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Coinduction functor and simple comodules

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Abstract Consider a coring with exact rational functor, and a finitely generated and projective right comodule. We construct a functor (*coinduction functor*) which is right adjoint to the hom-functor represented by this comodule. Using the coinduction functor, we establish a bijective map between the set of representative classes of torsion simple right comodules and the set of representative classes of simple right modules over the endomorphism ring. A detailed application to group-graded modules is also given.

Keywords Corings · Simple comodules · Simple graded modules

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0 Introduction

Let G be a group with neutral element e and $A = \oplus_{x \in G} A_x$ a G-graded ring. To each element $x \in G$, one can associate the restriction functor $(-)_x : \mathsf{gr}\text{-}A \to \mathsf{Mod}_{A_\mathsf{e}}$ from the category of G-graded right A-modules to the category of right A_e-modules. This

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functor sends an object $M \in \mathsf{gr}\text{-}A$ to its homogeneous component M_x. Since $(-)_x$ is right exact and commutes with direct sums, a classical result of P. Gabriel tell us that it has a right adjoint functor which we denote by $\mathsf{Coind}^x : \mathsf{Mod}_{A_\mathsf{e}} \to \mathsf{gr}\text{-}A$. The functor Coind^x is known in the literature as a *coinduction functor*, and was first introduced by Năstăsescu in [12], for $x = \mathsf{e}$. In [1], Abrams and Menini defined and studied Coind^x in the case of semigroup-graded rings. The use of the coinduction functor was crucial to study both simple and injective objects in either a group-graded or semigroup-graded module categories, see [1,11,12].

It was well established recently in [5] that the unified way of studying relative modules and in particular group (semigroup, set-group) graded modules, is the framework of the category of comodules over a suitable coring. The purpose of this paper fits in this direction. Our main aim is to introduce and study the coinduction functor, as well as simple objects, in the context of right comodules over a coring. Some restriction on the base coring is needed. Namely, we work with coring which is a member of rational pairing whose associated rational functor is exact. The case of (semi)group-graded modules fits exactly in this situation. A more general case where our methods can perfectly be applied is the case of entwined modules [5]. Of course in this case the factor algebra should be a flat module, while the factor coalgebra should be locally projective module with exact rational functor. Because of space and time, here we only give a detailed application to the case of group-graded modules.

We proceed as follows. In Sect. 1, we give a brief review on rational pairings and rational functors. Section 2 is devoted to construct the coinduction functor. We consider a coring with exact rational functor. To any finitely generated and projective right comodule, we associate a functor (*coinduction functor*) which is right adjoint to the hom-functor represented by this object (Proposition 2.1). By a classical result of P. Gabriel, we can then deduce that the kernel of this hom-functor is a TTF-class, and the quotient category is equivalent to the category of right modules over the endomorphism ring (Corollary 2.2). Section 3 presents the main observations on simple comodules. Using the results of Sect. 2, we are able to reconstruct, in a bijective way, any simple right comodule which is torsion with respect to a finitely generated and projective right comodule (Theorem 3.3, Corollary 3.4). In the last Section, we give a complete and detailed application to the category of group-graded modules, where we recover some results from [11,12].

Details on corings and their comodules are available in [5].

1 Rational pairing and rational functor

Rational pairings for coalgebras over commutative rings were introduced in [10] and used in [3] to study the category of right comodules over the finite dual coalgebra associated to certain algebras over noetherian commutative rings. This development was adapted for corings in [8], see also [2]. We recall from [8, Section 2] the definition of this notion:

Let P, Q be A–bimodules. Any balanced A–bilinear form

$$\langle -, - \rangle : P \times Q \longrightarrow A$$

provides in a canonical way two natural transformations $\beta : Q \otimes_A - \longrightarrow \mathrm{Hom}\,({}_AP\,,\,-)$ and $\alpha : - \otimes_A P \longrightarrow \mathrm{Hom}\,(Q_A\,,\,-)$. Moreover, if M is an A–bimodule then β_M and α_M are bimodule morphisms. The canonical isomorphisms provide two bimodule maps

$$\begin{array}{llll} \beta_A : Q \longrightarrow \mathrm{Hom}\,({}_AP\,,\,{}_AA) = {}^*P, & & \alpha_A : P \longrightarrow \mathrm{Hom}\,(Q_A\,,\,A_A) = Q^* \\ q \longrightarrow [p \mapsto \langle p, q\rangle] & & p \longrightarrow [q \mapsto \langle p, q\rangle] \end{array} \tag{1.1}$$

which are bimodule morphisms. So we can recover the balanced bilinear form if one of the natural transformations is given. The data $\mathcal{T} = (P, Q, \langle -, -\rangle)$ are called a *right rational system over* A if α_M is injective for each right A–module M, and a *left rational system* if β_N is injective for every left A–module N. As was mentioned in [8, Remark 2.2], if $\mathcal{T} = (P, Q, \langle -, -\rangle)$ is a right rational system, then ${}_AP$ is a flat module.

Example 1.1 Recall from [18, Theorem 2.1] that a left A–module P is said to be *locally projective* if for every finite number of elements $p_1, \ldots, p_k \in P$ there exists a finite system $\{(x_i, \varphi_i)\}_{1 \le i \le n} \subset P \times {}^*P$ such that

$$p_j = \sum_{i=1}^{n} \langle p_j, \varphi_i\rangle x_i, \quad \text{for } 1 \le j \le k \quad (\text{here } \varphi_i(p_j) = \langle p_j, \varphi_i\rangle).$$

For instance, if P is an A–bimodule which is locally projective as a left A–module, then one can easily show that the three-tuple $(P, {}^*P, \langle -, -\rangle)$ where $\langle -, -\rangle : P \times {}^*P \to A$ is the evaluation map, is a right rational system, see [2, Lemma 1.29].

A *right rational pairing over* A is a right rational system $\mathcal{T} = (\mathfrak{C}, B, \langle -, -\rangle)$ over A consisting of an A–coring $\mathfrak{C}$ and an A–ring B (i.e., B is a ring extension of A) such that $\beta_A : B \to {}^*\mathfrak{C}$ is a ring anti-morphism where ${}^*\mathfrak{C}$ is the left dual convolution ring of $\mathfrak{C}$ defined in [17, Proposition 3.2]. As one can easily observe, the rational pairings are particular instances of right coring measurings introduced recently in [4].

Example 1.2 Let $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ be an A–coring such that ${}_A\mathfrak{C}$ is a locally projective left module. Consider the left colinear endomorphism ring $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$ as a subring of the linear endomorphism ring $\mathrm{End}({}_A\mathfrak{C})$, that is with multiplication opposite to the usual composition. Since $\Delta_{\mathfrak{C}}$ is left $\mathfrak{C}$–colinear and right A–linear map, the canonical ring extension $A \to \mathrm{End}({}_A\mathfrak{C})$ factors throughout the extension $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) \hookrightarrow \mathrm{End}({}_A\mathfrak{C})$. Therefore, the three-tuple $\mathcal{T} = (\mathfrak{C}, \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}), \langle -, -\rangle)$ where the balanced A-bilinear $\langle -, -\rangle$ is defined by

$$\langle c, f\rangle = \varepsilon_{\mathfrak{C}}(f(c)), \quad \text{for every } (c, f) \in \mathfrak{C} \times \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$$

is a rational pairing since $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$ is already a ring anti-isomorphic to ${}^*\mathfrak{C}$ via the beta map associated to $\langle -, -\rangle$. We refer to $\mathcal{T}$ as *the right canonical pairing* associated to $\mathfrak{C}$.

Given $\mathcal{T} = (\mathfrak{C}, B, \langle -, - \rangle)$ any right rational pairing over A, one can define a functor called a *right rational functor* as follows. Let M be any right B–module. An element $m \in M$ is called *rational* if there exists a set of *right rational parameters* $\{(c_i, m_i)\} \subseteq \mathfrak{C} \times M$ such that $mb = \sum_i m_i \langle c_i, b \rangle$, for all $b \in B$. The set of all rational elements in M is denoted by $\mathrm{Rat}^{\mathcal{T}}(M)$. As it was explained in [8, Sect. 2], the proofs detailed in [10, Section 2] can be adapted in a straightforward way in order to get that $\mathrm{Rat}^{\mathcal{T}}(M)$ is a B–submodule of M and the assignment $M \mapsto \mathrm{Rat}^{\mathcal{T}}(M)$ is a well defined functor

$$\mathrm{Rat}^{\mathcal{T}} : \mathsf{Mod}_B \to \mathsf{Mod}_B,$$

which is in fact a left exact preradical. Therefore, the full subcategory $\mathrm{Rat}^{\mathcal{T}}(\mathsf{Mod}_B)$ of Mod_B whose objects are those B–modules M such that $\mathrm{Rat}^{\mathcal{T}}(M) = M$ is a closed subcategory. Furthermore, $\mathrm{Rat}^{\mathcal{T}}(\mathsf{Mod}_B)$ is a Grothendieck category which is shown to be isomorphic to the category of right comodules $\mathsf{Comod}_{\mathfrak{C}}$ as [8, Theorem 2.6'] asserts. In this way, we still denote

$$\mathrm{Rat}^{\mathcal{T}} : \mathsf{Mod}_B \longrightarrow \mathrm{Rat}^{\mathcal{T}}(\mathsf{Mod}_B) \cong \mathsf{Comod}_{\mathfrak{C}}.$$

2 Coinduction functor in corings

Let $\mathcal{T} = (\mathfrak{C}, B, \langle -, - \rangle)$ be a right rational pairing over A, and $\mathrm{Rat}^{\mathcal{T}} : \mathsf{Mod}_B \to \mathsf{Comod}_{\mathfrak{C}}$ the associated rational functor. We know that there is an adjunction

$$\mathrm{Rat}^{\mathcal{T}} : \mathsf{Mod}_B \rightleftarrows \mathsf{Comod}_{\mathfrak{C}} : i^{\mathcal{T}},$$

where $i^{\mathcal{T}}$ is left adjoint to $\mathrm{Rat}^{\mathcal{T}}$. Consider (Σ, ρ_Σ) a right $\mathfrak{C}$-comodule such that Σ_A is a finitely generated and projective module. Let us denote by $T_\Sigma = \mathrm{End}(\Sigma_{\mathfrak{C}})$ its colinear endomorphism ring. As we have seen in Sect. 1, we also have $T_\Sigma = \mathrm{End}(\Sigma_B)$. It is well known that the coinvariant functor $\mathrm{Hom}_{\mathfrak{C}}(\Sigma, -) : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Mod}_{T_\Sigma}$ is a right adjoint to the tensor product functor $- \otimes_{T_\Sigma} \Sigma : \mathsf{Mod}_{T_\Sigma} \longrightarrow \mathsf{Comod}_{\mathfrak{C}}$. We are interested in looking at a possible right adjoint of the functor $\mathrm{Hom}_{\mathfrak{C}}(\Sigma, -)$. So, if this right adjoint functor exists, then Σ should be a projective right $\mathfrak{C}$-comodule, since $\mathsf{Comod}_{\mathfrak{C}}$ is an abelian category (recall that ${}_A\mathfrak{C}$ is flat by the pairing $\mathcal{T}$). Henceforth, we assume that $\Sigma_{\mathfrak{C}}$ is a finitely generated and projective comodule. Since $i^{\mathcal{T}}$ is left adjoint to $\mathrm{Rat}^{\mathcal{T}}$, if $\mathrm{Rat}^{\mathcal{T}}$ is an exact functor, then $i^{\mathcal{T}}(\Sigma_{\mathfrak{C}}) = \Sigma_B$ is finitely generated and projective right B-module. We denote by $\Sigma^\star = \mathrm{Hom}_B(\Sigma, B)$ its right dual. Then by the rational pairing $\mathcal{T}$, the structure of $(T_\Sigma, \mathfrak{C})$-bicomodule on Σ is equivalent to the structure of (T_Σ, B)-bimodule, and so $\Sigma^\star$ becomes a (B, T_Σ)-bimodule.

Proposition 2.1 *Let $\mathcal{T} = (\mathfrak{C}, B, \langle -, - \rangle)$ be a right rational pairing over a ring A, and (Σ, ρ_Σ) a finitely generated and projective right $\mathfrak{C}$-comodule with endomorphism ring T_Σ. If the rational functor $\mathrm{Rat}^{\mathcal{T}} : \mathsf{Mod}_B \to \mathsf{Comod}_{\mathfrak{C}}$ is exact, then Σ_B is finitely generated and projective module. In particular, the functor*

$$\mathrm{Rat}^{\mathcal{T}} \circ \mathrm{Hom}_{T_\Sigma}\left(\Sigma^\star, -\right) : \mathsf{Mod}_{T_\Sigma} \longrightarrow \mathsf{Comod}_{\mathfrak{C}}$$

is right adjoint to the coinvariant functor $\mathrm{Hom}_{\mathfrak{C}}(\Sigma\,,\,-) : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Mod}_{T_\Sigma}$.

Proof We have seen in the preamble of this Section that Σ_B should be finitely generated and projective. Therefore, the stated adjunction follows from the following composition of adjoint pairs of functors

$$\mathsf{Mod}_{T_\Sigma} \underset{\mathrm{Hom}_B(\Sigma\,,\,-)}{\overset{\mathrm{Hom}_{T_\Sigma}(\Sigma^\star\,,\,-)}{\rightleftarrows}} \mathsf{Mod}_B \underset{i^{\mathcal{T}}}{\overset{\mathrm{Rat}^{\mathcal{T}}}{\rightleftarrows}} \mathsf{Comod}_{\mathfrak{C}},$$

since we know that $\mathrm{Hom}_B(\Sigma\,,\,-) \circ i^{\mathcal{T}} = \mathrm{Hom}_{\mathfrak{C}}(\Sigma\,,\,-)$. □

From now on, we set

$$\mathsf{Coind}^\Sigma(-) := \mathrm{Rat}^{\mathcal{T}} \circ \mathrm{Hom}_{T_\Sigma}\left(\Sigma^\star\,,\,-\right) : \mathsf{Mod}_{T_\Sigma} \longrightarrow \mathsf{Comod}_{\mathfrak{C}}$$

and refer to as *the coinduction functor* associated to the finitely generated and projective comodule $\Sigma_{\mathfrak{C}}$. Fix a right dual basis $\{(u_j,\, u_j^\star)\}_j$ for Σ_B. The unit and counit of the adjunction stated in Proposition 2.1 are given as follows:

$$\begin{aligned} &\xi_{Y_{T_\Sigma}} : \mathrm{Hom}_{\mathfrak{C}}\left(\Sigma\,,\, \mathsf{Coind}^\Sigma(Y)\right) \longrightarrow Y, \quad \left(f \longmapsto \sum_j f(u_j)\left(u_j^\star\right)\right), \\ &\eta_{X_{\mathfrak{C}}} : X \longrightarrow \mathsf{Coind}^\Sigma\left(\mathrm{Hom}_{\mathfrak{C}}(\Sigma\,,\,X)\right), \quad \left(x \longmapsto \left[u^\star \mapsto [v \mapsto xu^\star(v)]\right]\right) \end{aligned} \tag{2.1}$$

for every comodule $X_{\mathfrak{C}}$ and module Y_{T_Σ}. The counit ξ_- is actually a natural isomorphism. Effectively, for every right T_Σ-module Y, we have

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{C}}\left(\Sigma\,,\,\mathsf{Coind}^\Sigma(Y)\right) &= \mathrm{Hom}_B\left(\Sigma\,,\,\mathrm{Hom}_{T_\Sigma}\left(\Sigma^\star\,,\,Y\right)\right) \\ &\cong \mathrm{Hom}_{T_\Sigma}\left(\Sigma\otimes_B\Sigma^\star\,,\,Y\right), \quad T_\Sigma \cong \Sigma\otimes_B\Sigma^\star \\ &\cong \mathrm{Hom}_{T_\Sigma}(T_\Sigma\,,\,Y) \cong Y_{T_\Sigma}, \end{aligned}$$

and the composition of those isomorphisms is exactly ξ_Y defined in (2.1).

Let us set

$$\mathscr{C}_\Sigma = \mathrm{Ker}(\mathrm{Hom}_{\mathfrak{C}}(\Sigma\,,\,-)) = \{L \in \mathsf{Comod}_{\mathfrak{C}} \mid \mathrm{Hom}_{\mathfrak{C}}(\Sigma\,,\,L) = 0\}.$$

Consider

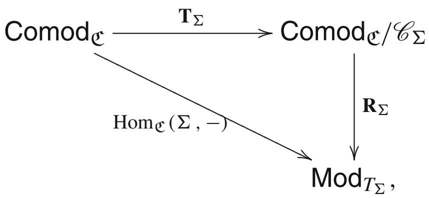
$$\mathscr{T}_\Sigma = \{M \in \mathsf{Comod}_{\mathfrak{C}} \mid \mathrm{Hom}_{\mathfrak{C}}(M\,,\,L) = 0,\ \forall L \in \mathscr{C}_\Sigma\}$$

the torsion class associated to the torsion-free class $\mathscr{C}_\Sigma$. The corresponding idempotent radical is

$$\mathfrak{r}_\Sigma : \mathsf{Comod}_{\mathfrak{C}} \longrightarrow \mathsf{Comod}_{\mathfrak{C}}, \quad \left(M \mapsto \mathfrak{r}_\Sigma(M) = \sum \{\mathrm{Im}(f) \mid f : N \to M,\ N \in \mathscr{T}_\Sigma\}\right). \tag{2.2}$$

The following corollary is a direct consequence of [9, Proposition 5, p. 374], since $\Sigma_{\mathfrak{C}}$ is projective and ξ_- is a natural isomorphism. For the definition of TTF-class in Grothendieck categories we refer the reader to [16, Chap. VI].

Corollary 2.2 *With assumptions as in* Proposition 2.1, *the full subcategory $\mathscr{C}_\Sigma$ is a TTF-class, and there is a commutative diagram*



where $\mathbf{T}_\Sigma$ is the localizing functor and $\mathbf{R}_\Sigma$ is an equivalence of categories with inverse $\mathbf{L}_\Sigma = \mathbf{T}_\Sigma \circ \mathsf{Coind}^\Sigma$.

3 Simple comodules

In this section, we make some observations on simple right $\mathfrak{C}$-comodules, which are also $\mathscr{T}_\Sigma$-torsion comodules. To this end the following two lemmata will be needed.

Lemma 3.1 *Let $\mathcal{G}$ be a Grothendieck category and $\mathscr{C}$ a TTF-class of $\mathcal{G}$ with torsion theory $(\mathscr{T}, \mathscr{C})$ and associated radical functor $\mathfrak{r} : \mathcal{G} \to \mathcal{G}$. Let $\mathbf{T} : \mathcal{G} \rightleftarrows \mathcal{G}/\mathscr{C} : \mathbf{S}$ be the canonical adjunction of localization. Consider an object N of $\mathcal{G}$ such that $\mathfrak{r}(N) \neq 0$. If $\mathbf{T}(N)$ is a simple object of $\mathcal{G}/\mathscr{C}$, then $\mathfrak{r}(N)$ is a simple object of $\mathcal{G}$ as well.*

Proof Let $\alpha : M \hookrightarrow \mathfrak{r}(N)$ be a non zero monomorphism of $\mathcal{G}$. Since $\mathbf{T}$ is an exact functor, we get a monomorphism $\mathbf{T}(\alpha) : \mathbf{T}(M) \hookrightarrow \mathbf{T}(\mathfrak{r}(N))$. Clearly $\mathbf{T}(\alpha) \neq 0$ since $\mathrm{Im}(\alpha)$ is not an object of $\mathscr{C}$. On the other hand, we know that $N/\mathfrak{r}(N) \in \mathscr{C}$, which means that $\mathbf{T}(\mathfrak{r}(N)) \cong \mathbf{T}(N)$. Hence $\mathbf{T}(\alpha)$ is an isomorphism, since $\mathbf{T}(N)$ is assumed to be simple. Consequently $\mathrm{Coker}(\alpha) = \mathfrak{r}(N)/\mathrm{Im}(\alpha) \in \mathscr{C}$ which implies that $N/\mathrm{Im}(\alpha) \in \mathscr{C}$. Whence $\mathfrak{r}(N) = \mathrm{Im}(\alpha)$, and so α is an isomorphism. Therefore, $\mathfrak{r}(N)$ is a simple object of $\mathcal{G}$. □

Lemma 3.2 *Let B be any ring and Σ_B a finitely generated and projective module with endomorphism ring $T = \mathrm{End}(\Sigma_B)$ and right dual module $\Sigma^\star = \mathrm{Hom}_B(\Sigma, B)$. Consider a maximal right ideal I_B of B_B and assume that $(I\Sigma^\star)_T \lneqq \Sigma^\star_T$. Then $(I\Sigma^\star)_T$ is a maximal submodule of $\Sigma^\star_T$.*

Proof It is sufficient to show that, for any element $u^\star \in \Sigma^\star \backslash I\Sigma^\star$, we have

$$I\Sigma^\star \;+\; u^\star T \;=\; \Sigma^\star. \tag{3.1}$$

Let $\{u_i, u_i^\star\}_{1\leq i\leq n}$ be a finite dual basis for Σ_B, and choose an arbitrary element $u^\star \in \Sigma^\star\backslash I\Sigma^\star$. We know that $u^\star = \sum_1^n u^\star(u_i)u_i^\star$ and $\{1,\ldots,n\} = \Lambda \uplus \Lambda'$ where $(i \in \Lambda \Rightarrow u^\star(u_i) \in I)$ and $(i \in \Lambda' \Rightarrow u^\star(u_i) \notin I)$ (by assumption we have $\Lambda \subsetneqq \{1,\ldots,n\}$). Since I_B is maximal, we have

$$I \;+\; \sum_{i\in\Lambda'} u^\star(u_i)\, B \;=\; B.$$

Hence

$$I\Sigma^\star \;+\; \sum_{i\in\Lambda'} u^\star(u_i)\Sigma^\star \;=\; \Sigma^\star \quad \text{in the category of modules } \mathsf{Mod}_T. \tag{3.2}$$

Consider $K := \sum_{i\in\Lambda'} u^\star(u_i)\Sigma^\star$ as right T-module. It is easily checked that the following T-linear map is a monomorphism

$$\varsigma:\; K \longrightarrow u^\star T, \qquad \left(\sum_{i\in\Lambda'} u^\star(u_i)v_i^\star \longmapsto u^\star\left(\sum_{i\in\Lambda'} t_i\right)\right),$$

where the t_i's are elements of T given by

$$t_i : \Sigma \longrightarrow \Sigma, \quad [x \mapsto u_i v_i^\star(x)], \;\; \forall i \in \Lambda'.$$

Therefore, equality (3.2) implies equality (3.1). □

Theorem 3.3 *Let $\mathcal{T} = (\mathfrak{C}, B, \langle -, -\rangle)$ be a right rational pairing over A with exact rational functor. Consider a finitely generated and projective right $\mathfrak{C}$-comodule Σ whose endomorphism ring is T_Σ, and with associated idempotent radical $\mathfrak{r}_\Sigma$ as in* Sect. 2.

(a) *If Y_{T_Σ} is a simple module, then $\mathfrak{r}_\Sigma(\mathsf{Coind}^\Sigma(Y))_\mathfrak{C}$ is a simple comodule, and*

$$Y_{T_\Sigma} \;\cong\; \mathrm{Hom}_\mathfrak{C}\Big(\Sigma\,,\; \mathfrak{r}_\Sigma\Big(\mathsf{Coind}^\Sigma(Y)\Big)\Big)_{T_\Sigma}.$$

Moreover, given two simple right T_Σ-modules Y and Y', we have

$$Y_{T_\Sigma} \;\cong Y'_{T_\Sigma} \;\Longleftrightarrow\; \mathfrak{r}_\Sigma\Big(\mathsf{Coind}^\Sigma(Y)\Big)_\mathfrak{C} \cong \mathfrak{r}_\Sigma\Big(\mathsf{Coind}^\Sigma(Y')\Big)_\mathfrak{C}.$$

(b) *If $S_\mathfrak{C}$ is a simple comodule such that* $\mathrm{Hom}_\mathfrak{C}(\Sigma\,,\, S) \neq 0$. *Then* $\mathrm{Hom}_\mathfrak{C}(\Sigma\,,\, S)_{T_\Sigma}$ *is a simple module. Furthermore, there is an isomorphism of right comodules*

$$S_\mathfrak{C} \;\cong\; \mathfrak{r}_\Sigma\Big(\mathsf{Coind}^\Sigma\Big(\mathrm{Hom}_\mathfrak{C}(\Sigma\,,\, S)\Big)\Big)_\mathfrak{C}.$$

Proof (a) Suppose that $\mathfrak{r}_\Sigma\left(\mathsf{Coind}^\Sigma(Y)\right) = 0$, that is, $\mathsf{Coind}^\Sigma(Y) \in \mathscr{C}_\Sigma$. So

$$Y_{T_\Sigma} \cong \mathrm{Hom}_{\mathfrak{C}}\left(\Sigma\,,\, \mathsf{Coind}^\Sigma(Y)\right)_{T_\Sigma} = 0, \quad (\text{via } \xi_Y,\ \text{Eq. } 2.1).$$

This is a contradiction since we know that $Y \neq 0$. Henceforth, $\mathfrak{r}_\Sigma(\mathsf{Coind}^\Sigma(Y)) \neq 0$, and clearly $Y \cong \mathrm{Hom}_{\mathfrak{C}}(\Sigma\mathfrak{r}_\Sigma(\mathsf{Coind}^\Sigma(Y)))$. In order to check that $\mathfrak{r}_\Sigma(\mathsf{Coind}^\Sigma(Y))$ is a simple right T_Σ-module, it suffices to show by Lemma 3.1 that $\mathbf{T}_\Sigma(\mathsf{Coind}^\Sigma(Y))$ is a simple object in the quotient category $\mathsf{Comod}_{\mathfrak{C}}/\mathscr{C}_\Sigma$. This is fulfilled since by Corollary 2.2, $\mathbf{L}_\Sigma = \mathbf{T}_\Sigma \circ \mathsf{Coind}^\Sigma$ is an equivalence between the categories Mod_{T_Σ} and $\mathsf{Comod}_{\mathfrak{C}}/\mathscr{C}_\Sigma$. The implication ($\Rightarrow$) of the stated equivalence is obvious. Let us check the converse one. So assume we are given Y and Y' two simple right T_Σ-modules such that $\mathfrak{r}_\Sigma(\mathsf{Coind}^\Sigma(Y)) \cong \mathfrak{r}_\Sigma(\mathsf{Coind}^\Sigma(Y'))$. Applying the functor $\mathbf{T}_\Sigma$ of Corollary 2.2, we obtain the following chain of isomorphisms

$$\begin{aligned}\mathbf{T}_\Sigma\left(\mathsf{Coind}^\Sigma(Y)\right) &\cong \mathbf{T}_\Sigma\left(\mathfrak{r}_\Sigma(\mathsf{Coind}^\Sigma(Y))\right) \cong \mathbf{T}_\Sigma\left(\mathfrak{r}_\Sigma(\mathsf{Coind}^\Sigma(Y'))\right)\\ &\cong \mathbf{T}_\Sigma\left(\mathsf{Coind}^\Sigma(Y')\right),\end{aligned}$$

where we have used the same argument as in the proof of Lemma 3.1 for the first and the third isomorphisms. Now, since $\mathbf{T}_\Sigma \circ \mathsf{Coind}^\Sigma$ is an equivalence of categories, we get $Y \cong Y'$ as right T_Σ-modules.

(b) Since Σ_B is a finitely generated and projective module by Proposition 2.1, we have $\mathrm{Hom}_{\mathfrak{C}}(\Sigma\,,\, S)_{T_\Sigma}$ is a simple module if and only if $(S \otimes_B \Sigma^\star)_{T_\Sigma}$ is so. On the other hand, $S_B \cong (B/I)_B$, for some maximal right ideal I_B of B_B. Hence,

$$(S \otimes_B \Sigma^\star)_{T_\Sigma} \cong \left((B/I) \otimes_B \Sigma^\star\right)_{T_\Sigma} \cong \left(\Sigma^\star/I\Sigma^\star\right)_{T_\Sigma},$$

and so $\mathrm{Hom}_B(\Sigma\,,\, S) \neq 0$ implies $(I\Sigma^\star)_{T_\Sigma} \lneqq \Sigma^\star_{T_\Sigma}$. The hypothesis of Lemma 3.2 is then fulfilled, which implies that $(\Sigma^\star/I\Sigma^\star)_{T_\Sigma}$ is a simple module and so is $(S \otimes_B \Sigma^\star)_{T_\Sigma}$. Therefore, $\mathfrak{r}_\Sigma(\mathsf{Coind}^\Sigma(\mathrm{Hom}_{\mathfrak{C}}(\Sigma\,,\, S)))$ is a simple right $\mathfrak{C}$-comodule, by item (a). Lastly, the stated isomorphism of comodules is deduced from the natural transformation η_- of Eq. 2.1 as follows

$$\mathfrak{r}_\Sigma(\eta_S) : S = \mathfrak{r}_\Sigma(S) \cong \mathfrak{r}_\Sigma\left(\mathsf{Coind}^\Sigma\left(\mathrm{Hom}_{\mathfrak{C}}(\Sigma\,,\, S)\right)\right).$$

□

Let us consider the following two sets: $\mathscr{S}_\Sigma$ the set of isomorphism classes $[S]$ represented by simple right $\mathfrak{C}$-comodules S such that $\mathrm{Hom}_{\mathfrak{C}}(\Sigma\,,\, S) \neq 0$ and $\mathscr{S}_{T_\Sigma}$ the set of isomorphism classes $[Y]$ represented by simple right T_Σ-modules Y. Using Theorem 3.3, we get

Corollary 3.4 *Let $\mathcal{T} = (\mathfrak{C}, B, \langle -, - \rangle)$ be a right rational pairing over A with exact rational functor. Consider a finitely generated and projective right $\mathfrak{C}$-comodule Σ*

whose endomorphism ring is T_Σ. Then there is a bijective map given by

$$\begin{array}{ccc} \mathscr{S}_\Sigma & \longrightarrow & \mathscr{S}_{T_\Sigma} \\ [S] & \longmapsto & [\mathrm{Hom}_{\mathfrak{C}}(\Sigma\,,\,S)] \\ \left[\mathfrak{r}_\Sigma \mathsf{Coind}^\Sigma(Y)\right] & \longleftarrow & [Y]. \end{array}$$

4 Application to group-graded modules

We show that the coinduction functor [12] in group-graded modules involves a rational functor associated to a suitable rational pairing. This pairing is constructed using the smash product [6,15] and the canonical coring arising from a group-graded base ring. Using this functor, we will apply the results of Sect. 3 to simple group-graded right modules.

In what follows, we consider a group G with neutral element e and a G-graded base ring $A = \oplus_{x\in G} A_x$. It is well known that the free left A-module with basis G, denoted by $\mathfrak{C} = AG$, admits a right A-action given by the rule $xa_y = a_y(xy)$ for every homogeneous element $a_y \in A_y$ and every $x, y \in G$. It turns out that $\mathfrak{C}$ is in fact an A-bimodule equipped with an A-coring structure with comultiplication $\Delta(ax) = ax \otimes_A x$ and counit $\varepsilon(ax) = a$, for every $x \in G$ and $a \in A$. Of course G is contained in the set of all grouplike elements of $\mathfrak{C}$; recall that a grouplike element of an A-coring $\mathfrak{C}$ is an element $g \in \mathfrak{C}$ such that $\varepsilon_{\mathfrak{C}}(g) = 1$ and $\Delta_{\mathfrak{C}}(g) = g \otimes_A g$.

In this way the category of all right graded A-module $\mathsf{gr}\text{-}A$ is isomorphic to the category of all right comodules $\mathsf{Comod}_{\mathfrak{C}}$. This isomorphism, over objects, is given as follows. For every $x \in G$, we consider the right $\mathfrak{C}$-comodule $([x]A, \rho_{[x]A})$ whose underlying right A-module coincides with A_A, i.e. $([x]A)_A = A_A$ and its coaction map is $\rho_{[x]A} : [x]A \to \mathfrak{C} \cong A \otimes_A \mathfrak{C}$ sending $a \mapsto xa$. Given a right comodule (M, ρ_M), M decomposes as $M = \oplus_{x\in G} M^{\mathrm{cov}(x)}$, where

$$M^{\mathrm{cov}(x)} = \{m \in M \,|\, \rho_M(m) = m \otimes_A x\}, \quad \text{for every } x \in G.$$

One can easily check that this gives M a structure of a right graded A-module, and there is an isomorphism of abelian groups

$$M^{\mathrm{cov}(x)} \cong \mathrm{Hom}_{\mathfrak{C}}([x]A\,,\,M)\,, \quad \text{for every } x \in G. \tag{4.1}$$

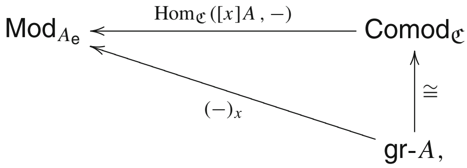
In particular, we have

$$\mathrm{Hom}_{\mathfrak{C}}([x]A\,,\,[y]A) = \left\{a \in A \,|\, ax = ya\right\} = A_{xy^{-1}}, \quad \text{for every } x, y \in G,$$

which implies that the endomorphism ring of each right comodules $[x]A$ is isomorphic to A_{e}. That is,

$$\mathrm{End}_{\mathfrak{C}}([x]A) \cong A_{\mathsf{e}}, \quad \text{for every } x \in G.$$

Conversely, given $N = \oplus_{x \in G} N_x$ a graded right A-module, one can endow this module with a right $\mathfrak{C}$-coaction given by $\rho_N(n_x) = n_x \otimes_A x$, for every homogeneous element $n_x \in N_x$, $x \in G$. The isomorphism of categories $\mathsf{Comod}_{\mathfrak{C}} \cong \mathsf{gr}\text{-}A$ is now established taking into account that its action on morphisms is an identity. Moreover, for every element $x \in G$, we have a commutative diagram of functors



where $(-)_x : \mathsf{gr}\text{-}A \to \mathsf{Mod}_{A_e}$ is the functor which sends every right graded A-module $N = \oplus_{x \in G} N_x$ to $(N)_x = N_x$, its homogeneous component of degree x.

On the other hand, for each $x \in G$, the left A-submodule $P_x = Ax$ of $\mathfrak{C}$ generated by x is clearly a left $\mathfrak{C}$-comodule with coaction sending $\lambda_{P_x}(ax) = ax \otimes_A x \in \mathfrak{C} \otimes_A P_x$, $a \in A$. Therefore, ${}_{\mathfrak{C}}\mathfrak{C} = \oplus_{x \in G} P_x$ is a direct sum of left $\mathfrak{C}$-comodules. For any $x \in G$, we denote by $e_x : \mathfrak{C} \twoheadrightarrow P_x \hookrightarrow \mathfrak{C}$ the composition of the canonical injection and projection in this direct sum. The e_x's form a set of orthogonal idempotent elements in the endomorphism ring $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$. The ring extension $\widetilde{(-)} : A \to \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$ which sends any element $a \in A$ to the map $\widetilde{a} : c \mapsto ca$, is a monomorphism as ${}_A\mathfrak{C}$ is a free module. We thus identify A with its image $\widetilde{A}$.

Proposition 4.1 *Let A be a G-graded ring by a group G. Consider the associated A-coring $\mathfrak{C} = AG$, and denote by B the subring of* $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$ *generated by $\widetilde{A}$ and the set of orthogonal idempotents $\{e_x\}_{x \in G}$. Then*

(i) *B_A is a free right module with basis $\{1, e_x\}_{x \in G}$, that is, $B = \widetilde{A} \oplus \big(\oplus_{x \in G}\, e_x \widetilde{A}\big)$.*
(ii) *There is a right rational pairing $\mathcal{T} = (\mathfrak{C}, B, \langle -, - \rangle)$, where the bilinear form is defined by the rule:*

$$\langle x, e_y \rangle \;=\; \delta_{x,y} \text{ (the Kronecker delta) and } \langle x, \widetilde{a} \rangle \;=\; a,$$

for every $x, \ y \in G$, $a \in A$, and satisfies

$$c \;=\; \sum_{z \in G} \langle c, e_z \rangle\, z, \text{ (finite sum) for every } c \in \mathfrak{C}.$$

(iii) *Let $\mathrm{Rat}^{\mathcal{T}} : \mathsf{Mod}_B \to \mathsf{Comod}_{\mathfrak{C}}$ be the rational functor associated to the pairing $\mathcal{T}$ of item (ii). Then the trace ideal is $\mathrm{Rat}^{\mathcal{T}}(B_B) = \oplus_{x \in G}\, e_x \widetilde{A}$, and the functor $\mathrm{Rat}^{\mathcal{T}}$ is exact. In particular, for every right B-module M, we have*

$$\mathrm{Rat}^{\mathcal{T}}(M_B) \;=\; M\, \mathrm{Rat}^{\mathcal{T}}(B_B) \;=\; \oplus_{x \in G}\, M e_x.$$

Proof (i) This was proved in [15] and [12], using the fact that $\mathrm{End}({}_A\mathfrak{C})$ is isomorphic to the ring of finite rows $|G| \times |G|$-matrices over A. Here, we can restrict ourselves to the colinear endomorphism ring $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$. The statement of this item follows then from the following two facts. The first is that, for every $x, y \in G$, $a \in A$, $a_y \in A_y$, we have

$$e_x\, \widetilde{a}\, e_y \;=\; \widetilde{\pi_{y^{-1}x}(a)}\, e_y, \qquad e_x\, \widetilde{a_y} \;=\; \widetilde{a_y}\, e_{xy}, \tag{4.2}$$

where $\pi_z : A \to A_z$, $z \in G$ are the canonical projections. The second is that, if $a \in A$ and $e_x \widetilde{a} = 0$, for some $x \in G$, then $a = 0$.

(ii) To show that $\langle -, - \rangle$ is well defined, we only prove that $\langle -, - \rangle$ is A-balanced. So, let $a = \sum_{z \in G} \pi_z(a) \in A$ and $x, y \in G$, we have

$$\langle xa, e_y \rangle \;=\; \sum_{z \in G} \pi_z(a) \langle xz, e_y \rangle \;=\; \sum_{z \in G} \pi_z(a) \delta_{xz,\, y} \;=\; \pi_{x^{-1}y}(a),$$

and

$$\begin{aligned}\langle x, \widetilde{a} e_y \rangle \;&=\; \sum_{z \in G} \langle x, \widetilde{\pi_z(a)} e_y \rangle \;=\; \sum_{z \in G} \langle x, e_{yz^{-1}} \rangle \pi_z(a) \;=\; \sum_{z \in G} \pi_z(a) \delta_{x,\, yz^{-1}} \\ &= \pi_{x^{-1}y}(a),\end{aligned}$$

where we have used Eq. 4.2. Hence, $\langle xa, e_y \rangle = \langle x, \widetilde{a} e_y \rangle$, for every $x, y \in G$ and $a \in A$ which implies that $\langle -, - \rangle$ is well defined. Given $c \in \mathfrak{C}$, we denote as usual by $\mathrm{Supp}(c) \subseteq G$ (finite subset) the support of c, that is, $z \in \mathrm{Supp}(c) \Leftrightarrow \kappa_z(c) \neq 0$, where $\kappa_x : \mathfrak{C} \to Ax$, $x \in G$ are the canonical projections. Obviously, $c = \sum_{z \in \mathrm{Supp}(c)} a^{c,\, z} z$ with $a^{c,\, z} \in A$, and if $z_0 \in \mathrm{Supp}(c)$, then

$$\langle c, e_{z_0} \rangle = \sum_{z \in \mathrm{Supp}(c)} a^{c,\, z} \delta_{z,\, z_0} = a^{c,\, z_0}.$$

Therefore, $c = \sum_{z \in \mathrm{Supp}(c)} \langle c, e_z \rangle z$ as claimed. Now, the natural transformation associated to this bilinear form is then given by

$$\alpha_M : M \otimes_A \mathfrak{C} \longrightarrow \mathrm{Hom}_A(B,\, M),\;\; (m \otimes_A c \longmapsto [b \mapsto m\langle c, b \rangle]),$$

for every right A-module M. Suppose that $\alpha_M\left(\sum_i^n m_i \otimes_A c_i\right) = 0$, and put $H = \cup_{i=1}^n \mathrm{Supp}(c_i)$, then $c_i = \sum_{z \in H} \langle c_i, e_z \rangle z$, for all $i = 1, \ldots, n$. We thus obtain

$$\sum_{i=1}^n m_i \otimes_A c_i \;=\; \sum_{i=1}^n \sum_{z \in H} m_i \otimes_A \langle c_i, e_z \rangle z \;=\; \sum_{z \in H} \left(\sum_{i=1}^n m_i \langle c_i, e_z \rangle \right) \otimes_A z \;=\; 0,$$

since $\alpha_M\left(\sum_i^n m_i \otimes_A c_i\right)(e_z) = \sum_i m_i \langle c_i, e_z \rangle = 0$, for all $z \in G$. This shows that α_M is injective for every right A-module M. Remaining axioms are obviously satisfied, which shows that $\mathcal{T}$ is a rational pairing.

(iii) By Eq. 4.2, we know that $\oplus_{x \in G} e_x \widetilde{A}$ is a two-sided idempotent ideal of B which is left and right pure B-submodule of B. Now, for every $x \in G$, $\left(e_x \widetilde{A}, \rho_{e_x \widetilde{A}}\right)$ is a right $\mathfrak{C}$-comodule with coaction $\rho_{e_x\widetilde{A}} : e_x\widetilde{A} \to e_x\widetilde{A} \otimes_A \mathfrak{C}$ sending $e_x\widetilde{a} \mapsto e_x \otimes_A xa$. Thus, we have an inclusion $\oplus_{x\in G} e_x\widetilde{A} \subseteq \mathrm{Rat}^{\mathcal{T}}(B_B)$. Using the definition of rational elements (Sect. 1), item (i) and the second equation of (4.2), one can show that $\mathrm{Rat}^{\mathcal{T}}(B_B) \cap \widetilde{A} = \{0\}$. Therefore, $\oplus_{x\in G} e_x\widetilde{A} = \mathrm{Rat}^{\mathcal{T}}(B_B)$. The rational functor $\mathrm{Rat}^{\mathcal{T}}$ is now exact by [7, Theorem 1.2] as $\mathfrak{C}\,\mathrm{Rat}^{\mathcal{T}}(B_B) = \mathfrak{C}$. Using again [7, Theorem 1.2] and Eq. 4.2, we obtain $\mathrm{Rat}^{\mathcal{T}}(M) = M\,\mathrm{Rat}^{\mathcal{T}}(B_B) = \oplus_{x\in G} Me_x$, for every right B-module M. □

Remark 4.2 By [7, Theorem 1.2] and Proposition 4.1, we know that the category

$$\mathrm{Rat}^{\mathcal{T}}(\mathsf{Mod}_B) = \{M \in \mathsf{Mod}_B |\ M = \oplus_{x\in G} Me_x\}$$

is a localizing sub-category of Mod_B, this is [13, Proposition 1.1]. The adjoint pair of functors constructed in [13, Section 1], can be easily obtained as follows. From the proof of Proposition 4.1, it is easily seen that each of the right A-modules $e_x\widetilde{A}$ admits a structure of right B-module coming from its right $\mathfrak{C}$-coaction. Moreover, $e_x\widetilde{A}$ and $[x]A$ are isomorphic as right $\mathfrak{C}$-comodules, for every $x \in G$. Thus the trace ideal $\mathfrak{a} = \mathrm{Rat}^{\mathcal{T}}(B_B) \cong \oplus_{x\in G}[x]A$ as right $\mathfrak{C}$-comodules. Therefore, the right adjoint functor $F = \mathrm{Hom}_B(\mathfrak{a}, -) \circ i^{\mathcal{T}} : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Mod}_B$ of $\mathrm{Rat}^{\mathcal{T}} : \mathsf{Mod}_B \to \mathsf{Comod}_{\mathfrak{C}}$ is then naturally isomorphic to the functor

$$M_{\mathfrak{C}} \longrightarrow \prod_{x\in G} M^{\mathrm{cov}(x)}, \quad \text{and} \quad f \longrightarrow \left(f_{M^{\mathrm{cov}(x)}}\right)_{x\in G},$$

where, for every $x \in G$, $f_{M^{\mathrm{cov}(x)}}$ is the restriction of f to $M^{\mathrm{cov}(x)}$.

Recall, from [1,12], the coinduction functor $\mathsf{Coind}^x : \mathsf{Mod}_{A_{\mathsf{e}}} \to \mathsf{gr}\text{-}A$ which is given, over objects, by

$$\mathsf{Coind}^x(N) = \bigoplus_{y\in G} \mathrm{Hom}_{A_{\mathsf{e}}}\left(A_{y^{-1}x}, N\right),$$

for every right A_{e}-module N. The following shows that the functor Coind^x coincides, up to the isomorphism $\mathsf{Comod}_{\mathfrak{C}} \cong \mathsf{gr}\text{-}A$, with the coinduction functor associated to the finitely generated projective right $\mathfrak{C}$-comodule $[x]A$. Of course this involves the rational functor $\mathrm{Rat}^{\mathcal{T}}$ constructed in Proposition 4.1.

Proposition 4.3 *Let A be a G-graded ring by a group G with neutral element* e, $\mathfrak{C} = AG$ *its associated A-coring, and* $\mathcal{T} = (\mathfrak{C}, B, \langle -, - \rangle)$ *the rational pairing of* Proposition 4.1(ii). *Consider for some $x \in G$ the right $\mathfrak{C}$-comodule $[x]A$ and its right B-dual* $([x]A)^{\star} = \mathrm{Hom}_B([x]A, B)$ *as a* (B, A_{e})*-bimodule. Then the functor* $\mathrm{Rat}^{\mathcal{T}} \circ \mathrm{Hom}_{A_{\mathsf{e}}}(([x]A)^{\star}, -) : \mathsf{Mod}_{A_{\mathsf{e}}} \to \mathsf{Comod}_{\mathfrak{C}}$ *is right adjoint to the coinvariant functor*

$\mathrm{Hom}_{\mathfrak{C}}([x]A\,,\,-) : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Mod}_{A_{\mathsf{e}}}$. *Moreover, we have a commutative diagram*

$$\begin{array}{ccc} \mathsf{Mod}_{A_{\mathsf{e}}} & \xrightarrow{\mathrm{Rat}^{\mathcal{T}} \circ \mathrm{Hom}_{A_{\mathsf{e}}}\left(([x]A)^{\star}\,,\,-\right)} & \mathsf{Comod}_{\mathfrak{C}} \\ & \searrow^{\mathsf{Coind}^{x}} & \downarrow \cong \\ & & \mathsf{gr}\text{-}A. \end{array}$$

Proof The first statement is a direct consequence of Propositions 2.1 and 4.1(iii), since we know by the isomorphism of Eq. 4.1 that $[x]A$ is a finitely generated and projective right $\mathfrak{C}$-comodule.

Let N be any right A_{e}-module and $x \in G$. Then, we have

$$\begin{aligned} \mathrm{Rat}^{\mathcal{T}}\left(\mathrm{Hom}_{A_{\mathsf{e}}}\left(([x]A)^{\star}\,,\,N\right)\right) &= \oplus_{y\in G}\mathrm{Hom}_{A_{\mathsf{e}}}\left(([x]A)^{\star}\,,\,N\right)e_{y}, \quad \text{by Proposition 4.1 (iii)} \\ &\cong \oplus_{y\in G}\mathrm{Hom}_{B}\left(e_{y}B\,,\,\mathrm{Hom}_{A_{\mathsf{e}}}\left(([x]A)^{\star}\,,\,N\right)\right) \\ &\cong \oplus_{y\in G}\mathrm{Hom}_{A_{\mathsf{e}}}\left(e_{y}B\otimes_{B}([x]A)^{\star}\,,\,N\right) \\ &\cong \oplus_{y\in G}\mathrm{Hom}_{A_{\mathsf{e}}}\left(e_{y}([x]A)^{\star}\,,\,N\right). \end{aligned}$$

On the other hand, one can prove that the mutually inverse maps

$$\begin{array}{ccc} e_{y}\,([x]A)^{\star} & \longrightarrow & A_{y^{-1}x} \\ \left[e_{y}\sigma \longmapsto \langle y, \sigma(1)\rangle\right] & & \left[\,[a \mapsto e_{y}\,\widetilde{a_{y^{-1}x}a}] \longleftarrow\!\shortmid\, a_{y^{-1}x}\right] \end{array}$$

establish an isomorphism of right A_{e}-modules. Therefore, we obtain a natural isomorphism

$$\mathrm{Rat}^{\mathcal{T}}\left(\mathrm{Hom}_{A_{\mathsf{e}}}\left(([x]A)^{\star}\,,\,N\right)\right) \;\cong\; \oplus_{y\in G}\mathrm{Hom}_{A_{\mathsf{e}}}\left(A_{y^{-1}x}\,,\,N\right) \;=\; \mathsf{Coind}^{x}(N).$$

This finishes the proof since the compatibility, in relation with arrows, is clear. □

Let $x \in G$ and consider its associated right $\mathfrak{C}$-comodule $[x]A$. Denote by $\mathscr{C}_x$ and $\mathscr{T}_x$, respectively, the torsion-free class and torsion class attached to this comodule, see Sect. 3. Up to the isomorphism of categories $\mathsf{Comod}_{\mathfrak{C}} \cong \mathsf{gr}\text{-}A$, it is clear that a G-graded module M belongs to $\mathscr{C}_x$ if and only if its x-homogeneous component vanishes. That is, $M \in \mathscr{C}_x$ if and only if $M_x = 0$. A more specific computation of the torsion part of any G-graded right module can be given as follows. First denote by $\mathfrak{r}_x$ the associated idempotent radical as in Eq. 2.2.

Lemma 4.4 *Let A be a graded ring by a group G with neutral element e. Then, for every element $x \in G$ and every G-graded right A-module M, we have*

$$\mathfrak{r}_x(M) \;=\; \mathrm{Im}\left(\oplus_{y\in G}\left(M_x \otimes_{A_{\mathsf{e}}} A_{x^{-1}y}\right) \longrightarrow M\right),$$

wherein the map is the graded morphism given by the right action of A.

Proof Fix an element x in the group G. Let us first check that $\oplus_{y \in G}(M_x \otimes_{A_{\mathsf{e}}} A_{x^{-1}y})$ is an object in $\mathscr{T}_x$, for every G-graded right A-module M. So given a G-graded right module Y in $\mathscr{C}_x$, and a graded map $f : \oplus_{y \in G}(M_x \otimes_{A_{\mathsf{e}}} A_{x^{-1}y}) \to Y$, we need to show that $f = 0$. Since $Y_x = 0$, it is obvious that $f(M_x \otimes_{A_{\mathsf{e}}} A_{\mathsf{e}}) = 0$. Taking an element in $M_x \otimes_{A_{\mathsf{e}}} A_{x^{-1}y}$ of the form $m \otimes_{A_{\mathsf{e}}} a$, we can find finite sets $\{a_i\} \subset A_{\mathsf{e}}$ and $\{a'_i\} \subset A_{x^{-1}y}$ such that $m \otimes_{A_{\mathsf{e}}} a = \sum_i m \otimes_{A_{\mathsf{e}}} a_i a'_i$. Thus

$$f(m \otimes_{A_{\mathsf{e}}} a) = \sum_i f(m \otimes_{A_{\mathsf{e}}} a_i) a'_i = 0,$$

by linearity, which implies that $f = 0$. Now, take an object $Z \in \mathscr{T}_x$, and a graded morphism $g : Z \to M$. We know that the canonical projection $Z \to Z / \oplus_{y \in G} (Z_x \otimes_{A_{\mathsf{e}}} A_{x^{-1}y})$ is zero since $Z / \oplus_{y \in G} (Z_x \otimes_{A_{\mathsf{e}}} A_{x^{-1}y}) \in \mathscr{C}_x$. That is, $Z \cong \oplus_{y \in G}(Z_x \otimes_{A_{\mathsf{e}}} A_{x^{-1}y})$. It clear then that g factors thoughout the canonical injection $\mathrm{Im}(\nu) \hookrightarrow M$, where $\nu : \oplus_{y \in G}(M_x \otimes_{A_{\mathsf{e}}} A_{x^{-1}y}) \to M$ is the obvious graded map, and this finishes the proof. □

Let us denote by $\mathscr{S}$ the set of isomorphisms classes $[Y]$ represented by simple right A_{e}-modules Y. $\mathscr{S}_x$ will denote the set of isomorphisms classes $[S]$ represented by simple G-graded right A-modules S such that $x \in \mathrm{Supp}(S)$.

The following is a direct consequence of Corollary 3.4, Propositions 4.1, 4.3, and Lemma 4.4 (compare with [11]).

Corollary 4.5 *Let A be a graded ring by a group G. For every element $x \in G$, there is a bijective map*

$$\begin{array}{ccc} \mathscr{S}_x & \longrightarrow & \mathscr{S} \\ [S] & \longmapsto & [S_x] \\ \left[\oplus_{y \in G} (Y \otimes_{A_{\mathsf{e}}} A_{x^{-1}y}) \right] & \longleftarrow\!\shortmid & [Y]. \end{array}$$

In particular every simple G-graded right A-module is semisimple right A_{e}-module.

Remark 4.6 The results presented in Sect. 3 can be also applied to semi-group graded modules [1], G-set graded modules [14], or more general to any category of entwined modules [5]. For instance, Corollary 3.4 in conjunction with semigroup-graded version of Propositions 4.1, 4.3, will leads to [1, Theorem 2.8].

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