

WIDE MORITA CONTEXTS IN BICATEGORIES[†]

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الخلاصة:

نقدم مفهوماً رسمياً لسياق Morita الواسع (الأيمن) بين زوج من خلايا-0 في أي فئة مزدوجة. ثم نقوم ببناء فئة مزدوجة جديدة لها نفس خلايا-0 الخاصة بالفئة السابقة، وتكون خلايا-1 الخاصة بها جميع سياقات Morita الواسعة (اليمنى). نقدم تطبيقاً على فئة Eilenberg-Moore المزدوجة (اليمنى) للمونادات المصاحبة المرتبطة بالفئة المزدوجة للمعايير المزدوجة.

ABSTRACT

We give a formal concept of (right) wide Morita context between two 0-cells in an arbitrary bicategory. We then construct a new bicategory with the same 0-cells as the previous one, and with 1-cells all these (right) wide Morita contexts. An application to the (right) Eilenberg–Moore bicategory of comonads associated to the bicategory of bimodules is also given.

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Key words: bimodules, corings, category of comodules, Morita contexts, bicategories

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WIDE MORITA CONTEXTS IN BICATEGORIES

1. INTRODUCTION

A Morita context (see [1]) connecting two associative rings with unit A and B , is a four-tuple $({}_B M_A, {}_A N_B, \varphi, \psi)$ consisting of two bimodules ${}_B M_A$ and ${}_A N_B$ and two bilinear maps $\varphi : M \otimes_A N \rightarrow B$ and $\psi : N \otimes_B M \rightarrow A$ satisfying compatibility conditions. A morphism between two Morita contexts connecting A and B is a pair of bilinear maps satisfying two equations. A basic result is that if φ and ψ are surjective, then they are in fact invertible, and so A and B have equivalent categories of modules. Part of the fundamental Morita theorem says that there is a one-to-one correspondence between the isomorphism types of category equivalences between categories of modules and the isomorphism types of Morita contexts with surjective maps. A natural question that was posed was how far is a Morita context, with not necessarily invertible maps, from an equivalence of categories. An answer was given by B. J. Müller in [2] (based on the works of T. Kato [3, 4]), which says that every Morita context induces an equivalence between the quotient categories of modules categories determined by the two trace ideals. Part of Müller's Theorem says that there is a one-to-one correspondence between the isomorphism types of maximal category equivalences between full subcategories of categories of modules containing the modules A_A and B_B , and the isomorphism types of right normalized Morita contexts connecting A and B (see [2, Theorem 7]). A companion to Müller's Theorem was given by W. K. Nicholson and J. F. Watters in [5, Theorem 5].

The notion of a Morita context was extended to an arbitrary Grothendieck category by F. Castaño Iglesias and J. Gómez-Torrecillas in [6, p. 602]. Let \mathcal{A} and \mathcal{B} be additive categories with cokernels and direct sums, consider a pair of additive covariant functors $\mathcal{F} : \mathcal{B} \rightleftarrows \mathcal{A} : \mathcal{G}$ together with two natural transformations $\eta : \mathcal{F}\mathcal{G} \rightarrow \mathbb{1}_{\mathcal{A}}$ and $\rho : \mathcal{G}\mathcal{F} \rightarrow \mathbb{1}_{\mathcal{B}}$ such that \mathcal{F}, \mathcal{G} preserve cokernels and $\eta_{\mathcal{F}(Y)} = \mathcal{F}\rho_Y, \rho_{\mathcal{G}(X)} = \mathcal{G}\eta_X$, for every pair of objects $Y \in \mathcal{B}$ and $X \in \mathcal{A}$. The four-tuple $(\mathcal{F}, \mathcal{G}, \eta, \rho)$ is referred to as a *wide right Morita context* between \mathcal{A} and \mathcal{B} . A *wide left Morita context* is dually defined, that is, in the dual categories \mathcal{A}^0 and \mathcal{B}^0 . Obviously if one of these natural transformations is invertible, then the context is nothing but an adjunction (with invertible unit or counit) between the categories \mathcal{A} and \mathcal{B} . This is the case, which corresponds to a wide left Morita context, when for example \mathcal{B} is a Grothendieck category and \mathcal{F} is a localizing functor.

In a series of papers [6–8], F. Castaño Iglesias and J. Gómez-Torrecillas proved several results concerning an equivalences of categories between full subcategories (quotient categories) that can be constructed from wide (right) Morita contexts. Stimulating applications to the categories of modules and graded modules were explicitly exhibited in the first two papers. In the third one, Morita-Takeuchi contexts [9] as wide left Morita contexts between categories of comodules over coalgebras over a field were particularly studied. A different approach of this case was also given by I. Berbec in [10]. Recently in [11] N. Chifan, S. Dăscălescu, and C. Năstăsescu unify all the previous results in the framework of abelian categories where they also added new examples, see [11, Section 5]. Morita contexts with injective bilinear maps were studied more recently by J. Y. Abuhlail and S. K. Nauman in [12], where several equivalences of categories were established [12, Section 5].

Bimodules are 1-cells in the bicategory \mathbf{Bim} (rings, bimodules, bilinear maps), while functors are 1-cells in the 2-category \mathbf{Cat} (categories, functors, natural transformations). Thus given an arbitrary bicategory (see the definition below), one can directly extend the concept of wide right Morita context between any pair of 0-cells. This means that the best and unifying setting of studying (wide) Morita contexts connecting rings or additive categories is the framework of bicategories. On the other hand, we sincerely think that this concept can be studied in arbitrary bicategory by its own right as was done for the notions of adjunction and of equivalence in bicategories.

In this paper we consider wide right Morita contexts in an arbitrary bicategory \mathbf{B} . In Section 2 we study the relationship between internal equivalences and wide right Morita contexts (Propositions 2.1 and 2.2). We propose a notion of morphism of right Morita contexts, and construct a new bicategory which we denote by $\mathbf{W}(\mathbf{B})$. The 0-cells of $\mathbf{W}(\mathbf{B})$ are all that of \mathbf{B} and its Hom-Categories are wide right Morita contexts and their morphisms (Proposition 2.9). We show that this construction is in fact functorial with respect to homomorphisms of bicategories (Proposition 2.10). In Section 3 we give an application of this construction taking $\mathbf{B} = \mathbf{REM}(\mathbf{Bim})$ the right Eilenberg-Moore bicategory of comonad attached to \mathbf{Bim} , whose 0-cells are all corings over rings with identity. This bicategory was studied in [13] and earlier introduced for general setting in [14]. We prove that

every 1-cell in $W(\text{REM}(\text{Bim}))$ (i.e. a wide right Morita context) induces a wide right Morita context between the categories of right comodules over corings in the sense of [6] (Proposition 3.3). Notice that the categories of comodules over corings are not in general abelian [15]. The converse of that proposition is also analyzed (Proposition 3.5).

Notations and Basic Notions

Since we will construct bicategories from other ones, it is convenient to include the general definition of a bicategory; more details can be found in the fundamental paper [16] (see also [17] for a basic definitions). A bicategory \mathbf{B} consists of the following data subject to the forthcoming axioms

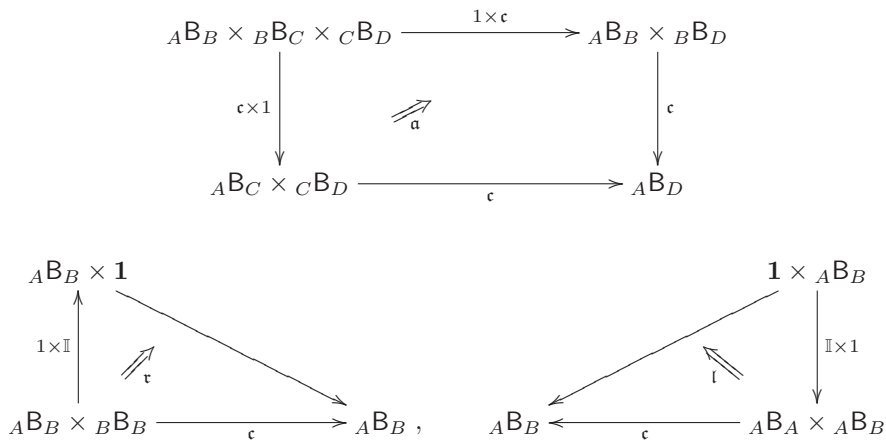
Data:

- *Objects (or 0-cells).* A class of objects $Ob(\mathbf{B})$ denoted by A, B, C, D, \dots .
- *Hom-Categories.* For each pair of 0-cells A and B , there is a category ${}_A\mathbf{B}_B$. The objects of this category are known as 1-cells from B to A , and morphisms as 2-cells from B to A .
- *Functors.* Defining the horizontal and vertical two sided multiplications

$$\begin{aligned} c_{ABC} : {}_A\mathbf{B}_B \times {}_B\mathbf{B}_C &\longrightarrow {}_A\mathbf{B}_C \\ (f, g) &\longrightarrow fg \\ (\alpha, \beta) &\longrightarrow \alpha\beta \end{aligned}$$

and $\mathbb{1} : \mathbf{1} \rightarrow {}_A\mathbf{B}_A$ defining an identity 1-cell $\mathbb{1}_A$, where $\mathbf{1}$ denotes the category with one object.

- *Natural isomorphism.* Defining associativity up to isomorphisms, and compatibility with left and right multiplications by identity 1-cells

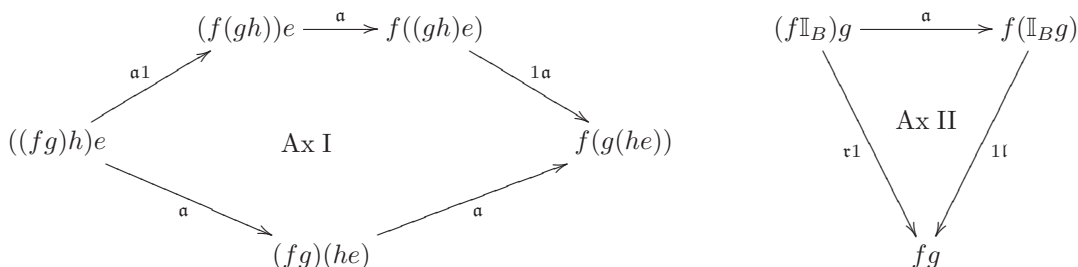


that is, 2-cells

$$\alpha_{fgh} : (fg)h \xrightarrow{\cong} f(gh), \quad \tau_f : f\mathbb{1}_B \xrightarrow{\cong} f, \quad \iota_f : \mathbb{1}_A f \xrightarrow{\cong} f.$$

Axioms:

The following diagrams commute



Using Axioms I and II, and the naturality of α , τ and ι we obtain the equality $\tau_{\mathbb{I}_-} = \iota_{\mathbb{I}_-}$, and that the following diagrams are commutative

$$\begin{array}{ccc}
 (fg)\mathbb{I}_C & \xrightarrow{\alpha} & f(g\mathbb{I}_C) \\
 \downarrow \tau & & \downarrow \iota\tau \\
 fg & & fg
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{I}_A(fg) & \xrightarrow{\alpha^{-1}} & (\mathbb{I}_A f)g \\
 \downarrow \iota & & \downarrow \iota\mathbb{I} \\
 fg & & fg
 \end{array}
 \tag{1.1}$$

$$\begin{array}{ccc}
 (\mathbb{I}_A f)\mathbb{I}_B & \xrightarrow{\alpha} & \mathbb{I}_A(f\mathbb{I}_B) \\
 \downarrow \iota\mathbb{I} & & \downarrow \iota\tau \\
 f\mathbb{I}_B & & \mathbb{I}_A f \\
 \downarrow \tau & & \downarrow \iota \\
 f & & f
 \end{array}
 \tag{1.2}$$

A morphisms (\mathcal{F}, Φ) from a bicategory $(\mathbf{B}, \mathbf{c}, \alpha, \tau, \iota, \mathbb{I})$ to $(\mathbf{B}', \mathbf{c}', \alpha', \tau', \iota', \mathbb{I}')$ consists of the following data subject to the forthcoming axioms

Data:

- *Function.* $\mathcal{F} : Ob(\mathbf{B}) \rightarrow Ob(\mathbf{B}')$.
- *Functors.* $\mathcal{F}_{A,B} : {}^A\mathbf{B}_B \rightarrow \mathcal{F}(A)\mathbf{B}'_{\mathcal{F}(B)}$.
- *Natural isomorphism.*

$$\begin{array}{ccc}
 {}^A\mathbf{B}_B \times {}^B\mathbf{B}_C & \xrightarrow{\mathbf{c}_{ABC}} & {}^A\mathbf{B}_C \\
 \downarrow (\mathcal{F}_{A,B}, \mathcal{F}_{B,C}) & \nearrow \Phi^1 & \downarrow \mathcal{F}_{A,C} \\
 \mathcal{F}(A)\mathbf{B}'_{\mathcal{F}(B)} \times \mathcal{F}(B)\mathbf{B}'_{\mathcal{F}(C)} & \xrightarrow{\mathbf{c}'_{\mathcal{F}(A),\mathcal{F}(B),\mathcal{F}(C)}} & \mathcal{F}(A)\mathbf{B}'_{\mathcal{F}(C)}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbf{1} & \xrightarrow{\mathbb{I}_A} & {}^A\mathbf{B}_A \\
 \parallel & \nearrow \Phi^0 & \downarrow \mathcal{F}_{A,A} \\
 \mathbf{1} & \xrightarrow{\mathbb{I}'_{\mathcal{F}(A)}} & \mathcal{F}(A)\mathbf{B}'_{\mathcal{F}(A)}
 \end{array}$$

that is, two 2-cells

$$\Phi_{f,g}^1 : \mathcal{F}(f)\mathcal{F}(g) \longrightarrow \mathcal{F}(fg) \quad \text{and} \quad \Phi_A^0 : \mathbb{I}'_{\mathcal{F}(A)} \longrightarrow \mathcal{F}(\mathbb{I}_A).$$

Axioms:

The following diagrams commute

$$\begin{array}{ccc}
 & (\mathcal{F}(f)\mathcal{F}(g))\mathcal{F}(h) & \\
 & \swarrow \alpha' & \searrow \Phi^1 \\
 \mathcal{F}(f)(\mathcal{F}(g)\mathcal{F}(h)) & & \mathcal{F}(fg)\mathcal{F}(h) \\
 \downarrow \mathbb{I}\Phi & & \downarrow \Phi \\
 \mathcal{F}(f)\mathcal{F}(gh) & & \mathcal{F}((fg)h) \\
 & \searrow \Phi & \swarrow \mathcal{F}(\alpha) \\
 & \mathcal{F}(f(gh)) &
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{F}(f)\mathbb{I}'_{\mathcal{F}(B)} & \xrightarrow{1\Phi} & \mathcal{F}(f)\mathcal{F}(\mathbb{I}_B) \\
 \downarrow \tau' & & \downarrow \Phi \\
 \mathcal{F}(f) & \xrightarrow{\mathcal{F}(\tau^{-1})} & \mathcal{F}(f\mathbb{I}_B)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{I}'_{\mathcal{F}(A)}\mathcal{F}(f) & \xrightarrow{\Phi 1} & \mathcal{F}(\mathbb{I}_A)\mathcal{F}(f) \\
 \downarrow \iota' & & \downarrow \Phi \\
 \mathcal{F}(f) & \xrightarrow{\mathcal{F}(\iota^{-1})} & \mathcal{F}(\mathbb{I}_A f)
 \end{array}$$

(\mathcal{F}, Φ) is called a *homomorphism* provided that Φ are natural isomorphisms, that is, $\mathcal{F}(f)\mathcal{F}(g) \cong \mathcal{F}(fg)$ and $\mathcal{F}(\mathbb{I}_A) \cong \mathbb{I}'_{\mathcal{F}(A)}$.

2. WIDE MORITA CONTEXTS AS 1-CELLS

Let \mathbf{B} be a bicategory with natural isomorphisms α, τ and ι . Given two 0-cells A and B , we define a *wide right Morita context* (WRMC for short) from B to A as a four-tuple $\Gamma = (f, g, \eta, \rho)$ consisting of two 1-cells $f \in {}_A\mathbf{B}_B$ and $g \in {}_B\mathbf{B}_A$, and two 2-cells $\eta : fg \rightarrow \mathbb{I}_A$ and $\rho : gf \rightarrow \mathbb{I}_B$ such that the following diagrams commute

$$\begin{array}{ccc}
 g(fg) & \xrightarrow{1\eta} & g\mathbb{I}_A \\
 \uparrow \alpha & & \searrow \tau \\
 (gf)g & \xrightarrow{\rho 1} & \mathbb{I}_B g \\
 & & \nearrow \iota \\
 & & g
 \end{array}
 \qquad
 \begin{array}{ccc}
 (fg)f & \xrightarrow{\eta 1} & \mathbb{I}_A f \\
 \downarrow \alpha & & \searrow \iota \\
 f(gf) & \xrightarrow{1\rho} & f\mathbb{I}_B \\
 & & \nearrow \tau \\
 & & f
 \end{array}
 \tag{2.1}$$

Wide left Morita contexts are dually defined in the conjugate category \mathbf{B}^{co} obtained from \mathbf{B} by reversing 2-cells.

Let A and B two 0-cells of \mathbf{B} . Recall (see [17]) that A is said to be *equivalent to B (inside \mathbf{B})*, if there exists a pair of 1-cells $f \in {}_A\mathbf{B}_B$ and $g \in {}_B\mathbf{B}_A$ together with two isomorphisms $\eta : fg \rightarrow \mathbb{I}_A$ in ${}_A\mathbf{B}_A$, and $\theta : \mathbb{I}_B \rightarrow gf$ in ${}_B\mathbf{B}_B$.

It is well known that any equivalence inside a bicategory induces an adjunction with invertible unit (or counit). Whence it induces a wide (right) Morita context. For sake of completeness, we give a complete proof of this fact.

Proposition 2.1. *Assume that A is equivalent to B inside \mathbf{B} via the isomorphisms $\eta : fg \rightarrow \mathbb{I}_A$ and $\theta : \mathbb{I}_B \rightarrow gf$.*

(i) *The following diagrams commute*

$$\begin{array}{ccc}
 (fg)(fg) & \xrightarrow{1\eta} & (fg)\mathbb{I}_A \\
 \downarrow \eta 1 & & \downarrow \tau \\
 \mathbb{I}_A(fg) & \xrightarrow{\iota} & fg
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{I}_B(gf) & \xrightarrow{\theta 1} & (gf)(gf) \\
 \uparrow \iota^{-1} & & \uparrow 1\theta \\
 gf & \xrightarrow{\tau^{-1}} & (gf)\mathbb{I}_B
 \end{array}$$

- (ii) *The functor $f- : {}_B\mathbf{B}_B \rightarrow {}_A\mathbf{B}_B$ (respectively $g- : {}_A\mathbf{B}_A \rightarrow {}_B\mathbf{B}_A$), defined by left multiplication by f (respectively by g), is fully faithful.*
- (iii) *There exists a 2-cell $\rho : gf \rightarrow \mathbb{I}_B$ such that (f, g, η, ρ) is WRMC from B to A .*

Proof. (i). This is immediate since τ, ι are natural and η, θ are monomorphisms.

(ii). Let $\sigma, \sigma' : g_1 \rightarrow g_2$ two morphisms in the category ${}_B\mathbf{B}_A$. We claim that, if $f\sigma = f\sigma'$, then $\sigma = \sigma'$, that is the left multiplication by f is faithful. This in fact follows from the following commutative diagram

$$\begin{array}{ccccccc}
 g_1 & \xrightarrow{\iota^{-1}} & \mathbb{I}_B g_1 & \xrightarrow{\theta_1} & (gf)g_1 & \xrightarrow{\alpha} & g(fg_1) \\
 \sigma \downarrow & & \downarrow 1\sigma & & \downarrow 1\sigma & & \downarrow 1(\sigma) \\
 \sigma' \downarrow & & \downarrow 1\sigma' & & \downarrow 1\sigma' & & \downarrow 1(\sigma') \\
 g_2 & \xrightarrow{\iota^{-1}} & \mathbb{I}_B g_2 & \xrightarrow{\theta_1} & (gf)g_2 & \xrightarrow{\alpha} & g(fg_2)
 \end{array}$$

whose horizontal maps are all isomorphisms. Similar arguments are used for the left multiplication by g . Consider now a morphism $\alpha : fh_1 \rightarrow fh_2$ in the category ${}_A\mathbf{B}_B$ where h_i are objects of the category ${}_B\mathbf{B}_B$, for $i = 1, 2$. So, we can define the composition

$$\sigma : h_1 \xrightarrow{\iota^{-1}} \mathbb{I}_B h_1 \xrightarrow{\theta_1} (gf)h_1 \xrightarrow{\alpha} g(fh_1) \xrightarrow{1\alpha} g(fh_2) \xrightarrow{\alpha^{-1}} (gf)h_2 \xrightarrow{\theta^{-1}_1} \mathbb{I}_B h_2 \xrightarrow{\iota} h_2$$

Using the diagrams stated in item (i) and in (1.1), Axioms I and II, in conjunction with the naturality of α , we can check that $g(f\sigma) = g\alpha$, and so $f\sigma = \alpha$ since $g-$ is faithful. This proves that the left multiplication by f is full. Similar arguments are used for left multiplication by g .

(iii). Consider the following morphism in the category ${}_A\mathbf{B}_B$

$$\kappa : f(gf) \xrightarrow{\alpha^{-1}} (fg)f \xrightarrow{\eta_1} \mathbb{I}_A f \xrightarrow{\iota} f \xrightarrow{\tau^{-1}} f\mathbb{I}_B$$

by item (ii) there exists $\rho : gf \rightarrow \mathbb{I}_B$ such that

$$f\rho = \kappa = \tau_f^{-1} \circ \iota_f \circ \eta f \circ \alpha_{fgf}^{-1},$$

that is $\tau_f \circ f\rho \circ \alpha_{fgf} = \iota_f \circ \eta f$. This last equality is exactly the commutativity of the first diagram in (2.1). To show the commutativity of the second diagram in (2.1), we multiply on the left by f to get

$$\begin{array}{ccccc}
 f((gf)g) & \xrightarrow{1\alpha} & & & f(g(fg)) \\
 \downarrow 1(\rho) & & \swarrow \alpha^{-1} & & \downarrow 1(\eta) \\
 & & (fg)(fg) & & \\
 & & \downarrow \alpha^{-1} & & \\
 & & ((fg)f)g & & \\
 \textcircled{1} & & \downarrow (\eta_1)1 & & \textcircled{2} \\
 & & (\mathbb{I}_A f)g & & \\
 \downarrow 1\iota & & \downarrow 1\iota & & \downarrow 1\tau \\
 f(\mathbb{I}_B g) & & (fg) & & f(g\mathbb{I}_A)
 \end{array}$$

Diagram ① commutes by applying consecutively Axiom I, the naturality of α , the equality satisfied by $(f\rho)g$ and lastly by applying Axiom II. The diagram ② commutes by using consecutively the naturality of α , the first diagram of item (i), and both diagram in Equation (1.1). Thus the total diagram commutes, which is sufficient since the left multiplication by f is faithful. \square

The converse is not in general true. Under some extra conditions we can prove

Proposition 2.2. *Let \mathbf{B} be a locally abelian bicategory whose left and right multiplications are right exact functors. Consider a WRMC $\Gamma = (f, g, \eta, \rho)$ from B to A . If η and ρ are epimorphisms, then they are isomorphisms. In particular if η and ρ are epimorphisms, then A and B are equivalent inside \mathbf{B} .*

Proof. We only prove that $\eta : fg \rightarrow \mathbb{I}_A$ is an isomorphism. Since ${}_A\mathbf{B}_A$ is an abelian category, it suffices to check that η has a null kernel *i.e.* $\text{Ker}(\eta) = 0$. From the sequence

$$0 \longrightarrow \text{Ker}(\eta) \xrightarrow{\eta^k} fg \xrightarrow{\eta} \mathbb{I}_A$$

we obtain a commutative diagram

$$\begin{array}{ccccc}
 (fg)\text{Ker}(\eta) & \xrightarrow{1\eta^k} & (fg)(fg) & \xrightarrow{1\eta} & (fg)\mathbb{I}_A \\
 \eta 1 \downarrow & & \eta 1 \downarrow & & \eta 1 \downarrow \\
 \mathbb{I}_A\text{Ker}(\eta) & \xrightarrow{1\eta^k} & \mathbb{I}_A(fg) & \xrightarrow{1\eta} & \mathbb{I}_A\mathbb{I}_A \\
 \wr \searrow & & \wr \searrow & & \wr \searrow \\
 \text{Ker}(\eta) & \xrightarrow{\eta^k} & (fg) & \xrightarrow{\eta} & \mathbb{I}_A
 \end{array}$$

which implies that $\eta^k \circ \wr_{\text{Ker}(\eta)} \circ (\eta\text{Ker}(\eta)) = 0$, and so $\eta\text{Ker}(\eta) = 0$. Whence $\text{Ker}(\eta) = 0$ since η is an epimorphism and the multiplication is by hypothesis a right exact functor. \square

Let $\Gamma' = (f', g', \eta', \rho')$ be another WRMC from B to A . A *morphism* from Γ to Γ' is a pair $(\alpha, \beta) : \Gamma \rightarrow \Gamma'$ consisting of two 2-cells $\alpha : f \rightarrow f'$ and $\beta : g \rightarrow g'$ rendering commutative the following diagrams

$$\begin{array}{ccc}
 fg & \xrightarrow{\eta} & \mathbb{I}_A \\
 \alpha\beta \downarrow & \nearrow & \\
 f'g' & \xrightarrow{\eta'} & \mathbb{I}_A
 \end{array}
 \qquad
 \begin{array}{ccc}
 gf & \xrightarrow{\rho} & \mathbb{I}_B \\
 \beta\alpha \downarrow & \nearrow & \\
 g'f' & \xrightarrow{\rho'} & \mathbb{I}_B
 \end{array}
 \tag{2.2}$$

called *compatibility conditions*. The identity morphism is given by the pair of identities 2-cells $(1_f, 1_g) : \Gamma \rightarrow \Gamma$. The composition is componentwise. We thus arrive to

Lemma 2.3. *The wide right Morita contexts from B to A and their morphisms form a category which we denote by ${}_A\mathbf{W}(\mathbf{B})_B$.*

Proof. We only need to check that the composition law is well defined. That is, if $(\alpha, \beta) : \Gamma_1 \rightarrow \Gamma_2$ and $(\alpha', \beta') : \Gamma_2 \rightarrow \Gamma_3$ are morphisms of WRMC, then $(\alpha' \circ \alpha, \beta' \circ \beta) : \Gamma_1 \rightarrow \Gamma_3$ is also a morphism of WRMC. Thus we need to show that the diagrams of Equation (2.2) corresponding to $(\alpha' \circ \alpha, \beta' \circ \beta)$ are commutative. This is clearly fulfilled as the following commutative diagrams shown

$$\begin{array}{ccc}
 f_1g_1 & \xrightarrow{\eta_1} & \mathbb{I}_A \\
 \alpha\beta \downarrow & \nearrow & \\
 f_2g_2 & \xrightarrow{\eta_2} & \mathbb{I}_A \\
 \alpha'\beta' \downarrow & \nearrow & \\
 f_3g_3 & \xrightarrow{\eta_3} & \mathbb{I}_A \\
 (\alpha' \circ \alpha)(\beta' \circ \beta) \curvearrowright & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 g_1f_1 & \xrightarrow{\rho_1} & \mathbb{I}_B \\
 \beta\alpha \downarrow & \nearrow & \\
 g_2f_2 & \xrightarrow{\rho_2} & \mathbb{I}_B \\
 \beta'\alpha' \downarrow & \nearrow & \\
 g_3f_3 & \xrightarrow{\rho_3} & \mathbb{I}_B \\
 (\beta' \circ \beta)(\alpha' \circ \alpha) \curvearrowright & &
 \end{array}$$

\square

Next, we prove that wide right Morita contexts are 1-cells in a suitable bicategory. We start by constructing the two sided multiplications and identities functors.

Diagram ① commutes by applying five times the naturality of \mathfrak{a} , by Axioms I , II, and the second diagram of (1.1), and also by the first diagram of Equation (2.1) satisfied by γ and μ . To check the commutativity of ②, we start by Axiom I, next apply the naturality of \mathfrak{a} (in five occasions), and use the second diagram of (1.1) together with the usual obvious fact about multiplying 2-cells *i.e.* $\alpha\beta = 1\beta \circ \alpha 1 = \alpha 1 \circ 1\beta$ (used three times). In diagram ③, we first apply the Axiom I, and continuously the naturality of \mathfrak{a} (five times), and ended by Axiom II and the first diagram of (2.1) satisfied by η and ρ . The total diagram is then commutative and corresponds to the first diagram of (2.1) taking the 2-cells $\eta * \gamma$ and $\mu * \rho$. \square

Given two morphisms $(\alpha, \beta) : \Gamma_1 \rightarrow \Gamma_2$ and $(\varsigma, \tau) : \Lambda_1 \rightarrow \Lambda_2$, we clearly have two 2-cells, namely, $\alpha\varsigma : f_1 p_1 \rightarrow f_2 p_2$ and $\tau\beta : q_1 g_1 \rightarrow q_2 g_2$, which satisfy

Lemma 2.5. *The pair of 2-cells $(\alpha\varsigma, \tau\beta)$ defines a morphism $(\alpha\varsigma, \tau\beta) : \Gamma_1 \Lambda_1 \rightarrow \Gamma_2 \Lambda_2$ in the category ${}_A W(B)_C$, where $\Gamma_i \Lambda_i$ are defined as in Lemma 2.4, $i = 1, 2$.*

Proof. The first diagram in Equation (2.2) corresponding to the stated pair of morphisms commutes as it is shown by the following diagram

$$\begin{array}{ccccccccc}
 (f_1 p_1)(q_1 g_1) & \xrightarrow{\mathfrak{a}} & f_1(p_1(q_1 g_1)) & \xrightarrow{1\mathfrak{a}^{-1}} & f_1((p_1 q_1)g_1) & \xrightarrow{1(\gamma_1 1)} & f_1(\mathbb{I}_B g_1) & \xrightarrow{1\mathbb{I}} & f_1 g_1 & \xrightarrow{\eta_1} & \mathbb{I}_A \\
 \downarrow (\alpha\varsigma)(\tau\beta) & & \downarrow \alpha(\varsigma(\tau\beta)) & & \downarrow \alpha((\varsigma\tau)\beta) & & \downarrow \alpha(1\beta) & & \downarrow \alpha\beta & \nearrow \eta_2 & \\
 (f_2 p_2)(q_2 g_2) & \xrightarrow{\mathfrak{a}} & f_2(p_2(q_2 g_2)) & \xrightarrow{1\mathfrak{a}^{-1}} & f_2((p_2 q_2)g_2) & \xrightarrow{1(\gamma_2 1)} & f_2(\mathbb{I}_B g_2) & \xrightarrow{1\mathbb{I}} & f_2 g_2 & &
 \end{array}$$

The commutativity of the second one is similarly obtained. \square

Since the multiplication \mathfrak{c} of two identities 2-cells gives an identity, the above multiplication respects also this rule. By Lemmata 2.4 and 2.5, we thus get the following

Corollary 2.6. *Given A, B and C three 0-cells of B , there is a covariant functor*

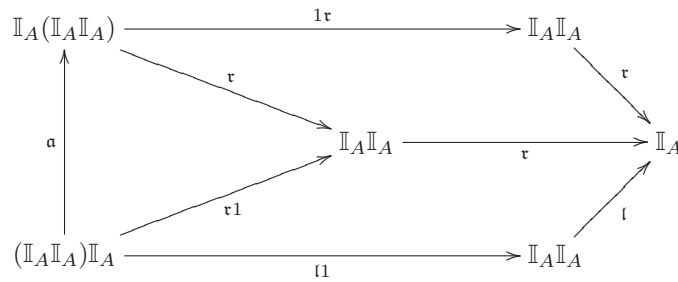
$$\begin{aligned}
 \bar{\mathfrak{c}}_{ABC} : {}_A W(B)_B \times {}_B W(B)_C &\longrightarrow {}_A W(B)_C \\
 ((\Gamma, \Lambda)) &\longrightarrow \Gamma\Lambda = (fp, qg, \eta * \gamma, \mu * \rho) \\
 ((\alpha, \beta), (\varsigma, \tau)) &\longrightarrow (\alpha, \beta)(\varsigma, \tau) = (\alpha\varsigma, \tau\beta)
 \end{aligned}$$

where $\Gamma = (f, g, \eta, \rho)$ and $\Lambda = (p, q, \gamma, \mu)$.

The identity functors are given as follows. Let A be any 0-cell of B , define the four-tuple $\bar{\mathbb{I}}_A = (\mathbb{I}_A, \mathbb{I}_A, \mathfrak{r}_{\mathbb{I}_A}, \mathfrak{l}_{\mathbb{I}_A})$, then we have the following diagrams

$$\begin{array}{ccc}
 \mathbb{I}_A(\mathbb{I}_A \mathbb{I}_A) & \xrightarrow{1\mathfrak{r}} & \mathbb{I}_A \mathbb{I}_A \\
 \uparrow \mathfrak{a} & & \searrow \mathfrak{r} \\
 (\mathbb{I}_A \mathbb{I}_A) \mathbb{I}_A & \xrightarrow{1\mathbb{I}} & \mathbb{I}_A \mathbb{I}_A \\
 & & \nearrow \mathfrak{l}
 \end{array}
 \qquad
 \begin{array}{ccc}
 (\mathbb{I}_A \mathbb{I}_A) \mathbb{I}_A & \xrightarrow{\mathfrak{r}1} & \mathbb{I}_A \mathbb{I}_A \\
 \downarrow \mathfrak{a} & & \searrow \mathfrak{l} \\
 \mathbb{I}_A(\mathbb{I}_A \mathbb{I}_A) & \xrightarrow{1\mathbb{I}} & \mathbb{I}_A \mathbb{I}_A \\
 & & \nearrow \mathfrak{r}
 \end{array}$$

where the second diagram commutes by Axiom II, and the first one decomposes as



which is commutative by the first diagram of (1.1), the naturality of τ , and the equality $\tau_{\mathbb{I}_A} = \iota_{\mathbb{I}_A}$.

Now, we look at the associativity of the two sided multiplications $\bar{\alpha}$.

2.2. The Natural Isomorphisms $\bar{\alpha}_{\Gamma\Lambda\Omega}$

Let $\Gamma = (f, g, \eta, \rho) \in {}_A\mathcal{W}(\mathbb{B})_B$, $\Lambda = (p, q, \gamma, \mu) \in {}_B\mathcal{W}(\mathbb{B})_C$, and $\Omega = (u, v, \theta, \sigma) \in {}_C\mathcal{W}(\mathbb{B})_D$. We then have the following 2-cells:

$$\begin{array}{lll}
 \eta : fg \longrightarrow \mathbb{I}_A & \gamma : pq \longrightarrow \mathbb{I}_B & \theta : uv \longrightarrow \mathbb{I}_C \\
 \rho : gf \longrightarrow \mathbb{I}_B, & \mu : qp \longrightarrow \mathbb{I}_C, & \sigma : vu \longrightarrow \mathbb{I}_D,
 \end{array}$$

where each vertical pair satisfies (2.1). By Corollary 2.6, we have two WRMC

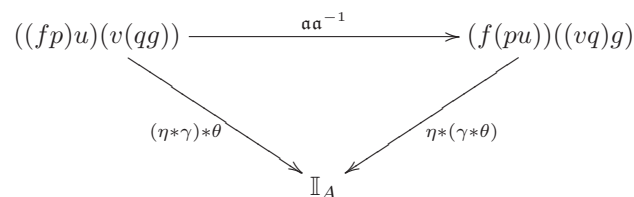
$$(\Gamma\Lambda)\Omega = (fp, qg, \eta * \gamma, \mu * \rho)(u, v, \theta, \sigma) = ((fp)u, v(qg), (\eta * \gamma) * \theta, \sigma * (\mu * \rho)) \in {}_A\mathcal{W}(\mathbb{B})_D$$

and

$$\Gamma(\Lambda\Omega) = (f, g, \eta, \rho)(pu, vq, \gamma * \theta, \sigma * \mu) = (f(pu), (vq)g, \eta * (\gamma * \theta), (\sigma * \mu) * \rho) \in {}_A\mathcal{W}(\mathbb{B})_D$$

Lemma 2.7. *The pair $\bar{\alpha}_{\Gamma\Lambda\Omega} := (\alpha, \alpha^{-1}) : (\Gamma\Lambda)\Omega \rightarrow \Gamma(\Lambda\Omega)$ defines an isomorphism in the category ${}_A\mathcal{W}(\mathbb{B})_D$ with inverse (α^{-1}, α) . Moreover, $\bar{\alpha}_{---} : (---) \rightarrow -(---)$ is a natural isomorphism.*

Proof. For the first statement, we need to show that the diagrams in Equation (2.2) associated to the pair (α, α^{-1}) are commutative. We only prove the commutativity of the first one the remainder is similarly deduced. This first diagram is



writing explicitly the maps, we get

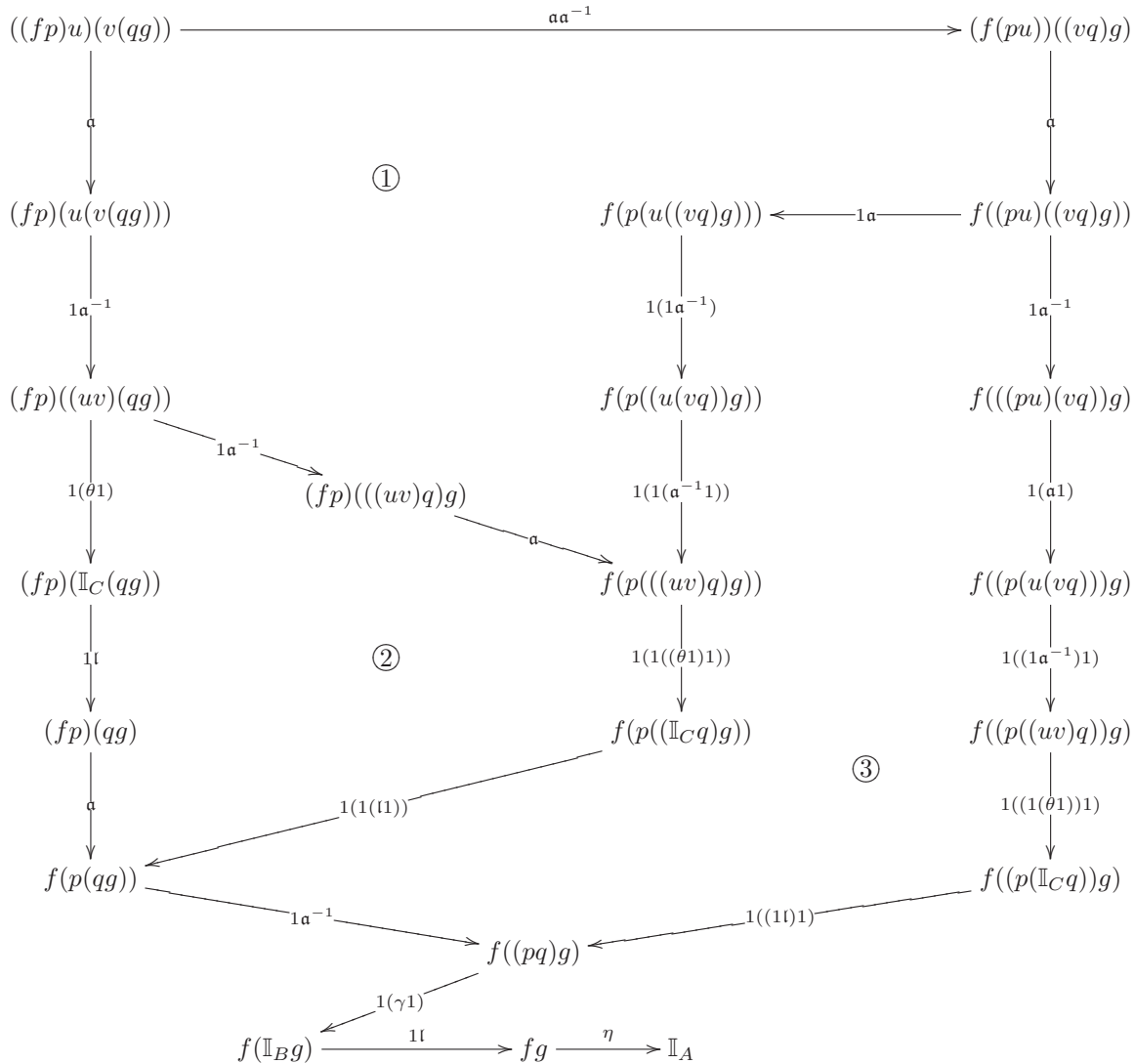


Diagram ① commutes by Axiom I (applied two times) and by naturality of α (used three times). Applying the second diagram of (1.1), and three times the naturality of α , we get the commutativity of ②. It is clear that ③ = $f \times$ ③', and ③' is commutative before applying Axiom I, and consecutively three times the naturality of α .

The rest of the proof is clearly obtained by using the natural isomorphism α . □

2.3. The Natural Isomorphisms $\bar{\tau}_\Gamma$ and \bar{l}_Γ

Let A and B two 0-cells of \mathbf{B} , we have shown that $\bar{\mathbb{I}}_A = (\mathbb{I}_A, \mathbb{I}_A, \tau_{\mathbb{I}_A}, l_{\mathbb{I}_A})$ and $\bar{\mathbb{I}}_B = (\mathbb{I}_B, \mathbb{I}_B, \tau_{\mathbb{I}_B}, l_{\mathbb{I}_B})$ are, respectively, objects of ${}_A\mathbf{W}(\mathbf{B})_A$ and ${}_B\mathbf{W}(\mathbf{B})_B$. Consider now $\Gamma = (f, g, \eta, \rho)$ any object of the category ${}_A\mathbf{W}(\mathbf{B})_B$, using the multiplication $\bar{\tau}$, we can define two 2-cells:

$$(\tau_f, l_g) : \Gamma \bar{\mathbb{I}}_B = (f \mathbb{I}_B, \mathbb{I}_B g, \eta * \tau, l * \rho) \longrightarrow \Gamma, \quad (l_g, \tau_f) : \bar{\mathbb{I}}_A \Gamma = (\mathbb{I}_A f, g \mathbb{I}_A, \tau * \eta, \rho * l) \longrightarrow \Gamma$$

Lemma 2.8. *Keep the above notation.*

- (i) *The pairs $(\tau_f, l_g) : \Gamma \bar{\mathbb{I}}_B \rightarrow \Gamma$ and $(l_g, \tau_f) : \bar{\mathbb{I}}_A \Gamma \rightarrow \Gamma$ are isomorphisms of the category ${}_A\mathbf{W}(\mathbf{B})_B$ with inverses, respectively, (τ_f^{-1}, l_g^{-1}) and (l_g^{-1}, τ_f^{-1}) .*

(ii) These in fact define natural isomorphisms $\bar{\tau}_-$ and $\bar{\iota}_-$ given, respectively, at the object $\Gamma = (f, g, \eta, \rho)$ by $\bar{\tau}_\Gamma = (\tau_f, \iota_g)$ and $\bar{\iota}_\Gamma = (\iota_g, \tau_f)$.

Proof. (i) We only prove that (τ_f, ι_g) is a morphisms of ${}_A W(B)_B$. The proof of the other morphism is similarly given. We thus need to show that the following diagrams are commutative

$$\begin{array}{ccc}
 (f\mathbb{I}_B)(\mathbb{I}_B g) & & (\mathbb{I}_B g)(f\mathbb{I}_B) \\
 \downarrow \tau\iota & \searrow \eta*\tau\iota_B & \downarrow \iota\tau \\
 & & \mathbb{I}_A \\
 & \nearrow \eta & \\
 fg & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & \mathbb{I}_B \\
 & \nearrow \rho & \\
 gf & &
 \end{array}
 \qquad (2.4)$$

The first diagram decomposes as

$$\begin{array}{ccccc}
 (f\mathbb{I}_B)(\mathbb{I}_B g) & \xrightarrow{\alpha} & f(\mathbb{I}_B(\mathbb{I}_B g)) & \xrightarrow{1\alpha^{-1}} & f((\mathbb{I}_B\mathbb{I}_B)g) \\
 \downarrow 1\iota & & \downarrow 1(1\iota) & \swarrow 1(\tau\iota) & \\
 (f\mathbb{I}_B)g & \xrightarrow{\alpha} & f(\mathbb{I}_B g) & & \\
 \searrow \tau\iota & & \downarrow 1\iota & & \\
 fg & & fg & \xrightarrow{\eta} & \mathbb{I}_A
 \end{array}$$

which is commutative by naturality of α , the first diagram in Equation (1.1), and Axiom II. The second diagram in (2.4) decomposes as

$$\begin{array}{ccccccc}
 (\mathbb{I}_B g)(f\mathbb{I}_B) & \xrightarrow{\alpha} & \mathbb{I}_B(g(f\mathbb{I}_B)) & \xrightarrow{1\alpha^{-1}} & \mathbb{I}_B((fg)\mathbb{I}_B) & \xrightarrow{1(\rho\iota)} & \mathbb{I}_B(\mathbb{I}_B\mathbb{I}_B) \\
 \downarrow 1\tau & & \swarrow 1\tau & & \downarrow \alpha^{-1} & & \downarrow \alpha^{-1} \\
 (\mathbb{I}_B g)f & & \mathbb{I}_B(fg) & & & & \\
 \downarrow 1\iota & \swarrow \iota & \downarrow \iota & & \downarrow \iota & & \downarrow 1\iota \\
 fg & & (gf)\mathbb{I}_B & \xrightarrow{\rho\iota} & \mathbb{I}_B\mathbb{I}_B & & \mathbb{I}_B\mathbb{I}_B \\
 & \searrow \tau & \downarrow \iota & & \downarrow \iota & & \downarrow \iota \\
 & & fg & & fg & & fg \\
 & & \swarrow \rho & & \swarrow \rho & & \swarrow \rho \\
 & & fg & & fg & & fg
 \end{array}$$

Diagram ① commutes by naturality of \mathbf{a} and by the second diagram in Equation (1.1). Applying Equation (1.2), we get the commutativity of ②. The diagrams ③, ④ and ⑥ are commutative, since \mathbf{a}^{-1} , \mathbf{l} and \mathbf{r} are natural transformations. Lastly, diagram ⑤ commutes by the second diagram in (1.1) and the equality $\mathbf{l}_{\mathbb{I}_A} = \mathbf{r}_{\mathbb{I}_A}$.
 (ii) It is componentwise derived from the naturality of \mathbf{r}_- and \mathbf{l}_- . \square

The Axioms $\bar{\text{I}}$ and $\bar{\text{II}}$ corresponding to the functors $\bar{\mathbf{c}}$, $\bar{\mathbb{I}}$ and the natural isomorphisms $\bar{\mathbf{a}}$, $\bar{\mathbf{r}}$, and $\bar{\mathbf{l}}$, are easily derived from Axiom I and II. We then have

Proposition 2.9. *Let \mathbf{B} be a bicategory. Then the following data form a bicategory $\mathbf{W}(\mathbf{B})$*

- 0-cells. Are all 0-cells of \mathbf{B}
- 1-cells. From B to A are four-tuples $\Gamma = (f, g, \eta, \rho)$ consisting of two 1-cells $f \in {}_A\mathbf{B}_B$ and $g \in {}_B\mathbf{B}_A$, and two 2-cells $\eta : fg \rightarrow \mathbb{I}_A$ and $\rho : gf \rightarrow \mathbb{I}_B$ satisfying the compatibility conditions (i.e. the diagrams of (2.1) are commutative). The identity 1-cell is given by the four-tuple $\bar{\mathbb{I}}_A = (\mathbb{I}_A, \mathbb{I}_A, \mathbf{r}_{\mathbb{I}_A}, \mathbf{l}_{\mathbb{I}_A})$.
- 2-cells. Are pairs $(\alpha, \beta) : \Gamma = (f, g, \eta, \rho) \rightarrow \Gamma' = (f', g', \eta', \rho')$ consisting of two 2-cells $\alpha : f \rightarrow f'$ and $\beta : g \rightarrow g'$ rendering commutative the diagrams of Equation (2.2).

The construction of wide right Morita contexts is in fact functorial.

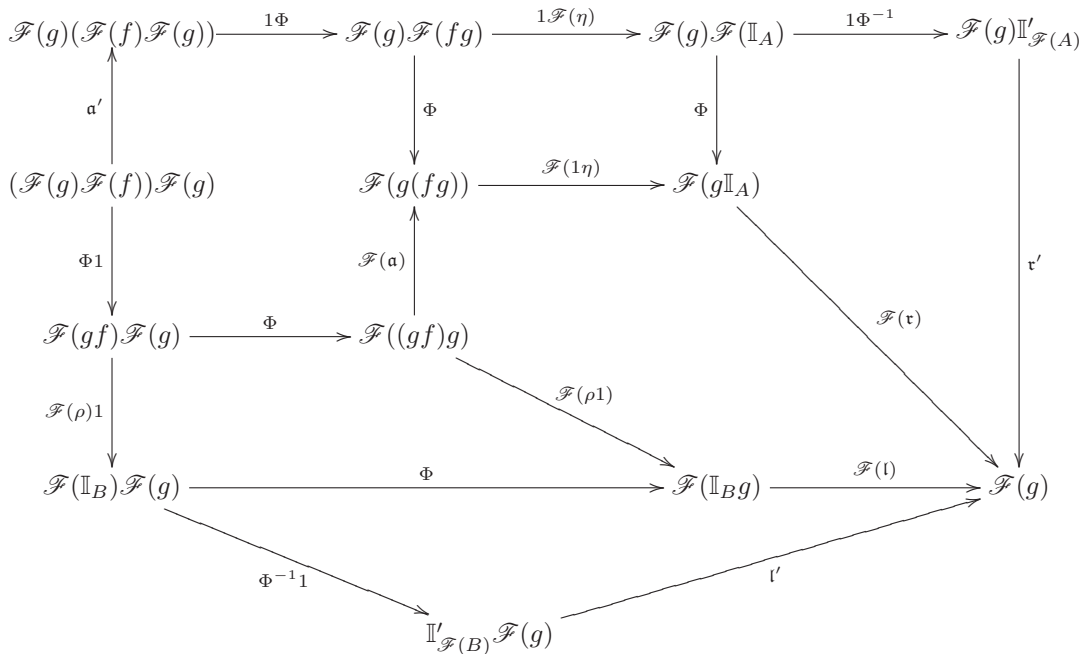
Proposition 2.10. *Let $(\mathcal{F}, \Phi) : \mathbf{B} \rightarrow \mathbf{B}'$ be a homomorphism of bicategories. Then there is a homomorphism of bicategories $\mathbf{W}(\mathcal{F}, \Phi) : \mathbf{W}(\mathbf{B}) \rightarrow \mathbf{W}(\mathbf{B}')$.*

Proof. The function on 0-cell coincides with that of \mathcal{F} . Let $\Gamma = (f, g, \eta, \rho)$ be any 1-cell in ${}_A\mathbf{W}(\mathbf{B})_B$. Define the following two 2-cells in ${}_{\mathcal{F}(A)}\mathbf{B}'_{\mathcal{F}(B)}$

$$\eta' : \mathcal{F}(f)\mathcal{F}(g) \xrightarrow{\Phi_{fg}^1} \mathcal{F}(fg) \xrightarrow{\mathcal{F}(\eta)} \mathcal{F}(\mathbb{I}_A) \xrightarrow{\Phi^{0-1}} \mathbb{I}'_{\mathcal{F}(A)},$$

$$\rho' : \mathcal{F}(g)\mathcal{F}(f) \xrightarrow{\Phi_{gf}^1} \mathcal{F}(gf) \xrightarrow{\mathcal{F}(\rho)} \mathcal{F}(\mathbb{I}_B) \xrightarrow{\Phi^{0-1}} \mathbb{I}'_{\mathcal{F}(B)}.$$

Set $\Gamma' = (\mathcal{F}(f), \mathcal{F}(g), \eta', \rho')$, we claim that $\Gamma' \in {}_{\mathcal{F}(A)}\mathbf{W}(\mathbf{B}')_{\mathcal{F}(B)}$. We need to check that the diagrams of Equation (2.1) corresponding to η' and ρ' in \mathbf{B}' are commutative. The first one decomposes as



which is clearly a commutative diagram. Similarly we check that the second one is also commutative.

Now, given a 2-cell $(\alpha, \beta) : \Gamma \rightarrow \Sigma = (h, e, \sigma, \tau)$ in ${}_A W(B)_B$, we get by the naturality of Φ and Equation (2.2) satisfied by the pair (α, β) , the following commutative diagrams

$$\begin{array}{ccccc}
 \mathcal{F}(f)\mathcal{F}(g) & \xrightarrow{\Phi} & \mathcal{F}(fg) & \xrightarrow{\mathcal{F}(\eta)} & \mathcal{F}(\mathbb{I}_A) \\
 \downarrow \mathcal{F}(\alpha)\mathcal{F}(\beta) & & \downarrow \mathcal{F}(\alpha\beta) & & \searrow \Phi^{-1} \\
 \mathcal{F}(h)\mathcal{F}(e) & \xrightarrow{\Phi} & \mathcal{F}(he) & \xrightarrow{\mathcal{F}(\sigma)} & \mathcal{F}(\mathbb{I}_A) \\
 & & & & \nearrow \Phi^{-1} \\
 & & & & \mathbb{I}'_{\mathcal{F}(A)}
 \end{array}$$

and

$$\begin{array}{ccccc}
 \mathcal{F}(g)\mathcal{F}(f) & \xrightarrow{\Phi} & \mathcal{F}(gf) & \xrightarrow{\mathcal{F}(\rho)} & \mathcal{F}(\mathbb{I}_B) \\
 \downarrow \mathcal{F}(\beta)\mathcal{F}(\alpha) & & \downarrow \mathcal{F}(\beta\alpha) & & \searrow \Phi^{-1} \\
 \mathcal{F}(e)\mathcal{F}(h) & \xrightarrow{\Phi} & \mathcal{F}(eh) & \xrightarrow{\mathcal{F}(\tau)} & \mathcal{F}(\mathbb{I}_B) \\
 & & & & \nearrow \Phi^{-1} \\
 & & & & \mathbb{I}'_{\mathcal{F}(B)}
 \end{array}$$

This means that $(\mathcal{F}(\alpha), \mathcal{F}(\beta)) : (\mathcal{F}(f), \mathcal{F}(g), \eta', \rho') \rightarrow (\mathcal{F}(h), \mathcal{F}(e), \sigma', \tau')$ is a 2-cell in ${}_{\mathcal{F}(A)} W(B')_{\mathcal{F}(B)}$. We have thus construct a family of functors

$$\begin{array}{ccc}
 {}_A W(B)_B & \xrightarrow{W(\mathcal{F})_{A, B}} & {}_{\mathcal{F}(A)} W(B')_{\mathcal{F}(B)} \\
 \Gamma = (f, g, \eta, \rho) & \xrightarrow{\quad\quad\quad} & W(\mathcal{F})(\Gamma) := (\mathcal{F}(f), \mathcal{F}(g), \eta', \rho') \\
 [(\alpha, \beta) : \Gamma \rightarrow \Sigma] & \xrightarrow{\quad\quad\quad} & [W(\mathcal{F})(\alpha, \beta) := (\mathcal{F}(\alpha), \mathcal{F}(\beta)) : W(\mathcal{F})(\Gamma) \rightarrow W(\mathcal{F})(\Sigma)]
 \end{array}$$

On the other hand, given $\Gamma = (f, g, \eta, \rho)$ and $\Lambda = (p, q, \mu, \gamma)$ two 1-cells, respectively, in ${}_A W(B)_B$ and ${}_B W(B)_C$, using the naturality of Φ , we obtain two 2-cells

$$\begin{array}{ccc}
 W(\mathcal{F})(\Gamma) W(\mathcal{F})(\Lambda) & & W(\mathcal{F})(\Gamma\Lambda) \\
 \parallel & \xrightarrow{(\Phi_{fp}^{-1}, \Phi_{qg}^{-1})} & \parallel \\
 (\mathcal{F}(f)\mathcal{F}(p), \mathcal{F}(q)\mathcal{F}(g), \eta' * \mu', \rho' * \gamma') & & (\mathcal{F}(fp), \mathcal{F}(qg), (\eta * \mu)', (\rho * \gamma)')
 \end{array}$$

and

$$\begin{array}{ccc}
 W(\mathcal{F}) \left(\mathbb{I}_A, \mathbb{I}_A, \mathfrak{r}_A, \mathfrak{l}_A \right) & & \left(\mathbb{I}'_{\mathcal{F}(A)}, \mathbb{I}'_{\mathcal{F}(A)}, \mathfrak{r}'_{\mathbb{I}'_{\mathcal{F}(A)}}, \mathfrak{l}'_{\mathbb{I}'_{\mathcal{F}(A)}} \right) \\
 \parallel & \xrightarrow{(\Phi_A^{-1}, \Phi_A^{-1})} & \parallel \\
 \left(\mathcal{F}(\mathbb{I}_A), \mathcal{F}(\mathbb{I}_A), \mathfrak{r}'_{\mathbb{I}_A}, \mathfrak{l}'_{\mathbb{I}_A} \right) & &
 \end{array}$$

The axioms satisfied by the natural transformations Φ show that $(W(\mathcal{F}), (\Phi^{-1}, \Phi^{-1})) : W(B) \rightarrow W(B')$ is a homomorphism of bicategories as claimed. \square

3. APPLICATION TO REM(BIM)

In what follows all rings are assumed to be associative with 1. Modules are unital modules, and bimodules are left and right unital modules. The identity linear map associated to any module X is denoted by the module itself X . Given A and B two rings, the category of (A, B) -bimodules (left A -modules and right B -modules) is denoted as usual by ${}_A \mathcal{M}_B$. The symbol $- \otimes_A -$ between bimodules and bilinear map denotes the tensor product over A . The bimodules bicategory Bim has the class of 0-cells all rings A, B, C, \dots , and Hom-categories the categories of bimodules ${}_A \mathcal{M}_B$. The vertical and horizontal multiplications are given by the tensor product over rings. For every $M \in {}_A \mathcal{M}_B$, we denote by ι_M the obvious natural isomorphism at M , $\iota_M : A \otimes_A M \rightarrow M \otimes_B B$, that is, $\iota_M = \mathbf{r}_M^{-1} \circ \mathbf{l}_M$, where \mathbf{r} and \mathbf{l} are respectively, the right and left identities natural isomorphism of Bim .

Let A be a ring, an A -coring [18] is a three-tuple $(\mathfrak{C}, \Delta, \varepsilon)$ consisting of an A -bimodule and two A -bilinear maps

$$\Delta : \mathfrak{C} \rightarrow \mathfrak{C} \otimes_A \mathfrak{C} \quad \text{and} \quad \varepsilon : \mathfrak{C} \rightarrow A,$$

known as the comultiplication and the counit of \mathfrak{C} , which satisfy

$$(\mathfrak{C} \otimes_A \Delta) \circ \Delta = (\Delta \otimes_A \mathfrak{C}) \circ \Delta, \quad (\mathfrak{C} \otimes_A \varepsilon) \circ \Delta = \mathfrak{C} = (\varepsilon \otimes_A \mathfrak{C}) \circ \Delta.$$

A right \mathfrak{C} -comodule is a pair (M, ϱ^M) with M a right A -module and $\varrho^M : M \rightarrow M \otimes_A \mathfrak{C}$ a right A -linear map (called right \mathfrak{C} -coaction) satisfying two equalities $(\varrho^M \otimes_A \mathfrak{C}) \circ \varrho^M = (M \otimes_A \Delta) \circ \varrho^M$, and $(M \otimes_A \varepsilon) \circ \varrho^M = M$. A morphism of right \mathfrak{C} -comodules $f : (M, \varrho^M) \rightarrow (M', \varrho^{M'})$ is a right A -linear map $f : M \rightarrow M'$ which is compatible with coactions: $f \circ \varrho^{M'} = (f \otimes_A \mathfrak{C}) \circ \varrho^M$ (f is right \mathfrak{C} -colinear). The category of all right \mathfrak{C} -comodules is denoted by $\mathcal{M}^{\mathfrak{C}}$. Left \mathfrak{C} -comodules are symmetrically defined, we use the Greek letter λ^- to denote their coactions. The category of (right) \mathfrak{C} -comodules is not in general an abelian category, it has cokernels and arbitrary direct sums which can be already computed in the category of A -modules. However, if ${}_A\mathfrak{C}$ is a flat module, then $\mathcal{M}^{\mathfrak{C}}$ becomes a Grothendieck category (see [15]). Let \mathfrak{D} be a B -coring, a $(\mathfrak{C}, \mathfrak{D})$ -bicomodule is a three-tuple $(M, \varrho^M, \lambda^M)$ consisting of an (A, B) -bimodule $(M \in {}_A\mathcal{M}_B)$ and $A - B$ -bilinear maps $\varrho^M : M \rightarrow M \otimes_B \mathfrak{D}$ and $\lambda^M : M \rightarrow \mathfrak{C} \otimes_A M$ such that (M, ϱ^M) is right \mathfrak{D} -comodule and (M, λ^M) is left \mathfrak{C} -comodule with compatibility condition $(\mathfrak{C} \otimes_A \varrho^M) \circ \lambda^M = (\lambda^M \otimes_B \mathfrak{D}) \circ \varrho^M$. A morphism of bicomodules is a left and right colinear map. The category of all $(\mathfrak{C}, \mathfrak{D})$ -bicomodule is denoted by ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$. Obviously any ring A can be endowed with a trivial structure of an A -coring with comultiplication the isomorphism $A \cong A \otimes_A A$ (i.e. ι_A) and counit the identity A . In this way an (A, \mathfrak{D}) -bicomodule is just a right \mathfrak{D} -comodule (M, ϱ^M) whose underlying module M is an $A - B$ -bimodule and whose coaction ϱ^M is an $A - B$ -bilinear map.

Recall from [13] (see [14] for general notions), that the bicategory $\text{REM}(\text{Bim})$ is defined by the following data

- 0-cells. Are all corings $(\mathfrak{C} : A)$ (i.e., \mathfrak{C} is an A -coring)
- 1-cells. From $(\mathfrak{D} : B)$ to $(\mathfrak{C} : A)$ are pairs (M, \mathfrak{m}) where $M \in {}_A\mathcal{M}_B$ and $\mathfrak{m} : \mathfrak{C} \otimes_A M \rightarrow M \otimes_B \mathfrak{D}$ is an $A - B$ -bilinear map satisfying

$$(M \otimes_B \varepsilon_{\mathfrak{D}}) \circ \mathfrak{m} = \varepsilon_{\mathfrak{C}} \otimes_A M, \tag{3.1}$$

$$(\mathfrak{m} \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \mathfrak{m}) \circ (\Delta_{\mathfrak{C}} \otimes_A M) = (M \otimes_B \Delta_{\mathfrak{D}}) \circ \mathfrak{m}$$

where the first equality was taken up to the isomorphism ι_M . The identity 1-cell of $(\mathfrak{C} : A)$ is given by the pair $\mathbb{I}_{(\mathfrak{C}:A)} = (A, \iota_{\mathfrak{C}}^{-1})$.

- 2-cells. (in their reduced form) $\alpha : (M, \mathfrak{m}) \rightarrow (M', \mathfrak{m}')$ is an $A - B$ -bilinear map $\alpha : \mathfrak{C} \otimes_A M \rightarrow M'$ satisfying

$$\mathfrak{m}' \circ (\mathfrak{C} \otimes_A \alpha) \circ (\Delta_{\mathfrak{C}} \otimes_A M) = (\alpha \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \mathfrak{m}) \circ (\Delta_{\mathfrak{C}} \otimes_A M). \tag{3.2}$$

Let (M, \mathfrak{m}) be a 1-cell from $(\mathfrak{D} : B)$ to $(\mathfrak{C} : A)$ and (W, \mathfrak{w}) a 1-cell from $(\mathfrak{E} : C)$ to $(\mathfrak{D} : B)$. The horizontal multiplication is defined by

$$(M, \mathfrak{m})(W, \mathfrak{w}) = \left(M \otimes_B W, (M \otimes_B \mathfrak{w}) \circ (\mathfrak{m} \otimes_B W) \right)$$

If $\alpha : \mathfrak{C} \otimes_A M \rightarrow M'$ and $\beta : \mathfrak{D} \otimes_B W \rightarrow W'$ are two 2-cells, then the vertical multiplication $\alpha\beta : \mathfrak{C} \otimes_A M \otimes_B W \rightarrow M' \otimes_B W'$, is given by the rule

$$\alpha\beta = (M' \otimes_B \beta) \circ (\mathfrak{m}' \otimes_B W) \circ (\mathfrak{C} \otimes_A \alpha \otimes_B W) \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B W).$$

The composition of 2-cells $\alpha : \mathfrak{C} \otimes_A M \rightarrow M'$ and $\alpha' : \mathfrak{C} \otimes_A M' \rightarrow M''$ is given by

$$\alpha' \circ \alpha = \alpha' \circ (\mathfrak{C} \otimes_A \alpha) \circ (\Delta_{\mathfrak{C}} \otimes_A M).$$

The left and right identity functors are given as follows. Consider (M, \mathfrak{m}) any 1-cell from $(\mathfrak{D} : B)$ to $(\mathfrak{C} : A)$, the left identity multiplication $\mathfrak{l}_{(M, \mathfrak{m})} : \mathbb{I}_{(\mathfrak{C}:A)}(M, \mathfrak{m}) \rightarrow (M, \mathfrak{m})$ is defined as the composition

$$\mathfrak{l}_{(M, \mathfrak{m})} : \mathfrak{C} \otimes_A A \otimes_A M \xrightarrow{\varepsilon \otimes M} A \otimes_A A \otimes_A M \xrightarrow{1} A \otimes_A M \xrightarrow{1} M$$

while the right identity multiplication $\mathfrak{r}_{(M, \mathfrak{m})} : (M, \mathfrak{m})\mathbb{I}_{(\mathfrak{D}:B)} \rightarrow (M, \mathfrak{m})$ is defined as the composition

$$\mathfrak{r}_{(M, \mathfrak{m})} : \mathfrak{C} \otimes_A M \otimes_B B \xrightarrow{\varepsilon \otimes M \otimes B} A \otimes_A M \otimes_B B \xrightarrow{1} M \otimes_B B \xrightarrow{\mathfrak{r}} M.$$

Let (M, ϱ^M) be an (A, \mathfrak{D}) -bicomodule, and consider the following composed map

$$\mathfrak{m} : \mathfrak{C} \otimes_A M \xrightarrow{\mathfrak{C} \otimes \varrho^M} \mathfrak{C} \otimes_A M \otimes_B \mathfrak{D} \xrightarrow{\varepsilon_{\mathfrak{C}} \otimes M \otimes \mathfrak{D}} A \otimes_A M \otimes_B \mathfrak{D} \cong M \otimes_B \mathfrak{D}$$

Lemma 3.1. *The pair (M, \mathfrak{m}) is a 1-cell from $(\mathfrak{D} : B)$ to $(\mathfrak{C} : A)$ in the bicategory $\text{REM}(\text{Bim})$.*

Proof. By definition \mathfrak{m} is an $A - B$ -bilinear map. Let us show that \mathfrak{m} satisfies the Equations of (3.1). First, we have

$$\begin{aligned} (M \otimes_B \varepsilon_{\mathfrak{D}}) \circ \mathfrak{m} &= (M \otimes_B \varepsilon_{\mathfrak{D}}) \circ (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A M) \circ (\mathfrak{C} \otimes_A M \otimes_B \varepsilon_{\mathfrak{D}}) \circ (\mathfrak{C} \otimes_A \varrho^M) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A M) \end{aligned}$$

and secondly, on one hand we have

$$\begin{aligned} (M \otimes_B \Delta_{\mathfrak{D}}) \circ \mathfrak{m} &= (M \otimes_B \Delta_{\mathfrak{D}}) \circ (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{C}) \circ (\mathfrak{C} \otimes_A \varrho^M) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D} \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A M \otimes_B \Delta_{\mathfrak{D}}) \circ (\mathfrak{C} \otimes_A \varrho^M) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{C} \otimes_B \mathfrak{D}) \circ \left(\mathfrak{C} \otimes_A \left((M \otimes_B \Delta_{\mathfrak{D}}) \circ \varrho^M \right) \right) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{C} \otimes_B \mathfrak{D}) \circ \left(\mathfrak{C} \otimes_A \left((\varrho^M \otimes_B \mathfrak{D}) \circ \varrho^M \right) \right) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D} \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M) \\ &= (\mathfrak{m} \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M) \end{aligned}$$

and on the other hand we have

$$\begin{aligned} (\mathfrak{m} \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \mathfrak{m}) \circ (\Delta_{\mathfrak{C}} \otimes_A M) &= \left(\left((\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M) \right) \otimes_B \mathfrak{D} \right) \\ &\quad \circ \left(\mathfrak{C} \otimes_A \left((\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M) \right) \right) \circ (\Delta_{\mathfrak{C}} \otimes_A M) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D} \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D}) \\ &\quad \circ (\mathfrak{C} \otimes_A \mathfrak{C} \otimes_A \varrho^M) \circ (\Delta_{\mathfrak{C}} \otimes_A M) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D} \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D}) \\ &\quad \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D} \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M) \\ &= (\mathfrak{m} \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M) \end{aligned}$$

Therefore, $(M \otimes_B \Delta_{\mathfrak{D}}) \circ \mathfrak{m} = (\mathfrak{m} \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \mathfrak{m}) \circ (\Delta_{\mathfrak{C}} \otimes_A M)$. □

The bicategory $\text{W}(\text{REM}(\text{Bim}))$ is then defined by the following data:

- 0-cells. Are all corings $(\mathfrak{C} : A)$.
- 1-cells. From $(\mathfrak{D} : B)$ to $(\mathfrak{C} : A)$, are four-tuples $\Gamma = \left((M, \mathfrak{m}), (N, \mathfrak{n}), \eta, \rho \right)$ with $M \in {}_A\mathcal{M}_B, N \in {}_B\mathcal{M}_A$, bilinear maps $\mathfrak{m} : \mathfrak{C} \otimes_A M \rightarrow M \otimes_B \mathfrak{D}, \mathfrak{n} : \mathfrak{D} \otimes_B N \rightarrow N \otimes_A \mathfrak{C}$ satisfying (3.1), and two 2-cells, $\eta : \mathfrak{C} \otimes_A M \otimes_B N \rightarrow A$, and $\rho : \mathfrak{D} \otimes_B N \otimes_A M \rightarrow B$ satisfying both them the following equalities (the first two ones are up to the natural isomorphism ι_{-})

$$(\eta \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \mathfrak{n}) \circ (\mathfrak{C} \otimes_A \mathfrak{m} \otimes_B N) \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B N) = (\mathfrak{C} \otimes_A \eta) \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B N), \quad (3.3)$$

$$(\rho \otimes_B \mathfrak{D}) \circ (\mathfrak{D} \otimes_B N \otimes_A \mathfrak{m}) \circ (\mathfrak{D} \otimes_B \mathfrak{n} \otimes_A M) \circ (\Delta_{\mathfrak{D}} \otimes_B N \otimes_A M) = (\mathfrak{D} \otimes_B \rho) \circ (\Delta_{\mathfrak{D}} \otimes_B N \otimes_A M), \quad (3.4)$$

$$\iota_M \circ (\eta \otimes_A M) = (M \otimes_B \rho) \circ (\mathfrak{m} \otimes_B N \otimes_A M), \quad (3.5)$$

$$\iota_N \circ (\rho \otimes_B N) = (N \otimes_A \eta) \circ (\mathfrak{n} \otimes_A M \otimes_B N). \quad (3.6)$$

The identity 1-cell of $(\mathfrak{C} : A)$ is given by the four-tuple $\left((A, \iota_{\mathfrak{C}}^{-1}), (A, \iota_{\mathfrak{C}}^{-1}), \mathfrak{r}_{(\mathfrak{C}:A)}, \iota_{(\mathfrak{C}:A)} \right)$

- 2-cells. Are pairs $(\alpha, \beta) : \Gamma = \left((M, \mathfrak{m}), (N, \mathfrak{n}), \eta, \rho \right) \rightarrow \Gamma' = \left((M', \mathfrak{m}'), (N', \mathfrak{n}'), \eta', \rho' \right)$ where $\alpha : \mathfrak{C} \otimes_A M \rightarrow M'$ and $\beta : \mathfrak{D} \otimes_B N \rightarrow N'$ are 2-cells satisfying the following equalities

$$\begin{aligned} \eta &= \eta' \circ (\mathfrak{C} \otimes_A M' \otimes_B \beta) \circ (\mathfrak{C} \otimes_A \mathfrak{m}' \otimes_B N) \circ (\mathfrak{C} \otimes_A \mathfrak{C} \otimes_A \alpha \otimes_B N) \\ &\quad \circ (\mathfrak{C} \otimes_A \Delta_{\mathfrak{C}} \otimes_A M \otimes_B N) \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B N) \end{aligned} \quad (3.7)$$

$$\begin{aligned} \rho &= \rho' \circ (\mathfrak{D} \otimes_B N' \otimes_A \alpha) \circ (\mathfrak{D} \otimes_B \mathfrak{n}' \otimes_A M) \circ (\mathfrak{D} \otimes_B \mathfrak{D} \otimes_B \beta \otimes_A M) \\ &\quad \circ (\mathfrak{D} \otimes_B \Delta_{\mathfrak{D}} \otimes_B N \otimes_A M) \circ (\Delta_{\mathfrak{D}} \otimes_B N \otimes_A M) \end{aligned} \tag{3.8}$$

Remark 3.2. Recall that a (classical) Morita context between two rings A and B (or from B to A) consists of two bimodules $P \in {}_A\mathcal{M}_B$ and $Q \in {}_B\mathcal{M}_A$ together with bilinear maps $\varphi : P \otimes_B Q \rightarrow A$ and $\psi : Q \otimes_A P \rightarrow B$ satisfying

$$y\varphi(x, y') = \psi(y, x)y', \quad x\psi(y, x') = \varphi(x, y)x'$$

for every $x, x' \in P$ and $y, y' \in Q$. This of course says exactly that the four-tuple (P, Q, φ, ψ) is a WRMC from B to A in the bicategory $\mathbf{W}(\mathbf{Bim})$. This can also be recognized in the bicategory $\mathbf{W}(\mathbf{REM}(\mathbf{Bim}))$ as follows. As was mentioned before we can consider trivially any ring A as an A -coring $(A : A)$. That is we can consider $(A : A)$ as a 0-cell in $\mathbf{REM}(\mathbf{Bim})$ for every ring A . So let $(A : A)$ and $(B : B)$ two trivial 0-cells in $\mathbf{REM}(\mathbf{Bim})$. The category of all 1-cells from $(B : B)$ to $(A : A)$ in this bicategory is clearly, using the natural isomorphism ι_- , isomorphic to the category of bimodules ${}_A\mathcal{M}_B$. Now, consider $\Gamma = ((M, \iota_M), (N, \iota_N), \eta, \rho)$ any 1-cell from $(B : B)$ to $(A : A)$ in the bicategory $\mathbf{W}(\mathbf{REM}(\mathbf{Bim}))$. This means that $M \in {}_A\mathcal{M}_B$, $N \in {}_B\mathcal{M}_A$, and $\eta : A \otimes_A M \otimes_B N \rightarrow A$, $\rho : B \otimes_B N \otimes_A M \rightarrow B$ are bilinear maps satisfying Equations (3.3)–(3.6). Define in an obvious way $\tilde{\eta} : M \otimes_B N \rightarrow A$ and $\tilde{\rho} : N \otimes_A M \rightarrow B$. It is easily seen that $(M, N, \tilde{\eta}, \tilde{\rho})$ is a Morita context between A and B . Conversely any WRMC from $(B : B)$ to $(A : A)$ in $\mathbf{W}(\mathbf{REM}(\mathbf{Bim}))$ is deduced from a classical Morita context from B to A .

Fix an arbitrary 1-cell $\Gamma = ((M, \mathfrak{m}), (N, \mathfrak{n}), \eta, \rho)$ in $(\mathfrak{C} : A)\mathbf{W}(\mathbf{REM}(\mathbf{Bim}))_{(\mathfrak{D} : B)}$. Since (M, \mathfrak{m}) and (N, \mathfrak{n}) are a 1-cells in $\mathbf{REM}(\mathbf{Bim})$, we can associate by [13, 4.2], the (right) push-out functors $\mathcal{M} : \mathcal{M}^{\mathfrak{C}} \rightarrow \mathcal{M}^{\mathfrak{D}}$ and $\mathcal{N} : \mathcal{M}^{\mathfrak{D}} \rightarrow \mathcal{M}^{\mathfrak{C}}$, explicitly acting on objects by

$$\begin{aligned} \mathcal{M}(X, \varrho^X) &= \left(X \otimes_A M, \varrho^{X \otimes_A M} = (X \otimes_A \mathfrak{m}) \circ (\varrho^X \otimes_A M) \right) \\ \mathcal{N}(Y, \varrho^Y) &= \left(Y \otimes_B N, \varrho^{Y \otimes_B N} = (Y \otimes_B \mathfrak{n}) \circ (\varrho^Y \otimes_B N) \right), \end{aligned} \tag{3.9}$$

for every right \mathfrak{C} -comodule (X, ϱ^X) and every right \mathfrak{D} -comodule (Y, ϱ^Y) . The actions on morphisms are obvious.

On the other hand, we can define the following right linear maps

$$\begin{aligned} \tilde{\eta}_X : X \otimes_A M \otimes_B N &\xrightarrow{\varrho^{X \otimes_A M \otimes_B N}} X \otimes_A \mathfrak{C} \otimes_A M \otimes_B N \xrightarrow{X \otimes \eta} X \otimes_A A \cong X \\ \tilde{\rho}_Y : Y \otimes_B N \otimes_A M &\xrightarrow{\varrho^{Y \otimes_B N \otimes_A M}} Y \otimes_B \mathfrak{D} \otimes_B N \otimes_A M \xrightarrow{Y \otimes \varrho} Y \otimes_B B \cong Y \end{aligned}$$

for every pair of right \mathfrak{C} -comodule (X, ϱ^X) and right \mathfrak{D} -comodule (Y, ϱ^Y) . We consider $\mathcal{NM}(X) = X \otimes_A M \otimes_B N$ and $\mathcal{MN}(Y) = Y \otimes_B N \otimes_A M$, respectively, as right \mathfrak{C} -comodule and as right \mathfrak{D} -comodule using coactions defined in (3.9).

Proposition 3.3. *Keeping the above notations, we have*

- (i) *The right linear maps $\tilde{\eta}_X$ and $\tilde{\rho}_Y$ are, respectively, right \mathfrak{C} -colinear and right \mathfrak{D} -colinear.*
- (ii) *They define a natural transformation $\tilde{\eta}_- : \mathcal{NM} \rightarrow \mathbb{1}_{\mathcal{M}^{\mathfrak{C}}}$ and $\tilde{\rho}_- : \mathcal{MN} \rightarrow \mathbb{1}_{\mathcal{M}^{\mathfrak{D}}}$ satisfying the following conditions*

$$\begin{aligned} \tilde{\eta}_{\mathcal{N}(Y)} &= \mathcal{N}\tilde{\rho}_Y, \quad \text{for every } (Y, \varrho^Y) \in \mathcal{M}^{\mathfrak{D}}, \\ \tilde{\rho}_{\mathcal{M}(X)} &= \mathcal{M}\tilde{\eta}_X, \quad \text{for every } (X, \varrho^X) \in \mathcal{M}^{\mathfrak{C}}. \end{aligned}$$

That is $(\mathcal{M}, \mathcal{N}, \tilde{\eta}, \tilde{\rho})$ is right wide Morita context, in the sense of [6], between the categories $\mathcal{M}^{\mathfrak{C}}$ and $\mathcal{M}^{\mathfrak{D}}$ (i.e. a WRMC in the 2-category \mathbf{Cat} with right exact functors).

Proof. (i). We only prove the colinearity of $\tilde{\eta}_X$, similar arguments are used for $\tilde{\rho}_Y$. By definition $\tilde{\eta}_X$ is right A -linear, and we have

$$\begin{aligned} \varrho^X \circ \tilde{\eta}_X &= \varrho^X \circ (X \otimes_A \eta) \circ (\varrho^X \otimes_A M \otimes_B N) \\ &= (X \otimes_A \mathfrak{C} \otimes_A \eta) \circ (\varrho^X \otimes_A \mathfrak{C} \otimes_A M \otimes_B N) \circ (\varrho^X \otimes_A M \otimes_B N) \\ &= (X \otimes_A \mathfrak{C} \otimes_A \eta) \circ \left((\varrho^X \otimes_A \mathfrak{C}) \circ \varrho^X \right) \otimes_A M \otimes_B N \end{aligned}$$

$$\begin{aligned}
 &= (X \otimes_A \mathfrak{C} \otimes_A \eta) \circ \left(\left((X \otimes_A \Delta_{\mathfrak{C}}) \circ \varrho^X \right) \otimes_A M \otimes_B N \right) \\
 &= (X \otimes_A \mathfrak{C} \otimes_A \eta) \circ (X \otimes_A \Delta_{\mathfrak{C}} \otimes_A M \otimes_B N) \circ (\varrho^X \otimes_A M \otimes_B N) \\
 &= \left(X \otimes_A \left((\mathfrak{C} \otimes_A \eta) \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B N) \right) \right) \circ (\varrho^X \otimes_A M \otimes_B N) \\
 &= \left(X \otimes_A \left((\eta \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \mathfrak{n}) \circ (\mathfrak{C} \otimes_A \mathfrak{m} \otimes_B N) \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B N) \right) \right) \\
 &\quad \circ (\varrho^X \otimes_A M \otimes_B N), \text{ by (3.3)} \\
 &= (X \otimes_A \eta \otimes_A \mathfrak{C}) \circ (X \otimes_A \mathfrak{C} \otimes_A M \otimes_B \mathfrak{n}) \circ (X \otimes_A \mathfrak{C} \otimes_A \mathfrak{m} \otimes_B N) \\
 &\quad \circ (X \otimes_A \Delta_{\mathfrak{C}} \otimes_A M \otimes_B N) \circ (\varrho^X \otimes_A M \otimes_B N) \\
 &= (X \otimes_A \eta \otimes_A \mathfrak{C}) \circ (X \otimes_A \mathfrak{C} \otimes_A M \otimes_B \mathfrak{n}) \circ (X \otimes_A \mathfrak{C} \otimes_A \mathfrak{m} \otimes_B N) \\
 &\quad \circ (\varrho^X \otimes_A \mathfrak{C} \otimes_A M \otimes_B N) \circ (\varrho^X \otimes_A M \otimes_B N) \\
 &= (X \otimes_A \eta \otimes_A \mathfrak{C}) \circ (X \otimes_A \mathfrak{C} \otimes_A M \otimes_B \mathfrak{n}) \circ (\varrho^X \otimes_A M \otimes_B \mathfrak{D} \otimes_B N) \\
 &\quad \circ (X \otimes_A \mathfrak{m} \otimes_A M \otimes_B N) \circ (\varrho^X \otimes_A M \otimes_B N) \\
 &= (X \otimes_A \eta \otimes_A \mathfrak{C}) \circ (\varrho^X \otimes_A M \otimes_B N \otimes_A \mathfrak{C}) \circ (X \otimes_A M \otimes_B \mathfrak{n}) \\
 &\quad \circ (X \otimes_A \mathfrak{m} \otimes_A M \otimes_B N) \circ (\varrho^X \otimes_A M \otimes_B N) \\
 &= (\tilde{\eta}_X \otimes_A \mathfrak{C}) \circ \varrho^{X \otimes_A M \otimes_B N},
 \end{aligned}$$

where $\varrho^{X \otimes_B M \otimes_A N} = (X \otimes_B M \otimes_A \mathfrak{n}) \circ (X \otimes_B \mathfrak{m} \otimes_B M \otimes_A N) \circ (\varrho^X \otimes_B M \otimes_A N)$ is the right coaction of $\mathcal{N}\mathcal{M}(X, \varrho^X)$ as defined in (3.9).

(ii). Let $f : (X, \varrho^X) \rightarrow (X', \varrho^{X'})$ be a right \mathfrak{C} -colinear map, then

$$\begin{aligned}
 f \circ \tilde{\eta}_X &= f \circ (X \otimes_A \eta) \circ (\varrho^X \otimes_A M \otimes_B N) \\
 &= (X' \otimes_A \eta) \circ (f \otimes_A \mathfrak{C} \otimes_A M \otimes_B N) \circ (\varrho^X \otimes_A M \otimes_B N), \text{ } f \text{ is linear} \\
 &= (X' \otimes_A \eta) \circ \left((f \otimes_A \mathfrak{C}) \circ \varrho^X \right) \otimes_A M \otimes_B N \\
 &= (X' \otimes_A \eta) \circ (\varrho^{X'} \otimes_A M \otimes_B N) \circ (f \otimes_A M \otimes_B N), \text{ } f \text{ is colinear} \\
 &= \tilde{\eta}_{X'} \circ \mathcal{N}\mathcal{M}(f).
 \end{aligned}$$

Hence $\tilde{\eta}_-$ is natural. Similarly we check that $\tilde{\rho}_-$ is also natural. Given now any right \mathfrak{D} -comodule (Y, ϱ^Y) , we know that $\varrho^{N(Y)} = (N \otimes_A \mathfrak{n}) \circ (\varrho^Y \otimes_B N)$, so

$$\begin{aligned}
 \tilde{\eta}_{N(Y)} &= (Y \otimes_B N \otimes_A \eta) \circ (Y \otimes_B \mathfrak{n} \otimes_A M \otimes_B N) \circ (\varrho^Y \otimes_B N \otimes_A M \otimes_B N) \\
 &= \left(Y \otimes_B \left((N \otimes_A \eta) \circ (\mathfrak{n} \otimes_A M \otimes_B N) \right) \right) \circ (\varrho^Y \otimes_B N \otimes_A M \otimes_B N) \\
 &= (Y \otimes_B \rho \otimes_B N) \circ (\varrho^Y \otimes_B N \otimes_A M \otimes_B N), \text{ by (3.6) (up to } \iota_-) \\
 &= \mathcal{N}\tilde{\rho}_Y,
 \end{aligned}$$

which gives the first condition of the stated item (ii). The second one is similarly proved. Since the push-out functors preserve cokernels, we have thus showed that $(\mathcal{M}, \mathcal{N}, \tilde{\eta}, \tilde{\rho})$ is right Morita context between $\mathcal{M}^{\mathfrak{D}}$ and $\mathcal{M}^{\mathfrak{C}}$ in the sense of [6]. □

Remark 3.4. A more conceptual proof of Proposition 3.3 can be given using Proposition 2.10, in the following way. One can show that the push-out defines a homomorphism of bicategories $\mathcal{P} : \text{REM}(\text{Bim}) \rightarrow \text{Cat}$, sending any 0-cell $(\mathfrak{C} : A)$ to its category of right \mathfrak{C} -comodules $\mathcal{M}^{\mathfrak{C}}$. Therefore, there is a homomorphism of bicategories $\mathcal{W}(\mathcal{P}) : \mathcal{W}(\text{REM}(\text{Bim})) \rightarrow \mathcal{W}(\text{Cat})$. Since the push-out functor preserves cokernels and direct sums, we immediately deduce the claim of Proposition 3.3.

On the other hand, push-out functors are in fact a lifting functors. Precisely, given a bimodule ${}_A M_B$, we know (see [14, p. 256] and the dual version of [19, Lemma 1]) that there is a 1-1 correspondence between

- (i) $A - B$ -bilinear maps $\mathfrak{m} : \mathfrak{C} \otimes_A M \rightarrow M \otimes_B \mathfrak{D}$ satisfying Equation (3.1);

(ii) Lifting functors $\mathcal{M} : \mathcal{M}^{\mathfrak{C}} \rightarrow \mathcal{M}^{\mathfrak{D}}$ of the functor $-\otimes_A M : \mathcal{M}_A \rightarrow \mathcal{M}_B$, that is, functors \mathcal{M} rendering commutative the following diagram

$$\begin{array}{ccc} \mathcal{M}^{\mathfrak{C}} & \xrightarrow{\mathcal{M}} & \mathcal{M}^{\mathfrak{D}} \\ \mathcal{O}_{\mathfrak{C}} \downarrow & & \downarrow \mathcal{O}_{\mathfrak{D}} \\ \mathcal{M}_A & \xrightarrow{-\otimes_A M} & \mathcal{M}_B \end{array}$$

where $\mathcal{O}_{\mathfrak{C}}$ and $\mathcal{O}_{\mathfrak{D}}$ are the forgetful functors.

In this way, it is possible to analyze the converse of Proposition 3.3. The forthcoming proposition discusses a special case of this analysis.

Suppose that a wide right Morita context in the sense of [6] between $\mathcal{M}^{\mathfrak{D}}$ and $\mathcal{M}^{\mathfrak{C}}$ is given. This consists of pair of covariant additive functors $\mathcal{M} : \mathcal{M}^{\mathfrak{D}} \rightleftarrows \mathcal{M}^{\mathfrak{C}} : \mathcal{N}$ which preserve cokernels, and two natural transformations $\tilde{\eta} : \mathcal{M}\mathcal{N} \rightarrow \mathbb{1}_{\mathcal{M}^{\mathfrak{C}}}$ and $\tilde{\rho} : \mathcal{N}\mathcal{M} \rightarrow \mathbb{1}_{\mathcal{M}^{\mathfrak{D}}}$ with compatible conditions as stated in Proposition 3.3(ii). Assume that \mathcal{M} and \mathcal{N} preserve direct sums, and they are lifting functors for some (A, \mathfrak{D}) -bicomodule (M, ϱ^M) and some (B, \mathfrak{C}) -bicomodule (N, ϱ^N) , respectively (here A and B are considered trivially as an A -coring and B -coring). In this case, we are assuming that there are commutative diagrams

$$\begin{array}{ccc} \mathcal{M}^{\mathfrak{C}} & \xrightarrow{\mathcal{M}} & \mathcal{M}^{\mathfrak{D}} \\ \mathcal{O}_{\mathfrak{C}} \searrow & \nearrow & \mathcal{M}_A \\ & & -\otimes_A M \end{array} \quad \begin{array}{ccc} \mathcal{M}^{\mathfrak{D}} & \xrightarrow{\mathcal{N}} & \mathcal{M}^{\mathfrak{C}} \\ \mathcal{O}_{\mathfrak{D}} \searrow & \nearrow & \mathcal{M}_B \\ & & -\otimes_B N \end{array} \tag{3.10}$$

We know from Lemma 3.1, that (M, \mathfrak{m}) is a 1-cell in $\text{REM}(\text{Bim})$ from $(\mathfrak{D} : B)$ to $(\mathfrak{C} : A)$, where

$$\mathfrak{m} : \mathfrak{C} \otimes_A M \xrightarrow{\mathfrak{C} \otimes \varrho^M} \mathfrak{C} \otimes_A M \otimes_B \mathfrak{D} \xrightarrow{\varepsilon_{\mathfrak{C}} \otimes M \otimes \mathfrak{D}} A \otimes_A M \otimes_B \mathfrak{D} \cong M \otimes_B \mathfrak{D}.$$

Similarly (N, \mathfrak{n}) is a 1-cell from $(\mathfrak{C} : A)$ to $(\mathfrak{D} : B)$ in $\text{REM}(\text{Bim})$ with

$$\mathfrak{n} : \mathfrak{D} \otimes_B N \xrightarrow{\mathfrak{D} \otimes \varrho^N} \mathfrak{D} \otimes_B N \otimes_A \mathfrak{C} \xrightarrow{\varepsilon_{\mathfrak{D}} \otimes N \otimes \mathfrak{C}} B \otimes_B N \otimes_A \mathfrak{C} \cong N \otimes_A \mathfrak{C}$$

Proposition 3.5. *Let \mathcal{M}, \mathcal{N} be the functors defined in (3.10), and $(M, \mathfrak{m}), (N, \mathfrak{n}), \tilde{\eta}$ and $\tilde{\rho}$ as above. Then $\Gamma = ((M, \mathfrak{m}), (N, \mathfrak{n}), \eta, \rho)$ is a 1-cell in $(\mathfrak{C} : A)\mathbf{W}(\text{REM}(\text{Bim}))_{(\mathfrak{D} : B)}$, where $\eta = \varepsilon_{\mathfrak{D}} \circ \tilde{\eta}_{\mathfrak{D}}$ and $\rho = \varepsilon_{\mathfrak{C}} \circ \tilde{\rho}_{\mathfrak{C}}$.*

Proof. We need to show that the stated maps η and ρ satisfy Equations (3.3)–(3.6).

Since $\tilde{\eta}_-$ and $\tilde{\rho}_-$ are natural transformations, it is clearly seen that, for every ring R and every pair of (R, \mathfrak{D}) -bicomodule (Y, ϱ^Y) and (R, \mathfrak{C}) -bicomodule (X, ϱ^X) , we have $\tilde{\eta}_Y : Y \otimes_B M \otimes_A N \rightarrow Y$ and $\tilde{\rho}_X : X \otimes_A N \otimes_B M \rightarrow X$ are, respectively, $R - B$ -bilinear and $R - A$ -bilinear maps. Furthermore, we can prove, using free presentations of right modules, that

$$\begin{aligned} \tilde{\eta}_{Y \otimes_B \mathfrak{D}} &= Y \otimes_B \tilde{\eta}_{\mathfrak{D}}, \quad \text{for every right module } Y \in \mathcal{M}_B \\ \tilde{\rho}_{X \otimes_A \mathfrak{C}} &= X \otimes_A \tilde{\rho}_{\mathfrak{C}}, \quad \text{for every right module } X \in \mathcal{M}_A. \end{aligned}$$

Keeping this in mind, we compute

$$\begin{aligned} &(\eta \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \mathfrak{n}) \circ (\mathfrak{C} \otimes_A \mathfrak{m} \otimes_B N) \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D} \otimes_B N) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ (\tilde{\eta}_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \mathfrak{n}) \circ (\mathfrak{C} \otimes_A \mathfrak{m} \otimes_B N) \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D} \otimes_B N) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ (\tilde{\eta}_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \varepsilon_{\mathfrak{D}} \otimes_B N \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \mathfrak{D} \otimes_B \varrho^N) \\ &\quad \circ (\mathfrak{C} \otimes_A \varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D} \otimes_B N) \circ (\mathfrak{C} \otimes_A \mathfrak{C} \otimes_A \varrho^M \otimes_B N) \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D} \otimes_B N) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ (\tilde{\eta}_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \varepsilon_{\mathfrak{D}} \otimes_B N \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \mathfrak{D} \otimes_B \varrho^N) \\ &\quad \circ (\mathfrak{C} \otimes_A \varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D} \otimes_B N) \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D} \otimes_B N) \circ (\mathfrak{C} \otimes_A \varrho^M \otimes_B N) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ (\tilde{\eta}_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \varepsilon_{\mathfrak{D}} \otimes_B N \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \mathfrak{D} \otimes_B \varrho^N) \circ (\mathfrak{C} \otimes_A \varrho^M \otimes_B N) \\ &= (\varepsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ (\tilde{\eta}_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \varepsilon_{\mathfrak{D}} \otimes_B N \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A \varrho^M \otimes_B N \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \varrho^N) \end{aligned}$$

$$\begin{aligned}
 &= (\varepsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ (\tilde{\eta}_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A M \otimes_B \varrho^N) \\
 &= (\varepsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ \Delta_{\mathfrak{C}} \circ \tilde{\eta}_{\mathfrak{C}}, \quad \tilde{\eta}_{\mathfrak{C}} \text{ is colinear} \\
 &= \tilde{\eta}_{\mathfrak{C}}.
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 (\mathfrak{C} \otimes_A \eta) \circ (\Delta_{\mathfrak{C}} \otimes_A \otimes_B N) &= (\mathfrak{C} \otimes_A \varepsilon_{\mathfrak{C}}) \circ (\mathfrak{C} \otimes_A \tilde{\eta}_{\mathfrak{C}}) \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B N) \\
 &= (\mathfrak{C} \otimes_A \varepsilon_{\mathfrak{C}}) \circ \tilde{\eta}_{\mathfrak{C} \otimes_A \mathfrak{C}} \circ (\Delta_{\mathfrak{C}} \otimes_A M \otimes_B N) \\
 &= (\mathfrak{C} \otimes_A \varepsilon_{\mathfrak{C}}) \circ \Delta_{\mathfrak{C}} \circ \tilde{\eta}_{\mathfrak{C}} = \tilde{\eta}_{\mathfrak{C}}
 \end{aligned}$$

Therefore, η satisfies (3.3). Similarly, we prove that ρ satisfies (3.4). Now, up to the isomorphism ι_M , we compute

$$\begin{aligned}
 &(M \otimes_B \varepsilon_{\mathfrak{D}}) \circ (M \otimes_B \tilde{\rho}_{\mathfrak{D}}) \circ (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D} \otimes_B N \otimes_A M) \circ (\mathfrak{C} \otimes_A \varrho^M \otimes_B N \otimes_A M) \\
 &= (M \otimes_B \varepsilon_{\mathfrak{D}}) \circ (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A M \otimes_B \tilde{\rho}_{\mathfrak{D}}) \circ (\mathfrak{C} \otimes_A \varrho^M \otimes_B N \otimes_A M) \\
 &= (M \otimes_B \varepsilon_{\mathfrak{D}}) \circ (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \tilde{\rho}_{M \otimes_B \mathfrak{D}}) \circ (\mathfrak{C} \otimes_A \varrho^M \otimes_B N \otimes_A M) \\
 &= (M \otimes_B \varepsilon_{\mathfrak{D}}) \circ (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D}) \circ \left(\mathfrak{C} \otimes_A \left(\tilde{\rho}_{M \otimes_B \mathfrak{D}} \circ (\varrho^M \otimes_B N \otimes_A M) \right) \right) \\
 &= (M \otimes_B \varepsilon_{\mathfrak{D}}) \circ (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D}) \circ \left(\mathfrak{C} \otimes_A \left(\varrho^M \circ \tilde{\rho}_M \right) \right) \\
 &= (M \otimes_B \varepsilon_{\mathfrak{D}}) \circ (\varepsilon_{\mathfrak{C}} \otimes_A M \otimes_B \mathfrak{D}) \circ (\mathfrak{C} \otimes_A \varrho^M) \circ (\mathfrak{C} \otimes_A \tilde{\rho}_M) \\
 &= (M \otimes_B \varepsilon_{\mathfrak{D}}) \circ \varrho^M \circ (\varepsilon_{\mathfrak{C}} \otimes_A M) \circ (\mathfrak{C} \otimes_A \tilde{\rho}_M) \\
 &= (\varepsilon_{\mathfrak{C}} \otimes_A M) \circ (\mathfrak{C} \otimes_A \tilde{\rho}_M) = (\varepsilon_{\mathfrak{C}} \otimes_A M) \circ \tilde{\rho}_{\mathfrak{C} \otimes_A M} = (\varepsilon_{\mathfrak{C}} \otimes_A M) \circ (\tilde{\eta}_{\mathfrak{C}} \otimes_A M),
 \end{aligned}$$

which implies the Equation (3.5). A similar argument proves Equation (3.6) and this finishes the proof. \square

Remarks 3.6. (1) We can study the case of special corings such as a corings constructed by an entwined structures (see [20]) and other kind of corings, all of them within the bicategory $\text{REM}(\text{Bim})$. This will lead in particular to the study of the cases of graded modules, comodules over coalgebras over fields, Hopf modules... etc.

- (2) The case of the 2-category whose 0-cells are all Grothendieck categories, and whose Hom-Categories consists of categories of continuous functors, *i.e.* right exact functors which preserve direct sums (see [21] for elementary details), could be of special interest since it gives another point of view for the treatments of the problem of equivalence theorems stated in [7, 8, 10, 11].
- (3) The non unital case, that is, the case concerning the bicategory of unital bimodules over rings with local units (see [21]), leads to the study of Morita contexts between rings with local units, as was developed in [22] and [23].

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