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Monoidal categories of corings

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SUNTO – Introduciamo una categoria monoidale di coanelli usando due diverse nozioni di morfismi di coanelli. La prima è l'estensione (destra) di coanelli recentemente introdotta da Brzeziński in [2], mentre la seconda è la nozione usuale di morfismo definita in [6] da J. Gómez-Torrecillas.

ABSTRACT – We introduce a monoidal category of corings using two different notions of corings morphisms. The first one is the (right) coring extensions recently introduced by T. Brzeziński in [2], and the other is the usual notion of morphisms defined in [6] by J. Gómez-Torrecillas.

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Introduction.

The word coring appeared for the first time in the literature in the paper of M. Sweedler [9], where he showed that this notion can be used to give a simple proof of the first Galois-correspondence theorem for division rings. It turns out that corings and their comodules unify many kinds of relative modules, such as graded modules, Doi-Hopf modules, and more general entwined modules. This was shown by T. Brzeziński in [1].

Corings are, in some sense, a generalization of coalgebras to the case of non-commutative scalar base rings. They have a bimodule structure rather than a module one. Thus a tensor product in the category of bimodules hampers any

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attempts to give a compatibility with multiplication and comultiplication. This is the well known problem of how to define bialgebras using bimodules. The first approach to generalize bialgebras to the case of bimodules was given by M. Sweedler in [10]. If we see that such bialgebras should be defined as monoids or comonoids in an appropriate monoidal category, then we should look first at the possible monoidal categories. In this note we prove that there is more than one monoidal category whose objects are corings.

We work over a unital commutative ring k. All algebras A, A', B, B', etc. are unital associative k-algebras. For any algebra we denote by $\mathcal{M}_A$ its category of all unital right A-modules; we use the notation ${}_A\mathcal{M}$ to denote the category of unital left A-modules. Bimodules are assumed to be central k-bimodules, and their category is denoted by ${}_A\mathcal{M}_B$. If ${}_AM_B$ and ${}_BN_C$ are, respectively, an (A, B)-bimodule and (B, C)-bimodule, then their tensor product $M \otimes_B N$ will be considered as an (A, C)-bimodule, in the canonical way.

An A-coring is a three-tuple $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ consisting of an A-bimodule $\mathfrak{C}$ and two A-bilinear maps

$$\mathfrak{C} \xrightarrow{\Delta_{\mathfrak{C}}} \mathfrak{C} \otimes_A \mathfrak{C}, \qquad \mathfrak{C} \xrightarrow{\varepsilon_{\mathfrak{C}}} A$$

such that $(\Delta_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ \Delta_{\mathfrak{C}} = (\mathfrak{C} \otimes_A \Delta_{\mathfrak{C}}) \circ \Delta_{\mathfrak{C}}$ and $(\varepsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ \Delta_{\mathfrak{C}} = (\mathfrak{C} \otimes_A \varepsilon_{\mathfrak{C}}) \circ$ $\circ \Delta_{\mathfrak{C}} = \mathfrak{C}$. A right $\mathfrak{C}$-comodule is a pair (M, ρ^M) consisting of a right A-module and a right A-linear map $\rho^M : M \to M \otimes_A \mathfrak{C}$, called right $\mathfrak{C}$-coaction, such that $(M \otimes_A \Delta_{\mathfrak{C}}) \circ \rho^M = (\rho^M \otimes_A \mathfrak{C}) \circ \rho^M$ and $(M \otimes_A \varepsilon_{\mathfrak{C}}) \circ \rho^M = M$. Left $\mathfrak{C}$-comodules are symmetrically defined. For instance, $(\mathfrak{C}, \Delta_{\mathfrak{C}})$ is a left and right $\mathfrak{C}$-comodule. We use Sweedler's notation for comultiplications, that is $\Delta_{\mathfrak{C}}(c) = c_{(1)} \otimes_A c_{(2)}$, for every $c \in \mathfrak{C}$ (the finite sums are understood). We also use lower index Sweedler notation for coactions: $\rho^M(m) = m_{(0)} \otimes_A m_{(1)}$, for all $m \in M$. However in the case of two different coactions, it is convenient to use upper indices: $\rho^N(n) = n^{[0]} \otimes_B n^{[1]}$, for a right $\mathfrak{D}$-comodule N. A source for the basic notions of corings, categories of comodules, bicomodules, and cotensor product is [4].

1. – Tensor product of corings.

In this section we recall the tensor product of two corings over different scalar base rings.

Let $\mathcal{C}$ and $\mathcal{C}'$ be, respectively, an A-bimodule and A'-bimodule. We consider the tensor product $\mathcal{C} \otimes_k \mathcal{C}'$ as an $A \otimes_k A'$-bimodule by the canonical bi-action

$$(a \otimes_k a')(c \otimes_k c')(b \otimes_k b') = (acb) \otimes_k (a'c'b'), \tag{1.1}$$

for all $(a, b) \in A \times A$, $(a', b') \in A' \times A'$, and $(c, c') \in \mathcal{C} \times \mathcal{C}'$. The following well known lemma will be used frequently; for completeness we include the proof.

LEMMA 1.1. *For every pair of modules $(M_A, N_{A'}) \in \mathcal{M}_A \times \mathcal{M}_{A'}$, there exists a right $A \otimes_k A'$-linear map*

$$\begin{aligned} \eta_{(M_A, N_{A'})} : \qquad & M \underset{A}{\otimes} \mathcal{C} \underset{k}{\otimes} N \underset{A'}{\otimes} \mathcal{C}' \longrightarrow (M \otimes_k N) \underset{A \otimes_k A'}{\otimes} (\mathcal{C} \otimes_k \mathcal{C}') \\ & m \underset{A}{\otimes} c \underset{k}{\otimes} n \underset{A'}{\otimes} c' \longmapsto (m \otimes_k n) \underset{A \otimes_k A'}{\otimes} (c \otimes_k c'), \end{aligned}$$

which becomes an $(A, A \otimes_k A')$-bilinear map if $M \in {}_A\mathcal{M}_A$. Furthermore,

$$\eta_{(-,-)} : - \otimes_A \mathcal{C} \otimes_k - \otimes_{A'} \mathcal{C}' \longrightarrow (- \otimes_k -) \otimes_{A \otimes_k A'} (\mathcal{C} \otimes_k \mathcal{C}')$$

is a natural isomorphism.

PROOF. It is clear that $A \otimes_A \mathcal{C} \otimes_k A' \otimes_{A'} \mathcal{C}' \cong (A \otimes_k A') \otimes_{A \otimes_k A'} (\mathcal{C} \otimes_k \mathcal{C}')$, via the map $a \otimes_A c \otimes_k a' \otimes_{A'} c' \mapsto (a \otimes_k a').(c \otimes_k c') = ac \otimes_k a'c'$. If $(f, g) : (A_A, A'_{A'}) \to$ $\to (A_A, A'_{A'})$ is any arrow in the product category $\mathcal{M}_A \times \mathcal{M}_{A'}$, then it is also clear that

$$\begin{array}{ccc} A \underset{A}{\otimes} \mathcal{C} \underset{k}{\otimes} A' \underset{A'}{\otimes} \mathcal{C}' & \xrightarrow{\; f \underset{A}{\otimes} \mathcal{C} \underset{k}{\otimes} g \underset{A'}{\otimes} \mathcal{C}' \;} & A \underset{A}{\otimes} \mathcal{C} \underset{k}{\otimes} A' \underset{A'}{\otimes} \mathcal{C}' \\ \Big\downarrow \cong & & \Big\downarrow \cong \\ (A \otimes_k A') \underset{A \otimes_k A'}{\otimes} (\mathcal{C} \otimes_k \mathcal{C}') & \xrightarrow{\; (f \otimes_k g) \underset{A \otimes_k A'}{\otimes} (\mathcal{C} \otimes_k \mathcal{C}') \;} & (A \otimes_k A') \underset{A \otimes_k A'}{\otimes} (\mathcal{C} \otimes_k \mathcal{C}') \end{array}$$

is a commutative diagram. Since $(A_A, A'_{A'})$ is a projective generator in $\mathcal{M}_A \times \mathcal{M}_{A'}$ and the tensor product commutes with direct limits, Mitchell's Theorem [8, Theorem 5.4, p. 109], implies that there exists a unique natural isomorphism

$$\eta_{(-,-)} : - \otimes_A \mathcal{C} \otimes_k - \otimes_{A'} \mathcal{C}' \longrightarrow (- \otimes_k -) \otimes_{A \otimes_k A'} (\mathcal{C} \otimes_k \mathcal{C}').$$

For all $m \in M$, $n \in N$, $c \in \mathcal{C}$ and $c' \in \mathcal{C}'$, the form of the image $\eta_{(M,N)}(m \otimes_A c \otimes_k n \otimes_{A'} c')$, is computed by using the morphisms $(A_A, A'_{A'}) \to$ $\to (mA, nA')$ and the naturality of η, where mA and nA denote the cyclic submodules. □

Let $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ and $(\mathfrak{C}', \Delta_{\mathfrak{C}'}, \varepsilon_{\mathfrak{C}'})$ be, respectively, an A-coring and A'-coring, and consider $\mathfrak{C} \otimes_k \mathfrak{C}'$ canonically as an $A \otimes_k A'$-bimodule.

PROPOSITION 1.2 ([7]). *The tensor product $\mathfrak{C} \otimes_k \mathfrak{C}'$ is an $A \otimes_k A'$-coring with comultiplication given by the composition map*

$$\mathfrak{C} \otimes_k \mathfrak{C}' \xrightarrow{\Delta_{\mathfrak{C}} \otimes_k \Delta_{\mathfrak{C}'}} (\mathfrak{C} \otimes_A \mathfrak{C}) \otimes_k (\mathfrak{C}' \otimes_{A'} \mathfrak{C}') \xrightarrow{\eta_{\mathfrak{C}, \mathfrak{C}'}} (\mathfrak{C} \otimes_k \mathfrak{C}') \otimes_{A \otimes_k A'} (\mathfrak{C} \otimes_k \mathfrak{C}'),$$

and counit by

$$\mathfrak{C} \otimes_k \mathfrak{C}' \xrightarrow{\varepsilon_{\mathfrak{C}} \otimes_k \varepsilon_{\mathfrak{C}'}} A \otimes_k A'.$$

2. – Tensor product of right coring extensions.

Let A and B be k-algebras, $\mathfrak{C}$ an A-coring and $\mathfrak{D}$ a B-coring. Recall from [2, Definition 2.1], that $\mathfrak{D}$ is called a *right extension* of $\mathfrak{C}$ provided $\mathfrak{C}$ is a $(\mathfrak{C},\mathfrak{D})$-bicomodule with the regular left coaction $\Delta_{\mathfrak{C}}$. This implies that $\mathfrak{C}$ is an (A,B)-bimodule and $\Delta_{\mathfrak{C}}$ is a right B-linear map.

PROPOSITION 2.1. *Let $\mathfrak{D}$ and $\mathfrak{D}'$ be, respectively, a B-coring and B'-coring. Assume that $\mathfrak{D}$ (resp. $\mathfrak{D}'$) is a right extension of $\mathfrak{C}$ (resp. of $\mathfrak{C}'$). Then $\mathfrak{D} \otimes_k \mathfrak{D}'$ is a right extension of $\mathfrak{C} \otimes_k \mathfrak{C}'$.*

PROOF. Denote by $\rho^{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C} \otimes_B \mathfrak{D}$ $(c \mapsto c_{(0)} \otimes_B c_{(1)})$ and $\rho^{\mathfrak{C}'} : \mathfrak{C}' \to \mathfrak{C}' \otimes_{B'} \mathfrak{D}'$ $(c' \mapsto c'_{(0)} \otimes_{B'} c'_{(1)})$ (sums are understood) the right coactions corresponding to the stated extensions. Define $\rho^{\mathfrak{C}\otimes_k\mathfrak{C}'}$ as the composition of maps

$$\begin{array}{ccc} \mathfrak{C} \otimes_k \mathfrak{C}' \xrightarrow{\rho^{\mathfrak{C}}\otimes_k \rho^{\mathfrak{C}'}} (\mathfrak{C} \otimes_B \mathfrak{D}) \otimes_k (\mathfrak{C}' \otimes_{B'} \mathfrak{D}') & \xrightarrow{\eta_{\mathfrak{C},\mathfrak{C}'}} & (\mathfrak{C} \otimes_k \mathfrak{C}') \otimes_{B\otimes_k B'} (\mathfrak{D} \otimes_k \mathfrak{D}'), \\ c \otimes_k c' & \longmapsto & (c_{(0)} \otimes_k c'_{(0)}) \otimes_{B\otimes_k B'} (c_{(1)} \otimes_k c'_{(1)}). \end{array}$$

Using the B-linearity of $\Delta_{\mathfrak{C}}$ and the B'-linearity of $\Delta_{\mathfrak{C}'}$, it is easily checked that this composition is a $(A \otimes_k A') - (B \otimes_k B')$-bilinear map. Furthermore, $\rho^{\mathfrak{C}}$, $\rho^{\mathfrak{C}'}$, and the comultiplications of $\mathfrak{C}$ and $\mathfrak{C}'$, enjoy the following four commutative diagrams.

$$\begin{array}{ccc} \mathfrak{C} \underset{k}{\otimes} \mathfrak{C}' & \xrightarrow{\rho^{\mathfrak{C}} \underset{k}{\otimes} \rho^{\mathfrak{C}'}} & (\mathfrak{C} \underset{B}{\otimes} \mathfrak{D}) \underset{k}{\otimes} (\mathfrak{C}' \underset{B'}{\otimes} \mathfrak{D}') \\ {\scriptstyle \Delta \underset{k}{\otimes} \Delta'} \downarrow & (1) & \downarrow {\scriptstyle (\Delta \otimes_B \mathfrak{D}) \otimes_k (\Delta' \otimes_{B'} \mathfrak{D}')} \\ (\mathfrak{C} \underset{A}{\otimes} \mathfrak{C}) \underset{k}{\otimes} (\mathfrak{C}' \underset{A'}{\otimes} \mathfrak{C}') & \xrightarrow{(\mathfrak{C} \underset{A}{\otimes} \rho^{\mathfrak{C}}) \underset{k}{\otimes} (\mathfrak{C}' \underset{A'}{\otimes} \rho^{\mathfrak{C}'})} & (\mathfrak{C} \underset{A}{\otimes} \mathfrak{C} \underset{B}{\otimes} \mathfrak{D}) \underset{k}{\otimes} (\mathfrak{C}' \underset{A'}{\otimes} \mathfrak{C}' \underset{B'}{\otimes} \mathfrak{D}') . \end{array}$$

The diagram (1) commutes by colinearity of $\rho^{\mathfrak{C}}$ and $\rho^{\mathfrak{C}'}$,

$$\begin{array}{ccc} (\mathfrak{C} \underset{B}{\otimes} \mathfrak{D}) \underset{k}{\otimes} (\mathfrak{C}' \underset{B'}{\otimes} \mathfrak{D}') & \xrightarrow{\cong} & (\mathfrak{C} \underset{k}{\otimes} \mathfrak{C}') \underset{B\otimes_k B'}{\otimes} (\mathfrak{D} \underset{k}{\otimes} \mathfrak{D}') \\ {\scriptstyle (\Delta \otimes_B \mathfrak{D}) \otimes_k (\Delta' \otimes_{B'} \mathfrak{D}')} \downarrow & (2) & \downarrow {\scriptstyle (\Delta \underset{k}{\otimes} \Delta') \underset{B\otimes_k B'}{\otimes} (\mathfrak{D} \underset{k}{\otimes} \mathfrak{D}')} \\ (\mathfrak{C} \underset{A}{\otimes} \mathfrak{C} \underset{B}{\otimes} \mathfrak{D}) \underset{k}{\otimes} (\mathfrak{C}' \underset{A'}{\otimes} \mathfrak{C}' \underset{B'}{\otimes} \mathfrak{D}') & \xrightarrow{\cong} & \left((\mathfrak{C} \underset{A}{\otimes} \mathfrak{C}) \underset{k}{\otimes} (\mathfrak{C}' \underset{A'}{\otimes} \mathfrak{C}') \right) \underset{B\otimes_k B'}{\otimes} (\mathfrak{D} \underset{k}{\otimes} \mathfrak{D}') . \end{array}$$

The diagram (2) commutes by the naturality of $\eta_{-,-}$,

$$\begin{array}{ccc}
(\mathfrak{C} \underset{A}{\otimes} \mathfrak{C}) \underset{k}{\otimes} (\mathfrak{C}' \underset{A'}{\otimes} \mathfrak{C}') & \xrightarrow{(\mathfrak{C}\underset{A}{\otimes}\rho^{\mathfrak{C}})\underset{k}{\otimes}(\mathfrak{C}'\underset{A'}{\otimes}\rho^{\mathfrak{C}'})} & (\mathfrak{C} \underset{A}{\otimes} \mathfrak{C} \underset{B}{\otimes} \mathfrak{D}) \underset{k}{\otimes} (\mathfrak{C}' \underset{A'}{\otimes} \mathfrak{C}' \underset{B'}{\otimes} \mathfrak{D}') \\
\downarrow \cong & (3) & \downarrow \cong \\
(\mathfrak{C} \underset{k}{\otimes} \mathfrak{C}') \underset{A\otimes_k A'}{\otimes} (\mathfrak{C} \underset{k}{\otimes} \mathfrak{C}') & \xrightarrow{(\mathfrak{C}\otimes\mathfrak{C}') \underset{A\otimes_k A'}{\otimes} (\rho^{\mathfrak{C}}\underset{k}{\otimes}\rho^{\mathfrak{C}'})} & (\mathfrak{C} \underset{k}{\otimes} \mathfrak{C}') \underset{A\otimes_k A'}{\otimes} \Big((\mathfrak{C} \otimes_B \mathfrak{D}) \underset{k}{\otimes} (\mathfrak{C}' \otimes_{B'} \mathfrak{D}')\Big) .
\end{array}$$

The diagram (3) commutes by the naturality of the left version of $\eta_{-,-}$

$$\begin{array}{ccc}
(\mathfrak{C} \underset{A}{\otimes} \mathfrak{C} \underset{B}{\otimes} \mathfrak{D}) \underset{k}{\otimes} (\mathfrak{C}' \underset{A'}{\otimes} \mathfrak{C}' \underset{B'}{\otimes} \mathfrak{D}') & \xrightarrow{\cong} & \Big((\mathfrak{C} \underset{A}{\otimes} \mathfrak{C}) \underset{k}{\otimes} (\mathfrak{C}' \underset{A'}{\otimes} \mathfrak{C}')\Big) \underset{B\otimes_k B'}{\otimes} (\mathfrak{D} \underset{k}{\otimes} \mathfrak{D}') \\
\downarrow \cong & (4) & \downarrow \cong \\
(\mathfrak{C} \underset{k}{\otimes} \mathfrak{C}') \underset{A\otimes_k A'}{\otimes} \Big((\mathfrak{C} \otimes_B \mathfrak{D}) \underset{k}{\otimes} (\mathfrak{C}' \otimes_{B'} \mathfrak{D}')\Big) & \xrightarrow{\cong} & (\mathfrak{C} \underset{k}{\otimes} \mathfrak{C}') \underset{A\otimes_k A'}{\otimes} (\mathfrak{C} \underset{k}{\otimes} \mathfrak{C}') \underset{B\otimes_k B'}{\otimes} (\mathfrak{D} \underset{k}{\otimes} \mathfrak{D}') .
\end{array}$$

The diagram (4) commutes by direct computation, where the isomorphisms in this diagram are defined by the natural isomorphism of Lemma 1.1. If we arrange those diagrams in the following form

(1)	(2)
(3)	(4)

,

we then obtain another commutative diagram, which shows that $\rho^{\mathfrak{C}\otimes_k\mathfrak{C}'}$ is left $\mathfrak{C} \otimes_k \mathfrak{C}'$-colinear with respect to the regular left coaction $\Delta_{\mathfrak{C}\otimes_k\mathfrak{C}'}$. The proof of the fact that $\rho^{\mathfrak{C}\otimes_k\mathfrak{C}'}$ is a right $\mathfrak{D} \otimes_k \mathfrak{D}'$-coaction can be done in a similar way. This completes the proof. □

3. – A monoidal category.

Let us recall from [2] the category of coring extensions $\mathbf{CrgExt}_k^r$. The objects in this category are corings understood as pairs $(\mathfrak{C} : A)$ (that is $\mathfrak{C}$ is an A-coring), and morphisms $(\mathfrak{C} : A) \to (\mathfrak{D} : B)$ are pairs $(\rho_{\mathfrak{C}}, \rho^{\mathfrak{C}})$, where $\rho_{\mathfrak{C}} : \mathfrak{C} \otimes_k B \to \mathfrak{C}$ is a left $\mathfrak{C}$-colinear right B-action, and $\rho^{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C} \otimes_B \mathfrak{D}$ is a left $\mathfrak{C}$-colinear right $\mathfrak{D}$-coaction (that is $\mathfrak{D}$ is a right extension of $\mathfrak{C}$). The identity arrow of an object $(\mathfrak{C} : A)$ is given by the pair $id_{(\mathfrak{C}:A)} = (\rho_{\mathfrak{C}}, \rho^{\mathfrak{C}})$ where $\rho_{\mathfrak{C}} = \iota_{\mathfrak{C}} : \mathfrak{C} \otimes_k A \to \mathfrak{C}$ is the initial right A-action and $\rho^{\mathfrak{C}} = \Delta_{\mathfrak{C}}$ is the comultiplication of $\mathfrak{C}$. The composition law is given as follows. If

$(\rho_{\mathfrak{E}}, \rho^{\mathfrak{E}}) : (\mathfrak{E} : C) \to (\mathfrak{C} : A)$ and $(\rho_{\mathfrak{C}}, \rho^{\mathfrak{C}}) : (\mathfrak{C} : A) \to (\mathfrak{D} : B)$, then

$$(\rho_{\mathfrak{C}}, \rho^{\mathfrak{C}}) \circ (\rho_{\mathfrak{E}}, \rho^{\mathfrak{E}}) = (\rho_{\mathfrak{C}} \bullet \rho_{\mathfrak{E}}, \rho^{\mathfrak{C}} \bullet \rho^{\mathfrak{E}}),$$

where

$$\rho_{\mathfrak{C}} \bullet \rho_{\mathfrak{E}} : \mathfrak{E} \otimes_k B \xrightarrow{\rho^{\mathfrak{E}} \otimes B} \mathfrak{E} \otimes_A \mathfrak{C} \otimes_k B \xrightarrow{\mathfrak{E} \otimes_A \rho_{\mathfrak{C}}} \mathfrak{E} \otimes_A \mathfrak{C} \xrightarrow{\mathfrak{E} \otimes_A \varepsilon} \mathfrak{E} \otimes_A A \cong \mathfrak{E}$$

and

$$\rho^{\mathfrak{C}} \bullet \rho^{\mathfrak{E}} : \mathfrak{E} \xrightarrow{\cong} \mathfrak{E} \square_{\mathfrak{C}} \mathfrak{C} \xrightarrow{\mathfrak{E} \square_{\mathfrak{C}} \rho^{\mathfrak{C}}} \mathfrak{E} \square_{\mathfrak{C}} (\mathfrak{C} \otimes_B \mathfrak{D}) \xrightarrow{\cong} \mathfrak{E} \otimes_B \mathfrak{D}.$$

Explicitly, the bullet compositions are given as follows: for $e \in \mathfrak{E}$ and $b \in B$

$$\rho_{\mathfrak{C}} \bullet \rho_{\mathfrak{E}}(e \otimes_k b) = e_{(0)} \varepsilon_{\mathfrak{C}}(e_{(1)} b),$$

$$\rho^{\mathfrak{C}} \bullet \rho^{\mathfrak{E}}(e) = e_{(0)} \varepsilon_{\mathfrak{C}}({e_{(1)}}^{[0]}) \otimes_B {e_{(1)}}^{[1]}, \tag{3.2}$$

where $\rho^{\mathfrak{E}}(e) = e_{(0)} \otimes_A e_{(1)} \in \mathfrak{E} \otimes_A \mathfrak{C}$, and, for all $c \in \mathfrak{C}$, $\rho^{\mathfrak{C}}(c) = c^{[0]} \otimes_B c^{[1]} \in \in \mathfrak{C} \otimes_B \mathfrak{D}$.

The tensor product of two morphisms $(\rho_{\mathfrak{C}}, \rho^{\mathfrak{C}}) : (\mathfrak{C} : A) \to (\mathfrak{D} : B)$ and $(\rho_{\mathfrak{C}'}, \rho^{\mathfrak{C}'}) : (\mathfrak{C}' : A') \to (\mathfrak{D}' : B')$ in $\mathbf{CrgExt}_k^r$, is defined as in the proof of Proposition 2.1; explicitly

$$(\rho_{\mathfrak{C} \otimes_k \mathfrak{C}'}, \rho^{\mathfrak{C} \otimes_k \mathfrak{C}'}) : (\mathfrak{C} \otimes_k \mathfrak{C}' : A \otimes_k A') \longrightarrow (\mathfrak{D} \otimes_k \mathfrak{D}' : B \otimes_k B'),$$

where the right multiplication is given by

$$\begin{aligned} \rho_{\mathfrak{C} \otimes_k \mathfrak{C}'} : (\mathfrak{C} \otimes_k \mathfrak{C}') \otimes_k (B \otimes_k B') &\longrightarrow \mathfrak{C} \otimes_k \mathfrak{C}', \\ (c \otimes_k c') \otimes_k (b \otimes_k b') &\longmapsto (cb) \otimes_k (c'b'). \end{aligned} \tag{3.3}$$

Here $\rho_{\mathfrak{C}}(c \otimes_k b) = cb$, $\rho_{\mathfrak{C}'}(c' \otimes_k b') = c'b'$. The right coaction is defined by

$$\begin{aligned} \rho^{\mathfrak{C} \otimes_k \mathfrak{C}'} : \mathfrak{C} \otimes_k \mathfrak{C}' &\longrightarrow (\mathfrak{C} \otimes_k \mathfrak{C}') \otimes_{B \otimes_k B'} (\mathfrak{D} \otimes_k \mathfrak{D}'), \\ c \otimes_k c' &\longmapsto (c_{(0)} \otimes_k c'_{(0)}) \otimes_{B \otimes_k B'} (c_{(1)} \otimes_k c'_{(1)}), \end{aligned} \tag{3.4}$$

where $\rho^{\mathfrak{C}}(c) = c_{(0)} \otimes_B c_{(1)}$ and $\rho^{\mathfrak{C}'}(c') = c'_{(0)} \otimes_{B'} c'_{(1)}$.

PROPOSITION 3.1. *Let k be an unital commutative ring. Consider the category* $\mathbf{CrgExt}_k^r$ *of corings with morphisms right coring extensions, and denote by* $\Bbbk := (k : k)$ *the trivial k-coring k. Then there exists a covariant bi-functor*

$$\begin{aligned} - \otimes_{\Bbbk} - : \mathbf{CrgExt}_k^r \times \mathbf{CrgExt}_k^r &\longrightarrow \mathbf{CrgExt}_k^r, \\ ((\mathfrak{C} : A), (\mathfrak{C}' : A')) &\longrightarrow (\mathfrak{C} \otimes_k \mathfrak{C}' : A \otimes_k A'), \\ ((\rho_{\mathfrak{C}}, \rho^{\mathfrak{C}}), (\rho_{\mathfrak{C}'}, \rho^{\mathfrak{C}'})) &\longrightarrow (\rho_{\mathfrak{C} \otimes_k \mathfrak{C}'}, \rho^{\mathfrak{C} \otimes_k \mathfrak{C}'}), \end{aligned}$$

where $(\rho_{\mathfrak{C}\otimes_k\mathfrak{C}'}, \rho^{\mathfrak{C}\otimes_k\mathfrak{C}'})$ *is the morphism defined in equations* (3.3) *and* (3.4). *Moreover,*

$$\Bbbk \otimes_{\Bbbk} (\mathfrak{C} : A) \cong (\mathfrak{C} : A) \text{ and } (\mathfrak{C} : A) \otimes_{\Bbbk} \Bbbk \cong (\mathfrak{C} : A)$$

are natural isomorphisms in $\mathbf{CrgExt}_k^r$. In particular, $\mathbf{CrgExt}_k^r$ is a monoidal category with unit $\Bbbk$.

PROOF. Propositions 1.2 and 2.1, imply that the stated functor is well defined. Now, by definition of the comultiplication of the tensor product of two corings, the identity arrow of any pair of corings is mapped by $- \otimes_{\Bbbk} -$ to the identity arrow of their tensor product; that is in the above notation, we have

$$id_{(\mathfrak{C}:A)} \otimes_{\Bbbk} id_{(\mathfrak{C}':A')} = (\imath_{\mathfrak{C}}, \Delta_{\mathfrak{C}}) \otimes_{\Bbbk} (\imath_{\mathfrak{C}'}, \Delta_{\mathfrak{C}'}) = (\imath_{\mathfrak{C}\otimes_k\mathfrak{C}'}, \Delta_{\mathfrak{C}\otimes_k\mathfrak{C}'}) = id_{(\mathfrak{C}\otimes_k\mathfrak{C}':A\otimes_k A')}.$$

Consider the following four morphisms in $\mathbf{CrgExt}_k^r$

$$(\mathfrak{E} : C) \xrightarrow{(\rho_{\mathfrak{E}}, \rho^{\mathfrak{E}})} (\mathfrak{C} : A) \xrightarrow{(\rho_{\mathfrak{C}}, \rho^{\mathfrak{C}})} (\mathfrak{D} : B)$$

$$(\mathfrak{E}' : C') \xrightarrow{(\rho_{\mathfrak{E}'}, \rho^{\mathfrak{E}'})} (\mathfrak{C}' : A') \xrightarrow{(\rho_{\mathfrak{C}'}, \rho^{\mathfrak{C}'})} (\mathfrak{D}' : B'),$$

and put $(\rho_{\mathfrak{C}} \bullet \rho_{\mathfrak{E}}, \rho^{\mathfrak{C}} \bullet \rho^{\mathfrak{E}}) \otimes_{\Bbbk} (\rho_{\mathfrak{C}'} \bullet \rho_{\mathfrak{E}'}, \rho^{\mathfrak{C}'} \bullet \rho^{\mathfrak{E}'}) = (\overline{\rho}_{\mathfrak{E}\otimes_k\mathfrak{E}'}, \overline{\rho}^{\mathfrak{E}\otimes_k\mathfrak{E}'})$. By definition $\overline{\rho}_{\mathfrak{E}\otimes_k\mathfrak{E}'} : (\mathfrak{E} \otimes_k \mathfrak{E}') \otimes_k (B \otimes_k B') \to \mathfrak{E} \otimes_k \mathfrak{E}'$ sends $(e \otimes_k e') \otimes_k (b \otimes_k b') \mapsto (eb) \otimes_k (e'b')$, where $eb = \rho_{\mathfrak{C}} \bullet \rho_{\mathfrak{E}}(e \otimes_k b) = e_{(0)}\varepsilon_{\mathfrak{C}}(e_{(1)}b)$ and $e'b' = \rho_{\mathfrak{C}'} \bullet \rho_{\mathfrak{E}'}(e' \otimes_k b') = e'_{(0)}\varepsilon_{\mathfrak{C}'}(e'_{(1)}b')$. That is

$$\begin{aligned}\overline{\rho}_{\mathfrak{E}\otimes_k\mathfrak{E}'}((e \otimes_k e') \otimes_k (b \otimes_k b')) &= e_{(0)}\varepsilon_{\mathfrak{C}}(e_{(1)}b) \otimes_k e'_{(0)}\varepsilon_{\mathfrak{C}'}(e'_{(1)}b') \\ &= \rho_{\mathfrak{C}\otimes_k\mathfrak{C}'} \bullet \rho_{\mathfrak{E}\otimes_k\mathfrak{E}'}((e \otimes_k e') \otimes_k (b \otimes_k b')),\end{aligned}$$

for every $e \in \mathfrak{E}$, $e' \in \mathfrak{E}'$, $b \in B$, and $b' \in B'$. Thus $\overline{\rho}_{\mathfrak{E}\otimes_k\mathfrak{E}'} = \rho_{\mathfrak{C}\otimes_k\mathfrak{C}'} \bullet \rho_{\mathfrak{E}\otimes_k\mathfrak{E}'}$.

On the other hand the map $\overline{\rho}^{\mathfrak{E}\otimes_k\mathfrak{E}'}$ is defined by the composition

$$\mathfrak{E} \otimes_k \mathfrak{E}' \xrightarrow{(\rho^{\mathfrak{C}} \bullet \rho^{\mathfrak{E}}) \otimes (\rho^{\mathfrak{C}'} \bullet \rho^{\mathfrak{E}'})} (\mathfrak{E} \otimes_B \mathfrak{D}) \otimes_k (\mathfrak{E}' \otimes_{B'} \mathfrak{D}') \xrightarrow{\eta_{\mathfrak{E},\mathfrak{E}'}} (\mathfrak{E} \otimes_k \mathfrak{E}') \otimes_{B\otimes_k B'} (\mathfrak{D} \otimes_k \mathfrak{D}'),$$

sending

$$e \otimes_k e' \longmapsto \left((e_{(0)}\varepsilon_{\mathfrak{C}}(e_{(1)}{}^{[0]})) \otimes_k (e'_{(0)}\varepsilon_{\mathfrak{C}'}(e'_{(1)}{}^{[0]}))\right) \otimes_{B\otimes_k B'} \left(e_{(1)}{}^{[1]} \otimes_k e'_{(1)}{}^{[1]}\right),$$

that is $\overline{\rho}^{\mathfrak{E}\otimes_k\mathfrak{E}'}(e \otimes_k e') = \rho^{\mathfrak{C}\otimes_k\mathfrak{C}'} \bullet \rho^{\mathfrak{E}\otimes_k\mathfrak{E}'}(e \otimes_k e')$, for every $(e, e') \in \mathfrak{E} \times \mathfrak{E}'$. Hence $\overline{\rho}^{\mathfrak{E}\otimes_k\mathfrak{E}'} = \rho^{\mathfrak{C}\otimes_k\mathfrak{C}'} \bullet \rho^{\mathfrak{E}\otimes_k\mathfrak{E}'}$. Therefore,

$$\begin{aligned}(\rho_{\mathfrak{C}} \bullet \rho_{\mathfrak{E}}, \rho^{\mathfrak{C}} \bullet \rho^{\mathfrak{E}}) \otimes_{\Bbbk} (\rho_{\mathfrak{C}'} \bullet \rho_{\mathfrak{E}'}, \rho^{\mathfrak{C}'} \bullet \rho^{\mathfrak{E}'}) &= (\rho_{\mathfrak{C}\otimes_k\mathfrak{C}'} \bullet \rho_{\mathfrak{E}\otimes_k\mathfrak{E}'}, \rho^{\mathfrak{C}\otimes_k\mathfrak{C}'} \bullet \rho^{\mathfrak{E}\otimes_k\mathfrak{E}'}) \\ &= (\rho_{\mathfrak{C}}, \rho^{\mathfrak{C}}) \otimes_{\Bbbk} (\rho_{\mathfrak{C}'}, \rho^{\mathfrak{C}'}) \circ (\rho_{\mathfrak{E}}, \rho^{\mathfrak{E}}) \otimes_{\Bbbk} (\rho_{\mathfrak{E}'}, \rho^{\mathfrak{E}'}) \\ &= \left((\rho_{\mathfrak{C}}, \rho^{\mathfrak{C}}) \circ (\rho_{\mathfrak{E}}, \rho^{\mathfrak{E}})\right) \otimes_{\Bbbk} \left((\rho_{\mathfrak{C}'}, \rho^{\mathfrak{C}'}) \circ (\rho_{\mathfrak{E}'}, \rho^{\mathfrak{E}'})\right).\end{aligned}$$

This shows that $-\otimes_{\Bbbk}-$ is a covariant functor. The last assertion is obvious.

For the particular assertion, we only show that the associative isomorphism

$$\alpha_{\mathfrak{C},\mathfrak{C}',\mathfrak{E}} : \big((\mathfrak{C}:A)\otimes_{\Bbbk}(\mathfrak{C}':A')\big)\otimes_{\Bbbk}(\mathfrak{E}:C) \longrightarrow (\mathfrak{C}:A)\otimes_{\Bbbk}\big((\mathfrak{C}':A')\otimes_{\Bbbk}(\mathfrak{E}:C)\big)$$

is a natural isomorphism, and we will do it only for the first factor. Firstly, it is easily seen that two objects $\big((\mathfrak{C}:A)\otimes_{\Bbbk}(\mathfrak{C}':A')\big)\otimes_{\Bbbk}(\mathfrak{E}:C)$ and $(\mathfrak{C}:A)\otimes_{\Bbbk}\big((\mathfrak{C}':A')\otimes_{\Bbbk}(\mathfrak{E}:C)\big)$ define, via α, the same coring in the category $\mathbf{CrgExt}_k^r$, where the coaction part in the corresponding identity map is $\Delta_{\mathfrak{C}\otimes_k\mathfrak{C}'\otimes_k\mathfrak{E}}$. Now, let $(\rho_{\mathfrak{C}},\rho^{\mathfrak{C}}):(\mathfrak{C}:A)\to(\mathfrak{D}:B)$ be a morphism in $\mathbf{CrgExt}_k^r$. It remains to be shown that the following diagram

$$\begin{array}{ccc}
\big((\mathfrak{C}:A)\otimes_{\Bbbk}(\mathfrak{C}':A')\big)\otimes_{\Bbbk}(\mathfrak{E}:C) & \xrightarrow{\alpha_{\mathfrak{C},\mathfrak{C}',\mathfrak{E}}} & (\mathfrak{C}:A)\otimes_{\Bbbk}\big((\mathfrak{C}':A')\otimes_{\Bbbk}(\mathfrak{E}:C)\big) \\
{\scriptstyle \left((\rho_{\mathfrak{C}},\rho^{\mathfrak{C}})\otimes_{\Bbbk}\mathfrak{C}'\right)\otimes_{\Bbbk}\mathfrak{E}}\big\downarrow & & \big\downarrow{\scriptstyle (\rho_{\mathfrak{C}},\rho^{\mathfrak{C}})\otimes_{\Bbbk}(\mathfrak{C}'\otimes_{\Bbbk}\mathfrak{E})} \\
\big((\mathfrak{D}:B)\otimes_{\Bbbk}(\mathfrak{C}':A')\big)\otimes_{\Bbbk}(\mathfrak{E}:C) & \xrightarrow{\alpha_{\mathfrak{D},\mathfrak{C}',\mathfrak{E}}} & (\mathfrak{D}:B)\otimes_{\Bbbk}\big((\mathfrak{C}':A')\otimes_{\Bbbk}(\mathfrak{E}:C)\big)
\end{array}$$

is commutative. The bullet composition concerning the coaction component of the morphisms reads:

$$\Delta_{\mathfrak{D}\otimes_k\mathfrak{C}'\otimes_k\mathfrak{E}} \bullet \rho^{(\mathfrak{C}\otimes_k\mathfrak{C}')\otimes_k\mathfrak{E}} = \rho^{\mathfrak{C}\otimes_k(\mathfrak{C}'\otimes_k\mathfrak{E})} \bullet \Delta_{\mathfrak{C}\otimes_k\mathfrak{C}'\otimes_k\mathfrak{E}},$$

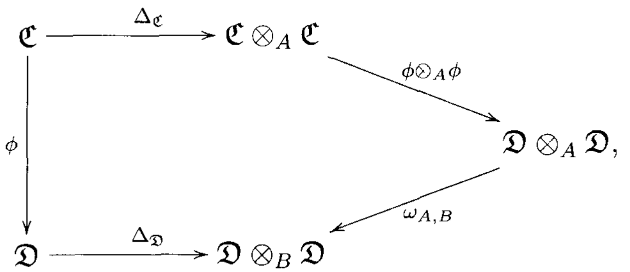
where $\rho^{(\mathfrak{C}\otimes_k\mathfrak{C}')\otimes_k\mathfrak{E}}$ and $\rho^{\mathfrak{C}\otimes_k(\mathfrak{C}'\otimes_k\mathfrak{E})}$ are defined by the equation (3.4) using the maps $\Delta_{\mathfrak{C}'\otimes_k\mathfrak{E}}$, $\Delta_{\mathfrak{C}'}$ and $\Delta_{\mathfrak{E}}$. If we denote $\rho^{\mathfrak{C}}(c)=c^{[0]}\otimes_B c^{[1]}$, $c\in\mathfrak{C}$, then this equality is satisfied by the equation (3.2) if and only if, for every $c\in\mathfrak{C}$, $c'\in\mathfrak{C}'$ and $e\in\mathfrak{E}$,

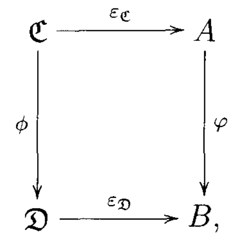
$$\begin{aligned}
&\big(c^{[0]}\varepsilon_{\mathfrak{C}}(c^{[1]}{}_{(1)})\otimes_k c'_{(1)}\otimes_k e_{(1)}\big)\otimes_{B\otimes_kA'\otimes_kC}\big(c^{[1]}{}_{(2)}\otimes_k c'_{(2)}\otimes_k e_{(2)}\big) \\
&\qquad = \big(c_{(1)}\varepsilon_{\mathfrak{C}}(c_{(2)}{}^{[0]})\otimes_k c'_{(1)}\otimes_k e_{(1)}\big)\otimes_{B\otimes_kA'\otimes_kC}\big(c_{(2)}{}^{[1]}\otimes_k c'_{(2)}\otimes_k e_{(2)}\big)
\end{aligned}$$

The equality follows since $\mathfrak{C}$ is a $(\mathfrak{C},\mathfrak{D})$-bicomodule. □

REMARK 3.2. Of course, we have a similar result for left coring extensions. That is, the category $\mathbf{CrgExt}_k^l$ whose objects are corings and morphisms are left coring extensions is also a monoidal category with the same unit $\Bbbk=(k:k)$. If we denote by $\mathbf{CrgExt}_k$ the category whose objects are all corings and morphisms are the left and right (at the same time) coring extensions, then the study of $\mathbf{CrgExt}_k$ can be also posed, viewing it as a subcategory of both $\mathbf{CrgExt}_k^r$ and $\mathbf{CrgExt}_k^l$. Examples of morphisms in this subcategory are morphisms between corings with the same scalar base ring.

Finally, we will consider the category of corings $\mathfrak{Corings}$, whose objects are corings understood also as pairs $(\mathfrak{C} : A)$ and morphisms are as in [6]; that is, a morphism is a pair of maps $(\phi, \varphi) : (\mathfrak{C} : A) \to (\mathfrak{D} : B)$, where $\varphi : A \to B$ is an algebra map and $\phi : \mathfrak{C} \to \mathfrak{D}$ is an A-bilinear map ($\mathfrak{D}$ is an A-bimodule by restriction of scalars) such that the following diagrams commute





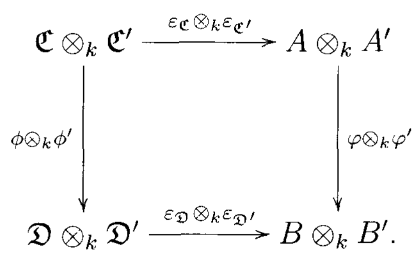
where $\omega_{A,B} : \mathfrak{D} \otimes_A \mathfrak{D} \to \mathfrak{D} \otimes_B \mathfrak{D}$ is the obvious map associated to φ. The identity arrow of an object $(\mathfrak{C} : A)$ is $id_{(\mathfrak{C}:A)} = (id_{\mathfrak{C}}, id_A)$, and the composition law is componentwise composition.

LEMMA 3.3. *Let* $(\phi, \varphi) : (\mathfrak{C} : A) \to (\mathfrak{D} : B)$ *and* $(\phi', \varphi') : (\mathfrak{C}' : A') \to (\mathfrak{D}' : B')$ *are coring morphisms. Then*

$$(\phi \otimes_k \phi', \varphi \otimes_k \varphi') : (\mathfrak{C} \otimes_k \mathfrak{C}' : A \otimes_k A') \to (\mathfrak{D} \otimes_k \mathfrak{D}' : B \otimes_k B')$$

is also a coring morphism.

PROOF. Analogous to that of [5, Proposición 1.1.20]. Obviously $\varphi \otimes_k \varphi'$ is an algebra map. Since ϕ and ϕ' are, respectively, A-bilinear and A'-bilinear, it is easily checked that $\phi \otimes_k \phi'$ is $A \otimes_k A'$-bilinear. The counit property of $\phi \otimes_k \phi'$ is given by the following commutative diagram



To prove the colinearity of $\phi \otimes_k \phi'$, first denote the natural isomorphisms of Lemma 1.1 by $\eta^{A,A'}_{-,-}$ and $\eta^{B,B'}_{-,-}$ to distinguish between the tensor product algebra $A \otimes_k A'$ and $B \otimes_k B'$. With this notation, we can compute

$$\begin{aligned}
&\omega_{A\otimes_k A', B\otimes_k B'} \circ \big((\phi \otimes_k \phi') \otimes_{A\otimes_k A'} (\phi \otimes_k \phi')\big) \circ \Delta_{\mathfrak{C}\otimes_k \mathfrak{C}'} \\
&\quad = \omega_{A\otimes_k A', B\otimes_k B'} \circ \big((\phi \otimes_k \phi') \otimes_{A\otimes_k A'} (\phi \otimes_k \phi')\big) \circ \eta^{A,A'}_{\mathfrak{C},\mathfrak{C}'} \circ (\Delta_{\mathfrak{C}} \otimes_k \Delta_{\mathfrak{C}'}) \\
&\quad = \omega_{A\otimes_k A', B\otimes_k B'} \circ \eta^{A,A'}_{\mathfrak{D},\mathfrak{D}'} \circ \big((\phi \otimes_A \phi) \otimes_k (\phi' \otimes_{A'} \phi')\big) \circ (\Delta_{\mathfrak{C}} \otimes_k \Delta_{\mathfrak{C}'}) \\
&\quad = \eta^{B,B'}_{\mathfrak{D},\mathfrak{D}'} \circ \big(\omega_{A,B} \otimes_k \omega_{A',B'}\big) \circ \big((\phi \otimes_A \phi) \otimes_k (\phi' \otimes_{A'} \phi')\big) \circ (\Delta_{\mathfrak{C}} \otimes_k \Delta_{\mathfrak{C}'}) \\
&\quad = \eta^{B,B'}_{\mathfrak{D},\mathfrak{D}'} \circ (\Delta_{\mathfrak{D}} \otimes_k \Delta_{\mathfrak{D}'}) \circ (\phi \otimes_k \phi') = \Delta_{\mathfrak{D}\otimes_k \mathfrak{D}'} \circ (\phi \otimes_k \phi'),
\end{aligned}$$

where we have used the naturality of $\eta^{-,-}_{-,-}$ and the colinearity of ϕ and ϕ'. This proves that $\phi \otimes_k \phi'$ is a colinear map as required. □

PROPOSITION 3.4. *Consider the category* $\mathfrak{Corings}$ *of all corings, and denote by* $\mathbb{k} = (k : k)$ *the trivial k-coring. The following*

$$\begin{aligned}
-\otimes_{\mathbb{k}} - : \mathfrak{Corings} \times \mathfrak{Corings} &\longrightarrow \mathfrak{Corings} \\
\big((\mathfrak{C} : A), (\mathfrak{C}' : A')\big) &\longrightarrow (\mathfrak{C} \otimes_k \mathfrak{C}' : A \otimes_k A') \\
\big((\phi, \varphi), (\phi', \varphi')\big) &\longrightarrow (\phi \otimes_k \phi', \varphi \otimes_k \varphi')
\end{aligned}$$

establishes a covariant bi-functor. Moreover,

$$\mathbb{k} \otimes_{\mathbb{k}} (\mathfrak{C} : A) \cong (\mathfrak{C} : A) \text{ and } (\mathfrak{C} : A) \otimes_{\mathbb{k}} \mathbb{k} \cong (\mathfrak{C} : A)$$

are natural isomorphisms in $\mathfrak{Corings}$. *In particular,* $\mathfrak{Corings}$ *is a monoidal category with unit* $\mathbb{k}$.

PROOF. Consequence of Lemma 3.3. □

REMARK 3.5. The relationship between morphisms in $\mathfrak{Corings}$ and those in $\mathbf{CrgExt}^r_k$ can be described as follows. Let $(\phi, \varphi) : (\mathfrak{C} : A) \to (\mathfrak{D} : B)$ be a morphism in $\mathfrak{Corings}$, and consider the B-coring $B \otimes_A \mathfrak{C} \otimes_A B$ called the base ring extension of $\mathfrak{C}$, see [4, 17.2]. Denote by $\rho_{B\otimes_A \mathfrak{C} \otimes_A B} : B \otimes_A \mathfrak{C} \otimes_A B \otimes_k B \to B \otimes_A \mathfrak{C} \otimes_A B$ the right B-multiplication map, and define

$$\begin{aligned}
\rho^{B\otimes_A \mathfrak{C}\otimes_A B} : B \otimes_A \mathfrak{C} \otimes_A B &\longrightarrow B \otimes_A \mathfrak{C} \otimes_A B \otimes_B \mathfrak{D} \cong B \otimes_A \mathfrak{C} \otimes_A \mathfrak{D}\,, \\
b \otimes_A c \otimes_A b' &\longmapsto b \otimes_A c_{(1)} \otimes_A \phi(c_{(2)})b'\,,
\end{aligned}$$

where $\Delta_{\mathfrak{C}}(c) = c_{(1)} \otimes_A c_{(2)}$. It is easily checked that the pair

$$(\rho_{B\otimes_A \mathfrak{C}\otimes_A B}\,, \rho^{B\otimes_A \mathfrak{C}\otimes_A B}) : (B \otimes_A \mathfrak{C} \otimes_A B : B) \longrightarrow (\mathfrak{D} : B)$$

is a morphism in the category $\mathbf{CrgExt}^r_k$.

In fact $B \otimes_A \mathfrak{C} \otimes_A B$ is a right and left coring extension of $\mathfrak{D}$. That is to any morphism (ϕ, φ) in $\mathfrak{Corings}$, one can associate a morphism in the category $\mathbf{CrgExt}_k$ described in Remark 3.2.

REMARK 3.6. G. Böhm kindly informed me that the monoidal structure in Proposition 3.4 can also be deduced by viewing $\mathfrak{Corings}$ as a sub-category of the monoidal bicategory of corings in the sense of [3]. Indeed, in the notation of [3, Section 4], if (Σ, σ) and (Σ', σ') are module-morphisms, respectively, in ${}_{(\mathfrak{D}:B)}\mathcal{M}_{(\mathfrak{C}:A)}$ and ${}_{(\mathfrak{D}':B')}\mathcal{M}_{(\mathfrak{C}':A')}$ (that is 2-cells in the stated bicategory), then the tensor product $(\Sigma \otimes_k \Sigma', \sigma \widetilde{\otimes_k} \sigma')$ is a module-morphism in ${}_{(\mathfrak{D}\otimes_k\mathfrak{D}':B\otimes_k B')}\mathcal{M}_{(\mathfrak{C}\otimes_k\mathfrak{C}':A\otimes_k A')}$, where the coring map

$$\sigma \widetilde{\otimes_k} \sigma' : (\Sigma \otimes_k \Sigma')^* \otimes_{B\otimes_k B'} (\mathfrak{D} \otimes_k \mathfrak{D}') \otimes_{B\otimes_k B'} (\Sigma \otimes_k \Sigma') \longrightarrow \mathfrak{C} \otimes_k \mathfrak{C}'$$

is defined using the natural transformation of Lemma 1.1. However, we believe that the proof of the fact that $\sigma \widetilde{\otimes_k} \sigma'$ is a coring morphism in Lemma 3.3 is more direct than the bicategory considerations.

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