counitary property) and $\Delta' \circ \phi = (\phi \otimes \phi) \circ \Delta$ (coassociativity property). Dually, a monoid in $\mathscr{M}$ is a three-tuple $(\mathbb{A}, \mu, \eta)$ consisting of an object $\mathbb{A} \in \mathscr{M}$ and two morphisms $\mu : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ (multiplication), $\eta : \mathbb{I} \to \mathbb{A}$ (unit) such that $\mu \circ (\eta \otimes \mathbb{A}) = \mathbb{A} = \mu \circ (\mathbb{A} \otimes \eta)$ and $\mu \circ (\mathbb{A} \otimes \mu) = \mu \circ$

$(\mu \otimes \mathbb{A})$. A morphism of monoids $\psi : (\mathbb{A}, \mu, \eta) \to (\mathbb{T}, \mu', \eta')$ is a morphism $\psi : \mathbb{A} \to \mathbb{T}$ such that $\psi \circ \eta = \eta'$ and $\psi \circ \mu = \mu' \circ (\psi \otimes \psi)$.

A right $\mathbb{C}$-comodule is a pair (X, ρ^X) consisting of an object $X \in \mathcal{M}$ and a morphism $\rho^X : X \to X \otimes \mathbb{C}$ (right $\mathbb{C}$-coaction) such that $(X \otimes \varepsilon) \circ \rho^X = X$ and $(\rho^X \otimes \mathbb{C}) \circ \rho^X = (X \otimes \Delta) \circ \rho^X$. A morphism of right $\mathbb{C}$-comodules $f : (X, \rho^X) \to (X', \rho^{X'})$ is a morphism $f : X \to X'$ in the category $\mathcal{M}$ such that $\rho^{X'} \circ f = (f \otimes \mathbb{C}) \circ \rho^X$. These form a category which we denote by $\mathcal{M}^{\mathbb{C}}$. The category ${}^{\mathbb{C}}\mathcal{M}$ of left $\mathbb{C}$-comodules is similarly defined; we use the Greek letter λ^- to denote their left $\mathbb{C}$-coactions. A $\mathbb{C}$-bicomodule is a three-tuple (X, ρ^X, λ^X) where (X, ρ^X) is a right $\mathbb{C}$-comodule and (X, λ^X) is a left $\mathbb{C}$-comodule such that $(\lambda^X \otimes \mathbb{C}) \circ \rho^X = (\mathbb{C} \otimes \rho^X) \circ \lambda^X$. A morphism of $\mathbb{C}$-bicomodules is a morphism of left and of right $\mathbb{C}$-comodules. We use the notation $\mathrm{Hom}_{\mathbb{C}-\mathbb{C}}(-, -)$ for the sets of all $\mathbb{C}$-bicomodules morphisms. Dually, a right $\mathbb{A}$-module is a pair (P, r_P) consisting of an object $P \in \mathcal{M}$ and a morphisms $\mathsf{r}_P : P \otimes \mathbb{A} \to P$ (right $\mathbb{A}$-action) such that $\mathsf{r}_P \circ (P \otimes \eta) = P$ and $\mathsf{r}_P \circ (P \otimes \mu) = \mathsf{r}_P \circ (\mathsf{r}_P \otimes \mathbb{A})$. A morphism of right $\mathbb{A}$-modules $g : (P, \mathsf{r}_P) \to (P', \mathsf{r}_{P'})$ is a morphism $g : P \to P'$ in $\mathcal{M}$ such that $g \circ \mathsf{r}_P = \mathsf{r}_{P'} \circ (g \otimes \mathbb{A})$. Right modules and their morphisms form a category which we denote by $\mathcal{M}_{\mathbb{A}}$. The category ${}_{\mathbb{A}}\mathcal{M}$ of left $\mathbb{A}$-module is similarly defined, and we use the letter l_- to denote their left $\mathbb{A}$-actions. An $\mathbb{A}$-bimodule is a three-tuple $(P, \mathsf{r}_P, \mathsf{l}_P)$, where (P, r_P) is a right $\mathbb{A}$-module and (P, l_P) is a left $\mathbb{A}$-module such that $\mathsf{r}_P \circ (\mathsf{l}_P \otimes \mathbb{A}) = \mathsf{l}_P \circ (\mathbb{A} \otimes \mathsf{r}_P)$. A morphism of $\mathbb{A}$-bimodules is a morphism of left and of right $\mathbb{A}$-modules. We denote by $\mathrm{Hom}_{\mathbb{A}-\mathbb{A}}(-, -)$ the set of all $\mathbb{A}$-bimodule morphisms. For details on comodules over corings, definitions, and basic properties of bicomodules over corings, the reader is referred to monograph of Brzeziński and Wisbauer (2003).

1. REVIEW ON (RIGHT) EILENBERG–MOORE MONOIDAL CATEGORIES

Let $\mathcal{M}$ denote a strict monoidal category with multiplication $- \otimes -$ and identity object $\mathbb{I}$. In this section, we review the Eilenberg–Moore monoidal categories (Lack and Street, 2002) associated to a monoid and to a comonoid both defined in $\mathcal{M}$. We start by considering a comonoid $\mathbb{C}$ in $\mathcal{M}$ with structure morphisms $\Delta : \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$ and $\varepsilon : \mathbb{C} \to \mathbb{I}$.

1.1. The Monoidal Category $\mathcal{R}^c_{\mathbb{C}}$

This is the right Eilenberg–Moore monoidal category associated to the comonoid $\mathbb{C}$ (see Lack and Street, 2002, for a general context) and defined as follows:

Objects of $\mathcal{R}^c_{\mathbb{C}}$: Are pairs $(X, \mathfrak{x})$ consisting of an object $X \in \mathcal{M}$ and a morphism $\mathfrak{x} : \mathbb{C} \otimes X \to X \otimes \mathbb{C}$ such that

$$(X \otimes \Delta) \circ \mathfrak{x} = (\mathfrak{x} \otimes \mathbb{C}) \circ (\mathbb{C} \otimes \mathfrak{x}) \circ (\Delta \otimes X) \tag{1.1}$$

$$(X \otimes \varepsilon) \circ \mathfrak{x} = \varepsilon \otimes X. \tag{1.2}$$

We have the following useful lemma.

Lemma 1.1. *For every object $X \in \mathcal{M}$, the following conditions are equivalent*:

(i) $\mathbb{C} \otimes X$ *is a* $\mathbb{C}$*-bicomodule with a left* $\mathbb{C}$*-coaction* $\lambda^{\mathbb{C}\otimes X} = \Delta \otimes X$;
(ii) *There is a morphism* $\mathfrak{x} : \mathbb{C} \otimes X \to X \otimes \mathbb{C}$ *satisfying equalities* (1.1) *and* (1.2).

Proof. (ii) $\Rightarrow$ (i) Take the right $\mathbb{C}$-coaction $\rho^{\mathbb{C}\otimes X} = (\mathbb{C} \otimes \mathfrak{x}) \circ (\Delta \otimes X)$.

(i) $\Rightarrow$ (ii) Take $\mathfrak{x} = (\varepsilon \otimes X \otimes \mathbb{C}) \circ \rho^{\mathbb{C}\otimes X}$, where $\rho^{\mathbb{C}\otimes X}$ is the given right $\mathbb{C}$-coaction. □

In this way, the morphisms in $\mathcal{R}^c_{\mathbb{C}}$ are defined in their unreduced form as follows:

Morphisms in $\mathcal{R}^c_{\mathbb{C}}$:

$$\mathrm{Hom}_{\mathcal{R}^c_{\mathbb{C}}}((X, \mathfrak{x}), (X', \mathfrak{x}')) := \mathrm{Hom}_{\mathbb{C}-\mathbb{C}}(\mathbb{C} \otimes X, \mathbb{C} \otimes X'),$$

where $\mathbb{C} \otimes X$ and $\mathbb{C} \otimes X'$ are endowed with the structure of $\mathbb{C}$-bicomodule defined in Lemma 1.1. That is, a morphism $\alpha : (X, \mathfrak{x}) \to (X', \mathfrak{x})$ in $\mathcal{R}^c_{\mathbb{C}}$ is morphism $\alpha : \mathbb{C} \otimes X \to \mathbb{C} \otimes X'$ in $\mathcal{M}$ satisfying

$$(\Delta \otimes X') \circ \alpha = (\mathbb{C} \otimes \alpha) \circ (\Delta \otimes X), \tag{1.3}$$
$$(\mathbb{C} \otimes \mathfrak{x}') \circ (\Delta \otimes X') \circ \alpha = (\alpha \otimes \mathbb{C}) \circ (\mathbb{C} \otimes \mathfrak{x}) \circ (\Delta \otimes X). \tag{1.4}$$

The Multiplication of $\mathcal{R}^c_{\mathbb{C}}$: Let $\alpha : (X, \mathfrak{x}) \to (X', \mathfrak{x}')$ and $\beta : (Y, \mathfrak{y}) \to (Y', \mathfrak{y}')$ two morphisms in $\mathcal{R}^c_{\mathbb{C}}$. One can easily prove that

$$(X, \mathfrak{x}) \otimes_{\mathbb{C}} (Y, \mathfrak{y}) := (X \otimes Y, (X \otimes \mathfrak{y}) \circ (\mathfrak{x} \otimes Y))$$
$$(X', \mathfrak{x}') \otimes_{\mathbb{C}} (Y', \mathfrak{y}') := (X' \otimes Y', (X' \otimes \mathfrak{y}') \circ (\mathfrak{x}' \otimes Y'))$$

are also objects of the category $\mathcal{R}^c_{\mathbb{C}}$, which defines the horizontal multiplication. The vertical one is defined by the composition

$$\begin{aligned}(\alpha \otimes_{\mathbb{C}} \beta) &:= (\mathbb{C} \otimes X' \otimes \varepsilon \otimes Y') \circ (\mathbb{C} \otimes X' \otimes \beta) \\ &\quad \circ (\mathbb{C} \otimes \mathfrak{x}' \otimes Y) \circ (\mathbb{C} \otimes \alpha \otimes Y) \circ (\Delta \otimes X \otimes Y) \\ &= (\mathbb{C} \otimes X' \otimes \varepsilon \otimes Y') \circ (\alpha \otimes \beta) \circ (\mathbb{C} \otimes \mathfrak{x} \otimes Y) \circ (\Delta \otimes X \otimes Y).\end{aligned} \tag{1.5}$$

Lastly, the identity object of the multiplication $- \otimes_{\mathbb{C}} -$ is given by the pair $(\mathbb{I}, \mathbb{C})$ (here $\mathbb{C}$ denotes the identity morphism of $\mathbb{C}$ in $\mathcal{M}$).

1.2. The Monoidal Category $\mathscr{L}_{\mathbb{C}}^{c}$

This is the left Eilenberg–Moore category associated to $\mathbb{C}$ and defined as follows:

Objects of $\mathscr{L}_{\mathbb{C}}^{c}$: Are pairs $(\mathfrak{p}, P)$ consisting of an object $P \in \mathscr{M}$ and a morphism $\mathfrak{p} : P \otimes \mathbb{C} \to \mathbb{C} \otimes P$ such that

$$(\Delta \otimes P) \circ \mathfrak{p} = (\mathbb{C} \otimes \mathfrak{p}) \circ (\mathfrak{p} \otimes \mathbb{C}) \circ (P \otimes \Delta) \tag{1.6}$$

$$(\varepsilon \otimes P) \circ \mathfrak{p} = P \otimes \varepsilon. \tag{1.7}$$

As before one can easily check the following lemma.

Lemma 1.2. *For every object $P \in \mathscr{M}$, the following conditions are equivalent*:

(i) *$P \otimes \mathbb{C}$ is a $\mathbb{C}$-bicomodule with a right $\mathbb{C}$-coaction $\rho^{P \otimes \mathbb{C}} = P \otimes \Delta$;*
(ii) *There is a morphism $\mathfrak{p} : P \otimes \mathbb{C} \to \mathbb{C} \otimes P$ satisfying equalities* (1.6) *and* (1.7).

In this way the morphisms in $\mathscr{L}_{\mathbb{C}}^{c}$ are defined in their unreduced form as follows:

Morphisms in $\mathscr{L}_{\mathbb{C}}^{c}$:

$$\mathrm{Hom}_{\mathscr{L}_{\mathbb{C}}^{c}}((\mathfrak{p}, P), (\mathfrak{p}', P')) := \mathrm{Hom}_{\mathbb{C}-\mathbb{C}}(P \otimes \mathbb{C}, P' \otimes \mathbb{C}),$$

where $P \otimes \mathbb{C}$ and $P' \otimes \mathbb{C}$ are endowed with the structure of $\mathbb{C}$-bicomodule defined in Lemma 1.2. That is, a morphism $\gamma : (\mathfrak{p}, P) \to (\mathfrak{p}', P')$ in $\mathscr{L}_{\mathbb{C}}^{c}$ is morphism $\gamma : P \otimes \mathbb{C} \to P' \otimes \mathbb{C}$ in $\mathscr{M}$ satisfying

$$(P' \otimes \Delta) \circ \gamma = (\gamma \otimes \mathbb{C}) \circ (P \otimes \Delta), \tag{1.8}$$

$$(\mathfrak{p}' \otimes \mathbb{C}) \circ (P' \otimes \Delta) \circ \gamma = (\mathbb{C} \otimes \gamma) \circ (\mathfrak{p} \otimes \mathbb{C}) \circ (P \otimes \Delta). \tag{1.9}$$

The Multiplication of $\mathscr{L}_{\mathbb{C}}^{c}$: Let $\gamma : (\mathfrak{p}, P) \to (\mathfrak{p}', P')$ and $\sigma : (\mathfrak{q}, Q) \to (\mathfrak{q}', Q')$ two morphisms in $\mathscr{L}_{\mathbb{C}}^{c}$. One can easily prove that

$$(\mathfrak{p}, P) \otimes^{\mathbb{C}} (\mathfrak{q}, Q) := ((\mathfrak{p} \otimes Q) \circ (P \otimes \mathfrak{q}), P \otimes Q)$$
$$(\mathfrak{p}', P') \otimes^{\mathbb{C}} (\mathfrak{q}', Q') := ((\mathfrak{p}' \otimes Q') \circ (P' \otimes \mathfrak{q}'), P' \otimes Q')$$

are also objects of the category $\mathscr{L}_{\mathbb{C}}^{c}$, which leads to the horizontal multiplication. The vertical one is defined as the composed morphism

$$\begin{aligned}(\gamma \otimes^{\mathbb{C}} \sigma) &:= (P' \otimes \varepsilon \otimes Q' \otimes \mathbb{C}) \circ (\gamma \otimes Q' \otimes \mathbb{C}) \circ (P \otimes \mathfrak{q}' \otimes \mathbb{C}) \\ &\quad \circ (P \otimes \sigma \otimes \mathbb{C}) \circ (P \otimes Q \otimes \Delta) \\ &= (P' \otimes \varepsilon \otimes Q' \otimes \mathbb{C}) \circ (\gamma \otimes \sigma) \circ (P \otimes \mathfrak{q} \otimes \mathbb{C}) \circ (P \otimes Q \otimes \Delta). \end{aligned} \tag{1.10}$$

The identity object of the multiplication $- \otimes^{\mathbb{C}} -$ is given by the pair $(\mathbb{C}, \mathbb{I})$ (here $\mathbb{C}$ denotes the identity morphism of $\mathbb{C}$ in $\mathscr{M}$).

1.3. Cowreath and Their Products

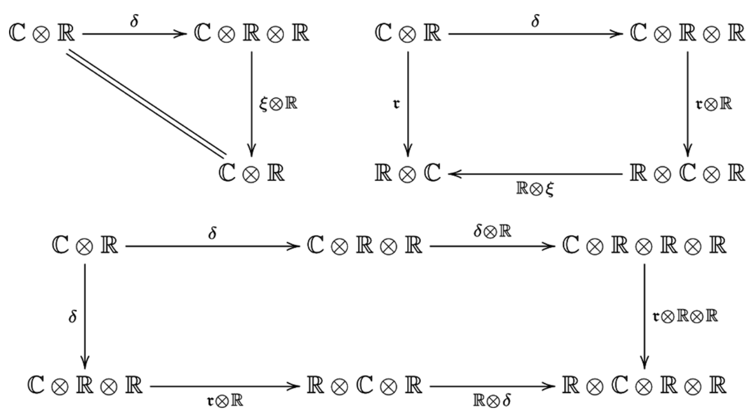
$\mathbb{C}$ still denotes a comonoid in $\mathscr{M}$, $\mathscr{R}^c_{\mathbb{C}}$, and $\mathscr{L}^c_{\mathbb{C}}$ are the monoidal categories defined, respectively, in Subsections 1.1 and 1.2. The notion of wreath was introduced in Lack and Street (2002) in the general context of 2-categories, in the monoidal case they are defined as follows.

Definition 1.3. Let $\mathbb{C}$ be a comonoid in a strict monoidal category $\mathscr{M}$. A *right cowreath over* $\mathbb{C}$ (or *right* $\mathbb{C}$*-cowreath*) is a comonoid in the monoidal category $\mathscr{R}^c_{\mathbb{C}}$. A *right wreath over* $\mathbb{C}$ (or *right* $\mathbb{C}$*-wreath*) is a monoid in the monoidal category $\mathscr{R}^c_{\mathbb{C}}$. The left versions of these definitions are obtained in the monoidal category $\mathscr{L}^c_{\mathbb{C}}$.

The following gives, in terms of the multiplication of $\mathscr{M}$, a simplest and equivalent definition of cowreath.

Proposition 1.4. *Let* $\mathbb{C}$ *be a comonoid in a strict monoidal category* $\mathscr{M}$ *and* $(\mathbb{R}, \mathfrak{r})$ *an object of the category* $\mathscr{R}^c_{\mathbb{C}}$. *The following statements are equivalent*:

(i) $(\mathbb{R}, \mathfrak{r})$ *is a right* $\mathbb{C}$*-cowreath*;
(ii) *There are morphisms of* $\mathbb{C}$*-bicomodules* $\xi : \mathbb{C} \otimes \mathbb{R} \to \mathbb{C}$ *and* $\delta : \mathbb{C} \otimes \mathbb{R} \to \mathbb{C} \otimes \mathbb{R} \otimes \mathbb{R}$ *making commutative the following diagrams*



Proof. Analogue to that of El Kaoutit (2006, Proposition 2.2). □

The cowreath product was introduced in Lack and Street (2002) for comonoids in a general 2-category. In the particular case of strict monoidal categories, this product is expressed by the following proposition.

Proposition 1.5. *Let* $\mathbb{C}$ *be a comonoid in* $\mathscr{M}$ *and* $(\mathbb{R}, \mathfrak{r})$ *a right* $\mathbb{C}$*-cowreath with structure morphisms* $\xi : \mathbb{C} \otimes \mathbb{R} \to \mathbb{C}$ *and* $\delta : \mathbb{C} \otimes \mathbb{R} \to \mathbb{C} \otimes \mathbb{R} \otimes \mathbb{R}$. *The object* $\mathbb{C} \otimes \mathbb{R}$ *admits a structure of comonoid with comultiplication and counit given by*

$$\boldsymbol{\Delta} : \mathbb{C} \otimes \mathbb{R} \xrightarrow{\Delta \otimes \mathbb{R}} \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{R} \xrightarrow{\mathbb{C} \otimes \delta} \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{R} \otimes \mathbb{R} \xrightarrow{\mathbb{C} \otimes \mathfrak{r} \otimes \mathbb{R}} \mathbb{C} \otimes \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{R},$$

$$\boldsymbol{\varepsilon} : \mathbb{C} \otimes \mathbb{R} \xrightarrow{\xi} \mathbb{C} \xrightarrow{\varepsilon} \mathbb{I}.$$

Moreover, with this comonoid structure the morphism $\xi : \mathbb{C} \otimes \mathbb{R} \to \mathbb{C}$ *becomes a morphism of comonoids.*

Proof. Straightforward. □

The comonoid $\mathbb{C} \otimes \mathbb{R}$ of the previous proposition is referred to as *the cowreath product* of $\mathbb{C}$ by $\mathbb{R}$.

Remark 1.6. Notice that the object $\mathbb{R}$ occurring in Proposition 1.4 need not to be a comonoid. However, if $\mathbb{R}$ is itself a comonoid with structure morphisms $\Delta' : \mathbb{R} \to \mathbb{R} \otimes \mathbb{R}$, $\varepsilon' : \mathbb{R} \to \mathbb{I}$ such that the pair $(\mathrm{r}, \mathbb{C})$ belongs to the left monoidal category $\mathscr{L}^c_{\mathbb{R}}$, then the morphisms $\xi := \mathbb{C} \otimes \varepsilon'$ and $\delta := \mathbb{C} \otimes \Delta'$ endow $(\mathbb{R}, \mathrm{r})$ with a structure of right $\mathbb{C}$-cowreath while $\xi' := \varepsilon \otimes \mathbb{R}$, and $\delta' := \Delta \otimes \mathbb{R}$ give to $(\mathrm{r}, \mathbb{C})$ a structure of left $\mathbb{R}$-cowreath. Furthermore, by Proposition 1.5, the morphisms $\xi : \mathbb{C} \otimes \mathbb{R} \to \mathbb{C}$ and $\xi' : \mathbb{C} \otimes \mathbb{R} \to \mathbb{R}$ are in fact morphisms of comonoids.

In this way the morphism $\mathrm{r} : \mathbb{C} \otimes \mathbb{R} \to \mathbb{R} \otimes \mathbb{C}$ satisfies the following equalities:

$$(\mathbb{R} \otimes \Delta) \circ \mathrm{r} = (\mathrm{r} \otimes \mathbb{C}) \circ (\mathbb{C} \otimes \mathrm{r}) \circ (\Delta \otimes \mathbb{R}) \tag{1.11}$$

$$(\mathbb{R} \otimes \varepsilon) \circ \mathrm{r} = \varepsilon \otimes \mathbb{R} \tag{1.12}$$

$$(\Delta' \otimes \mathbb{C}) \circ \mathrm{r} = (\mathbb{R} \otimes \mathrm{r}) \circ (\mathrm{r} \otimes \mathbb{R}) \circ (\mathbb{C} \otimes \Delta') \tag{1.13}$$

$$(\varepsilon' \otimes \mathbb{C}) \circ \mathrm{r} = \mathbb{C} \otimes \varepsilon'. \tag{1.14}$$

A morphism satisfying the four previous equalities is called a *comonoid distributive law* from $\mathbb{C}$ to $\mathbb{R}$, see Beck (1969) for the original definition.

A cowreath product satisfies a universal property in the following sense.

Proposition 1.7. *Let* $(\mathbb{C}, \Delta, \varepsilon)$ *be a comonoid in a strict monoidal category* $\mathscr{M}$ *and* $(\mathbb{R}, \mathrm{r})$ *a* $\mathbb{C}$*-cowreath with a structure morphisms* $\xi : \mathbb{C} \otimes \mathbb{R} \to \mathbb{C}$ *and* $\delta : \mathbb{C} \otimes \mathbb{R} \to \mathbb{C} \otimes \mathbb{R} \otimes \mathbb{R}$.

Let $(\mathbb{D}, \Delta', \varepsilon')$ *be a comonoid with a comonoid morphism* $\alpha : \mathbb{D} \to \mathbb{C}$ *and with morphism* $\beta : \mathbb{D} \to \mathbb{R}$ *satisfying*

$$\xi \circ (\mathbb{C} \otimes \beta) = \mathbb{C} \otimes \varepsilon' \tag{1.15}$$

$$\delta \circ (\mathbb{C} \otimes \beta) = (\mathbb{C} \otimes \beta \otimes \beta) \circ (\mathbb{C} \otimes \Delta'). \tag{1.16}$$

Assume that α *and* β *satisfy the equality*

$$\mathrm{r} \circ (\alpha \otimes \beta) \circ \Delta' = (\beta \otimes \alpha) \circ \Delta', \tag{1.17}$$

then there exists a unique comonoid morphism $\gamma : \mathbb{D} \to \mathbb{C} \otimes \mathbb{R}$ *such that* $\xi \circ \gamma = \alpha$ *and* $(\varepsilon \otimes \mathbb{R}) \circ \gamma = \beta$.

Proof. If there exists such morphism, then it should be unique by the following computations:

$$\begin{aligned}(\alpha \otimes \beta) \circ \Delta' &= ((\xi \circ \gamma) \otimes ((\varepsilon \otimes \mathbb{R}) \circ \gamma)) \circ \Delta' \\ &= (\xi \otimes \varepsilon \otimes \mathbb{R}) \circ (\gamma \otimes \gamma) \circ \Delta'\end{aligned}$$

$$
\begin{aligned}
&= (\xi \otimes \varepsilon \otimes \mathbb{R}) \circ (\mathbb{C} \otimes \mathrm{r} \otimes \mathbb{R}) \circ (\mathbb{C} \otimes \delta) \circ (\Delta \otimes \mathbb{R}) \circ \gamma \\
&= (\xi \otimes \mathbb{R}) \circ (\mathbb{C} \otimes \mathbb{R} \otimes \varepsilon \otimes \mathbb{R}) \circ (\mathbb{C} \otimes \mathrm{r} \otimes \mathbb{R}) \circ (\mathbb{C} \otimes \delta) \circ (\Delta \otimes \mathbb{R}) \circ \gamma \\
&\overset{(1.2)}{=} (\xi \otimes \mathbb{R}) \circ (\mathbb{C} \otimes \varepsilon \otimes \mathbb{R} \otimes \mathbb{R}) \circ (\mathbb{C} \otimes \delta) \circ (\Delta \otimes \mathbb{R}) \circ \gamma \\
&\overset{(1.3)}{=} (\xi \otimes \mathbb{R}) \circ (\mathbb{C} \otimes \varepsilon \otimes \mathbb{R} \otimes \mathbb{R}) \circ (\Delta \otimes \mathbb{R} \otimes \mathbb{R}) \circ \delta \circ \gamma \\
&= (\xi \otimes \mathbb{R}) \circ \delta \circ \gamma \overset{1.4(\mathrm{ii})}{=} \gamma.
\end{aligned}
$$

Since, by hypothesis, $\xi \circ (\alpha \otimes \beta) \circ \Delta' = \alpha$ and $(\varepsilon \otimes \mathbb{R}) \circ (\alpha \otimes \beta) \circ \Delta' = \beta$, it suffices to show that $(\alpha \otimes \beta) \circ \Delta'$ is a morphism of comonoids. The counitary property comes out as

$$
\begin{aligned}
\varepsilon \circ \xi \circ (\alpha \otimes \beta) \circ \Delta' &= \varepsilon \circ \xi \circ (\mathbb{C} \otimes \beta) \circ (\alpha \otimes \mathbb{D}) \circ \Delta' \\
&\overset{(1.15)}{=} \varepsilon \circ (\mathbb{C} \otimes \varepsilon') \circ (\alpha \otimes \mathbb{D}) \circ \Delta' \\
&= \varepsilon \circ \alpha \circ (\mathbb{D} \otimes \varepsilon') \circ \Delta' = \varepsilon'.
\end{aligned}
$$

Now, the coassociativity property is obtained by the following computations:

$$
\begin{aligned}
\boldsymbol{\Delta} \circ (\alpha \otimes \beta) \circ \Delta' &= (\mathbb{C} \otimes \mathrm{r} \otimes \mathbb{R}) \circ (\mathbb{C} \otimes \delta) \circ (\Delta \otimes \mathbb{R}) \circ (\alpha \otimes \beta) \circ \Delta' \\
&\overset{(1.3)}{=} (\mathbb{C} \otimes \mathrm{r} \otimes \mathbb{R}) \circ (\Delta \otimes \mathbb{R} \otimes \mathbb{R}) \circ \delta \circ (\alpha \otimes \beta) \circ \Delta' \\
&= (\mathbb{C} \otimes \mathrm{r} \otimes \mathbb{R}) \circ (\Delta \otimes \mathbb{R} \otimes \mathbb{R}) \circ \delta \circ (\mathbb{C} \otimes \beta) \circ (\alpha \otimes \mathbb{D}) \circ \Delta' \\
&\overset{(1.16)}{=} (\mathbb{C} \otimes \mathrm{r} \otimes \mathbb{R}) \circ (\Delta \otimes \mathbb{R} \otimes \mathbb{R}) \circ (\mathbb{C} \otimes \beta \otimes \beta) \\
&\quad \circ (\mathbb{C} \otimes \Delta') \circ (\alpha \otimes \mathbb{D}) \circ \Delta' \\
&= (\mathbb{C} \otimes \mathrm{r} \otimes \mathbb{R}) \circ (\Delta \otimes \mathbb{R} \otimes \mathbb{R}) \circ (\alpha \otimes \mathbb{R} \otimes \mathbb{R}) \\
&\quad \circ (\mathbb{D} \otimes \beta \otimes \beta) \circ (\mathbb{D} \otimes \Delta') \circ \Delta' \\
&= (\mathbb{C} \otimes \mathrm{r} \otimes \mathbb{R}) \circ (\alpha \otimes \alpha \otimes \mathbb{R} \otimes \mathbb{R}) \circ (\Delta' \otimes \mathbb{R} \otimes \mathbb{R}) \\
&\quad \circ (\mathbb{D} \otimes \beta \otimes \beta) \circ (\mathbb{D} \otimes \Delta') \circ \Delta' \\
&= (\alpha \otimes \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{R}) \circ (\mathbb{D} \otimes (\mathrm{r} \circ (\alpha \otimes \beta)) \otimes \mathbb{R}) \circ (\Delta' \otimes \mathbb{D} \otimes \mathbb{R}) \\
&\quad \circ (\mathbb{D} \otimes \mathbb{D} \otimes \beta) \circ (\Delta' \otimes \mathbb{D}) \circ \Delta' \\
&= (\alpha \otimes \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{R}) \circ (\mathbb{D} \otimes (\mathrm{r} \circ (\alpha \otimes \beta)) \otimes \mathbb{R}) \circ (\Delta' \otimes \mathbb{D} \otimes \mathbb{R}) \\
&\quad \circ (\Delta' \otimes \mathbb{R}) \circ (\mathbb{D} \otimes \beta) \circ \Delta' \\
&= (\alpha \otimes \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{R}) \circ (\mathbb{D} \otimes (\mathrm{r} \circ (\alpha \otimes \beta) \circ \Delta') \otimes \mathbb{R}) \\
&\quad \circ (\Delta' \otimes \mathbb{R}) \circ (\mathbb{D} \otimes \beta) \circ \Delta' \\
&\overset{(1.17)}{=} (\alpha \otimes \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{R}) \circ (\mathbb{D} \otimes \beta \otimes \alpha \otimes \mathbb{R}) \circ (\mathbb{D} \otimes \Delta' \otimes \mathbb{R}) \\
&\quad \circ (\Delta' \otimes \mathbb{R}) \circ (\mathbb{D} \otimes \beta) \circ \Delta' \\
&= (\alpha \otimes \beta \otimes \alpha \otimes \mathbb{R}) \circ (\mathbb{D} \otimes \Delta' \otimes \mathbb{R}) \circ (\mathbb{D} \otimes \mathbb{D} \otimes \beta) \circ (\Delta' \otimes \mathbb{D}) \circ \Delta'
\end{aligned}
$$

$$= (\alpha \otimes \beta \otimes \alpha \otimes \beta) \circ (\mathbb{D} \otimes \Delta' \otimes \mathbb{D}) \circ (\Delta' \otimes \mathbb{D}) \circ \Delta'$$
$$= (\alpha \otimes \beta \otimes \alpha \otimes \beta) \circ (\Delta' \otimes \Delta') \circ \Delta' = (((\alpha \otimes \beta) \circ \Delta') \otimes ((\alpha \otimes \beta) \circ \Delta')) \circ \Delta'.$$
□

In what follows we announce the analogue notion for a given monoid in a strict monoidal category. So consider a monoid $(\mathbb{A}, \mu, \eta)$ in $\mathcal{M}$. We start by defining the right and left Eilenberg–Moore monoidal categories attached to $\mathbb{A}$.

1.4. The Monoidal Category $\mathcal{R}^a_{\mathbb{A}}$

This is the right Eilenberg–Moore monoidal category associated to the monoid $\mathbb{A}$ (see Lack and Street, 2002), and defined as follows:

Objects of $\mathcal{R}^a_{\mathbb{A}}$: Are pairs $(U, \mathfrak{u})$ consisting of an object $U \in \mathcal{M}$ and a morphism $\mathfrak{u} : \mathbb{A} \otimes U \to U \otimes \mathbb{A}$ such that:

$$\mathfrak{u} \circ (\mu \otimes U) = (U \otimes \mu) \circ (\mathfrak{u} \otimes \mathbb{A}) \circ (\mathbb{A} \otimes \mathfrak{u}) \tag{1.18}$$
$$\mathfrak{u} \circ (\eta \otimes U) = U \otimes \eta. \tag{1.19}$$

We have the following lemma.

Lemma 1.8. *For every object $U \in \mathcal{M}$, the following conditions are equivalent*:

(i) *$U \otimes \mathbb{A}$ is an $\mathbb{A}$-bimodule with a right $\mathbb{A}$-action* $\mathsf{r}_{U\otimes\mathbb{A}} = U \otimes \mu$;
(ii) *There is a morphism $\mathfrak{u} : \mathbb{A} \otimes U \to U \otimes \mathbb{A}$ satisfying equalities* (1.18) *and* (1.19).

Proof. (ii) ⇒ (i) Take the left $\mathbb{A}$-action $\mathsf{l}_{U\otimes\mathbb{A}} = (U \otimes \mu) \circ (\mathfrak{u} \otimes \mathbb{A})$.

(i) ⇒ (ii) Take $\mathfrak{u} = \mathsf{l}_{U\otimes\mathbb{A}} \circ (\mathbb{A} \otimes U \otimes \eta)$, where $\mathsf{l}_{U\otimes\mathbb{A}}$ is the given left action. □

In this way, the morphisms in $\mathcal{R}^a_{\mathbb{A}}$ are defined in their unreduced form as follows:

Morphisms in $\mathcal{R}^a_{\mathbb{A}}$:

$$\mathrm{Hom}_{\mathcal{R}^a_{\mathbb{A}}}((U, \mathfrak{u}), (U', \mathfrak{u}')) := \mathrm{Hom}_{\mathbb{A}-\mathbb{A}}(U \otimes \mathbb{A}, U' \otimes \mathbb{A}),$$

where $U \otimes \mathbb{A}$ and $U' \otimes \mathbb{A}$ are endowed with the structure of $\mathbb{A}$-bimodule defined in Lemma 1.8. That is, a morphism $v : (U, \mathfrak{u}) \to (U', \mathfrak{u}')$ in $\mathcal{R}^a_{\mathbb{A}}$ is a morphism $v : U \otimes \mathbb{A} \to U' \otimes \mathbb{A}$ satisfying

$$v \circ (U \otimes \mu) = (U' \otimes \mu) \circ (v \otimes \mathbb{A}), \tag{1.20}$$
$$v \circ (U \otimes \mu) \circ (\mathfrak{u} \otimes \mathbb{A}) = (U' \otimes \mu) \circ (\mathfrak{u}' \otimes \mathbb{A}) \circ (\mathbb{A} \otimes v). \tag{1.21}$$

The Multiplication of $\mathcal{R}^a_{\mathbb{A}}$: Let $v : (U, \mathfrak{u}) \to (U', \mathfrak{u}')$ and $\upsilon : (V, \mathfrak{v}) \to (V', \mathfrak{v}')$ two morphisms in $\mathcal{R}^a_{\mathbb{A}}$. One can easily check that

$$(U, \mathfrak{u}) \otimes_{\mathbb{A}} (V, \mathfrak{v}) := (U \otimes V, (U \otimes \mathfrak{v}) \circ (\mathfrak{u} \otimes V))$$
$$(U', \mathfrak{u}') \otimes_{\mathbb{A}} (V', \mathfrak{v}') := (U' \otimes V', (U' \otimes \mathfrak{v}') \circ (\mathfrak{u}' \otimes V'))$$

are also objects of the category $\mathscr{R}^a_{\mathbb{A}}$, which gives the horizontal multiplication. The vertical one is defined by the composition

$$\begin{aligned}(\nu \otimes_{\mathbb{A}} \upsilon) &:= (U' \otimes V' \otimes \mu) \circ (U' \otimes \upsilon \otimes \mathbb{A}) \circ (U' \otimes \mathfrak{v} \otimes \mathbb{A}) \circ (\nu \otimes V \otimes \mathbb{A}) \\ &\quad \circ (U \otimes \eta \otimes V \otimes \mathbb{A}) \\ &= (U' \otimes V' \otimes \mu) \circ (U' \otimes \mathfrak{v}' \otimes \mathbb{A}) \circ (\nu \otimes \upsilon) \circ (U \otimes \eta \otimes V \otimes \mathbb{A}). \quad (1.22)\end{aligned}$$

The identity object of the multiplication $- \otimes_{\mathbb{A}} -$ is given by the pair $(\mathbb{I}, \mathbb{A})$ (here $\mathbb{A}$ denotes the identity morphism of $\mathbb{A}$ in $\mathscr{M}$).

1.5. The Monoidal Category $\mathscr{L}^a_{\mathbb{A}}$

This is the left Eilenberg–Moore category associated to $\mathbb{A}$, and it is defined as follows:

Objects of $\mathscr{L}^a_{\mathbb{A}}$: Are pairs $(\mathfrak{m}, M)$ consisting of an object $M \in \mathscr{M}$ and a morphism $\mathfrak{m} : M \otimes \mathbb{A} \to \mathbb{A} \otimes M$ such that

$$\mathfrak{m} \circ (M \otimes \mu) = (\mu \otimes M) \circ (\mathbb{A} \otimes \mathfrak{m}) \circ (\mathfrak{m} \otimes \mathbb{A}) \quad (1.23)$$

$$\mathfrak{m} \circ (M \otimes \eta) = \eta \otimes M. \quad (1.24)$$

As before one can easily check the following lemma.

Lemma 1.9. *For every object $M \in \mathscr{M}$, the following conditions are equivalent*:

(i) *$\mathbb{A} \otimes M$ is an $\mathbb{A}$-bimodule with a left $\mathbb{A}$-action $\mathfrak{l}_{\mathbb{A}\otimes M} = \mu \otimes M$*;
(ii) *There is a morphism $\mathfrak{m} : M \otimes \mathbb{A} \to \mathbb{A} \otimes M$ satisfying equalities* (1.23) *and* (1.24).

In this way, the morphisms in $\mathscr{L}^a_{\mathbb{A}}$ are defined in their unreduced form as follows:

Morphisms in $\mathscr{L}^a_{\mathbb{A}}$:

$$\mathrm{Hom}_{\mathscr{L}^a_{\mathbb{A}}}((\mathfrak{m}, M), (\mathfrak{m}', M')) := \mathrm{Hom}_{\mathbb{A}-\mathbb{A}}(\mathbb{A} \otimes M, \mathbb{A} \otimes M'),$$

where $\mathbb{A} \otimes M$ and $\mathbb{A} \otimes M'$ are endowed with the structure of $\mathbb{A}$-bimodule defined in Lemma 1.9. That is, a morphism $\theta : (\mathfrak{m}, M) \to (\mathfrak{m}', M')$ in $\mathscr{L}^a_{\mathbb{A}}$ is a morphism $\theta : \mathbb{A} \otimes M \to \mathbb{A} \otimes M'$ such that

$$\theta \circ (\mu \otimes M) = (\mu \otimes M') \circ (\mathbb{A} \otimes \theta), \quad (1.25)$$

$$\theta \circ (\mu \otimes M) \circ (\mathbb{A} \otimes \mathfrak{m}) = (\mu \otimes M') \circ (\mathbb{A} \otimes \mathfrak{m}') \circ (\theta \otimes \mathbb{A}). \quad (1.26)$$

The Multiplication of $\mathscr{L}^a_{\mathbb{A}}$: Let $\theta : (\mathfrak{m}, M) \to (\mathfrak{m}', M')$ and $\vartheta : (\mathfrak{n}, N) \to (\mathfrak{n}', N')$ be two morphisms in $\mathscr{L}^a_{\mathbb{A}}$. One can easily show that

$$(\mathfrak{m}, M) \otimes^{\mathbb{A}} (\mathfrak{n}, N) := ((\mathfrak{m} \otimes N) \circ (M \otimes \mathfrak{n}), M \otimes N)$$

$$(\mathfrak{m}', M') \otimes^{\mathbb{A}} (\mathfrak{n}', N') := ((\mathfrak{n}' \otimes N') \circ (M' \otimes \mathfrak{n}'), M' \otimes N')$$

are also objects of the category $\mathscr{L}^a_{\mathbb{A}}$, which leads to the horizontal multiplication. The vertical one is defined as the composed morphism

$$\begin{aligned}(\theta \otimes^{\mathbb{A}} \vartheta) &:= (\mu \otimes M' \otimes N') \circ (\mathbb{A} \otimes \theta \otimes N') \circ (\mathbb{A} \otimes \mathfrak{m} \otimes N') \circ (\mathbb{A} \otimes M \otimes \vartheta) \\ &\quad \circ (\mathbb{A} \otimes M \otimes \eta \otimes N) \\ &= (\mu \otimes M' \otimes N') \circ (\mathbb{A} \otimes \mathfrak{m}' \otimes N') \circ (\theta \otimes \vartheta) \circ (\mathbb{A} \otimes M \otimes \eta \otimes N). \quad (1.27)\end{aligned}$$

Lastly, the identity object of the multiplication $- \otimes^{\mathbb{A}} -$ is given by the pair $(\mathbb{A}, \mathbb{I})$.

1.6. Wreaths and Their Products

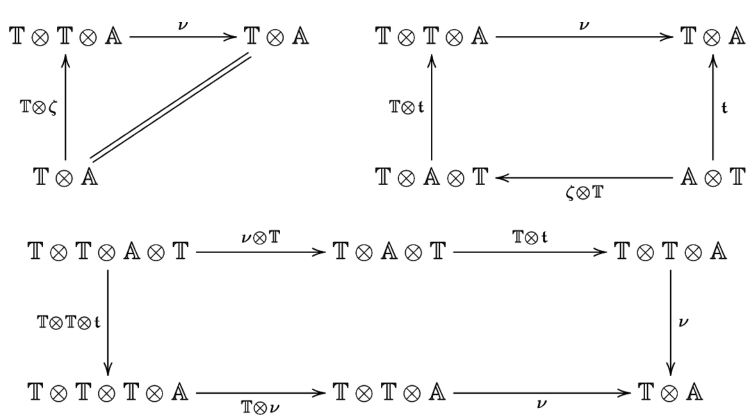
$\mathbb{A}$ still denotes a monoid in $\mathscr{M}$, $\mathscr{R}^a_{\mathbb{A}}$, and $\mathscr{L}^a_{\mathbb{A}}$, are the monoidal categories defined, respectively, in Subsections 1.4 and 1.5. As in the comonoid case, we have the following definition.

Definition 1.10. Let $\mathbb{A}$ be a monoid in a strict monoidal category $\mathscr{M}$. A *right wreath over* $\mathbb{A}$ (or *right* $\mathbb{A}$*-wreath*) is a monoid in the monoidal category $\mathscr{R}^a_{\mathbb{A}}$. A *right cowreath over* $\mathbb{A}$ (or *right* $\mathbb{A}$*-cowreath*) is a comonoid in the monoidal category $\mathscr{R}^a_{\mathbb{A}}$. The left versions of these notions are defined in the monoidal category $\mathscr{L}^c_{\mathbb{A}}$.

The following gives, in terms of the multiplication of $\mathscr{M}$, a simplest and equivalent definition of wreath.

Proposition 1.11. *Let* $\mathbb{A}$ *be a monoid in a strict monoidal category* $\mathscr{M}$ *and* $(\mathbb{T}, \mathfrak{t})$ *an object of the category* $\mathscr{R}^a_{\mathbb{A}}$. *The following statements are equivalent*:

(i) $(\mathbb{T}, \mathfrak{t})$ *is a right* $\mathbb{A}$*-wreath*;
(ii) *There are morphisms of* $\mathbb{A}$*-bimodules* $\zeta : \mathbb{A} \to \mathbb{T} \otimes \mathbb{A}$ *and* $\nu : \mathbb{T} \otimes \mathbb{T} \otimes \mathbb{A} \to \mathbb{T} \otimes \mathbb{A}$ *rendering commutative the following diagrams*:



Proof. It is left to the reader. □

The wreath product is expressed by the following proposition.

Proposition 1.12. *Let* $(\mathbb{A}, \mu, \eta)$ *be a monoid in* $\mathcal{M}$ *and* $(\mathbb{T}, \mathfrak{t})$ *a right* $\mathbb{A}$*-wreath with structure morphisms* $\zeta : \mathbb{A} \to \mathbb{T} \otimes \mathbb{A}$ *and* $v : \mathbb{T} \otimes \mathbb{T} \otimes \mathbb{A} \to \mathbb{T} \otimes \mathbb{A}$. *The object* $\mathbb{T} \otimes \mathbb{A}$ *admits a structure of monoid with multiplication and unit given by*

$$\mu' : \mathbb{T} \otimes \mathbb{A} \otimes \mathbb{T} \otimes \mathbb{A} \xrightarrow{\mathbb{T}\otimes\mathfrak{t}\otimes\mathbb{A}} \mathbb{T} \otimes \mathbb{T} \otimes \mathbb{A} \otimes \mathbb{A} \xrightarrow{v\otimes\mathbb{A}} \mathbb{T} \otimes \mathbb{A} \otimes \mathbb{A} \xrightarrow{\mathbb{T}\otimes\mu} \mathbb{T} \otimes \mathbb{A},$$

$$\eta' : \mathbb{I} \xrightarrow{\eta} \mathbb{A} \xrightarrow{\zeta} \mathbb{T} \otimes \mathbb{A}.$$

Moreover, with this monoid structure the morphism $\zeta : \mathbb{A} \to \mathbb{T} \otimes \mathbb{A}$ *becomes a morphism of monoides.*

Proof. Straightforward. □

The monoid $\mathbb{T} \otimes \mathbb{A}$ of the previous proposition is referred to as *the wreath product* of $\mathbb{A}$ by $\mathbb{T}$.

Remark 1.13. Notice that the object $\mathbb{T}$ occurring in Proposition 1.11 need not to be a monoid. However, if $\mathbb{T}$ is itself a monoid with structure morphisms $\mu' : \mathbb{T} \otimes \mathbb{T} \to \mathbb{T}$, $\eta' : \mathbb{I} \to \mathbb{T}$ such that the pair $(\mathfrak{t}, \mathbb{A})$ belongs to the left monoidal category $\mathcal{L}^a_{\mathbb{T}}$, then the morphisms $\zeta := \eta' \otimes \mathbb{A}$ and $v := \mu' \otimes \mathbb{A}$ endow $(\mathbb{T}, \mathfrak{t})$ with a structure of right $\mathbb{A}$-wreath, while $\zeta' := \eta \otimes \mathbb{T}$, and $v' := \mu \otimes \mathbb{A}$ give to $(\mathfrak{t}, \mathbb{A})$ a structure of left $\mathbb{T}$-wreath. Furthermore, by Proposition 1.12, the morphisms $\zeta : \mathbb{A} \to \mathbb{T} \otimes \mathbb{A}$ and $\zeta' : \mathbb{T} \to \mathbb{T} \otimes \mathbb{A}$ are in fact morphisms of monoids.

In this way the morphism $\mathfrak{t} : \mathbb{T} \otimes \mathbb{A} \to \mathbb{A} \otimes \mathbb{T}$ should satisfy the following equalities:

$$\mathfrak{t} \circ (\mu \otimes \mathbb{T}) = (\mathbb{T} \otimes \mu) \circ (\mathfrak{t} \otimes \mathbb{A}) \circ (\mathbb{A} \otimes \mathfrak{t}) \tag{1.28}$$

$$\mathfrak{t} \circ (\eta \otimes \mathbb{T}) = \mathbb{T} \otimes \eta \tag{1.29}$$

$$\mathfrak{t} \circ (\mathbb{A} \otimes \mu') = (\mu' \otimes \mathbb{A}) \circ (\mathbb{T} \otimes \mathfrak{t}) \circ (\mathfrak{t} \otimes \mathbb{T}) \tag{1.30}$$

$$\mathfrak{t} \circ (\mathbb{A} \otimes \eta') = \eta' \otimes \mathbb{A}. \tag{1.31}$$

A morphism satisfying the previous four equalities is called a *monoid distributive law* from $\mathbb{A}$ to $\mathbb{T}$, see also Beck (1969) for the original definition.

The universal property of wreath products is expressed as follows.

Proposition 1.14. *Let* $(\mathbb{A}, \mu, \eta)$ *be a monoid in a strict monoidal category* $\mathcal{M}$ *and* $(\mathbb{T}, \mathfrak{t})$ *right* $\mathbb{A}$*-wreath with a structure morphisms* $\zeta : \mathbb{A} \to \mathbb{T} \otimes \mathbb{A}$, $v : \mathbb{T} \otimes \mathbb{T} \otimes \mathbb{A} \to \mathbb{T} \otimes \mathbb{A}$.

Let $(\mathbb{L}, \mu', \eta')$ *be a monoid with a monoid morphism* $\varphi : \mathbb{A} \to \mathbb{L}$ *and with morphism* $\psi : \mathbb{T} \to \mathbb{L}$ *satisfying*

$$(\psi \otimes \mathbb{A}) \circ \zeta = \eta' \otimes \mathbb{A} \tag{1.32}$$

$$(\psi \otimes \mathbb{A}) \circ v = (\mu' \otimes \mathbb{A}) \circ (\psi \otimes \psi \otimes \mathbb{A}). \tag{1.33}$$

Assume that φ *and* ψ *satisfy the equality*

$$\mu' \circ (\varphi \otimes \psi) = \mu' \circ (\psi \otimes \alpha) \circ \mathfrak{t}, \tag{1.34}$$

then there exists a unique monoid morphism $\phi : \mathbb{T} \otimes \mathbb{A} \to \mathbb{L}$ *such that* $\phi \circ \zeta = \varphi$ *and* $\phi \circ (\mathbb{T} \otimes \eta) = \psi$.

Proof. Dual to that of Proposition 1.7. □

Remark 1.15. Take $\mathscr{M}$ the monoidal category $\mathscr{M}_{\Bbbk}$ (see Remark 2.4), let $\mathbb{A}$ be a $\Bbbk$-algebra and $\mathbb{C}$ a $\Bbbk$-coalgebra. If in Proposition 1.12 the right $\mathbb{A}$-wreath is induced by a $\Bbbk$-algebra $\mathbb{T}$, then the wreath product $\mathbb{A} \otimes_{\Bbbk} \mathbb{T}$ is the well-known smash product $\mathbb{A}\sharp_{\mathfrak{t}}\mathbb{T}$ as it was proved in Caenepeel et al. (2002, Theorem 2.5) (see also the references cited there). Dually, if in Proposition 1.5 the right $\mathbb{C}$-cowreath is induced by a $\Bbbk$-coalgebra $\mathbb{R}$, then the cowreath product $\mathbb{C} \otimes_{\Bbbk} \mathbb{R}$ is the well-known smash coproduct $\mathbb{C}_{\mathfrak{r}} \ltimes \mathbb{R}$ (Caenepeel et al., 2002, Theorem 3.4). In this way, the universal properties of smash product and smash coproduct stated, respectively, in Caenepeel et al. (2002, Propositions 2.12 and 3.8), are in fact a particular cases of Propositions 1.14 and 1.7.

On the other hand, the factorization problem (Caenepeel et al., 2002, Theorem 4.5) between two $\Bbbk$-bialgebras can be reformulated using the double distributive law. Explicitly, consider two $\Bbbk$-bialgebras $\mathbb{A}$ and $\mathbb{C}$ together with two $\Bbbk$-linear maps $\mathfrak{c} : \mathbb{A} \otimes \mathbb{C} \to \mathbb{C} \otimes \mathbb{A}$ and $\mathfrak{a} : \mathbb{C} \otimes \mathbb{A} \to \mathbb{A} \otimes \mathbb{C}$ such that $(\mathbb{C}, \mathfrak{c}) \in \mathscr{R}^a_{\mathbb{A}}$, $(\mathfrak{a}, \mathbb{C}) \in \mathscr{L}^a_{\mathbb{A}}$, and $(\mathbb{A}, \mathfrak{a}) \in \mathscr{R}^c_{\mathbb{C}}$, $(\mathfrak{c}, \mathbb{A}) \in \mathscr{L}^c_{\mathbb{C}}$. So as above, $\mathbb{A} \otimes \mathbb{C}$ is a $\Bbbk$-coalgebra and $\Bbbk$-algebra which is not necessarily a $\Bbbk$-bialgebra (this is the factorization problem). Using Corollary 4.1, one can give as in Caenepeel et al. (2002, Theorem 4.5) a necessary and sufficient conditions for a double distributive law $\hbar : (\mathbb{A} \otimes \mathbb{C}) \otimes (\mathbb{A} \otimes \mathbb{C}) \to (\mathbb{A} \otimes \mathbb{C}) \otimes (\mathbb{A} \otimes \mathbb{C})$ in order to get a structure of bimonoid on $\mathbb{A} \otimes \mathbb{C}$ in the monoidal category $\mathscr{M}_{\Bbbk}$.

Remark 1.16. Using Lemma 1.1, we can prove that an object $(X, \mathfrak{x})$ belongs to the category $\mathscr{R}^c_{\mathbb{C}}$ if and only if the functor $- \otimes X : \mathscr{M} \to \mathscr{M}$ is lifted to the category of right $\mathbb{C}$-comodules, in the sense that there exists a functor $\overline{- \otimes X} : \mathscr{M}^{\mathbb{C}} \to \mathscr{M}^{\mathbb{C}}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M}^{\mathbb{C}} & \xrightarrow{\ \overline{-\otimes X}\ } & \mathcal{M}^{\mathbb{C}} \\ {\scriptstyle \mathcal{O}_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle \mathcal{O}_{\mathbb{C}}} \\ \mathcal{M} & \xrightarrow{\ -\otimes X\ } & \mathcal{M}, \end{array}$$

where $\mathscr{O}_{\mathbb{C}} : \mathscr{M}^{\mathbb{C}} \to \mathscr{M}$ is the forgetful functor. Lifting functor of this type and their morphisms form a strict monoidal category. Under some suitable assumptions on the multiplication $- \otimes -$ and the category $\mathscr{M}$, one can show that this category is monoidal equivalent to $\mathscr{R}^c_{\mathbb{C}}$ (e.g., El Kaoutit, 2006, Proposition 2.2). Notice that lifting functors were first studied in Beck (1969, Proposition, p. 122) (see also Appelgate and Tierney, 1969; Johnstone, 1975); recent treatments can be found in Wisbauer (2008) and Gomez-Torrecillas (2006).

2. BIMONOID IN A GENERAL MONOIDAL CATEGORY

The letter $\mathscr{M}$ sill denotes a strict monoidal category with multiplication $- \otimes -$ and identity object $\mathbb{I}$. Consider $\mathbb{B}$ an object of $\mathscr{M}$ such that $(\mathbb{B}, \Delta, \varepsilon)$ is a comonoid in $\mathscr{M}$ and $(\mathbb{B}, \mu, \eta)$ is also a monoid in $\mathscr{M}$. Assume that there is a morphism $\hbar : \mathbb{B} \otimes \mathbb{B} \to \mathbb{B} \otimes \mathbb{B}$ which satisfies the following equalities:

$$\hbar \circ (\eta \otimes \mathbb{B}) = \mathbb{B} \otimes \eta \tag{2.1}$$

$$\hbar \circ (\mu \otimes \mathbb{B}) = (\mathbb{B} \otimes \mu) \circ (\hbar \otimes \mathbb{B}) \circ (\mathbb{B} \otimes \hbar) \tag{2.2}$$

$$\hbar \circ (\mathbb{B} \otimes \eta) = \eta \otimes \mathbb{B} \tag{2.3}$$

$$\hbar \circ (\mathbb{B} \otimes \mu) = (\mu \otimes \mathbb{B}) \circ (\mathbb{B} \otimes \hbar) \circ (\hbar \otimes \mathbb{B}) \tag{2.4}$$

$$(\mathbb{B} \otimes \varepsilon) \circ \hbar = \varepsilon \otimes \mathbb{B} \tag{2.5}$$

$$(\mathbb{B} \otimes \Delta) \circ \hbar = (\hbar \otimes \mathbb{B}) \circ (\mathbb{B} \otimes \hbar) \circ (\Delta \otimes \mathbb{B}) \tag{2.6}$$

$$(\varepsilon \otimes \mathbb{B}) \circ \hbar = \mathbb{B} \otimes \varepsilon \tag{2.7}$$

$$(\Delta \otimes \mathbb{B}) \circ \hbar = (\mathbb{B} \otimes \hbar) \circ (\hbar \otimes \mathbb{B}) \circ (\mathbb{B} \otimes \Delta). \tag{2.8}$$

Observe that the equalities (2.1)–(2.2) mean that $(\mathbb{B}, \hbar) \in \mathscr{R}^a_{\mathbb{B}}$, and (2.2)–(2.3) mean that $(\hbar, \mathbb{B}) \in \mathscr{L}^a_{\mathbb{B}}$; while $(\mathbb{B}, \hbar) \in \mathscr{R}^c_{\mathbb{B}}$ by equalities (2.5)–(2.6), and $(\hbar, \mathbb{B}) \in \mathscr{L}^c_{\mathbb{B}}$ by equalities (2.6)–(2.8). Moreover, (2.1)–(2.4) say that $\hbar$ is a monoid distributive law from $\mathbb{B}$ to $\mathbb{B}$, and (2.5)–(2.8) say that $\hbar$ is a comonoid distributive law form $\mathbb{B}$ to $\mathbb{B}$.

Definition 2.1. The morphism $\hbar : \mathbb{B} \otimes \mathbb{B} \to \mathbb{B} \otimes \mathbb{B}$ satisfying equalities (2.1)–(2.8), is called a *double distributive law* between the monoid $(\mathbb{B}, \mu, \eta)$ and the comonoid $(\mathbb{B}, \Delta, \varepsilon)$.

Observe that if $\hbar : \mathbb{B} \otimes \mathbb{B} \to \mathbb{B} \otimes \mathbb{B}$ is a double distributive law, then $(\mathbb{B} \otimes \mathbb{B}, (\mathbb{B} \otimes \hbar \otimes \mathbb{B}) \circ (\Delta \otimes \Delta), \varepsilon \circ (\mathbb{B} \otimes \varepsilon))$ is by Remark 1.6 and Proposition 1.5 a comonoid in $\mathscr{M}$, and $(\mathbb{B} \otimes \mathbb{B}, (\mu \otimes \mu) \circ (\mathbb{B} \otimes \hbar \otimes \mathbb{B}), (\eta \otimes \mathbb{B}) \circ \eta)$ is by Remark 1.13 and Proposition 1.12 a monoid in $\mathscr{M}$. Using both structures, we can now state our main result.

Proposition 2.2. *Let $\mathscr{M}$ be a strict monoidal category with multiplication $- \otimes -$ and identity object $\mathbb{I}$. Consider a* 6*-tuple* $(\mathbb{B}, \Delta, \varepsilon, \mu, \eta, \hbar)$ *where* $(\mathbb{B}, \Delta, \varepsilon)$ *is a comonoid in $\mathscr{M}$,* $(\mathbb{B}, \mu, \eta)$ *is a monoid in $\mathscr{M}$ and* $\hbar : \mathbb{B} \otimes \mathbb{B} \to \mathbb{B} \otimes \mathbb{B}$ *is a double distributive law between them* (*i.e., satisfies equalities* (2.1)–(2.8)). *The following statements are equivalent*:

(i) Δ *and* ε *are morphisms of monoids*;
(ii) μ *and* η *are morphisms of comonoids*;
(iii) Δ, ε, μ, *and* η *satisfy*:

   (a) $\Delta \circ \eta = \eta \otimes \eta$;
   (b) $(\mu \otimes \mu) \circ (\mathbb{B} \otimes \hbar \otimes \mathbb{B}) \circ (\Delta \otimes \Delta) = \Delta \circ \mu$;
   (c) $\varepsilon \circ \eta = \mathbb{I}$;
   (d) $\varepsilon \circ \mu = \varepsilon \otimes \varepsilon$.

Proof. It is clear from the definitions that Δ is morphism of monoid if and only if the equalities (iii)(a)–(b) are satisfied, and that ε is a morphism of monoids if and only if the equalities (iii)(c)–(d) are verified. This shows that (iii) $\Leftrightarrow$ (i). On the other hand, (iii)(b)–(d) is equivalent to say that μ is a comonoid morphism, and (iii)(a)–(c) is equivalent to say that η is a comonoid morphism. This leads to the equivalence (iii) $\Leftrightarrow$ (ii). □

Definition 2.3. Let $\mathscr{M}$ be a strict monoidal category with multiplication $- \otimes -$ and identity object $\mathbb{I}$. A *bimonoid* is a 6-tuple $(\mathbb{B}, \Delta, \varepsilon, \mu, \eta, \hbar)$ satisfying the equivalent conditions of Proposition 2.2.

Remark 2.4. The results stated in Sections 1 and 2 can be extended to the case of not necessary strict monoidal category by using the multiplicative equivalence between any monoidal category and a strict one, see Joyal and Street (1993, Corollary 1.4).

3. RIGHT TWISTED $\mathbb{B}$-MODULES

In this section, we show that the category of right modules over a bimonoid admits a class of right module which is a monoidal category with a strict monoidal forgetful functor (after forgetting all structures). This is the category of right twisted modules. In case of the monoidal category of bimodules, a twisted right module, is a twisted module in the sense of Čap et al. (1995, Section 3), see El Kaoutit (2006, 4.4) for more details. In the case of braided monoidal category, any right module is in fact a twisted right one, that is, the class of right twisted module coincides with the category of all right modules. The proper bimonoid is a right twisted module if and only if the attached double distributive law satisfies the Yang–Baxter equation, and this is the case of the double distributive laws of Examples 4.2, 4.5, and 4.6 of the forthcoming section.

The converse of the above implication is not so clear. That is, if there is a monoidal category $\mathscr{C}$ whose objects are right modules over a monoid $(\mathbb{B}, \mu, \eta)$ and contain the regular module $(\mathbb{B}, \mu)$ such that the forgetful functor $\mathscr{U} : \mathscr{C} \to \mathscr{M}$ is a strict monidal functor, then it is not clear that $\mathbb{B}$ admits a structure of bimonoid in $\mathscr{M}$.

Let $(\mathbb{B}, \Delta, \varepsilon, \mu, \eta, \hbar)$ be a bimonoid in a strict monoidal category $(\mathscr{M}, \otimes, \mathbb{I})$. Define the *category of right twisted modules* $\mathsf{Twist}_{\mathbb{B}}$ as follows:

- *Objects of* $\mathsf{Twist}_{\mathbb{B}}$: They are 3-tuples $(\mathfrak{x}, X, \mathsf{r}_X)$ where $(\mathfrak{x}, X)$ is an object of the category $\mathscr{L}^a_{\mathbb{B}}$ and (X, r_X) is a right $\mathbb{B}$-module with the following compatibility conditions:

$$(\mathbb{B} \otimes \mathsf{r}_X) \circ (\mathfrak{x} \otimes \mathbb{B}) \circ (X \otimes \hbar) = \mathfrak{x} \circ (\mathsf{r}_X \otimes \mathbb{B}) \tag{3.1}$$

$$(\mathbb{B} \otimes \mathfrak{x}) \circ (\mathfrak{x} \otimes \mathbb{B}) \circ (X \otimes \hbar) = (\hbar \otimes X) \circ (\mathbb{B} \otimes \mathfrak{x}) \circ (\mathfrak{x} \otimes \mathbb{B}) \tag{3.2}$$

- *Morphisms in* $\mathsf{Twist}_{\mathbb{B}}$: A morphism $f : (\mathfrak{x}, X, \mathsf{r}_X) \to (\mathfrak{x}', X', \mathsf{r}_{X'})$ in $\mathsf{Twist}_{\mathbb{B}}$ is a morphism $f : X \to X'$ of right $\mathbb{B}$-modules such that

$$(\mathbb{B} \otimes f) \circ \mathfrak{x} = \mathfrak{x}' \circ (f \otimes \mathbb{B}). \tag{3.3}$$

Using the properties of $\mathbb{B}$, we will show later that $(\mathbb{B}, \mathbb{I}, \varepsilon)$ is an object of the category $\mathsf{Twist}_{\mathbb{B}}$ (here $\mathbb{B}$ denotes the identity morphism of the object $\mathbb{B}$). Moreover, $(\hbar, \mathbb{B}, \mu)$ is a right twisted module if and only if $\hbar$ satisfies the Yang–Baxter equation, i.e.,

$$(\hbar \otimes \mathbb{B}) \circ (\mathbb{B} \otimes \hbar) \circ (\hbar \otimes \mathbb{B}) = (\mathbb{B} \otimes \hbar) \circ (\hbar \otimes \mathbb{B}) \circ (\mathbb{B} \otimes \hbar). \tag{3.4}$$

Note that $\hbar$ need not to be an isomorphism.

Let $(\mathfrak{x}, X, \mathsf{r}_X)$ and $(\mathfrak{y}, Y, \mathsf{r}_Y)$ two right twisted modules, define the following morphism:

$$\begin{array}{ccccc} X \otimes Y \otimes \mathbb{B} & \xrightarrow{X\otimes Y\otimes \Delta} & X \otimes Y \otimes \mathbb{B} \otimes \mathbb{B} & \xrightarrow{X\otimes \mathfrak{y}\otimes \mathbb{B}} & X \otimes \mathbb{B} \otimes Y \otimes \mathbb{B} \\ & \searrow^{\mathsf{r}_{X\otimes Y}} & & & \downarrow{\scriptstyle \mathsf{r}_X\otimes \mathsf{r}_Y} \\ & & & & X \otimes Y \end{array}$$

Lemma 3.1. *The* 3*-tuple* $((\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y}), X \otimes Y, \mathsf{r}_{X\otimes Y})$ *is an object of the category* $\mathsf{Twist}_{\mathbb{B}}$. *Moreover, if* $f : (\mathfrak{x}, X, \mathsf{r}_X) \to (\mathfrak{x}', X', \mathsf{r}_{X'})$ *and* $g : (\mathfrak{y}, Y, \mathsf{r}_Y) \to (\mathfrak{y}', Y', \mathsf{r}_{Y'})$ *are two morphisms in* $\mathsf{Twist}_{\mathbb{B}}$, *then*

$$f \otimes g : ((\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y}), X \otimes Y, \mathsf{r}_{X\otimes Y}) \to ((\mathfrak{x}' \otimes Y') \circ (X' \otimes \mathfrak{y}'), X' \otimes Y', \mathsf{r}_{X'\otimes Y'})$$

is also a morphism in $\mathsf{Twist}_{\mathbb{B}}$.

Proof. It was shown, in subsection 1.5, that $((\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y}), X \otimes Y)$ is an object of the category $\mathscr{L}^a_{\mathbb{B}}$. Let us first show that $\mathsf{r}_{X\otimes Y}$ is a right module structure on $X \otimes Y$; we have

$$\begin{aligned} \mathsf{r}_{X\otimes Y} \circ (X \otimes Y \otimes \eta) &= (\mathsf{r}_X \otimes \mathsf{r}_Y) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta) \circ (X \otimes Y \otimes \eta) \\ &\overset{\text{(iii)(a)}}{=} (\mathsf{r}_X \otimes \mathsf{r}_Y) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \eta \otimes \eta) \\ &\overset{(1.24)}{=} (\mathsf{r}_X \otimes \mathsf{r}_Y) \circ (X \otimes \eta \otimes Y \otimes \mathbb{B}) \circ (X \otimes Y \otimes \eta) \\ &= (X \otimes \mathsf{r}_Y) \circ (X \otimes Y \otimes \eta) \\ &= X \otimes Y. \end{aligned}$$

The associativity property is given as follows,

$$\begin{aligned} \mathsf{r}_{X\otimes Y} \circ (\mathsf{r}_{X\otimes Y} \otimes \mathbb{B}) &= (\mathsf{r}_X \otimes \mathsf{r}_Y) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta) \circ (\mathsf{r}_X \otimes \mathsf{r}_Y \otimes \mathbb{B}) \\ &\quad \circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta \otimes \mathbb{B}) \\ &= (\mathsf{r}_X \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathsf{r}_Y) \circ (\mathsf{r}_X \otimes \mathbb{B} \otimes Y \otimes \mathbb{B}) \\ &\quad \circ (X \otimes \mathbb{B} \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes \mathbb{B} \otimes \mathsf{r}_Y \otimes \mathbb{B} \otimes \mathbb{B}) \\ &\quad \circ (X \otimes \mathbb{B} \otimes Y \otimes \mathbb{B} \otimes \Delta) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \end{aligned}$$

$$
\begin{aligned}
&\circ (X \otimes Y \otimes \Delta \otimes \mathbb{B}) \\
= \;& (\mathsf{r}_X \otimes Y) \circ (\mathsf{r}_X \otimes \mathbb{B} \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathsf{r}_Y) \\
&\circ (X \otimes \mathbb{B} \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes \mathbb{B} \otimes \mathsf{r}_Y \otimes \mathbb{B} \otimes \mathbb{B}) \\
&\circ (X \otimes \mathbb{B} \otimes Y \otimes \mathbb{B} \otimes \Delta) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \\
&\circ (X \otimes Y \otimes \Delta \otimes \mathbb{B}) \\
= \;& (\mathsf{r}_X \otimes Y) \circ (X \otimes \mu \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathsf{r}_Y) \\
&\circ (X \otimes \mathbb{B} \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes \mathbb{B} \otimes \mathsf{r}_Y \otimes \mathbb{B} \otimes \mathbb{B}) \\
&\circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \mathbb{B} \otimes \mathbb{B} \otimes \Delta) \\
&\circ (X \otimes Y \otimes \Delta \otimes \mathbb{B}) \\
= \;& (\mathsf{r}_X \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathsf{r}_Y) \circ (X \otimes \mu \otimes Y \otimes \mathbb{B}) \\
&\circ (X \otimes \mathbb{B} \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes \mathbb{B} \otimes \mathsf{r}_Y \otimes \mathbb{B} \otimes \mathbb{B}) \\
&\circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta \otimes \Delta) \\
\overset{(3.1)}{=} \;& (\mathsf{r}_X \otimes \mathsf{r}_Y) \circ (X \otimes \mu \otimes Y \otimes \mathbb{B}) \circ (X \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathsf{r}_Y \otimes \mathbb{B}) \\
&\circ (X \otimes \mathbb{B} \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes \mathbb{B} \otimes Y \otimes \hbar \otimes \mathbb{B}) \\
&\circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta \otimes \Delta) \\
= \;& (\mathsf{r}_X \otimes \mathsf{r}_Y) \circ (X \otimes \mathbb{B} \otimes \mathsf{r}_Y \otimes \mathbb{B}) \circ (X \otimes \mu \otimes Y \otimes \mathbb{B} \otimes \mathbb{B}) \\
&\circ (X \otimes \mathbb{B} \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathbb{B}) \\
&\circ (X \otimes Y \otimes \mathbb{B} \otimes \hbar \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta \otimes \Delta) \\
\overset{(1.23)}{=} \;& (\mathsf{r}_X \otimes \mathsf{r}_Y) \circ (X \otimes \mathbb{B} \otimes \mathsf{r}_Y \otimes \mathbb{B}) \\
&\circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \mu \otimes \mathbb{B} \otimes \mathbb{B}) \\
&\circ (X \otimes Y \otimes \mathbb{B} \otimes \hbar \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta \otimes \Delta) \\
= \;& (\mathsf{r}_X \otimes \mathsf{r}_Y) \circ (X \otimes \mathbb{B} \otimes Y \otimes \mu) \\
&\circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \mu \otimes \mathbb{B} \otimes \mathbb{B}) \\
&\circ (X \otimes Y \otimes \mathbb{B} \otimes \hbar \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta \otimes \Delta) \\
= \;& (\mathsf{r}_X \otimes \mathsf{r}_Y) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \\
&\circ (X \otimes Y \otimes \mathbb{B} \otimes \mu) \circ (X \otimes Y \otimes \mu \otimes \mathbb{B} \otimes \mathbb{B}) \\
&\circ (X \otimes Y \otimes \mathbb{B} \otimes \hbar \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta \otimes \Delta) \\
= \;& (\mathsf{r}_X \otimes \mathsf{r}_Y) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes Y \otimes ((\mu \otimes \mu) \\
&\circ (\mathbb{B} \otimes \hbar \otimes \mathbb{B}) \circ (\Delta \otimes \Delta))) \\
\overset{\text{(iii)(b)}}{=} \;& (\mathsf{r}_X \otimes \mathsf{r}_Y) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta) \circ (X \otimes Y \otimes \mu) \\
= \;& \mathsf{r}_{X \otimes Y} \circ (X \otimes Y \otimes \mu).
\end{aligned}
$$

We need to show Eqs. (3.1) and (3.2) for the twist $(\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y})$, so we have

$$
\begin{aligned}
&(\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y}) \circ (\mathsf{r}_{X\otimes Y} \otimes \mathbb{B}) \\
&\quad = (\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y}) \circ (\mathsf{r}_X \otimes \mathsf{r}_Y \otimes \mathbb{B}) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta \otimes \mathbb{B}) \\
&\quad = (\mathfrak{x} \otimes Y) \circ (\mathsf{r}_X \otimes \mathbb{B} \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathfrak{y}) \circ (X \otimes \mathbb{B} \otimes \mathsf{r}_Y \otimes \mathbb{B}) \\
&\qquad \circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta \otimes \mathbb{B}) \\
&\quad \overset{(3.1)}{=} (\mathfrak{x} \otimes Y) \circ (\mathsf{r}_X \otimes \mathbb{B} \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathsf{r}_Y) \circ (X \otimes \mathbb{B} \otimes \mathfrak{y} \otimes \mathbb{B}) \\
&\qquad \circ (X \otimes \mathbb{B} \otimes Y \otimes \hbar) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta \otimes \mathbb{B}) \\
&\quad \overset{(3.1)}{=} (\mathbb{B} \otimes \mathsf{r}_X \otimes Y) \circ (\mathfrak{x} \otimes \mathbb{B} \otimes Y) \circ (X \otimes \hbar \otimes Y) \\
&\qquad \circ (X \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathsf{r}_Y) \circ (X \otimes \mathbb{B} \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes \mathbb{B} \otimes Y \otimes \hbar) \\
&\qquad \circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta \otimes \mathbb{B}) \\
&\quad = (\mathbb{B} \otimes \mathsf{r}_X \otimes Y) \circ (\mathfrak{x} \otimes \mathbb{B} \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathsf{r}_Y) \circ (X \otimes \hbar \otimes Y \otimes \mathbb{B}) \\
&\qquad \circ (X \otimes \mathbb{B} \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \mathbb{B} \otimes \hbar) \circ (X \otimes Y \otimes \Delta \otimes \mathbb{B}) \\
&\quad \overset{(3.2)}{=} (\mathbb{B} \otimes \mathsf{r}_X \otimes Y) \circ (\mathfrak{x} \otimes \mathbb{B} \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathsf{r}_Y) \circ (X \otimes \mathbb{B} \otimes \mathfrak{y} \otimes \mathbb{B}) \\
&\qquad \circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \hbar \otimes \mathbb{B}) \circ (X \otimes Y \otimes \mathbb{B} \otimes \hbar) \circ (X \otimes Y \otimes \Delta \otimes \mathbb{B}) \\
&\quad \overset{(2.6)}{=} (\mathbb{B} \otimes \mathsf{r}_X \otimes Y) \circ (\mathfrak{x} \otimes \mathbb{B} \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathsf{r}_Y) \circ (X \otimes \mathbb{B} \otimes \mathfrak{y} \otimes \mathbb{B}) \\
&\qquad \circ (X \otimes \mathfrak{y} \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \mathbb{B} \otimes \Delta) \circ (X \otimes Y \otimes \hbar) \\
&\quad = (\mathbb{B} \otimes \mathsf{r}_X \otimes Y) \circ (\mathfrak{x} \otimes \mathbb{B} \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathsf{r}_Y) \circ (X \otimes \mathbb{B} \otimes \mathfrak{y} \otimes \mathbb{B}) \\
&\qquad \circ (X \otimes \mathbb{B} \otimes Y \otimes \Delta) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \hbar) \\
&\quad = (\mathbb{B} \otimes \mathsf{r}_X \otimes \mathsf{r}_Y) \circ (\mathbb{B} \otimes X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (\mathfrak{x} \otimes Y \otimes \mathbb{B} \otimes \mathbb{B}) \\
&\qquad \circ (X \otimes \mathbb{B} \otimes Y \otimes \Delta) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \hbar) \\
&\quad = (\mathbb{B} \otimes \mathsf{r}_X \otimes \mathsf{r}_Y) \circ (\mathbb{B} \otimes X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (\mathbb{B} \otimes X \otimes Y \otimes \Delta) \\
&\qquad \circ (\mathfrak{x} \otimes Y \otimes \mathbb{B}) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \hbar) \\
&\quad = (\mathbb{B} \otimes \mathsf{r}_{X\otimes Y}) \circ (((\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y})) \otimes \mathbb{B}) \circ (X \otimes Y \otimes \hbar),
\end{aligned}
$$

this gives us Eq. (3.1). For Eq. (3.2), we have

$$
\begin{aligned}
&(\hbar \otimes X \otimes Y) \circ (\mathbb{B} \otimes ((\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y}))) \circ (((\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y})) \otimes \mathbb{B}) \\
&\quad = (\hbar \otimes X \otimes Y) \circ (\mathbb{B} \otimes \mathfrak{x} \otimes Y) \circ (\mathbb{B} \otimes X \otimes \mathfrak{y}) \circ (\mathfrak{x} \otimes Y \otimes \mathbb{B}) \circ (X \otimes \mathfrak{y}) \otimes \mathbb{B}) \\
&\quad = (\hbar \otimes X \otimes Y) \circ (\mathbb{B} \otimes \mathfrak{x} \otimes Y) \circ (\mathfrak{x} \otimes \mathbb{B} \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathfrak{y}) \circ (X \otimes \mathfrak{y}) \otimes \mathbb{B}) \\
&\quad \overset{(3.2)}{=} (\mathbb{B} \otimes \mathfrak{x} \otimes Y) \circ (\mathfrak{x} \otimes \mathbb{B} \otimes Y) \circ (X \otimes \hbar \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathfrak{y}) \circ (X \otimes \mathfrak{y}) \otimes \mathbb{B}) \\
&\quad \overset{(3.2)}{=} (\mathbb{B} \otimes \mathfrak{x} \otimes Y) \circ (\mathfrak{x} \otimes \mathbb{B} \otimes Y) \circ (X \otimes \mathbb{B} \otimes \mathfrak{y}) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \hbar)
\end{aligned}
$$

$$= (\mathbb{B} \otimes \mathfrak{x} \otimes Y) \circ (\mathbb{B} \otimes X \otimes \mathfrak{y}) \circ (\mathfrak{x} \otimes Y \otimes \mathbb{B}) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \hbar)$$
$$= (\mathbb{B} \otimes ((\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y}))) \circ (((\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y})) \otimes \mathbb{B}) \circ (X \otimes Y \otimes \hbar).$$

Let f and g be the stated morphisms, we have

$$\begin{aligned}(f \otimes g) \circ \mathsf{r}_{X\otimes Y} &= (f \otimes g) \circ (\mathsf{r}_X \otimes \mathsf{r}_Y) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta) \\ &= (\mathsf{r}_{X'} \otimes \mathsf{r}_{Y'}) \circ (f \otimes \mathbb{B} \otimes g \otimes \mathbb{B}) \circ (X \otimes \mathfrak{y} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta) \\ &\overset{(3.3)}{=} (\mathsf{r}_{X'} \otimes \mathsf{r}_{Y'}) \circ (f \otimes \mathfrak{y}' \otimes \mathbb{B}) \circ (X \otimes g \otimes \mathbb{B} \otimes \mathbb{B}) \circ (X \otimes Y \otimes \Delta) \\ &= (\mathsf{r}_{X'} \otimes \mathsf{r}_{Y'}) \circ (X' \otimes \mathfrak{y}' \otimes \mathbb{B}) \circ (X' \otimes Y' \otimes \Delta) \circ (f \otimes g \otimes \mathbb{B}) \\ &= \mathsf{r}_{X'\otimes Y'} \circ (f \otimes g \otimes \mathbb{B}),\end{aligned}$$

this proves that $f \otimes g$ is a morphism of right modules. Equation (3.3) for $f \otimes g$ is given as follows,

$$\begin{aligned}(\mathbb{B} \otimes f \otimes g) \circ (\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y}) &= (\mathbb{B} \otimes f \otimes Y') \circ (\mathbb{B} \otimes X \otimes g) \circ (\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y}) \\ &= (\mathbb{B} \otimes f \otimes Y') \circ (\mathfrak{x} \otimes Y') \circ (X \otimes \mathbb{B} \otimes g) \circ (X \otimes \mathfrak{y}) \\ &\overset{(3.3)}{=} (\mathfrak{x}' \otimes Y') \circ (f \otimes \mathbb{B} \otimes Y') \circ (X \otimes \mathfrak{y}') \circ (X \otimes g \otimes \mathbb{B}) \\ &= (\mathfrak{x}' \otimes Y') \circ (X' \otimes \mathfrak{y}') \circ (f \otimes g \otimes \mathbb{B}).\end{aligned}$$ □

Proposition 3.2. *If* $(\mathbb{B}, \Delta, \varepsilon, \mu, \eta, \hbar)$ *is a bimonoid in a strict monoidal category* $(\mathscr{M}, \otimes, \mathbb{I})$, *then the category of twisted right modules* $\mathsf{Twist}_{\mathbb{B}}$ *is a strict monoidal category with a strict monoidal forgetful functor* $\mathscr{U} : \mathsf{Twist}_{\mathbb{B}} \to \mathscr{M}$.

Proof. By Lemma 3.1, we know that the multiplication $- \otimes -$ of the category $\mathscr{M}$ induces a well-defined associative multiplication in the category of twisted modules $\mathsf{Twist}_{\mathbb{B}}$. By Proposition 2.2(iii)(c)–(d), ε is actually a right $\mathbb{B}$-action on $\mathbb{I}$. This action satisfies trivially Eq. (3.2), and Eq. (3.1) is fulfilled since $\hbar$ satisfies Eq. (2.5). The identity object of the above multiplication in $\mathsf{Twist}_{\mathbb{B}}$, is then given by the 3-tuple $(\mathbb{B}, \mathbb{I}, \varepsilon)$ (here $\mathbb{B}$ is the identity morphism of $\mathbb{B}$ in $\mathscr{M}$). By construction, it is clear that the forgetful functor $\mathscr{U} : \mathsf{Twist}_{\mathbb{B}} \to \mathscr{M}$ is a strict monoidal functor. □

Remark 3.3. There are also two functors $\mathscr{O} : \mathsf{Twist}_{\mathbb{B}} \to \mathscr{M}_{\mathbb{B}}$ and $\mathscr{O}' : \mathsf{Twist}_{\mathbb{B}} \to \mathscr{L}^a_{\mathbb{B}}$. The first one sends every twisted right $\mathbb{B}$-module $(\mathfrak{x}, X, \mathsf{r}_X)$ to its underlying right $\mathbb{B}$-module (X, r_X) and acts by identity on morphisms. The second functor sends every twisted right module $(\mathfrak{x}, X, \mathsf{r}_X)$ to its underlying object $(\mathfrak{x}, X) \in \mathscr{L}^a_{\mathbb{B}}$ and any morphism f in $\mathsf{Twist}_{\mathbb{B}}$ to the morphism $\mathbb{B} \otimes f$ in $\mathscr{L}^a_{\mathbb{B}}$. The functor $\mathscr{O}' : \mathsf{Twist}_{\mathbb{B}} \to \mathscr{L}^a_{\mathbb{B}}$ is in fact strict monoidal. Namely, if $f : (\mathfrak{x}, X, \mathsf{r}_X) \to (\mathfrak{x}', X', \mathsf{r}_{X'})$ and $g : (\mathfrak{y}, Y, \mathsf{r}_Y) \to (\mathfrak{y}', Y', \mathsf{r}_{Y'})$ are two morphisms in $\mathsf{Twist}_{\mathbb{B}}$, then by Lemma 3.1, we have $\mathscr{O}'((\mathfrak{x}, X, \mathsf{r}_X) \otimes (\mathfrak{y}, Y, \mathsf{r}_Y)) = ((\mathfrak{x} \otimes Y) \circ (X \otimes \mathfrak{y}), X \otimes Y) = \mathscr{O}'(\mathfrak{x}, X) \otimes^{\mathbb{B}} \mathscr{O}'(\mathfrak{y}, Y)$. Moreover, since f and g satisfy Eq. (3.3), an easy computation shows that $\mathscr{O}'(f \otimes g) = \mathbb{B} \otimes f \otimes g = f \otimes^{\mathbb{B}} g$, where $- \otimes^{\mathbb{B}} -$ is the multiplication of $\mathscr{L}^a_{\mathbb{B}}$ defined in Eq. (1.27).

Remark 3.4. Of course there is a dual result of Proposition 3.2, concerning right $\mathbb{B}$-comodules. That is, we can define the category of *right twisted comodules* $\mathsf{Twist}^{\mathbb{B}}$ as follows:

- *Objects of* $\mathsf{Twist}^{\mathbb{B}}$: They are 3-tuples $(\mathfrak{x}, X, \rho^X)$, where $(X, \mathfrak{x})$ is an object of the category $\mathscr{R}^c_{\mathbb{B}}$, and (X, ρ^X) is a right $\mathbb{B}$-comodule with the following compatibility conditions:

$$(X \otimes \hbar) \circ (\mathfrak{x} \otimes \mathbb{B}) \circ (X \otimes \rho^X) = (\rho^X \otimes \mathbb{B}) \circ \mathfrak{x}$$
$$(\mathfrak{x} \otimes \mathbb{B}) \circ (\mathbb{B} \otimes \mathfrak{x}) \circ (\hbar \otimes X) = (X \otimes \hbar) \circ (\mathfrak{x} \otimes \mathbb{B}) \circ (\mathbb{B} \otimes \mathfrak{x}).$$

- *Morphisms in* $\mathsf{Twist}^{\mathbb{B}}$: A morphism $f : (\mathfrak{x}, X, \rho^X) \to (\mathfrak{x}', X', \rho^{X'})$ in $\mathsf{Twist}^{\mathbb{B}}$ is a morphism $f : X \to X'$ of right $\mathbb{B}$-comodules such that

$$(f \otimes \mathbb{B}) \circ \mathfrak{x} = \mathfrak{x}' \circ (\mathbb{B} \otimes f).$$

We can prove, as in Proposition 3.2, that $\mathsf{Twist}^{\mathbb{B}}$ is a monoidal category with identity object $(\mathbb{B}, \mathbb{I}, \eta)$ and with a strict monoidal forgetful functor $\mathscr{V} : \mathsf{Twist}^{\mathbb{B}} \to \mathscr{M}$. We also have analogue results on both categories of left modules and left comodules.

4. SOME APPLICATIONS

In what follows, $\mathbb{k}$ denotes a commutative ring with 1 and $\mathscr{M}_{\mathbb{k}}$ its category of modules. The unadorned symbol $- \otimes -$ stands for the tensor product over $\mathbb{k}$.

4.1. Compatibility Conditions Between Rings and Corings

Let R be $\mathbb{k}$-algebra, all bimodules are assumed to be central $\mathbb{k}$-bimodules, and their category will be denoted by ${}_R\mathscr{M}_R$. This is a monoidal category with multiplication $- \otimes_R -$ the tensor product over R, and with identity object the regular R-bimodule ${}_RR_R$. Let $(\mathfrak{C}, \Delta, \varepsilon)$ be an R-coring (Sweedler, 1975a) (i.e., a comonoid in ${}_R\mathscr{M}_R$), we use Sweedler's notation for the comultiplication, that is, $\Delta(c) = c_{(1)} \otimes_R c_{(2)}$, for every $c \in \mathfrak{C}$ (summation understood). Given an R-bilinear morphism $\hbar : \mathfrak{C} \otimes_R \mathfrak{C} \to \mathfrak{C} \otimes_R \mathfrak{C}$, we denote the image of $x \otimes y \in \mathfrak{C} \otimes_R \mathfrak{C}$ by $\hbar(x \otimes y) = y^{\hbar} \otimes x^{\hbar}$ (summation understood).

Now taking into account Remark 2.4, we can state the compatibility conditions in the monoidal category ${}_R\mathscr{M}_R$.

Corollary 4.1. *Let R be a $\mathbb{k}$-algebra, and $\iota : R \to \mathfrak{C}$ a ring extension. Consider $\mathfrak{C}$ as an R-bimodule by restricting ι. Assume that this R-bimodule admits a structure of an R-coring with comultiplication and counit, respectively, Δ and ε. If $\hbar : \mathfrak{C} \otimes_R \mathfrak{C} \to \mathfrak{C} \otimes_R \mathfrak{C}$ is a double distributive law* (*i.e., an R-bilinear map satisfying* (2.1)–(2.8)), *then the* 6-*tuple $(\mathfrak{C}, \Delta, \varepsilon, \mu, 1_{\mathfrak{C}}, \hbar)$ is a bimonoid in the monoidal category ${}_R\mathscr{M}_R$* (*Definition* 2.3) *if and only if*:

(a) $\Delta(1_{\mathfrak{C}}) = 1_{\mathfrak{C}} \otimes_R 1_{\mathfrak{C}}$;
(b) $\Delta(xy) = x_{(1)}{y_{(1)}}^{\hbar} \otimes {x_{(2)}}^{\hbar} y_{(2)}$, *for every* $x, y \in \mathfrak{C}$;

(c) $\varepsilon(1_{\mathfrak{C}}) = 1_R$;
(d) $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$, *for every* $x, y \in \mathfrak{C}$.

In particular, R is a trivial bimonoid in ${}_R\mathscr{M}_R$.

Example 4.2. Let G be any group and consider the group algebra $\Bbbk[G]$. We denote by $\mathbb{B} := R \otimes \Bbbk[G]$ the tensor product algebra, and consider it as an R-ring by the canonical extension $R \to \mathbb{B}$ sending $r \mapsto (r \otimes e)$ (here e is the neutral element of G). This endows $\mathbb{B}$ with a structure of an R-bimodule given by the rule

$$r(s \otimes x)t = ((rst) \otimes x) \qquad \text{for every } r, s, t \in R \quad \text{and} \quad x \in G.$$

This bimodule is clearly an R-coring with structure maps

$$\begin{aligned} \Delta : \mathbb{B} &\longrightarrow \mathbb{B} \otimes_R \mathbb{B}, & \varepsilon : \mathbb{B} &\longrightarrow R \\ (r \otimes x) &\longmapsto (r \otimes x) \otimes_R (1_R \otimes x), & (r \otimes x) &\longmapsto r. \end{aligned}$$

Now define $\hbar : \mathbb{B} \otimes_R \mathbb{B} \to \mathbb{B} \otimes_R \mathbb{B}$ by the rule

$$\hbar((r \otimes x) \otimes_R (s \otimes y)) = ((rs \otimes y) \otimes_R (1_R \otimes x)) \qquad \text{for every } r, s \in R \quad \text{and} \quad x, y \in G.$$

It is clear that $\hbar$ is an R-bilinear map. Up to the canonical isomorphism $\mathbb{B} \otimes_R \mathbb{B} \cong R \otimes \Bbbk[G] \otimes \Bbbk[G]$, this map can be rewritten as $\hbar = R \otimes \tau$, where τ is the usual flip $\tau : x \otimes y \mapsto y \otimes x$ on $\Bbbk[G]$. In this way, it is easily checked that $\hbar$ satisfies Eq. (2.1)–(2.2) and (2.5)–(2.6). The rest of equations, that is, (2.3)–(2.4) and (2.7)–(2.8) are similarly deduced. Henceforth, $\hbar$ is a double distributive law between the R-ring $\mathbb{B}$ and the R-coring $\mathbb{B}$. Moreover, for every $r, s \in R$ and $x, y \in G$, we have

$$\begin{aligned} (r \otimes x)_{(1)}(t \otimes y)_{(1)}{}^{\hbar} \otimes_R (r \otimes x)_{(2)}{}^{\hbar}(t \otimes y)_{(2)} &= (rs \otimes xy) \otimes_R (1_R \otimes xy) \\ &= \Delta \circ \mu((r \otimes x) \otimes_R (t \otimes y)), \end{aligned}$$

and clearly $\Delta(1_{\mathbb{B}}) = 1_{\mathbb{B}} \otimes_R 1_{\mathbb{B}}$, $\varepsilon(1_{\mathbb{B}}) = 1_R$ (here $1_{\mathbb{B}} = 1_R \otimes e$ the unit of the ring $\mathbb{B}$), and

$$\varepsilon \circ \mu((r \otimes x) \otimes_R (t \otimes y)) = rs = (\varepsilon \otimes_R \varepsilon)((r \otimes x) \otimes_R (t \otimes y)).$$

Therefore, by Corollary 4.1, we conclude that the 6-tuple $(R \otimes \Bbbk[G], \Delta, \varepsilon, \mu, 1_R \otimes e, \hbar)$ is a bimonoid in the monoidal category of bimodules ${}_R\mathscr{M}_R$.

Example 4.3. Consider the Ore extension $\mathbb{B} := R[X; \sigma]$, with σ a ring endomorphism of R. The elements of $\mathbb{B}$ are left polynomials in the indeterminate X. This is an R-bimodule with the biaction

$$r'(rX^n)s = r'r\sigma^n(s)X^n, \qquad \forall r', r, s \in R, \quad \text{and} \quad \forall n \in \mathbb{N}.$$

$\mathbb{B}$ is in fact an R-ring with a canonical injection $R \hookrightarrow \mathbb{B}$ the identity map. On the other hand, $\mathbb{B}$ is an R-coring with structure maps

$$\begin{aligned} &\Delta : \mathbb{B} \to \mathbb{B} \otimes_R \mathbb{B}, && \varepsilon : \mathbb{B} \to R \\ &X^n \mapsto X^n \otimes_R 1 + 1 \otimes_R X^n, \quad \text{if } n \neq 0 && X^n \mapsto 0, \quad \text{if } n \neq 0 \\ &r \mapsto r \otimes_R 1 && r \mapsto r. \end{aligned}$$

Define $\hbar : \mathbb{B} \otimes_R \mathbb{B} \to \mathbb{B} \otimes_R \mathbb{B}$ by the rule

$$\hbar(rX^n \otimes_R sX^m) = r\sigma^n(s)X^m \otimes_R X^n, \qquad \text{for every } r, s \in R \quad \text{and} \quad n, m \in \mathbb{N}.$$

A routine computation shows that $\hbar$ is in fact a double distributive law.

4.2. Algebras and Coalgebras

Example 4.4. Let (B, Δ, ε) be a $\Bbbk$-coalgebra and $(B, \mu, 1_B)$ a $\Bbbk$-algebra. Denote by $\tau : B \otimes B \to B \otimes B$ the usual flip map, i.e., $\tau(x \otimes y) = y \otimes x$, for all $x, y \in B$. One can easily check that τ satisfies all Eqs. (2.1)–(2.4) and (2.5)–(2.8) with respect to Δ, ε, μ, and 1_B. That is, in our terminology, τ is a double distributive law. Therefore, by Corollary 4.1, $(B, \Delta, \varepsilon, \mu, 1_B, \tau)$ is a bimonoid in the monoidal category $\mathscr{M}_{\Bbbk}$ if and only if B is a bialgebra in the usual sense (Sweedler, 1969).

Example 4.5 (Takeuchi, 2002, Definition 5.1)**.** A *braided* $\Bbbk$*-bialgebra* is a 6-tuple $(\mathbb{H}, \Delta, \varepsilon, \mu, \eta, \mathscr{R})$, where $\mathbb{H}$ is a $\Bbbk$-module and the following are $\Bbbk$-linear maps

$$\Delta : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}, \qquad \varepsilon : \mathbb{H} \to \Bbbk, \qquad \mu : \mathbb{H} \otimes \mathbb{H} \to \mathbb{H},$$
$$\eta : \Bbbk \to \mathbb{H}, \qquad \mathscr{R} : \mathbb{H} \otimes \mathbb{H} \xrightarrow{\cong} \mathbb{H} \otimes \mathbb{H}$$

satisfying the following conditions:

(1) $(\mathbb{H}, \Delta, \varepsilon)$ is a $\Bbbk$-coalgebra;
(2) $(\mathbb{H}, \mu, \eta)$ is a $\Bbbk$-algebra;
(3) $\mathscr{R}$ is a Yang–Baxter operator on $\mathbb{H}$;
(4) The structures Δ, ε, μ and η commute with $\mathscr{R}$ (in the sense of Takeuchi, 2002, Lemma 1.7);
(5) ε is an algebra map and $\eta : \Bbbk \to \mathbb{H}$ is a coalgebra map;
(6) We have

$$\Delta \circ \mu = (\mu \otimes \mu) \circ (\mathbb{H} \otimes \mathscr{R} \otimes \mathbb{H}) \circ (\Delta \otimes \Delta).$$

An easy verification shows that the Yang–Baxter operator satisfies condition (4) if and only if it is a double distributive law. Conditions (5) and (6) are exactly the conditions (a)–(d) of Corollary 4.1. Thus, if $(\mathbb{H}, \Delta, \varepsilon, \mu, \eta, \mathscr{R})$ is a braided $\Bbbk$-bialgebra, then it is a bimonoid in the category of modules $\mathscr{M}_{\Bbbk}$. Notice, that in this case the double distributive law in an isomorphism and satisfies a further condition, namely, condition (3). An example of braided algebras are bialgebras in the braided monoidal category of right comodules over coquasitriangular bialgebra over a field (Takeuchi, 2002, Proposition 5.2).

The following gives an example of bimonoid in the monoidal category $\mathscr{M}_{\Bbbk}$ which is not necessarily a $\Bbbk$-bialgebra.

Example 4.6. Let L be a $\Bbbk$-module, and set $B := \Bbbk \oplus L$. An element belonging to B will be denoted by a pair (k, x), where $k \in \Bbbk$ and $x \in L$. We consider B as a $\Bbbk$-algebra with multiplication and unit defined by

$$\mu((k,x) \otimes (l,y)) = (k,x)(l,y) = (kl, ky + lx), \qquad 1_B = (1,0), \quad \forall (k,x), (l,y) \in B,$$

and also as a $\Bbbk$-coalgebra with comultiplication and counit defined by

$$\Delta(k,x) = (k,x) \otimes (1,0) + (1,0) \otimes (0,x), \qquad \varepsilon(k,x) = (k,0), \quad \forall (k,x) \in B.$$

It is well known that B is not a $\Bbbk$-bialgebra, since Δ is not in general a multiplicative map. Now consider the $\Bbbk$-linear map

$$\hbar : B \otimes B \longrightarrow B \otimes B,$$
$$\big((k,x) \otimes (l,y) \longmapsto (l,y) \otimes (k,x) - (0,y) \otimes (0,x) - (0,x) \otimes (0,y)\big).$$

We claim that $\hbar$ is a double distributive law. To this end, it suffices to check a half of conditions (2.1)–(2.8). That is, we only need to show (2.1), (2.2), (2.5), and (2.6), and the rest follow immediately by symmetry. Fix $(k,x), (l,y), (l',y') \in B$. First, we have

$$\begin{aligned} \hbar((1,0) \otimes (l,y)) &= (l,y) \otimes (1,0), \qquad \text{and} \\ (B \otimes \varepsilon) \circ \hbar((k,x) \otimes (l,y)) &= B \otimes \varepsilon((l,y) \otimes (k,x) - (0,y) \otimes (0,x) - (0,x) \otimes (0,y)) \\ &= (l,y)k = \varepsilon \otimes B((k,x) \otimes (l,y)), \end{aligned}$$

that is, $\hbar$ satisfies (2.1) and (2.5). On the other hand, we have

$$\begin{aligned} &(B \otimes \mu) \circ (\hbar \otimes B) \circ (B \otimes \hbar)((k,x) \otimes (l',y') \otimes (l,y)) \\ &\quad = (B \otimes \mu) \circ (\hbar \otimes B)((k,x) \otimes (l,y) \otimes (l',y') \\ &\qquad - (k,x) \otimes (0,y') \otimes (0,y) - (k,x) \otimes (0,y) \otimes (0,y')) \\ &\quad = (B \otimes \mu)\,[((l,y) \otimes (k,x) - (0,y) \otimes (0,x) - (0,x) \otimes (0,y)) \otimes (l',y') \\ &\qquad - ((0,y') \otimes (k,x) - (0,y') \otimes (0,x) - (0,x) \otimes (0,y')) \otimes (0,y) \\ &\qquad - ((0,y) \otimes (k,x) - (0,y) \otimes (0,x) - (0,x) \otimes (0,y)) \otimes (0,y')] \\ &\quad = (l,y) \otimes (kl', ky' + xl') - (0,y) \otimes (0,l'x) - (0,x) \otimes (0,l'y) \\ &\qquad - (0,y') \otimes (0,ky) - (0,y) \otimes (0,ky') \\ &\quad = (l,y) \otimes (kl', ky' + xl') - (0,y) \otimes (0, xl' + ky') - (0, l'x + ky') \otimes (0,y) \\ &\quad = \hbar \circ (\mu \otimes B)((k,x) \otimes (l',y') \otimes (l,y)), \end{aligned}$$

and

$$
\begin{aligned}
&(\hbar \otimes B) \circ (B \otimes \hbar) \circ (\Delta \otimes B)((k, x) \otimes (l, y)) \\
&\quad = (\hbar \otimes B) \circ (B \otimes \hbar)((k, x) \otimes (1, 0) \otimes (l, y) + (1, 0) \otimes (0, x) \otimes (l, y)) \\
&\quad = (\hbar \otimes B)((k, x) \otimes (l, y) \otimes (1, 0) + (1, 0) \otimes (l, y) \otimes (0, x) \\
&\qquad - (1, 0) \otimes (0, x) \otimes (0, y) - (1, 0) \otimes (0, y) \otimes (0, x)) \\
&\quad = (l, y) \otimes (k, x) \otimes (1, 0) - (0, y) \otimes (0, x) \otimes (1, 0) \\
&\qquad - (0, x) \otimes (0, y) \otimes (1, 0) + (l, y) \otimes (1, 0) \otimes (0, x) \\
&\qquad - (0, x) \otimes (1, 0) \otimes (0, y) - (0, y) \otimes (1, 0) \otimes (0, x) \\
&\quad = (B \otimes \Delta) \circ \hbar((k, x) \otimes (l, y)),
\end{aligned}
$$

which gives Eqs. (2.2) and (2.6) for $\hbar$. This finishes the proof of the claim. Next we show that the 6-tuple $(B, \Delta, \varepsilon, \mu, 1_B, \hbar)$ is by Proposition 2.2, a bimonoid in the monoidal category $\mathscr{M}_{\Bbbk}$. The equalities 2.2(iii)(a), 2.2(iii)(c), and 2.2(iii)(d) are easily checked, and 2.2(iii)(b) is derived from the following computation:

$$
\begin{aligned}
&(\mu \otimes \mu) \circ (B \otimes \hbar \otimes B) \circ (\Delta \otimes \Delta)((k, x) \otimes (l, y)) \\
&\quad = (\mu \otimes \mu) \circ (B \otimes \hbar \otimes B)\big((k, x) \otimes (1, 0) \otimes (l, y) \otimes (1, 0) \\
&\qquad + (k, x) \otimes (1, 0) \otimes (1, 0) \otimes (0, y) + (1, 0) \otimes (0, x) \otimes (l, y) \otimes (1, 0) \\
&\qquad + (1, 0) \otimes (0, x) \otimes (1, 0) \otimes (0, y)\big) \\
&\quad = (\mu \otimes \mu)\big((k, x) \otimes (l, y) \otimes (1, 0) \otimes (1, 0) + (k, x) \otimes (1, 0) \otimes (1, 0) \otimes (0, y) \\
&\qquad + (1, 0) \otimes ((l, 0) \otimes (0, x) - (0, x) \otimes (0, y)) \otimes (1, 0) \\
&\qquad + (1, 0) \otimes (1, 0) \otimes (0, x) \otimes (0, y)\big),
\end{aligned}
$$

and so

$$
\begin{aligned}
&(\mu \otimes \mu) \circ (B \otimes \hbar \otimes B) \circ (\Delta \otimes \Delta)((k, x) \otimes (l, y)) \\
&\quad = (kl, ky + lx) \otimes (1, 0) + (k, 0) \otimes (0, y) + (l, 0) \otimes (0, x) \\
&\quad = (kl, ky + lx) \otimes (1, 0) + (1, 0) \otimes (0, ky + lx) \\
&\quad = \Delta((kl, ky + lx)) = \Delta \circ \mu((k, x) \otimes (l, y)).
\end{aligned}
$$

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