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Comatrix Corings and Invertible Bimodules (*)

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SUNTO – Estendiamo il Teorema di Masuoka [11] riguardante l'isomorfismo tra il gruppo dei bimoduli invertibili su un'estensione di anelli non commutativa e il gruppo di automorfismi del coanello canonico associato di Sweedler, alla classe dei coanelli di comatrici finiti introdotta in [6].

ABSTRACT – We extend Masuoka's Theorem [11] concerning the isomorphism between the group of invertible bimodules in a non-commutative ring extension and the group of automorphisms of the associated Sweedler's canonical coring, to the class of finite comatrix corings introduced in [6].

Introduction.

Comatrix corings were introduced by the authors in [6] to give a structure theorem of all cosemisimple corings. This construction generalizes Sweedler's canonical corings [15], and provides a version of descent theory for modules [6, Theorem 3.10]. Sweedler's canonical corings and their automorphisms were the key tool in [11] to give a non-commutative version of the fact that the relative Picard group attached to any commutative ring extension is isomorphic to the Amistur 1-cohomology for the units-functor due to Grothendieck's faithfully flat descent.

In this note we extend, by using different methods, the main result of [11, §2]

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to the context of comatrix corings. In fact, we apply ideas and recent results from [7] and [6], and the present paper can be already seen as natural continuation of the theory developed in [6].

The first section is rather technical, and it is devoted to prove that there is an adjoint pair of functors between the category of comodules over a given comatrix coring and the category of comodules over its associated Sweedler's canonical coring. This adjunction will have a role in the proof of the main result. Section 2 is the core of the paper, as it contains the aforementioned isomorphism of groups (Theorem 2.5). The maps connecting bimodules and coring automorphisms are at a first glance different from the maps constructed in [11]. However, they are neatly related, as Proposition 2.6 shows.

All rings considered in this note are algebras with 1 over a commutative ground base ring K. A right or left module, means a unital module. All bimodules over rings are central K-bimodules. If A is any ring, then we denote by $\mathcal{M}_A$ (respectively ${}_A\mathcal{M}$) the category of all right (respectively left) A-modules. The opposite ring of A will be denoted by A^o, its multiplication is defined by $a_2^o a_1^o = (a_1 a_2)^o$, $a_1^o, a_2^o \in A^o$ (i.e. $a_1, a_2 \in A$). As usual, some special convention will be understood for the case of endomorphism rings of modules. Thus, if X_A is an object of $\mathcal{M}_A$, then its endomorphism ring will be denoted by $\mathrm{End}(X_A)$, while if ${}_AY$ is left A-module, then its endomorphism ring, denoted by $\mathrm{End}({}_AY)$, is, by definition, the opposite of the endomorphism ring of Y as an object of the category ${}_A\mathcal{M}$. In this way X is an $(\mathrm{End}(X_A), A)$-bimodule, while Y is an $(A, \mathrm{End}({}_AY))$-bimodule. The opposite left A^o-module of X_A, will be denoted by X^o, the action is given by $a^o x^o = (xa)^o$, $a^o \in A^o$, $x^o \in X^o$. Of course, if $f : X \to W$ is right A-linear map, then its opposite map $f^o : X^o \to W^o$ is left A^o-linear which is defined by $f^o(x^o) = (f(x))^o$, for all $x^o \in X^o$. The same process will be applied on bimodules and bilinear maps. For any (B,A)-bimodule M we denote by $M^* = \mathrm{Hom}(M_A, A_A)$ its right dual and by ${}^*M = \mathrm{Hom}({}_BM, {}_BB)$ its left dual. M^* and ${}^*M$ are considered, in a natural way, as (A, B)-bimodules.

Recall from [15] that an A-coring is a three-tuple $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ consisting of an A-bimodule $\mathfrak{C}$ and two A-bilinear maps

$$\mathfrak{C} \xrightarrow{\Delta_{\mathfrak{C}}} \mathfrak{C} \otimes_A \mathfrak{C}, \quad \mathfrak{C} \xrightarrow{\varepsilon_{\mathfrak{C}}} A$$

such that $(\Delta_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ \Delta_{\mathfrak{C}} = (\mathfrak{C} \otimes_A \Delta_{\mathfrak{C}}) \circ \Delta_{\mathfrak{C}}$ and $(\varepsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ \Delta_{\mathfrak{C}} = (\mathfrak{C} \otimes_A \varepsilon_{\mathfrak{C}}) \circ \Delta_{\mathfrak{C}} = \mathfrak{C}$. A morphism of A-corings is an A-bilinear map $\phi : \mathfrak{C} \to \mathfrak{D}$ which satisfies: $\varepsilon_{\mathfrak{D}} \circ \phi = \varepsilon_{\mathfrak{C}}$ and $\Delta_{\mathfrak{D}} \circ \phi = (\phi \otimes_A \phi) \circ \Delta_{\mathfrak{C}}$. A right $\mathfrak{C}$-comodule is a pair (M, ρ_M) consisting of a right A-module M and a right A-linear map $\rho_M : M \to M \otimes_A \mathfrak{C}$, called right $\mathfrak{C}$-coaction, such that $(M \otimes_A \Delta_{\mathfrak{C}}) \circ \rho_M = (\rho_M \otimes_A \mathfrak{C}) \circ \rho_M$ and $(M \otimes_A \varepsilon_{\mathfrak{C}}) \circ \rho_M = M$. Left $\mathfrak{C}$-comodules are symmetrically defined, and we will use the Greek letter λ_- to denote their coactions. For more details on comodules, definitions and basic properties of bicomodules and the cotensor product, the reader is referred to [3] and its bibliograpy.

1. – Comatrix coring and adjunctions.

Throughout this section Σ will be a fixed (B,A)-bimodule which is finitely generated and projective as a right A-module with a fixed dual basis $\{(e_i, e_i^*)\}_{1 \leq i \leq n} \subset \Sigma \times \Sigma^*$. Let $S = \mathrm{End}(\Sigma_A)$ be its right endomorphism ring, and let $\lambda : B \to S$ be the canonical associated ring extension. It is known that there is an S-bimodule isomorphism

$$\begin{aligned} \xi : \Sigma \otimes_A \Sigma^* &\longrightarrow S = \mathrm{End}(\Sigma_A) \\ u \otimes_A v^* &\longmapsto [x \mapsto uv^*(x)] \\ \textstyle\sum_i e_i \otimes_A e_i^* s = \sum_i s(e_i) \otimes_A e_i^* &\longleftarrow s \end{aligned} \tag{1}$$

With this identification the product of S (the composition) satisfies

$$\begin{aligned} s(u \otimes_A u^*) &= s(u) \otimes_A u^*, \\ (u \otimes_A u^*)s &= u \otimes_A u^* s, \\ (u \otimes_A u^*)(v \otimes_A v^*) &= uu^*(v) \otimes_A v^* = u \otimes_A u^*(v)v^*, \end{aligned} \tag{2}$$

for every $s \in S$, $u, v \in \Sigma$, $v^*, u^* \in \Sigma^*$. By [6, Proposition 2.1], the A-bimodule $\Sigma^* \otimes_B \Sigma$ is an A-coring with the following comultiplication and counit

$$\Delta_{\Sigma^* \otimes_B \Sigma}(u^* \otimes_B u) = \sum_i u^* \otimes_B e_i \otimes_A e_i^* \otimes_B u, \quad \varepsilon_{\Sigma^* \otimes_B \Sigma}(u^* \otimes_B u) = u^*(u).$$

The map $\Delta_{\Sigma^* \otimes_B \Sigma}$ is independent of the choice of the right dual basis of Σ_A, see [6, Remark 2.2]. This coring is known as *the comatrix coring* associated to the (B,A)-bimodule Σ.

REMARK 1.1. One can define a comatrix coring using a bimodule which is a finitely generated and projective left module. However, the resulting coring is isomorphic to the comatrix coring defined by the left dual module. To see this, consider any bimodule ${}_A\Lambda_B$ such that ${}_A\Lambda$ is a finitely generated and projective module with a fixed left dual basis $\{f_j, {}^*f_j\}_j$. Put ${}_B\Sigma_A = {}_B{}^*\Lambda_A$, the set $\{{}^*f_j, f_j^*\}_j$ where $f_j^* \in \Sigma^*$ are defined by $f_j^*(u) = u(f_j)$, for all $u \in \Sigma$ and j; form a right dual basis for Σ_A. The isomorphism of A-corings is given by

$$\begin{aligned} \Sigma^* \otimes_B \Sigma &\xrightarrow{\cong} \Lambda \otimes_B {}^*\Lambda \\ u^* \otimes_B {}^*v &\longmapsto \left(\textstyle\sum_j u^*({}^*f_j) f_j\right) \otimes_B {}^*v \end{aligned}$$

The proof is direct, using the above dual bases, and we leave it to the reader.

Keeping the notations before the Remark 1.1, we have that the right (respectively left) A-module Σ (respectively Σ^*) is a right (respectively left) $\Sigma^* \otimes_B \Sigma$-comodule with right A-linear (respectively left A-linear) maps:

$$\rho_\Sigma : \Sigma \longrightarrow \Sigma \otimes_A \Sigma^* \otimes_B \Sigma, \quad \left(u \mapsto \sum_i e_i \otimes_A e_i^* \otimes_B u\right),$$

for every $u \in \Sigma$, and

$$\lambda_{\Sigma^*} : \Sigma^* \longrightarrow \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^*, \quad \left(u^* \mapsto \sum_i u^* \otimes_B e_i \otimes_A e_i^*\right),$$

for every $u^* \in \Sigma^*$. Since $\sum_i be_i \otimes_A e_i^* = \sum_i e_i \otimes_A e_i^* b$ for every $b \in B$, we get that ρ_Σ is left B-linear and λ_{Σ^*} is right B-linear. Furthermore, the natural right A-linear isomorphism $\Sigma \cong {}^*(\Sigma^*)$ turns out to be a right $\Sigma^* \otimes_B \Sigma$-colinear isomorphism. Associated to the ring extension $\lambda : B \to S$, we consider also the canonical Sweedler S-coring $S \otimes_B S$ whose comultiplication is given by $\Delta_{S \otimes_B S}(s \otimes_B s') = s \otimes_B 1 \otimes_S 1 \otimes_B s'$, $s, s' \in S$, and the counit is the usual multiplication.

The aim of this section is to establish an adjunction between the category of right $\Sigma^* \otimes_B \Sigma$-comodules and the category of right $S \otimes_B S$-comodules. Recall first that this last category is isomorphic to the category of descent data associated to the extension $B \to S$, (cf. [13], [1]). This isomorphism of categories will be implicitly used in the sequel. For every right S-module Y and every left S-module Z, we denote by $\iota_Z : Z \to S \otimes_S Z$, and $\iota'_Y : Y \to Y \otimes_S S$ the obvious natural S-linear isomorphisms.

The functor $- \otimes_S \Sigma : \mathcal{M}^{S \otimes_B S} \to \mathcal{M}^{\Sigma^* \otimes_B \Sigma}$.

Let $(Y, \rho_Y) \in \mathcal{M}^{S \otimes_B S}$, and consider the following right S-linear map

$$Y \xrightarrow{\rho_Y} Y \otimes_S S \otimes_B S \xrightarrow{Y \otimes_S \xi^{-1} \otimes_B S} Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \tag{3}$$

where ξ is the S-bilinear map given in (1). Applying $- \otimes_S \Sigma$ to (3), we get

$$\begin{tikzcd}[column sep=large] Y \otimes_S \Sigma \arrow[r, "\rho_Y \otimes_S \Sigma"] \arrow[drr, dashed, "\rho_{Y \otimes_S \Sigma}"'] & Y \otimes_S S \otimes_B S \otimes_S \Sigma \arrow[r, "Y \otimes_S \xi^{-1} \otimes_B S \otimes_S \Sigma"] & Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma \arrow[d, "Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \iota^{-1}"] \\ & & Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma, \end{tikzcd} \tag{4}$$

explicitly,

$$\rho_{Y \otimes_S \Sigma}(y \otimes_S u) = \sum_{i,(y)} y_{(0)} \otimes_S e_i \otimes_A e_i^* \otimes_B y_{(1)} u,$$

where $\rho_Y(y) = \sum_{(y)} y_{(0)} \otimes_S 1 \otimes_B y_{(1)}$. It is clear that $\rho_{Y \otimes_S \Sigma}$ is a right A-linear map and satisfies the counitary property. To check the coassociativity, first consider the diagram

(5)
$$\begin{tikzcd}
Y \otimes_S \Sigma \arrow[r, "\rho_Y \otimes_S \Sigma"] \arrow[d, "\rho_Y \otimes_S \Sigma"'] & Y \otimes_S S \otimes_B S \otimes_S \Sigma \arrow[d, "Y \otimes_S \Delta \otimes_S \Sigma"] \\
Y \otimes_S S \otimes_B S \otimes_S \Sigma \arrow[r, "\rho_Y \otimes_S S \otimes_B S \otimes_S \Sigma"] \arrow[d, "Y \otimes_S \xi^{-1} \otimes_B S \otimes_S \Sigma"'] & Y \otimes_S S \otimes_B S \otimes_S S \otimes_B S \otimes_S \Sigma \arrow[d, "Y \otimes_S S \otimes_B S \otimes_S \xi^{-1} \otimes_B S \otimes_S \Sigma"] \\
Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma \arrow[r, "\rho_Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma"] \arrow[d, "Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \iota^{-1}"'] & Y \otimes_S S \otimes_B S \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma \arrow[d, "Y \otimes_S S \otimes_B S \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \iota^{-1}"] \\
Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma \arrow[r, "\rho_Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma"] & Y \otimes_S S \otimes_B S \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma
\end{tikzcd}$$

It is commutative because ρ_Y is a coaction for the right $S \otimes_B S$-comodule Y. Now, look at the following diagram

(6)
$$\begin{tikzcd}
Y \otimes_S S \otimes_B S \otimes_S \Sigma \arrow[r, "Y \otimes_S \xi^{-1} \otimes_B S \otimes_S \Sigma"] \arrow[d, "Y \otimes_S \Delta \otimes_S \Sigma"'] & Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma \arrow[d, "Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \iota^{-1}"] \\
Y \otimes_S S \otimes_B S \otimes_S S \otimes_B S \otimes_S \Sigma \arrow[d, "Y \otimes_S S \otimes_B S \otimes_S \xi^{-1} \otimes_B S \otimes_S \Sigma"'] & Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma \arrow[d, "Y \otimes_S \Sigma \otimes_A \Delta"] \\
Y \otimes_S S \otimes_B S \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma \arrow[d, "Y \otimes_S S \otimes_B S \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \iota^{-1}"'] & Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \otimes_B \Sigma^* \\
Y \otimes_S S \otimes_B S \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma \arrow[r, "Y \otimes_S \xi^{-1} \otimes_B S \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma"] & Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma \arrow[u, "Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \iota^{-1} \otimes_A \Sigma^* \otimes_B \Sigma"']
\end{tikzcd}$$

which is easily shown to be commutative. By concatenating diagrams (5) and (6) we see that the map $\rho_{Y \otimes_S \Sigma}$ endows $Y \otimes_S \Sigma$ with a structure of right $\Sigma^* \otimes_B \Sigma$-comodule.

Now, let $f : Y \to Y'$ be a morphism in $\mathcal{M}^{S \otimes_B S}$, and consider the right A-linear map $f \otimes_S \Sigma : Y \otimes_S \Sigma \to Y' \otimes_S \Sigma$. Then we have the following commutative diagram

$$\begin{tikzcd}
Y \otimes_S \Sigma \arrow[dr, "\rho_Y \otimes_S \Sigma"] \arrow[dd, "f \otimes_S \Sigma"'] & & Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma \arrow[dr, "Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \iota^{-1}"] \arrow[dd, "f \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma"] & \\
& Y \otimes_S S \otimes_B S \otimes_S \Sigma \arrow[ur, "Y \otimes_S \xi^{-1} \otimes_B S \otimes_S \Sigma"] \arrow[dd, "f \otimes_S S \otimes_B S \otimes_S \Sigma"] & & Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma \arrow[dd, "f \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma"] \\
Y' \otimes_S \Sigma \arrow[dr, "\rho_{Y'} \otimes_S \Sigma"] & & Y' \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma \arrow[dr, "Y' \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \iota^{-1}"] & \\
& Y' \otimes_S S \otimes_B S \otimes_S \Sigma \arrow[ur, "Y' \otimes_S \xi^{-1} \otimes_B S \otimes_S \Sigma"] & & Y' \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma,
\end{tikzcd}$$

which means that $f \otimes_S \Sigma$ is a morphism in $\mathcal{M}^{\Sigma^* \otimes_B \Sigma}$, with the coaction (4). Therefore, we have constructed a well defined functor $- \otimes_S \Sigma : \mathcal{M}^{S \otimes_B S} \to \mathcal{M}^{\Sigma^* \otimes_B \Sigma}$.

The functor $- \otimes_A \boldsymbol{\Sigma}^* : \mathcal{M}^{\boldsymbol{\Sigma}^* \otimes_B \boldsymbol{\Sigma}} \to \mathcal{M}^{S \otimes_B S}$.

Let $(X, \rho_X) \in \mathcal{M}^{\Sigma^* \otimes_B \Sigma}$, and consider the right S-linear map

$$\begin{tikzcd}
X \otimes_A \Sigma^* \arrow[r, "\rho_X \otimes_A \Sigma^*"] \arrow[drr, dashed, "\rho_{X \otimes_A \Sigma^*}"'] & X \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \arrow[r, "X \otimes_A \Sigma^* \otimes_B \xi"] & X \otimes_A \Sigma^* \otimes_B S \arrow[d, "X \otimes_A \iota' \otimes_B S"] \\
& & X \otimes_A \Sigma^* \otimes_S S \otimes_B S.
\end{tikzcd} \tag{7}$$

Direct verifications, using elements, and the coassociativity of ρ_X, give a commutative diagram:

$$\begin{tikzcd}
X \otimes_A \Sigma^* \arrow[dr, "\rho_X \otimes_A \Sigma^*"] \arrow[dd, "\rho_X \otimes_A \Sigma^*"'] & & X \otimes_A \Sigma^* \otimes_B S \arrow[dddd, "X \otimes_A \Sigma^* \otimes_B \mu^r"] \\
& X \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \arrow[ur, "X \otimes_A \Sigma^* \otimes_B \xi"] \arrow[dd, "X \otimes_A \Delta \otimes_A \Sigma^*"] & \\
X \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \arrow[dr, "\rho_X \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^*"] \arrow[dd, "X \otimes_A \Sigma^* \otimes_B \xi"'] & & \\
& X \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \arrow[dd, "X \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \otimes_B \xi"] & \\
X \otimes_A \Sigma^* \otimes_B S \arrow[dr, "\rho_X \otimes_A \Sigma^* \otimes_B S"] & & X \otimes_A \Sigma^* \otimes_B S \otimes_B S \\
& X \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \otimes_B S, \arrow[ur, "X \otimes_A \Sigma^* \otimes_B \xi \otimes_B S"] &
\end{tikzcd}$$

where μ^r is the $(B-S)$-bilinear map defined by $\mu^r(s) = 1 \otimes_B s$, for all $s \in S$. That is, the right S-linear map $f := (X \otimes_A \Sigma^* \otimes_B \xi) \circ (\rho_X \otimes_A \Sigma^*)$ verify the cocycle condition (see [13, Definition 3.5(2)]). Since $\rho_{X \otimes_A \Sigma^*}$ satisfies the counitary property, f is actually a descent datum on $X \otimes_A \Sigma^*$ (see [5], [13]). Henceforth, $\rho_{X \otimes_A \Sigma^*} = (X \otimes_A \iota' \otimes_B S) \circ f$ is a right $S \otimes_B S$-coaction on $X \otimes_A \Sigma^*$.

Given any right $\Sigma^* \otimes_B \Sigma$-colinear map $g : X \to X'$, we easily get a right $S \otimes_B S$-colinear map $g \otimes_A \Sigma^* : X \otimes_A \Sigma^* \to X' \otimes_A \Sigma^*$, with the coactions (7). Therefore, $- \otimes_A \Sigma^* : \mathcal{M}^{\Sigma^* \otimes_B \Sigma} \to \mathcal{M}^{S \otimes_B S}$ is a well defined functor.

The precedent discussion serves to state the following proposition.

Proposition 1.2. *For every pair of comodules* $((Y_{S \otimes_B S}, \rho_Y); (X_{\Sigma^* \otimes_B \Sigma}, \rho_X))$, *the following K-linear map*

$$\begin{aligned} \Psi_{Y,X} : \mathrm{Hom}_{\Sigma^* \otimes_B \Sigma}(Y \otimes_S \Sigma, X) &\longrightarrow \mathrm{Hom}_{S \otimes_B S}(Y, X \otimes_A \Sigma^*) \\ f &\longmapsto (f \otimes_A \Sigma^*) \circ (Y \otimes_S \xi^{-1}) \circ \iota'_Y \\ (X \otimes_A \varepsilon') \circ (g \otimes_S \Sigma) &\longleftarrow\!\shortmid g \end{aligned}$$

(where ε' is the counit of the comatrix S-coring $\Sigma^* \otimes_S \Sigma$), is a natural isomorphism. In other words, $- \otimes_S \Sigma$ is left adjoint to $- \otimes_A \Sigma^*$.

Proof. We only prove that $\Psi_{Y,X}$ and its inverse are well defined maps, the rest is straightforward. Clearly $\Psi_{Y,X}(f)$ is S-linear, for every $f \in \mathrm{Hom}_{\Sigma^* \otimes_B \Sigma}(Y \otimes_S \Sigma, X)$. The colinearity of $\Psi_{Y,X}(f)$ follows if we show that

(8)
$$\begin{tikzcd}[column sep=huge, row sep=huge] Y \arrow[r, "\Psi(f)"] \arrow[d, "\rho'_Y"'] & X \otimes_A \Sigma^* \arrow[d, "(X \otimes_A \Sigma^* \otimes_B \xi) \circ (\rho_X \otimes_A \Sigma^*)"] \\ Y \otimes_B S \arrow[r, "\Psi(f) \otimes_B S"] & X \otimes_A \Sigma^* \otimes_B S \end{tikzcd}$$

is a commutative diagram, where $\rho'_Y = (\iota_Y^{-1} \otimes_B S) \circ \rho_Y$. Put

$$\boldsymbol{f} = \Psi_{Y,X}(f) \circ \rho'_Y = (f \otimes_A \Sigma^* \otimes_B \xi) \circ (Y \otimes_S \xi^{-1} \otimes_B S) \circ \rho_Y.$$

Using that the map $\boldsymbol{f}$ is colinear, we easily prove that the following diagram is commutative

$$\begin{tikzcd}
Y \arrow[r, "\iota'_Y"] \arrow[dd, "\rho_Y"] \arrow[dddd, bend right, "f"'] & Y \otimes_S S \arrow[r, "Y \otimes_S \xi^{-1}"] & Y \otimes_S \Sigma \otimes_A \Sigma^* \arrow[d, "\rho_{Y \otimes_S \Sigma \otimes_A \Sigma^*}"] \arrow[dddd, bend left, "(\rho_X \circ f) \otimes_A \Sigma^*"] \\
 & & Y \otimes_S S \otimes_B S \otimes_S \Sigma \otimes_A \Sigma^* \arrow[d, "Y \otimes_S \xi^{-1} \otimes_B S \otimes_S \Sigma \otimes_A \Sigma^*"] \\
Y \otimes_S S \otimes_B S \arrow[d, "Y \otimes_S \xi^{-1} \otimes_B S"] & & Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma \otimes_A \Sigma^* \arrow[d, "Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \iota^{-1} \otimes_A \Sigma^*"] \\
Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \arrow[rr, "Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \xi^{-1}"] \arrow[d, "f \otimes_A \Sigma^* \otimes_B S"] & & Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \arrow[d, "f \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^*"] \\
X \otimes_A \Sigma^* \otimes_B S & & X \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \arrow[ll, "X \otimes_A \Sigma^* \otimes_B \xi"']
\end{tikzcd}$$

which is exactly the diagram (8). Now, let $g \in \mathrm{Hom}_{S \otimes_B S}(Y, X \otimes_A \Sigma^*)$, so the following diagram is easily shown to be commutative

$$\begin{tikzcd}
Y \otimes_S \Sigma \arrow[rr, "g \otimes_S \Sigma"] \arrow[ddd, "\rho_Y \otimes_S \Sigma"] & & X \otimes_A \Sigma^* \otimes_S \Sigma \arrow[d, "\rho_X \otimes_A \Sigma^* \otimes_S \Sigma"] \\
 & & X \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \otimes_S \Sigma \arrow[d, "X \otimes_A \Sigma^* \otimes_B \xi \otimes_S \Sigma"] \\
 & & X \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma \arrow[d, "X \otimes_A \Sigma^* \otimes_B \iota_\Sigma^{-1}"] \\
Y \otimes_S S \otimes_B S \otimes_S \Sigma \arrow[dr, "\iota_\Sigma^{-1} \otimes_B S \otimes_S \Sigma"] \arrow[dd, "Y \otimes_S \xi^{-1} \otimes_B S \otimes_S \Sigma"] & & X \otimes_A \Sigma^* \otimes_B \Sigma \\
 & Y \otimes_B S \otimes_S \Sigma \arrow[uur, "g \otimes_B S \otimes_S \Sigma"] & \\
Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B S \otimes_S \Sigma \arrow[d, "Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \iota_\Sigma^{-1}"] & & X \otimes_A A \otimes_A \Sigma^* \otimes_B \Sigma \arrow[uu, "\cong"] \\
Y \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma \arrow[rr, "g \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma"] & & X \otimes_A \Sigma^* \otimes_S \Sigma \otimes_A \Sigma^* \otimes_B \Sigma \arrow[u, "X \otimes_A \varepsilon' \otimes_A \Sigma^* \otimes_B \Sigma"']
\end{tikzcd}$$

On the other hand, we have

$$\rho_X \circ (X \otimes_A \varepsilon') = (X \otimes_A \Sigma^* \otimes_B \iota^{-1}) \circ (X \otimes_A \Sigma^* \otimes_B \xi \otimes_S \Sigma) \circ (\rho_X \otimes_A \Sigma^* \otimes_S \Sigma),$$

putting this in the above diagram, we get that $(X \otimes_A \varepsilon') \circ (g \otimes_S \Sigma)$ is $\Sigma^* \otimes_B \Sigma$-colinear; and this finishes the proof. $\square$

REMARK 1.3. 1) Applying Proposition 1.2, we get (up to natural isomorphisms) the following commutative diagram of functors

(9)
$$\begin{tikzcd}
 & \mathcal{M}^{S\otimes_B S} \arrow[dl, shift left, "-\otimes_S \Sigma"] \arrow[dr, shift left, "\mathrm{Hom}_{S\otimes_B S}(S,-)"] & \\
\mathcal{M}^{\Sigma^*\otimes_B \Sigma} \arrow[ur, shift left, "-\otimes_A \Sigma^*"] \arrow[rr, shift left, "\mathrm{Hom}_{\Sigma^*\otimes_B\Sigma}(\Sigma,-)"] & & \mathcal{M}_B, \arrow[ul, shift left, "-\otimes_B S"] \arrow[ll, shift left, "-\otimes_B \Sigma"]
\end{tikzcd}$$

where the sideways pairs represent adjunctions.

2) Symmetrically, one can define a pair of adjoint functors relating the categories of left comodules: $\Sigma^* \otimes_S - : {}^{S\otimes_B S}\mathcal{M} \leftrightarrows {}^{\Sigma^*\otimes_B\Sigma}\mathcal{M} : \Sigma \otimes_A -$, which turns the diagram

(10)
$$\begin{tikzcd}
 & {}^{S\otimes_B S}\mathcal{M} \arrow[dl, shift left, "\Sigma^*\otimes_S -"] \arrow[dr, shift left, "\mathrm{Hom}_{S\otimes_B S}(S,-)"] & \\
{}^{\Sigma^*\otimes_B \Sigma}\mathcal{M} \arrow[ur, shift left, "\Sigma\otimes_A -"] \arrow[rr, shift left, "\mathrm{Hom}_{\Sigma^*\otimes_B\Sigma}(\Sigma^*,-)"] & & {}_B\mathcal{M}, \arrow[ul, shift left, "S\otimes_B -"] \arrow[ll, shift left, "\Sigma^*\otimes_B -"]
\end{tikzcd}$$

commutative.

2. – A group isomorphism.

Let $B \subset S$ be ring extension. The set $\mathbf{I}_B(S)$ of all B-sub-bimodules of S is a monoid with the obvious product. For $I, J \in \mathbf{I}_B(S)$, consider the multiplication map:

$$\boldsymbol{m} : I \otimes_B J \to IJ, \quad \boldsymbol{m}(x \otimes_B y) = xy.$$

$\mathbf{I}^l_B(S)$ (respectively $\mathbf{I}^r_B(S)$) denotes the submonoid consisting of all B-sub-bimodules $I \subset S$ such that

$$S \otimes_B I \cong S \quad (\text{respectively } I \otimes_B S \cong S) \quad \text{through } \boldsymbol{m}.$$

$\mathbf{Inv}_B(S)$ denote the group of invertible B-sub-bimodules of S. By [11, Proposition 1.1], $\mathbf{Inv}_B(S) \subset \mathbf{I}^l_B(S) \cap \mathbf{I}^r_B(S)$.

From now on fix a bimodule ${}_B\Sigma_A$ with Σ_A finitely generated and projective, consider the endomorphism ring $S = \mathrm{End}(\Sigma_A)$, and assume that ${}_B\Sigma$ is faithful, i.e., the canonical ring extension $\lambda : B \to S$ is injective (B will be identified then with its image). Consider the comatrix A-coring $\mathfrak{C} := \Sigma^* \otimes_B \Sigma$, and denote by $\mathrm{End}_{A-cor}(\mathfrak{C})$ the monoid of the coring endomorphisms of $\mathfrak{C}$. We denote by $\mathrm{Aut}_{A-cor}(\mathfrak{C})$ its group of units, that is, the group of all coring automorphisms of $\mathfrak{C}$. The canonical Sweedler S-coring $S \otimes_B S$ associated to the ring extension $B \subset S$, will be also considered.

REMARK 2.1. Keeping the previous notations, we make the following remarks.

1) As we have seen, the (B,A)-bimodule Σ is actually a $(B,\mathfrak{C})$-bicomodule (B is considered as a trivial B-coring), while Σ^* becomes a $(\mathfrak{C},B)$-bicomodule. Given $g \in \mathrm{End}_{A-cor}(\mathfrak{C})$, and a right comodule $X_{\mathfrak{C}}$ (respectively left comodule ${}_{\mathfrak{C}}X$), we denote by X_g the associated induced right (respectively left) $\mathfrak{C}$-comodule. That is, $\rho_{X_g} = (X \otimes_A g) \circ \rho_X$ (respectively $\lambda_{X_g} = (g \otimes_A X) \circ \lambda_X$). If (X, ρ_X) is any right $\mathfrak{C}$-comodule such that X_A is finitely generated and projective module, then it is well known that the right dual module X^* admits a structure of left $\mathfrak{C}$-comodule with coaction

$$\lambda_{X^*}(x^*) = \sum \big((x^* \otimes_A \mathfrak{C}) \circ \rho_X(x_j)\big) \otimes_A x_j^*,\ x^* \in X^*,$$

where $\{x_j, x_j^*\}_j$ is any right dual basis of X_A. In this way $(\Sigma_g)^*$ and $(\Sigma^*)_g$ have the same left $\mathfrak{C}$-coaction, that is, they are equal as a left $\mathfrak{C}$-comodules, then we can remove the brackets $\Sigma_g^* = (\Sigma_g)^* = (\Sigma^*)_g$.

2) Given $g, h \in \mathrm{End}_{A-cor}(\mathfrak{C})$, the B-subbimodule $\Sigma_h \square_{\mathfrak{C}} \Sigma_g^*$ of $\Sigma \otimes_A \Sigma^*$ is identified, via the isomorphism given in (1), with $\mathrm{Hom}_{\mathfrak{C}}(\Sigma_g, \Sigma_h)$. Another way to obtain this identification is given as follows. Recall, from [7, Example 3.4] or [6, Example 6], that $(\Sigma_g^*)_B$ is a quasi-finite $(\mathfrak{C}, B)$-bicomodule with adjunction $- \otimes_B \Sigma_g \dashv -\square_{\mathfrak{C}} \Sigma_g^*$, so the cotensor functor $-\square_{\mathfrak{C}} \Sigma_g^*$ is naturally isomorphic to the hom-functor $\mathrm{Hom}_{\mathfrak{C}}(\Sigma_g, -)$. Moreover, this isomorphism can be chosen to be just the restriction of $- \otimes_A \Sigma_g^* \cong \mathrm{Hom}_A(\Sigma_g, -)$. Applying this isomorphism to Σ_h, for any $h \in \mathrm{End}_{A-cor}(\mathfrak{C})$, we arrive to the desired identification.

3) Let $g \in \mathrm{End}_{A-cor}(\mathfrak{C})$. The following multiplication

$$\overline{m} : \Sigma^* \otimes_B \mathrm{Hom}_{\mathfrak{C}}(\Sigma_g, \Sigma) \longrightarrow \Sigma_g^* \quad (u^* \otimes_B t \mapsto u^* t)$$

is a left $\mathfrak{C}$-comodule map. Furthermore, we have a commutative diagram

$$\begin{tikzcd}
\Sigma \otimes_A \Sigma^* \otimes_B \mathrm{Hom}_{\Sigma^* \otimes_B \Sigma}(\Sigma_g, \Sigma) \arrow[r, "\Sigma \otimes_A \overline{\mathbf{m}}"] \arrow[d, "\xi \otimes_A \mathrm{Hom}_{\Sigma^* \otimes_B \Sigma}(\Sigma_g, \Sigma)"'] & \Sigma \otimes_A \Sigma_g^* \arrow[d, "\xi"] \\
S \otimes_B \mathrm{Hom}_{\Sigma^* \otimes_B \Sigma}(\Sigma_g, \Sigma) \arrow[r, "\mathbf{m}"] & S,
\end{tikzcd}$$

where m is the usual multiplication of S.

We define the following two maps:

$$\mathsf{F}^r : \mathrm{End}_{A-cor}(\mathfrak{C}) \longrightarrow \mathbf{I}_B(S) \quad (g \longmapsto \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma_g)),$$

and

$$\mathsf{F}^l : \mathrm{End}_{A-cor}(\mathfrak{C}) \longrightarrow \mathbf{I}_B(S) \quad (g \longmapsto \mathrm{Hom}_{\mathfrak{C}}(\Sigma_g, \Sigma)).$$

These maps obey the following lemma. First, recall from [14] (cf. [4]), that a (B,A)-bimodule M is called a *separable bimodule* or B is said to be *M-separable over A* provided the evaluation map

$$M \otimes_A {}^*M \to B, \quad m \otimes_A \varphi \mapsto \varphi(m)$$

is a split epimorphism of (B,B)-bimodules. As shown in [14] (cf. [10, Theorem 3.1]), if M is a separable (B,A)-bimodule and $S := \mathrm{End}(M_A)$, then $B \to S$ is a *split extension*, i.e., there is a B-linear map $a : S \to B$ such that $a(1_S) = 1_B$. Conversely, if ${}_BM_A$ is such that M_A is finitely generated and projective module, and $B \to S$ is a splits extension, then ${}_BM_A$ is a separable bimodule.

LEMMA 2.2. *Let $g \in \mathrm{End}_{A-cor}(\mathfrak{C})$. Then*

(i) $\mathsf{F}^r(g) \in \mathbf{I}^r_B(S)$ if and only if ${}_BS$ preserves the equalizer of $(\rho_{\Sigma_g} \otimes_A \Sigma^*, \Sigma_g \otimes_A \lambda_{\Sigma^*})$ (cf. [7, Section 2.4]). *In particular, if either ${}_B\Sigma$ is flat module or ${}_B\Sigma_A$ is a separable bimodule, then $\mathsf{F}^r(g) \in \mathbf{I}^r_B(S)$.*

(ii) $\mathsf{F}^l(g) \in \mathbf{I}^l_B(S)$ if and only if S_B preserves the equalizer of $(\rho_\Sigma \otimes_A \Sigma^*_g, \Sigma \otimes_A \lambda_{\Sigma^*_g})$. In particular, if either Σ^*_B is flat module or ${}_B\Sigma_A$ is a separable bimodule, then $\mathsf{F}^l(g) \in \mathbf{I}^l_B(S)$.

(iii) If $g \in \mathrm{Aut}_{A-cor}(\mathfrak{C})$, then $\mathsf{F}^l(g) = \mathsf{F}^r(g^{-1})$.

PROOF. *(i)* and *(ii)* We only prove *(i)* because *(ii)* is symmetric. Following the identifications made in Remark 2.1, we have $\mathsf{F}^r(g) \cong \Sigma_g \square_{\mathfrak{C}} \Sigma^*$. Taking this isomorphism into account, the first statement in *(i)* is reduced to the problem of compatibility between tensor and cotensor products. Effectively, by [7, Lemma 2.2], ${}_BS \cong \Sigma \otimes_A \Sigma^*$ preserves the equalizer of $(\rho_{\Sigma_g} \otimes_A \Sigma^*, \Sigma_g \otimes_A \lambda_{\Sigma^*})$ if and only if

$$(\Sigma_g \square_{\mathfrak{C}} \Sigma^*) \otimes_B \Sigma \otimes_A \Sigma^* \cong \Sigma_g \square_{\mathfrak{C}} (\Sigma^* \otimes_B \Sigma \otimes_A \Sigma^*) = (\Sigma_g \square_{\mathfrak{C}} \mathfrak{C}) \otimes_A \Sigma^* \cong \Sigma \otimes_A \Sigma^* \cong S,$$

if and only if $(\Sigma_g \square_{\mathfrak{C}} \Sigma^*) \in \mathbf{I}^r_B(S)$, since by Remark 2.1 (3) this composition coincides with the multiplication of the monoid $\mathbf{I}_B(S)$. If ${}_B\Sigma$ is a flat module, then clearly ${}_BS$ is also flat. Hence, it preserves the stated equalizer. Now, if we assume that ${}_B\Sigma_A$ is a separable bimodule, then [2, Theorem 3.5] implies that $\mathfrak{C} = \Sigma^* \otimes_B \Sigma$ is a coseparable A-coring (cf. [9], [8] for definition). Therefore, equalizers split by [9, Proposition 1.2], and so they are preserved by any module.
(iii) A straightforward computation shows that $\mathrm{Hom}_{\mathfrak{C}}(\Sigma_g, \Sigma) = \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \Sigma_{g^{-1}})$.
□

THEOREM 2.3. *Let ${}_B\Sigma_A$ be a bimodule such that ${}_B\Sigma$ is faithful and Σ_A is finitely generated and projective. Consider $\mathfrak{C} = \Sigma^* \otimes_B \Sigma$ its associated comatrix A-coring. If either*

(*a*) *Σ^*_B is a faithfully flat module, or*
(*b*) *${}_B\Sigma_A$ is a separable bimodule,*

then $\mathsf{F}^l : \mathrm{End}_{A-cor}(\mathfrak{C}) \to \mathbf{I}^l_B(S)$ is a monoid isomorphism with inverse

$$\Gamma^l : \mathbf{I}^l_B(S) \longrightarrow \mathrm{End}_{A-cor}(\mathfrak{C}) \qquad (11)$$
$$I \longmapsto [u^* \otimes_B u \mapsto \textstyle\sum_k u^* s_k \otimes_B x_k u],$$

where $\boldsymbol{m}^{-1}(1) = \sum_k s_k \otimes_B x_k \in S \otimes_B I$.

PROOF. Under the hypothesis (*a*), we have, by the left version of the generalized Descent Theorem for modules [6, Theorem 2], that $\Sigma^* \otimes_B - : {}_B\mathcal{M} \to {}^{\Sigma^* \otimes_B \Sigma}\mathcal{M}$ is an equivalence of categories with inverse $\mathrm{Hom}_{\Sigma^* \otimes_B \Sigma}(\Sigma^*, -)$. Applying the diagram (10) of Remark 1.3, we obtain that $S \otimes_B - : {}_B\mathcal{M} \to {}^{S \otimes_B S}\mathcal{M}$ is a separable functor (cf. [12] for definition). Now, assume (*b*), then the ring extension $B \to S$ splits as a B-bimodule map. By [12, Proposition 1.3], the functor $S \otimes_B - : {}_B\mathcal{M} \to {}_S\mathcal{M}$ is separable, and by [12, Lemma 1.1(3)], the functor $S \otimes_B - : {}_B\mathcal{M} \to {}^{S \otimes_B S}\mathcal{M}$ is separable. In conclusion, under the hypothesis (*a*) or (*b*), the functor $S \otimes_B - : {}_B\mathcal{M} \to {}^{S \otimes_B S}\mathcal{M}$ reflects isomorphisms. Therefore, any inclusion $I \subseteq J$ in $\mathbf{I}^l_B(S)$, implies equality $I = J$. This fact will be used implicitly in the remainder of the proof.

The map Γ^l is easily shown to be well defined, while Lemma 2.2 implies that F^l is also well defined. Let us first show that F^l is a monoid map. The image of the unit is mapped to B, $\mathsf{F}^l(1_{\mathrm{End}_{A-cor}(\mathfrak{C})}) = \mathrm{End}_{\mathfrak{C}}(\Sigma) = B$, since by [6, Proposition 2] the inclusion $B \subseteq \mathrm{End}(\Sigma_{\Sigma^* \otimes_B \Sigma})$ is always true. Let $g, h \in \mathrm{End}_{A-cor}(\mathfrak{C})$, and $t \in \mathsf{F}^l(g)$, $s \in \mathsf{F}^l(h)$, that is

$$\sum_i e_i \otimes_A e_i^* \otimes_B tu = \sum_i te_i \otimes_A g(e_i^* \otimes_B u)$$

$$\sum_i e_i \otimes_A e_i^* \otimes_B su = \sum_i se_i \otimes_A h(e_i^* \otimes_B u)$$

for every element $u \in \Sigma$. So, for every $u \in \Sigma$, we have

$$\begin{aligned}
\rho_\Sigma(tsu) &= \sum_i e_i \otimes_A e_i^* \otimes_B tsu \\
&= \sum_i te_i \otimes_A g(e_i^* \otimes_B su) \\
&= (t \otimes_A \mathfrak{C}) \circ (\Sigma \otimes_A g)\left(\sum_i e_i \otimes_A e_i^* \otimes_B su \right)
\end{aligned}$$

$$= (t \otimes_A \mathfrak{C}) \circ (\Sigma \otimes_A g)\left(\sum_i se_i \otimes_A h(e_i^* \otimes_B u)\right)$$

$$= \sum_i tse_i \otimes_A gh(e_i^* \otimes_B u)$$

$$= (ts \otimes_A \mathfrak{C}) \circ \rho_{\Sigma_{gh}}(u)$$

which means that $ts \in \mathrm{Hom}_{\mathfrak{C}}(\Sigma_{gh}, \Sigma) = F^l(gh)$, and so $F^l(g)F^l(h) = F^l(gh)$. Now, let $I \in \mathbf{I}_B^l(S)$ with $\boldsymbol{m}^{-1}(1) = \sum_k s_k \otimes_B t_k \in S \otimes_B I$. If s is any element in I, then $1 \otimes_B s = \sum_k ss_k \otimes_B t_k \in S \otimes_B I$. Henceforth,

$$(s \otimes_A \mathfrak{C}) \circ \rho_{\Sigma_{\Gamma^l(I)}}(u) = (s \otimes_A \mathfrak{C})\left(\sum_i e_i \otimes_A \Gamma^l(I)(e_i^* \otimes_B u)\right)$$

$$= \sum_{i,k} se_i \otimes_A e_i^* s_k \otimes_B t_k u$$

$$= \sum_{i,k} e_i \otimes_A e_i^* ss_k \otimes_B t_k u$$

$$= \sum_i e_i \otimes_A e_i^* \otimes_B su = \rho_\Sigma(su)$$

for every $u \in \Sigma$, that is $s : \Sigma_{\Gamma^l(I)} \to \Sigma \in I$ is a $\mathfrak{C}$-colinear map. Therefore, $I = F^l(\Gamma^l(I))$, for every $I \in \mathbf{I}_B^l(S)$. Conversely, let $g \in \mathrm{End}_{A-cor}(\Sigma^* \otimes_B \Sigma)$, and put $I := F^l(g) = \mathrm{Hom}_{\mathfrak{C}}(\Sigma_g, \Sigma)$ with $\boldsymbol{m}^{-1}(1) = \sum_k s_k \otimes_B x_k \in S \otimes_B I$. For every $t \in I$, we have

$$\begin{aligned} &\textstyle\sum_i g(u^* t \otimes_B e_i) \otimes_A e_i^* = \sum_i u^* \otimes_B e_i \otimes_A e_i^* t, && \forall u^* \in \Sigma^* \\ &\textstyle\sum_i e_i \otimes_A e_i^* \otimes_B tu = \sum_i te_i \otimes_A g(e_i^* \otimes_B u), && \forall u \in \Sigma. \end{aligned} \tag{12}$$

Computing, using equations (12) we get

$$(\Gamma^l(I) \otimes_A \lambda_{\Sigma^*}(u^*) = \sum_{i,k} u^* s_k \otimes_B t_k e_i \otimes_A e_i^*$$

$$= \sum_k u^* s_k \otimes_B \left(\sum_i t_k e_i \otimes_A e_i^*\right)$$

$$= \sum_k u^* s_k \otimes_B \left(\sum_i e_i \otimes_A e_i^* t_k\right)$$

$$= \sum_k \left(\sum_i u^* s_k \otimes_B e_i \otimes_A e_i^* t_k\right)$$

$$= \sum_k \left(\sum_i g(u^* s_k t_k \otimes_B e_i) \otimes_A e_i^* \right)$$

$$= \sum_i g(u^* \otimes_B e_i) \otimes_A e_i^*$$

$$= (g \otimes_A \Sigma^*) \circ \lambda_{\Sigma^*}(u^*)$$

for every $u^* \in \Sigma^*$, that is $(\Gamma(I) \otimes_A \Sigma^*) \circ \lambda_{\Sigma^*} = (g \otimes_A \Sigma^*) \circ \lambda_{\Sigma^*}$. Whence,

$$(\Gamma(I) \otimes_A \Sigma^* \otimes_B \Sigma) \circ \Delta = (g \otimes_A \Sigma^* \otimes_B \Sigma) \circ \Delta, \tag{13}$$

because $\Delta_{\mathfrak{C}} = \lambda_{\Sigma^*} \otimes_B \Sigma$. On the other hand

$$\begin{aligned}
\Delta \circ \Gamma^l(I)(u^* \otimes_B u) &= \sum_k u^* s_k \otimes_B \left(\sum_i e_i \otimes_A e_i^* \otimes_B t_k u \right) \\
&= \sum_k u^* s_k \otimes_B \left(\sum_i t_k e_i \otimes_A g(e_i^* \otimes_B u) \right), \text{ by (12)} \\
&= \Sigma^* \otimes_B \Sigma \otimes_A g \left(\sum_{i,k} u^* s_k \otimes_B t_k e_i \otimes_A e_i^* \otimes_B u \right) \\
&= (\Sigma^* \otimes_B \Sigma \otimes_A g) \circ (\Gamma^l(I) \otimes_A \Sigma^* \otimes_B \Sigma) \circ \Delta(u^* \otimes_B u) \\
&= (\Sigma^* \otimes_B \Sigma \otimes_A g) \circ (g \otimes_A \Sigma^* \otimes_B \Sigma) \circ \Delta(u^* \otimes_B u), \text{ by (13)} \\
&= (g \otimes_A g) \circ \Delta(u^* \otimes_B u) \\
&= \Delta \circ g(u^* \otimes_B u), \; g \in \mathrm{End}_{A-cor}(\mathfrak{C}),
\end{aligned}$$

for every $u^* \in \Sigma^*$, $u \in \Sigma$. Therefore, $\Delta \circ \Gamma^l(I) = \Delta \circ g$, thus $\Gamma^l(I) = \Gamma^l(\mathsf{F}^l(g)) = g$, for every $g \in \mathrm{End}_{A-cor}(\mathfrak{C})$ since Δ is injective. □

Symmetrically we have the anti-homomorphism of monoids

$$\begin{aligned}
\Gamma^r : \mathbf{I}_B^r(S) &\longrightarrow \mathrm{End}_{A-cor}(\Sigma^* \otimes_B \Sigma) \\
I &\longmapsto [u^* \otimes_B u \mapsto \textstyle\sum_k u^* t_k \otimes_B s_k u],
\end{aligned} \tag{14}$$

where $\boldsymbol{m}^{-1}(1) = \sum_k t_k \otimes_B s_k \in I \otimes_B S$. Let $B^o \subset S^o$ denote the opposite ring extension of $B \subset S$, and identify S^o with $\mathrm{End}((\Sigma^*)^o{}_{A^o})$, where the notation X^o, for any left A-module X, means the opposite right A^o-module. Put ${}_{B^o}W_{A^o} := ({}_A\Sigma_B^*)^o$ the opposite bimodule, and consider its right dual W^*, with respect to A^o, i.e. $W^* = \mathrm{Hom}(W_{A^o}, A^o_{A^o})$. Obviously W_{A^o} is finitely generated and projective module, and we can consider its associated comatrix A^o-coring $W^* \otimes_{B^o} W$. By the

Remark 1.1, there is an A-coring isomorphism

$$(W^* \otimes_{B^o} W)^o \cong \Sigma^* \otimes_B \Sigma, \qquad \left((w^* \otimes_{B^o} w)^o \mapsto \sum_i w \otimes_B e_i w^*((e_i^*)^o)^o \right),$$

where $(W^* \otimes_{B^o} W)^o$ is the opposite A-coring of the A^o-coring $W^* \otimes_{B^o} W$. Therefore, we have an isomorphism of monoids $\mathrm{End}_{A^o-cor}(W^* \otimes_{B^o} W) \cong$ $\cong \mathrm{End}_{A-cor}(\Sigma^* \otimes_B \Sigma)$. Finally, using this last isomorphism together with the equality $\mathbf{I}^r_B(S) = \mathbf{I}^l_{B^o}(S^o)$, we can identify the Γ^r-map of equation (14) with the Γ^l-map (11) associated to the new data: A^o, $B^o \subset S^o$, and ${}_{B^o}W_{A^o}$. Henceforth, Theorem 2.3 yields

THEOREM 2.4. *Let ${}_B\Sigma_A$ be a bimodule such that ${}_B\Sigma$ is faithful and Σ_A is finitely generated and projective. Consider $\mathfrak{C} = \Sigma^* \otimes_B \Sigma$ its associated comatrix A-coring. If either*

(a) ${}_B\Sigma$ is faithfully flat module, or
(b) ${}_B\Sigma_A$ is a separable bimodule,

then $\Gamma^r : \mathrm{End}_{A-cor}(\Sigma^* \otimes_B \Sigma) \to \mathbf{I}^r_B(S)$ is an anti-isomorphism of monoids with inverse map

$$\Gamma^r : \mathbf{I}^r_B(S) \longrightarrow \mathrm{End}_{A-cor}(\Sigma^* \otimes_B \Sigma)$$
$$I \longmapsto [u^* \otimes_B u \mapsto \sum_k u^* t_k \otimes_B s_k u],$$

where $\boldsymbol{m}^{-1}(1) = \sum_k t_k \otimes_B s_k \in I \otimes_B S$.

The isomorphism Γ^l given in (11) gives, by restriction, an isomorphism of groups $\Gamma : \mathbf{Inv}_B(S) \to \mathrm{Aut}_{A-cor}(\Sigma^* \otimes_B \Sigma)$. Analogously, the anti-isomorphism Γ^r defined in (14), gives, by restriction, an anti-isomorphism of groups $\Gamma' : \mathbf{Inv}_B(S) \to \to \mathrm{Aut}_{A-cor}(\Sigma^* \otimes_B \Sigma)$. Moreover, when both Γ^r and Γ^l are bijective, Lemma 2.2. (iii) says that $\Gamma = (-)^{-1} \circ \Gamma'$, where $(-)^{-1}$ denotes the antipode map in the group of automorphisms. We can thus say that, either in the hypotheses of Theorem 2.3 or in the hypotheses of Theorem 2.4, we have an isomorphism of groups $\Gamma : \mathbf{Inv}_B(S) \to \mathrm{Aut}_{A-cor}(\Sigma^* \otimes_B \Sigma)$ defined either as Γ^l or as $(-)^{-1} \circ \Gamma^r$, respectively. We can then state our main theorem as follows:

THEOREM 2.5. *Let ${}_B\Sigma_A$ be a bimodule such that ${}_B\Sigma$ is faithful and Σ_A is finitely generated and projective. Consider $\mathfrak{C} = \Sigma^* \otimes_B \Sigma$ its associated comatrix A-coring. If either*

(a) ${}_B\Sigma$ or Σ^*_B is a faithfully flat module, or
(b) ${}_B\Sigma_A$ is a separable bimodule,

then there is an isomorphism of groups $\Gamma : \mathbf{Inv}_B(S) \to \mathrm{Aut}_{A-cor}(\Sigma^* \otimes_B \Sigma)$.

To finish, we want to compare Masuoka's maps [11, Theorem 2.2(2.3)] with our F-maps, using the adjunction of Section 1.

PROPOSITION 2.6. *Let ${}_B\Sigma_A$ be a bimodule such that ${}_B\Sigma$ is faithful and Σ_A is finitely generated and projective. Let $S = \mathrm{End}(\Sigma_A)$ be its ring of right linear endomorphisms. Then*

(1) *the map*

$$\widehat{(\ -\ )} : \mathrm{End}_{A-cor}(\Sigma^* \otimes_B \Sigma) \longrightarrow \mathrm{End}_{S-cor}(S \otimes_B S)$$

$$g \longmapsto \widehat{g} = (\xi \otimes_B \xi) \circ (\Sigma \otimes_A g \otimes_A \Sigma^*) \circ (\xi^{-1} \otimes_B \xi^{-1})$$

is an injective homomorphism of monoids which turns the following diagram commutative

$$\begin{tikzcd} \mathbf{I}^l_B(S) \arrow[r, "\Gamma^l"] \arrow[d, "\overline{\Gamma}^l"'] & \mathrm{End}_{A-cor}(\Sigma^* \otimes_B \Sigma) \arrow[dl, "\widehat{(-)}"] \\ \mathrm{End}_{S-cor}(S \otimes_B S) & \end{tikzcd}$$

where $\overline{\Gamma}^l$ is the Gamma map associated to the bimodule ${}_BS_S$ and the comatrix S-coring $S \otimes_B S$ (see [11, (2.1)]*);*

2) *for every $g \in \mathrm{End}_{A-cor}(\Sigma^* \otimes_B \Sigma)$, we have*

$$\mathrm{Hom}_{\Sigma^* \otimes_B \Sigma}(\Sigma_g, \Sigma) = \mathrm{Hom}_{S \otimes_B S}(S_{\widehat{g}}, S) = \{s \in S \mid \widehat{g}(s \otimes_B 1) = 1 \otimes_B s\}$$

PROOF. (1) We only show that $\widehat{(\ -\ )}$ is a well defined map, the compatibilities with the multiplication and unit are easy computations. So let $g \in \mathrm{End}_{A-cor}(\Sigma^* \otimes_B \Sigma)$. By definition $\widehat{g}$ is an S-bilinear map, and preserves the counit. Denote by Δ' the comultiplication of $S \otimes_B S$, i.e. $\Delta' : S \otimes_B S \to \to S \otimes_B S \otimes_B S$ sending $s \otimes_B s' \mapsto s \otimes_B 1 \otimes_B s'$, $s, s' \in S$. Then $\widehat{g}$ is coassociative if and only if

$$\Delta' \circ \widehat{g} = (\widehat{g} \otimes_B S) \circ (S \otimes_B \widehat{g}) \circ \Delta'. \tag{15}$$

Now, direct computations give the following equations

$$(\widehat{g} \otimes_B S) \circ (S \otimes_B \widehat{g}) = (\xi \otimes_B \xi \otimes_B \xi) \circ (\Sigma \otimes_A g \otimes_A g \otimes_A \Sigma) \circ (\xi^{-1} \otimes_B \xi^{-1} \otimes_B \xi^{-1}),$$

$$(\Sigma \otimes_A \Delta \otimes_A \Sigma^*) \circ (\xi^{-1} \otimes_B \xi^{-1}) = (\xi^{-1} \otimes_B \xi^{-1} \otimes_B \xi^{-1}) \circ \Delta,$$

$$\Delta' \circ (\xi \otimes_B \xi) = (\xi \otimes_B \xi \otimes_B \xi) \circ (\Sigma \otimes_A \Delta \otimes_A \Sigma^*),$$

which in conjunction with the coassociativity of g imply the equality of equation (15). (2) The second stated equality is a direct consequence of the identification of the B-

bimodule $\mathrm{Hom}_{S\otimes_B S}(S_{\hat{g}}, S)$ with a B-sub-bimodule of S. Now, observe that the canonical right A-linear and a right S-linear isomorphisms $S_{\hat{g}} \otimes_S \Sigma \cong \Sigma_g$ and $S \cong \Sigma \otimes_A \Sigma^*$ are, respectively, a right $\Sigma^* \otimes_B \Sigma$-colinear map and a right $S \otimes_B S$-colinear map, with respect to the coactions defined in equations (4) and (7). Whence,

$$\mathrm{Hom}_{\Sigma^*\otimes_B\Sigma}(\Sigma_g, \Sigma) \cong \mathrm{Hom}_{\Sigma^*\otimes_B\Sigma}(S_{\hat{g}} \otimes_S \Sigma, \Sigma) \cong$$

$$\cong \mathrm{Hom}_{S\otimes_B S}(S_{\hat{g}}, \Sigma \otimes_A \Sigma^*) \cong \mathrm{Hom}_{S\otimes_B S}(S_{\hat{g}}, S),$$

where the second isomorphism is given by Proposition 1.2. The desired first equality is now obtained using the inclusion $\mathrm{Hom}_{\Sigma^*\otimes_B\Sigma}(\Sigma_g, \Sigma) \subseteq$ $\subseteq \mathrm{Hom}_{S\otimes_B S}(S_{\hat{g}}, S) \subset S$ which we show as follows. An element $s \in S$ belongs to $\mathrm{Hom}_{\Sigma^*\otimes_B\Sigma}(\Sigma_g, \Sigma)$ if and only if

$$\sum_i e_i \otimes_A e_i^* \otimes_B su = \sum_i se_i \otimes_A g(e_i^* \otimes_B u), \quad \forall u \in \Sigma.$$

This implies

$$\sum_{i,j} e_i \otimes_A e_i^* \otimes_B se_j \otimes_A e_j^* = \sum_{i,j} se_i \otimes_A g(e_i^* \otimes_B e_j) \otimes_A e_j^*$$

Using the isomorphism ξ of equation (1) and the definition of the map $\widehat{(-)}$, we obtain $s \in \mathrm{Hom}_{\Sigma^*\otimes_B\Sigma}(\Sigma_g, \Sigma)$ implies $1 \otimes_B s = \widehat{g}(s \otimes_B 1)$. $\square$

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