Algebr Represent Theor (2009) 12:343–356
DOI 10.1007/s10468-009-9153-6

Corings with Decomposition and Semiperfect Corings

L. El Kaoutit · J. Gómez-Torrecillas

Received: 17 March 2008 / Accepted: 16 September 2008 / Published online: 4 March 2009

Abstract We give a characterization, in terms of Galois infinite comatrix corings, of the corings that decompose as a direct sum of left comodules which are finitely generated as left modules. Then we show that the associated rational functor is exact. This is the case of a right semiperfect coring which is locally projective and whose Galois comodule is a projective left unital module with superfluous radical.

Keywords Corings · Graded modules · Doi–Koppinen Hopf modules

Mathematics Subject Classifications (2000) 16W30 · 16W50 · 18E15

1 Introduction

When studying generalized module categories like Doi–Koppinen Hopf modules or, more generally, entwined modules (see [6] or [3] for detailed accounts of their fundamental properties), it is reasonable, after the observation that they are

---

Presented by Stefaan Caenepeel and Alain Verschoren.

Opgedragen aan Freddy Van Oystaeyen ter gelegenheid van zijn zestigste verjaardag.

This research is supported by the grants MTM2004-01406 and MTM2007-61673 from the Ministerio de Educación y Ciencia of Spain, and P06-FQM-01889 from the Consejería de Innovación, Ciencia y Empresa of Andalucía, with funds from FEDER (Unión Europea).

L. El Kaoutit (✉)
Departamento de Álgebra, Facultad de Educación y Humanidades,
Universidad de Granada, El Greco N^o 10, 51002 Ceuta, Spain
e-mail: kaoutit@ugr.es

J. Gómez-Torrecillas
Departamento de Álgebra, Facultad de Ciencias,
Universidad de Granada, 18071 Granada, Spain
e-mail: gomezj@ugr.es

categories of comodules [4], to formulate our questions in terms of comodules over corings. However, it happens often that the answer is given in terms of abstract categorical concepts, which do not say much more on the concrete comodule algebra or entwining structure we are dealing with. In a more favorable situation, we could characterize categorical properties of the category of relative modules in terms of structural properties of the associated coring. But, sometimes, the structure of the corings is described in terms which have no direct relationship with the entwined algebra and coalgebra. One example of the sketched situation is the following: assume we want to study under which conditions the category $\mathcal{M}(\Psi)_A^C$ of entwined right modules over an entwining structure $(A, C)_\Psi$ for an algebra A and a coalgebra C has a generating set of small projectives. Of course, when $\mathcal{M}(\Psi)_A^C$ is a Grothendieck category, a first answer is given by Freyd's theorem: it has to be equivalent to a category of modules over a ring with enough idempotents. Looking at $\mathcal{M}(\Psi)_A^C$ as the category of right comodules over a suitable coring [4], we have a characterization in terms of infinite comatrix corings in [8, Theorem 2.7]. However, none of these results give answers directly expressable in terms of A or C. In this paper, we describe a class of A-corings for which there exists a generating set of small projective objects for their category of right comodules. As a consequence, we will obtain that if C is a right semiperfect coalgebra over a field, then $\mathcal{M}(\Psi)_A^C$ has a generating set of small projective objects for every entwining structure $(A, C)_\Psi$ (Corollary 2.3). Of course, we think that the interest of our results go beyond the theory of entwined modules, as the categories of comodules over corings deserve to be investigated in their own right.

The idea comes from the theory of coalgebras. It is well known [17] that if C is a coalgebra over a field, then C is right semiperfect (or, equivalently, its category of right comodules Comod_C has a generating set consisting of finite-dimensional projectives) if and only if C decomposes as a direct sum of finite-dimensional left subcomodules. We give a generalization of this characterization to the case of corings over an arbitrary ring A (Theorem 2.2). It is then natural to try to understand right semiperfect corings. This is done in Section 3. Our point of view here, in contrast with [9] and [5], deliberatively avoids the assumption of conditions on the ground ring A. Our approach rests upon the study of general semiperfect categories due to Harada [15], and on the study of corings having a generating set of small projective comodules developed in [8]. We also discuss the notion of a (right) local coring and the exactness of the rational functor.

Notations and basic notions We work over a commutative ground base ring with 1 denoted by K. The letters A, B are reserved to denote associative K-algebras with unit, which will referred to as rings. A module over a ring with unit means an unital module, and all bimodules are assumed to be central K-bimodules. The category of all right A-modules is denoted by Mod_A. A linear morphism acts on the left, so some conventions should be established. That is, if ${}_AN$ is a left A-module then its endomorphism ring $\mathrm{End}({}_AN)$ is considered as ring with the opposite multiplication of the usual composition law. In this way N is an $(A, \mathrm{End}({}_AN))$-bimodule. While, if N_A is a right A-module, then its endomorphism ring $\mathrm{End}(N_A)$ has a multiplication the usual composition, and N becomes obviously an $(\mathrm{End}(N_A), A)$-bimodule. For any (B, A)-bimodule M, we consider in a canonical way its right and left dual modules $M^* = \mathrm{Hom}\,(M_A\,,\ A_A), {}^*M = \mathrm{Hom}\,({}_BM\,,\ {}_BB)$ as (A, B)-bimodules.

We will also consider some rings without unit. When this is the case will be clear from the context.

For any category $\mathcal{G}$, the notation $X \in \mathcal{G}$ means that X is an object of $\mathcal{G}$, and the identity morphism of any object will be represented by the object itself.

Recall from [20] that an A-coring is a three-tuple $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ consisting of an A-bimodule $\mathfrak{C}$ and two homomorphisms of A-bimodules

$$\mathfrak{C} \xrightarrow{\Delta_{\mathfrak{C}}} \mathfrak{C} \otimes_A \mathfrak{C}\ , \qquad \mathfrak{C} \xrightarrow{\varepsilon_{\mathfrak{C}}} A$$

such that $(\Delta_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ \Delta_{\mathfrak{C}} = (\mathfrak{C} \otimes_A \Delta_{\mathfrak{C}}) \circ \Delta_{\mathfrak{C}}$ and $(\varepsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}) \circ \Delta_{\mathfrak{C}} = (\mathfrak{C} \otimes_A \varepsilon_{\mathfrak{C}}) \circ \Delta_{\mathfrak{C}} = \mathfrak{C}$. A homomorphism of A-corings is an A-bilinear map $\phi : \mathfrak{C} \to \mathfrak{C}'$ which satisfies $\varepsilon_{\mathfrak{C}'} \circ \phi = \varepsilon_{\mathfrak{C}}$ and $\Delta_{\mathfrak{C}'} \circ \phi = (\phi \otimes_A \phi) \circ \Delta_{\mathfrak{C}}$.

A right $\mathfrak{C}$-comodule is a pair (M, ρ_M) consisting of a right A-module M and a right A-linear map $\rho_M : M \to M \otimes_A \mathfrak{C}$, called right $\mathfrak{C}$-coaction, such that $(M \otimes_A \Delta_{\mathfrak{C}}) \circ \rho_M = (\rho_M \otimes_A \mathfrak{C}) \circ \rho_M$ and $(M \otimes_A \varepsilon_{\mathfrak{C}}) \circ \rho_M = M$. A morphism of right $\mathfrak{C}$-comodules is a right A-linear map $f : M \to M'$ satisfying $\rho_{M'} \circ f = (f \otimes_A \mathfrak{C}) \circ \rho_M$. The K-module of all homomorphisms of right $\mathfrak{C}$-comodules from a comodule $M_{\mathfrak{C}}$ to a comodule $M'_{\mathfrak{C}}$ is denoted by $\mathrm{Hom}_{\mathfrak{C}}(M, M')$. Right $\mathfrak{C}$-comodules and their morphisms form a K-linear category $\mathsf{Comod}_{\mathfrak{C}}$ which is a Grothendieck category provided ${}_A\mathfrak{C}$ is a flat module, see [11, Section 1]. Left $\mathfrak{C}$-comodules and their morphisms are symmetrically defined. If P is a right $\mathfrak{C}$-comodule such that P_A is finitely generated and projective (*profinite*, for short), then its right dual P^* admits, in a natural way, a structure of left $\mathfrak{C}$-comodule [3, 19.19]. The same arguments are pertinent for left $\mathfrak{C}$-comodules. The natural isomorphism $P \cong {}^*(P^*)$ of right A-modules is in fact an isomorphism of $\mathfrak{C}$-comodules. Furthermore, if P and Q are two right $\mathfrak{C}$-comodules profinite as right A-modules, then the right dual functor induces a K-module isomorphism $\mathrm{Hom}_{\mathfrak{C}}(P, Q) \cong \mathrm{Hom}_{\mathfrak{C}}(Q^*, P^*)$.

2 Comatrix Corings and Corings with Decompositions

Let $\mathcal{P}$ be a set of profinite modules over a ring with unit A. Consider $\Sigma = \oplus_{P \in \mathcal{P}} P$ and, for each $P \in \mathcal{P}$, let $\iota_P : P \to \Sigma$, $\pi_P : \Sigma \to P$ be, respectively, the canonical inclusion and the canonical projection. Consider the set $\{u_P = \iota_P \circ \pi_P : P \in \mathcal{P}\}$ of orthogonal idempotents of $\mathrm{End}(\Sigma_A)$, and let T be a unital subring of $\mathrm{End}(\Sigma_A)$ that contains the idempotents u_P. Write

$$R = \bigoplus_{P,Q} u_Q T u_P,$$

a ring with enough idempotents. Its category of right unital R-modules is denoted by Mod_R. Unital here means $MR = M$, for M a right R-module. Let us recall one of the three constructions given in [8] for the (infinite) comatrix A-coring associated to $\mathcal{P}$ an T (or R). We have now the (R, A)-bimodule Σ and the (A, R)-bimodule $\Sigma^\dagger = \oplus_{P \in \mathcal{A}} P^*$. Both $\Sigma^\dagger$ and Σ are unital R-modules. In the A-bimodule $\Sigma^\dagger \otimes_R \Sigma$ we have that $\phi \otimes_R x = 0$ whenever $\phi \in P^*$, $x \in Q$ for $P \neq Q$. In this way, the formula

$$\Delta(\phi \otimes_R x) = \sum \phi \otimes_R e_{P,i} \otimes_A e^*_{P,i} \otimes_R x \qquad (\phi \in P^*,\ x \in P),$$

where $\{e_{P,i}, e^*_{P,i}\}$ is a dual basis for the profinite right A-module P, determines a comultiplication

$$\Delta : \Sigma^\dagger \otimes_R \Sigma \longrightarrow \Sigma^\dagger \otimes_R \Sigma \otimes_A \Sigma^\dagger \otimes_R \Sigma$$

which, according to [8, Proposition 5.2], endows $\Sigma^\dagger \otimes_R \Sigma$ with a structure of an A-coring referred to as the *infinite comatrix coring* associated to the set $\mathcal{P}$ and the ring R. Its counit is given by evaluation of the forms $\phi \in P^*$ at the elements $x \in P$, when P runs through $\mathcal{P}$.

When $\mathcal{P}$ consists of right comodules over an A-coring $\mathfrak{C}$, we have that $\Sigma = \bigoplus_{P\in\mathcal{P}} P$ is a right $\mathfrak{C}$-comodule. We consider then the infinite comatrix coring by putting $T = \mathrm{End}(\Sigma_{\mathfrak{C}})$. We have that

$$R = \bigoplus_{P,Q\in\mathcal{P}} u_Q \mathrm{End}(\Sigma_{\mathfrak{C}}) u_P \cong \bigoplus_{P,Q\in\mathcal{P}} \mathrm{Hom}_{\mathfrak{C}}(P, Q) \tag{2.1}$$

It follows from [8, Lemma 4.7] and [8, diagram (5.12)] that there is a canonical homomorphism of A-corings

$$\mathsf{can} : \Sigma^\dagger \otimes_R \Sigma \to \mathfrak{C}, \quad \mathsf{can}(\varphi \otimes_R x) = \varphi \otimes_A \mathfrak{C})\rho_\Sigma(x), \tag{2.2}$$

where $\varphi \in \Sigma^\dagger$ acts on $y \in \Sigma$ by evaluation in the obvious way.

Definition 2.1 The comodule Σ is said to be $R - \mathfrak{C}$-*Galois* if the canonical map can is bijective.

Recall from [11, Section 2] that a three-tuple $\mathfrak{T} = (\mathfrak{C}, B, \langle -, - \rangle)$ consisting of an A-coring $\mathfrak{C}$, an A-ring B (i.e., B is an algebra extension of A) and a balanced A-bilinear form $\langle -, - \rangle : \mathfrak{C} \times B \to A$, is said to be *a right rational pairing over* A provided

(1) $\beta_A : B \to {}^*\mathfrak{C}$ is a ring anti-homomorphism, where ${}^*\mathfrak{C}$ is the left dual convolution ring of $\mathfrak{C}$ defined in [20, Proposition 3.2], and
(2) α_M is an injective map, for each right A-module M,

where α_- and β_- are the following natural transformations

$$\beta_N : B \otimes_A N \longrightarrow \mathrm{Hom}({}_A\mathfrak{C}, {}_AN), \qquad \alpha_M : M \otimes_A \mathfrak{C} \longrightarrow \mathrm{Hom}(B_A, M_A)$$
$$b \otimes_A n \longrightarrow [c \mapsto \langle c, b \rangle n] \qquad m \otimes_A c \longrightarrow [b \mapsto m\langle c, b \rangle].$$

Given a right rational pairing $\mathfrak{T} = (\mathfrak{C}, B, \langle -, - \rangle)$ over A, we can define a functor called the *right rational functor* as follows. An element m of a right B-module M is called *rational* if there exists a set of *right rational parameters* $\{(c_i, m_i)\} \subseteq \mathfrak{C} \times M$ such that $mb = \sum_i m_i \langle c_i, b \rangle$, for all $b \in B$. The set of all rational elements in M is denoted by $\mathrm{Rat}^{\mathfrak{T}}(M)$. As it was explained in [11, Section 2], the proofs detailed in [12, Section 2] can be adapted in a straightforward way in order to get that $\mathrm{Rat}^{\mathfrak{T}}(M)$ is a B-submodule of M and the assignment $M \mapsto \mathrm{Rat}^{\mathfrak{T}}(M)$ is a well defined functor

$$\mathrm{Rat}^{\mathfrak{T}} : \mathsf{Mod}_B \to \mathsf{Mod}_B,$$

which is in fact a left exact preradical [19, Ch. VI]. Therefore, the full subcategory $\mathrm{Rat}^{\mathfrak{T}}(\mathsf{Mod}_B)$ of Mod_B whose objects are those B-modules M such that $\mathrm{Rat}^{\mathfrak{T}}(M) = M$ is a closed subcategory. Furthermore, $\mathrm{Rat}^{\mathfrak{T}}(\mathsf{Mod}_B)$ is a Grothendieck category which is shown to be isomorphic to the category of right comodules $\mathsf{Comod}_{\mathfrak{C}}$ as [11, Theorem 2.6'] asserts (see also [1, Proposition 2.8]). We say that a set $\mathcal{S}$ of objects of a Grothendieck category $\mathcal{A}$ is a *generating set* of $\mathcal{A}$ if the coproduct $\bigoplus_{X\in\mathcal{S}} X$ is a generator of $\mathcal{A}$.

We are ready to state and prove our main theorem.

Theorem 2.2 *The following statements are equivalent for an A-coring $\mathfrak{C}$:*

(i) *$\mathfrak{C} = \bigoplus_{E\in\mathcal{E}} E$, for a family of subcomodules $\mathcal{E}$ of ${}_{\mathfrak{C}}\mathfrak{C}$ such that ${}_AE$ is profinite for every $E \in \mathcal{E}$;*

(ii) *${}_A\mathfrak{C}$ is projective and there exists a generating set $\mathcal{P}$ of small projective objects in $\mathsf{Comod}_{\mathfrak{C}}$ such that the right comodule $\Sigma = \bigoplus_{P\in\mathcal{P}} P$, considered as a left module over $R = \bigoplus_{P,Q\in\mathcal{P}} \mathrm{Hom}_{\mathfrak{C}}(P, Q)$, admits a decomposition as direct sum of finitely generated R-submodules;*

(iii) *${}_A\mathfrak{C}$ is projective and there exists a generating set $\mathcal{P}$ of $\mathsf{Comod}_{\mathfrak{C}}$, whose members are profinite as right A-modules, such that $\Sigma = \bigoplus_{P\in\mathcal{P}} P$ admits, as a left module over $R = \bigoplus_{P,Q\in\mathcal{P}} \mathrm{Hom}_{\mathfrak{C}}(P, Q)$, a decomposition as direct sum of finitely generated R-submodules;*

(iv) *${}_A\mathfrak{C}$ is projective and there exists a set $\mathcal{P}$ of right $\mathfrak{C}$-comodules profinite as right A-modules such that $\mathfrak{C}$ is $R-\Sigma$-Galois for $\Sigma = \bigoplus_{P\in\mathcal{P}} P$ and $R = \bigoplus_{P,Q\in\mathcal{P}} \mathrm{Hom}_{\mathfrak{C}}(P, Q)$, and Σ admits, as a left R-module, a decomposition as direct sum of finitely generated R-submodules.*

Proof

(i) $\Rightarrow$ (ii) Associated to the given decomposition of left comodules $\mathfrak{C} = \oplus_{E\in\mathcal{E}} E$, there is a family of orthogonal idempotents $\{e_E : E \in \mathcal{E}\}$ in $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$, where $e_E = \iota_E \circ \pi_E$, for $\iota_E : E \to \mathfrak{C}$ the canonical injection and $\pi_E : \mathfrak{C} \to E$ the canonical projection for each $E \in \mathcal{E}$. The ring $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$ is endowed with the multiplication opposite to the composition. Since ${}_A\mathfrak{C}$ is projective we have the canonical rational pairing $\mathfrak{T} = (\mathfrak{C}, \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}), \langle -, - \rangle)$, where $\langle c, f \rangle = \varepsilon(f(c))$, for $c \in \mathfrak{C}$, $f \in \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$. Thus each right $\mathfrak{C}$-comodule admits a right $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$-action, and so is in particular for the right $\mathfrak{C}$-comodules ${}^*E$, with $E \in \mathcal{E}$.

Consider the set of right $\mathfrak{C}$-comodules $\mathcal{P} = \{{}^*E : E \in \mathcal{E}\}$ and the right $\mathfrak{C}$-comodule $\Sigma = \oplus_{E\in\mathcal{E}} {}^*E$. Each of the maps

$$\begin{array}{ccc} e_E\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) & \longrightarrow & \mathrm{Hom}_{\mathfrak{C}}(E, \mathfrak{C}) \cong {}^*E \\ e_E f & \longmapsto & \varepsilon \circ f \circ e_E \circ \iota_E \end{array} \tag{2.3}$$

is an isomorphism of right $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$-modules, which means that each $e_E\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$ is actually a rational right $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$-module, and so a right $\mathfrak{C}$-comodule. In this way, we get an isomorphism of right $\mathfrak{C}$-comodules

$$\bigoplus_{E\in\mathcal{E}} e_E\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) \cong \bigoplus_{E\in\mathcal{E}} {}^*E = \Sigma. \tag{2.4}$$

Since $e_E\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$ is a small object in the category $\mathsf{Comod}_{\mathfrak{C}}$ for every $E \in \mathcal{E}$, we deduce that

$$e_E\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) \;=\; \bigoplus_{F\in\mathcal{E}} e_E\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})e_F. \tag{2.5}$$

Consider in the ring $\mathrm{End}(\Sigma_{\mathfrak{C}})$ (with multiplication the usual composition), the set of idempotents $u_{^*E} = \iota_{^*E} \circ \pi_{^*E}$, for $\iota_{^*E} : {}^*E \to \Sigma$ the canonical injection and $\pi_{^*E} : \Sigma \to {}^*E$ the canonical projection for each $E \in \mathcal{E}$. We have the ring with enough orthogonal idempotents

$$R \;=\; \bigoplus_{E,\,F\in\mathcal{E}} u_{^*E}\mathrm{End}(\Sigma_{\mathfrak{C}})u_{^*F}.$$

Clearly we already have, for every pair of comodules $E, F \in \mathcal{E}$, K-linear isomorphisms

$$e_E\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})e_F \cong \mathrm{Hom}_{\mathfrak{C}}(E, F), \tag{2.6}$$

$$u_{^*E}\mathrm{End}(\Sigma_{\mathfrak{C}})u_{^*F} \cong \mathrm{Hom}_{\mathfrak{C}}({}^*F, {}^*E). \tag{2.7}$$

Using Eqs. 2.6 and 2.7 we define, taking into account the canonical K-linear isomorphism $\mathrm{Hom}_{\mathfrak{C}}(E, F) \cong \mathrm{Hom}_{\mathfrak{C}}({}^*F, {}^*E)$, an isomorphism

$$e_E\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})e_F \;\cong\; u_{^*E}\mathrm{End}(\Sigma_{\mathfrak{C}})u_{^*F} \tag{2.8}$$

for each pair $E, F \in \mathcal{E}$.
In view of equality (Eq. 2.5) and the family of isomorphisms (Eq. (2.8)), we deduce a K-linear isomorphism

$$\bigoplus_{E\in\mathcal{E}} e_E\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) \;\cong\; \bigoplus_{E\in\mathcal{E}} u_{^*E}\mathrm{End}(\Sigma_{\mathfrak{C}})u_{^*F} = R \tag{2.9}$$

Now, if we compose the isomorphisms given in Eqs. 2.4 and 2.9 we get an isomorphism $f : \Sigma \longrightarrow R$. In order to give an explicit expression for the isomorphism f, we should take into account that the inverse map of Eq. 2.4 is defined as follows: to each element $\theta_E \in {}^*E$ it corresponds $(\mathfrak{C} \otimes_A \theta_E) \circ \Delta \circ e_E \in e_E\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$. From this, given $\theta = \sum_{E\in\mathcal{E}} \theta_E \in \oplus_{E\in\mathcal{E}} {}^*E = \Sigma$, we get a map

$$f(\theta) : \bigoplus_{F\in\mathcal{E}} {}^*F \longrightarrow \bigoplus_{F\in\mathcal{E}} {}^*F$$

defined by

$$f(\theta)\left(\sum_{F\in\mathcal{E}} \varphi_F\right) = \sum_{E,\,F\in\mathcal{E}} \varphi_E \circ \pi_E \circ (\mathfrak{C} \otimes_A \theta_E) \circ \Delta \circ \iota_E,$$

$$\left(\sum_{F\in\mathcal{E}} \varphi_F \;\in\; \bigoplus_{F\in\mathcal{E}} {}^*F\right) \tag{2.10}$$

Using the expression 2.10 we can easily show that $f : \Sigma \to R$ is in fact a left R-module isomorphism. In particular, we deduce from the

decomposition $R = \oplus_{F \in \mathcal{E}} Ru_{*F}$ that ${}_R\Sigma$ admits a decomposition as direct sum of finitely generated R-submodules.
Consider now the infinite comatrix coring $\Sigma^\dagger \otimes_R \Sigma$, for $\Sigma^\dagger = \oplus_{E \in \mathcal{E}} ({}^*E)^*$. We have in fact an isomorphism $\Sigma^\dagger \cong \mathfrak{C}$ of left comodules. A routine computation show that the following diagram

$$\begin{tikzcd} \Sigma^\dagger \otimes_R \Sigma \arrow[r, "\mathsf{can}"] \arrow[d, "\Sigma^\dagger \otimes_R f"'] & \mathfrak{C} \\ \Sigma^\dagger \otimes_R R \arrow[r, "\cong"] & \Sigma^\dagger \arrow[u, "\cong"'] \end{tikzcd}$$

is commutative, and so can is an isomorphism. Since ${}_R\Sigma$ is a faithfully flat module because ${}_R\Sigma \cong {}_RR$, we can apply [8, Theorem 5.7((*iii*) $\Rightarrow$ (*i*))] to deduce that $\{{}^*E : E \in \mathcal{E}\}$ is a generating set of a small projectives for $\mathsf{Comod}_{\mathfrak{C}}$.

(ii) $\Rightarrow$ (iii) This is clear, since each $P \in \mathcal{P}$ is, as right A-module, finitely generated and projective (see [8, Theorem 5.7]).

(iii) $\Rightarrow$ (iv) This is deduced from [8, Theorem 4.8] and [8, diagram (5.12)].

(iv) $\Rightarrow$ (i) We can consider R as the (no unital) as a subring (without unit) of $\mathrm{End}(\Sigma_{\mathfrak{C}})$ given by

$$R = \bigoplus_{P, Q \in \mathcal{E}} u_P \mathrm{End}(\Sigma_{\mathfrak{C}}) u_Q,$$

where $u_P = \iota_P \circ \pi_P$ for $\pi_P : \Sigma \to P$ (resp. $\iota_P : P \to \Sigma$) is the canonical projection (resp. injection). Consider the decomposition ${}_R\Sigma = \oplus_{i \in I} \Sigma_i$ as direct sum of finitely generated R-submodules ${}_R\Sigma_i$. We have the following decomposition of the infinite comatrix coring as a left comodule

$$\Sigma^\dagger \otimes_R \Sigma \cong \bigoplus_{i \in I} \Sigma^\dagger \otimes_R \Sigma_i \tag{2.11}$$

For each $i \in I$ there exists a presentation of the left R-module Σ_i

$$F_i \longrightarrow \Sigma_i \longrightarrow 0, \tag{2.12}$$

where $F_i = \oplus_{P \in \mathcal{P}_i} Ru_p$, with a finite subset $\mathcal{P}_i \subset \mathcal{P}$. Applying the functor $\Sigma^\dagger \otimes_R -$ to the sequence 2.12, we obtain an exact sequence of left A-modules

$$\Sigma^\dagger \otimes_R F_i \longrightarrow \Sigma^\dagger \otimes_R \Sigma_i \longrightarrow 0. \tag{2.13}$$

Since there is an A-module isomorphism

$$\Sigma^\dagger \otimes_R F_i \cong \oplus_{P \in \mathcal{P}_i} P^*$$

for each $i \in I$, we deduce from the sequence 2.13 that $\Sigma^\dagger \otimes_R \Sigma_i$ is a finitely generated A-module for each $i \in I$.
Since $\mathfrak{C}$ is $R - \Sigma$-Galois, the map we have $\mathsf{can} : \Sigma^\dagger \otimes_R \Sigma \to \mathfrak{C}$ is an isomorphism of an A-corings. Therefore, the decomposition in Eq. 2.11 can be transferred via can to a decomposition of ${}_{\mathfrak{C}}\mathfrak{C} = \oplus_{i \in I} E_i$ as a

direct sum of subcomodules which are finitely generated as left A-modules. Each one of the E_i's is of course a projective A-module as ${}_A\mathfrak{C}$ is projective, and this finishes the proof. □

According to [17, Theorem 10] a coalgebra C over a field admits a decomposition as a direct sum of finite-dimensional left subcomodules if and only if Comod_C has a generating set of finite-dimensional projective right comodules (C is already right semiperfect). Thus, in the coalgebra case, the condition on Σ in statement (*ii*) of Theorem 2.2, namely that ${}_R\Sigma$ is a direct sum of finitely generated R-submodules, may be deleted. We do not know if this is also the case for corings over a general ring (for Quasi-Frobenius ground rings, the answer is positive [9, Theorem 3.5]). Note that the additional condition on the left R-module structure of Σ cannot be avoided in statement (*iii*) even in the coalgebra case, as there is a generating set for Comod_C of finite-dimensional comodules for any coalgebra C over a field. This is also the case for statement (*iv*), since any coalgebra is $R - \Sigma$-Galois (by [8, Theorem 4.8] and [8, diagram (5.12)]).

As an application of Theorem 2.2, we obtain the following remarkable fact concerning the existence of enough projectives for categories of entwined modules. For the definition of an entwining structure and a discussion of their properties and their relationships with corings, we refer to [3].

Corollary 2.3 *Let $\Psi : A \otimes C \to C \otimes A$ be an entwining structure between an algebra A and a coalgebra C over a commutative ring K. If C admits a decomposition as a direct sum of left subcomodules profinite as K-modules (e.g., if K is a field and C is right semiperfect), then the category of right entwined modules $\mathcal{M}(\Psi)_A^C$ has a generating set of small projective objects.*

Proof By [4, Proposition 2.2], $A \otimes C$ is endowed with the structure of an A-coring such that $\mathsf{Comod}_{\mathfrak{C}}$ is isomorphic to the category of right entwined $A - C$-modules. The comultiplication on $A \otimes C$ is given by

$$A \otimes C \xrightarrow{A \otimes \Delta_C} A \otimes C \otimes C \cong A \otimes C \otimes_A A \otimes C$$

so that every decomposition of C as direct sum of left C-comodules leads to such a decomposition of the A-coring $A \otimes C$. Obviously, if the direct summands in C are profinite K-modules, then the corresponding direct summands of $A \otimes C$ are profinite as left modules over A. Now, the Corollary is a consequence of Theorem 2.2. □

Right semiperfect coalgebras over a field are characterized by the fact that the rational functor $\mathrm{Rat} : {}_{{}^*C}\mathsf{Mod} \to {}_{{}^*C}\mathsf{Mod}$ is exact [13, Theorem 3.3]. The exactness of the rational functor associated to a general rational pairing in the context of corings has been considered recently in [10]. The rational functor canonically associated to a coring becomes exact for the corings characterized in Theorem 2.2, as the following Proposition shows. The exactness of the rational functor was given in [3] (see Remark 2.6 below).

Proposition 2.4 *Let $\mathfrak{C}$ be an A-coring admitting a direct sum decomposition $\mathfrak{C} = \bigoplus_{E \in \mathcal{E}} E$, for a family of subcomodules $\mathcal{E}$ of ${}_{\mathfrak{C}}\mathfrak{C}$ such that ${}_A E$ is profinite for every*

$E \in \mathcal{E}$. Consider the canonical right rational pairing $\mathfrak{T} = (\mathfrak{C}, \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}), \langle -, - \rangle)$, and denote by $\mathfrak{a} := \mathrm{Rat}^{\mathfrak{T}}\left(\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})_{\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})}\right)$ the rational ideal. Then

(a) *R is a right rational $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$-module injected in $\mathfrak{a}$.*
(b) *$\mathfrak{a}$ is generated as a bimodule by $\{e_E\}_{E \in \mathcal{E}}$, that is $\mathfrak{a} = \sum_{E \in \mathcal{E}} \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) e_E \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$, and $\mathfrak{a}$ is a pure left submodule of $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$.*
(c) *The functor $\mathrm{Rat}^{\mathfrak{T}} : \mathsf{Mod}_{\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})} \to \mathsf{Mod}_{\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})}$ is exact.*

Proof

(a) Follows directly from the isomorphisms 2.3 and 2.9.
(b) From the isomorphisms 2.3 we get that $\sum_{E \in \mathcal{E}} e_E \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$ is a rational right ideal, which implies that $\sum_{E \in \mathcal{E}} \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) e_E \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) \subseteq \mathfrak{a}$. Thus, we only need to check the reciprocal inclusion. So let $b \in \mathfrak{a} = \mathrm{Rat}^{\mathfrak{T}}\left(\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})_{\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})}\right)$ an arbitrary element with a right rational system of parameters $\{(b_i, c_i)\}_{i=1,\cdots,n} \subseteq \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) \times \mathfrak{C}$. Let $\mathcal{E}' \subset \mathcal{E}$ be a finite subset such that $c_i \in \oplus_{E \in \mathcal{E}'} E$ for every $i = 1, \ldots, n$, and take $e = \sum_{E \in \mathcal{E}'} e_E$. One easily checks that $c_i e = c_i$, for all $i = 1, \cdots, n$. Therefore,

$$b\, e \,=\, \sum_i b_i \langle c_i, e \rangle \,=\, \sum_i b_i \varepsilon_{\mathfrak{C}}(c_i\, e) \,=\, \sum_i b_i \varepsilon(c_i) \,=\, b, \tag{2.14}$$

which gives the needed inclusion. Equation 2.14 also implies that $\mathfrak{a}$ is a pure left $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$-submodule of $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})$.
(c) In view of (b) and the equality $\mathfrak{C}\mathfrak{a} = \mathfrak{C}$, we can apply [10, Theorem 1.2] to get the exactness of the rational functor. □

Example 2.5 (compare with [5, Example 5.2]) Let $\mathfrak{C}$ be a cosemisimple A-coring. By [11, Theorem 3.1], ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are projective modules. So we can consider its right canonical rational pairing $\mathfrak{T} = (\mathfrak{C}, \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}), \langle -, - \rangle)$. The structure Theorem of cosemisimple corings [7, Theorem 4.4] implies that $\mathfrak{C}$ is a direct sum of left $\mathfrak{C}$-comodules where each of them is a finitely generated and projective left A-module. Thus for a cosemisimple coring $\mathrm{Rat}^{\mathfrak{T}}$ has to be exact.

Remark 2.6 Under the anti-isomorphism of rings $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) \cong {}^*\mathfrak{C}$ the rational pairing considered in Proposition 2.4 goes to a rational pairing $(({}^*\mathfrak{C})^{op}, \mathfrak{C})$ that gives rise to the "more usual" rational functor $\mathrm{Rat} : {}_{*\mathfrak{C}}\mathsf{Mod} \to {}_{*\mathfrak{C}}\mathsf{Mod}$ considered for instance in [3, Section 20], where its exactness was studied. In particular, one deduces from [3, 20.8, 20.12] that $\mathrm{Rat} : {}_{*\mathfrak{C}}\mathsf{Mod} \to {}_{*\mathfrak{C}}\mathsf{Mod}$ is exact whenever ${}_{\mathfrak{C}}\mathfrak{C}$ admits a direct decomposition as assumed in Proposition 2.4. Their arguments run on a different road than ours.

3 Local Corings and Semiperfect Corings

Let $\mathfrak{C}$ be an A-coring such that its category $\mathsf{Comod}_{\mathfrak{C}}$ of all right $\mathfrak{C}$-comodules is a Grothendieck category. The coring $\mathfrak{C}$ is said to be a *right semiperfect coring* if every finitely generated right $\mathfrak{C}$-comodule has a projective cover. That is, $\mathfrak{C}$ is right semiperfect if and only if (by definition) $\mathsf{Comod}_{\mathfrak{C}}$ is a Grothendieck semiperfect category in the sense of M. Harada [15, Section 3, page 334]. Following [18, page 347] and [15, Section 1, page 330], a right $\mathfrak{C}$-comodule P is said to be a *semiperfect right comodule* if $P_{\mathfrak{C}}$ is a projective comodule and every factor comodule of P has a projective cover. A right $\mathfrak{C}$-comodule P is said to be a *completely indecomposable comodule* if its endomorphisms ring $\mathrm{End}(P_{\mathfrak{C}})$ is a local ring (i.e., its quotient by the Jacobson radical is a division ring).

Recall from [15, Corollary 1] that if $\mathcal{A}$ is a locally finitely generated Grothendieck category, then $\mathcal{A}$ is semiperfect if and only if $\mathcal{A}$ has a generating set of completely indecomposable projective objects. Of course, this result can be applied in particular to $\mathsf{Comod}_{\mathfrak{C}}$, whenever it is a locally finitely generated Grothendieck category.

Remark 3.1 Assume that $\mathsf{Comod}_{\mathfrak{C}}$ is a Grothendieck category. It seems to be an open question if it is locally finitely generated. In the case when ${}_A\mathfrak{C}$ is locally projective in the sense of [21], $\mathsf{Comod}_{\mathfrak{C}}$ is isomorphic to the category of all rational left ${}^*\mathfrak{C}$-modules (by [1, Lemma 1.29] and [11, Theorem 2.6']) and, therefore, the set of all cyclic rational left ${}^*\mathfrak{C}$-comodules generates the category of right $\mathfrak{C}$-comodules. In fact, in this case, a right $\mathfrak{C}$-comodule is finitely generated if and only if it is finitely generated as a right A-module (see, e.g., [9, Lemma 2.2]). Thus, $\mathsf{Comod}_{\mathfrak{C}}$ has a generating set of comodules that are finitely generated as right A-modules, whenever ${}_A\mathfrak{C}$ is locally projective.

The following theorem is a consequence of [15, Corollary 2], [15, Theorem 3] and [8, Theorem 5.7] (see also [14, Theorem 6.2]).

Theorem 3.2 *Let $\mathfrak{C}$ be an A-coring and $\mathcal{P}$ a set of right $\mathfrak{C}$-comodules. The following statements are equivalent.*

(i) *${}_A\mathfrak{C}$ is a flat module and $\mathcal{P}$ is a generating set of small completely indecomposable projective comodules for $\mathsf{Comod}_{\mathfrak{C}}$;*
(ii) *${}_A\mathfrak{C}$ is a flat module, $\mathsf{Comod}_{\mathfrak{C}}$ has a generating set consisting of finitely generated objects, $\mathfrak{C}$ is a right semiperfect A-coring and $\mathcal{P}$ contains a set of representatives of all semiperfect completely indecomposable right $\mathfrak{C}$-comodules*;
(iii) *each comodule in $\mathcal{P}$ is finitely generated and projective as a right A-module, $\mathfrak{C}$ is (R, Σ)-Galois, where $\Sigma = \bigoplus_{P\in\mathcal{P}} P$ and $R = \bigoplus_{P,Q\in\mathcal{P}} \mathrm{Hom}_{\mathfrak{C}}(P, Q)$, ${}_R\Sigma$ is faithfully flat, and* $\mathrm{End}(P_{\mathfrak{C}})$ *is a local ring for every $P \in \mathcal{P}$.*

Proof

(i) $\Rightarrow$ (iii) This is a consequence of [8, Theorem 5.7].
(iii) $\Rightarrow$ (ii) By [8, Theorem 5.7], ${}_A\mathfrak{C}$ is a flat module and $-\otimes_R \Sigma : \mathsf{Mod}_R \to \mathsf{Comod}_{\mathfrak{C}}$ is an equivalence of categories. From Eq. 2.1 we get the decomposition $R = \bigoplus_{P\in\mathcal{P}} u_P R$, and, since $\mathrm{End}(u_P R_R) \cong u_P R u_P \cong \mathrm{End}(P_{\mathfrak{C}})$ for every $P \in \mathcal{P}$, we deduce that $\{u_P R : P \in \mathcal{P}\}$ is a generating set of completely indecomposable finitely generated projective objects for Mod_R. Therefore, $\{u_P R \otimes_R \Sigma : P \in \mathcal{P}\}$ becomes a generating set of completely indecomposable finitely generated projective objects for $\mathsf{Comod}_{\mathfrak{C}}$. Now, $u_P R \otimes_R \Sigma \cong P$ as a right $\mathfrak{C}$-comodule, for every

$P \in \mathcal{P}$. This implies that $\mathcal{P}$ is a generating set of finitely generated completely indecomposable projective comodules. By [15, Corollary 1], $\mathfrak{C}$ is right semiperfect. Now, if U is any completely indecomposable projective right $\mathfrak{C}$-comodule, then U is an epimorphic image of a direct sum of comodules in $\mathcal{P}$. We get thus that U is isomorphic to a direct summand of a sum of comodules in $\mathcal{P}$. By the Krull-Schmidt-Azumaya theorem, we obtain that U is isomorphic to one comodule in $\mathcal{P}$.

(ii) $\Rightarrow$ (i) From the proof of [15, Theorem 3], it follows that there is a set of finitely generated projective completely indecomposable generators of $\mathsf{Comod}_{\mathfrak{C}}$. This obviously implies that $\mathcal{P}$ is a generating set of projective finitely generated indecomposable comodules. □

Let us record some information deduced from Theorem 2.2 as a kind of "Structure Theorem" of right semiperfect A-corings.

Corollary 3.3 *Assume ${}_A\mathfrak{C}$ to be locally projective. Then $\mathfrak{C}$ is right semiperfect if and only if there is a set $\mathcal{P}$ of right $\mathfrak{C}$-comodules such that*

1. *Every $P \in \mathcal{P}$ is profinite as a right A-module,*
2. *$\mathfrak{C}$ is $R - \Sigma$-Galois, where $\Sigma = \bigoplus_{P \in \mathcal{P}} P$ and $R = \bigoplus_{P,Q \in \mathcal{P}} \mathrm{Hom}_{\mathfrak{C}}(P, Q)$,*
3. *${}_R\Sigma$ is faithfully flat, and*
4. *$\mathrm{End}(P_{\mathfrak{C}})$ is a local ring, for every $P \in \mathcal{P}$.*

As mentioned before, a coalgebra over a field is right semiperfect if and only if its rational functor is exact. This characterization has been extended to the case of corings over QuasiFrobenius rings, see [9, Theorem 4.2], [5, Theorem 4.3], [3, 20.11]. For a general ground ring A, we obtain the following result.

Corollary 3.4 *Let $\mathfrak{C}$ be a right semiperfect A-coring such that ${}_A\mathfrak{C}$ is a locally projective module. Consider the canonical right rational pairing $\mathcal{T} = (\mathfrak{C}, \mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}), \langle -, - \rangle)$ and let Σ and R be as in Corollary 3.3. Assume that ${}_R\Sigma$ is projective and its Jacobson radical $\mathrm{Rad}({}_R\Sigma)$ is a superfluous R-submodule of ${}_R\Sigma$. Then the rational functor $\mathrm{Rat}^{\mathcal{T}} : \mathsf{Mod}_{\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})} \to \mathsf{Mod}_{\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C})}$ is exact.*

Proof By Corollary 3.3, $R = \bigoplus_{P \in P} u_P R$, where $\{u_P : P \in \mathcal{P}\}$ is a set of orthogonal local idempotents. By [16, Theorem 2], $\{Ru_P : P \in \mathcal{P}\}$ becomes a generating set of completely indecomposable (finitely generated) projective objects for ${}_R\mathsf{Mod}$. Since ${}_R\Sigma$ is projective, it must be a direct summand of a direct sum of left R-modules of the form Ru_P. Now, the non-unital version of [18, Theorem 5.5] gives that ${}_R\Sigma$ has to be a direct sum modules of the form Ru_P. By Theorem 2.2, ${}_{\mathfrak{C}}\mathfrak{C}$ is a direct sum of comodules, profinite as left A-modules. Proposition 2.4 gives the exactness of $\mathrm{Rat}^{\mathcal{T}}$. □

Next, we deal with the case where $\mathcal{P}$ contains only one comodule. This naturally leads to the notion of a local coring.

Definition 3.5 A right semiperfect A-coring $\mathfrak{C}$ whose category of right comodules has a unique (up to isomorphism) type of completely indecomposable semiperfect comodule, will be called a *right local coring*.

The following is the version of Theorem 3.2 for the case where $\mathcal{P}$ is a singleton.

Corollary 3.6 *Let $\mathfrak{C}$ be an A-coring and $(\Sigma_{\mathfrak{C}}, \rho_{\Sigma})$ a right $\mathfrak{C}$-comodule with endomorphism ring $T = \mathrm{End}(\Sigma_{\mathfrak{C}})$. The following statements are equivalent*

(i) *${}_A\mathfrak{C}$ is a flat module and $\Sigma_{\mathfrak{C}}$ is a small completely indecomposable projective generator in the category of right comodules $\mathsf{Comod}_{\mathfrak{C}}$;*
(ii) *${}_A\mathfrak{C}$ is a flat module, the category $\mathsf{Comod}_{\mathfrak{C}}$ is locally finitely generated, $\mathfrak{C}$ is a right local A-coring and $\Sigma_{\mathfrak{C}}$ the unique semiperfect completely indecomposable right comodule (up to isomorphism);*
(iii) *Σ_A is a finitely generated and projective module, $\mathfrak{C}$ is $T-\Sigma$-Galois, ${}_T\Sigma$ is a faithfully flat module, and T is a local ring.*

Our last proposition gives a connection between right local corings and simple cosemisimple corings. Every cosemisimple coring is a right and left semiperfect coring by [11, Theorem 3.1]. A *simple cosemisimple coring* is a cosemisimple coring with one type of simple right comodule, or equivalently, left comodule. The structure of these corings was given in [7, Theorem 4.3].

Proposition 3.7 *Let Σ_A be a non zero finitely generated and projective right A-module and $T \subseteq S = \mathrm{End}(\Sigma_A)$ a local subring of its endomorphism ring such that ${}_T\Sigma$ is a faithfully flat module. Consider the comatrix A-coring $\mathfrak{C} := \Sigma^* \otimes_T \Sigma$, and denote by* J *the Jacobson radical of T, and by $D := T/\mathrm{J}$ the division factor ring. Then we have*

(i) *$\Sigma/\mathrm{J}\Sigma$ admits the structure of a simple right $(\Sigma^* \otimes_T \Sigma)$-comodule whose endomorphism ring is isomorphic to the division ring D, that is $\mathrm{End}((\Sigma/\mathrm{J}\Sigma)_{\mathfrak{C}}) \cong D$.*
(ii) *If $(\Sigma/\mathrm{J}\Sigma)_A$ is a projective module, then the canonical map*

$$\mathsf{can}_{\Sigma/\mathrm{J}\Sigma} : (\Sigma/\mathrm{J}\Sigma)^* \otimes_D (\Sigma/\mathrm{J}\Sigma) \longrightarrow \Sigma^* \otimes_T \Sigma$$

is a monomorphism of A-corings with domain a simple cosemisimple coring.
(iii) *There is a short exact sequence of $\Sigma^* \otimes_T \Sigma$-bicomodules*

$$0 \longrightarrow \Sigma^* \otimes_T (\mathrm{J}\Sigma) \longrightarrow \Sigma^* \otimes_T \Sigma \longrightarrow \Sigma^* \otimes_T D \otimes_T \Sigma \longrightarrow 0$$

whose cokernel is an A-coring without counit.
(iv) *${}_TS$ is a faithfully flat module and the canonical Sweedler S-coring $S \otimes_T S$ is a right local S-coring with $S_{S \otimes_T S}$ the unique (up to isomorphism) semiperfect completely indecomposable comodule.*

Proof We first make some useful observations. Recall that Σ is a right $\mathfrak{C}$-comodule with coaction $\rho_{\Sigma}(u) = \sum_i u_i \otimes_A u_i^* \otimes_T u$, for every $u \in \Sigma$, where $\{(u_i, u_i^*)\}_i \subseteq \Sigma \times \Sigma^*$ is any finite dual basis for Σ_A, while Σ^* is a left $\mathfrak{C}$-comodule with coaction $\lambda_{\Sigma^*}(u^*) = \sum_i u^* \otimes_T u_i \otimes_A u_i^*$, for every $u^* \in \Sigma^*$. It is clear that ρ_{Σ} is left T-linear while λ_{Σ^*} is right T-linear. Since ${}_T\Sigma$ is a faithfully flat module, [7, Theorem 3.10] implies that $T = \mathrm{End}(\Sigma_{\mathfrak{C}})$ and $- \otimes_T \Sigma : \mathsf{Mod}_T \to \mathsf{Comod}_{\mathfrak{C}}$ is an equivalence of categories with inverse the cotensor functor $-\Box_{\mathfrak{C}} \Sigma^* : \mathsf{Comod}_{\mathfrak{C}} \to \mathsf{Mod}_T$. This means, by Corollary 3.6, that $\mathfrak{C}$ is a right local A-coring with $\Sigma_{\mathfrak{C}}$ the unique type of semiperfect completely indecomposable right comodule (up to isomorphisms).

(i) The right $\mathfrak{C}$-coaction of the right A-module $\Sigma/\mathrm{J}\Sigma$ is given by the functor $-\otimes_T \Sigma$, since $\Sigma/\mathrm{J}\Sigma \cong D \otimes_T \Sigma$ is an isomorphism of right A-modules. This comodule is the image under the equivalence $-\otimes_T \Sigma$ of the simple right T-module D_T, so it is necessarily a simple right $\mathfrak{C}$-comodule. Its endomorphism ring is computed using the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{C}}(D\otimes_T \Sigma\,,\ D\otimes_T \Sigma) &\cong \mathrm{Hom}_T\Big(D,\ \mathrm{Hom}_{\mathfrak{C}}(\Sigma\,,\ D\otimes_T \Sigma)\Big)\\ &\cong \mathrm{Hom}_T\Big(D,\ \mathrm{Hom}_T\big((\Sigma\square_{\mathfrak{C}}\Sigma^*)\,,\ D\big)\Big)\\ &\cong \mathrm{Hom}_T(D\,,\ D)\,,\end{aligned}$$

where we have used the adjunction $-\square_{\mathfrak{C}}\Sigma \dashv -\otimes_T \Sigma$ in the second isomorphism and the right T-linear isomorphism $\Sigma\square_{\mathfrak{C}}\Sigma^* \cong T$ in the last one.

(ii) Since $(\Sigma/\mathrm{J}\Sigma)_A$ is a finitely generated and projective module, we can construct the canonical map $\mathsf{can}_{\Sigma/\mathrm{J}\Sigma}$. This map is a monomorphism by (i) and [2, Theorem 3.1]. Finally, [7, Proposition 4.2] implies that $(\Sigma/\mathrm{J}\Sigma)^* \otimes_D (\Sigma/\mathrm{J}\Sigma)$ is simple cosemisimple A-coring, since D is already is division ring.

(iii) The stated sequence is obtained by applying $-\otimes_T \Sigma$ to the following short exact sequence of right T-modules

$$0 \longrightarrow \Sigma^*\mathrm{J} \longrightarrow \Sigma^* \longrightarrow \Sigma^* \otimes_T D \longrightarrow 0.$$

This gives in fact an exact sequence of right $\mathfrak{C}$-comodules; the $\mathfrak{C}$-bicomodule structure is then completed by the compatible left $\mathfrak{C}$-coaction of ${}_{\mathfrak{C}}\Sigma^*$. The $\mathfrak{C}$-bicomodule $\Sigma^* \otimes_T D \otimes_T \Sigma$ admits a coassociative comultiplication given by

$$\begin{aligned}\Delta : \Sigma^* \otimes_T D \otimes_T \Sigma &\longrightarrow (\Sigma^* \otimes_T D \otimes_T \Sigma) \otimes_A (\Sigma^* \otimes_T D \otimes_T \Sigma)\\ u^* \otimes_T d \otimes_T v &\longmapsto \textstyle\sum_i u^* \otimes_T d \otimes_T u_i \otimes_A u_i^* \otimes_T 1 \otimes_T v.\end{aligned}$$

(iv) This is an immediate consequence of [7, Theorem 3.10] and Corollary 3.6. □

References

1. Abuhlail, J.Y.: Rational modules for corings. Comm. Algebra **31**(12), 5793–5840 (2003)
2. Brzeziński, T.: Galois comodules. J. Algebra **290**, 503–537 (2005)
3. Brzeziński, T., Wisbauer, R.: Corings and Comodules, LMS, vol. 309. Cambridge University Press, Cambridge (2003)
4. Brzeziński, T.: The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties. Algebr. Represent. Theory **5**, 389–410 (2002)
5. Caenepeel, S., Iovanov, M.C.: Comodules over semiperfect corings. In: Kalla, S.L., Chawla, M.M. (eds.) Proceedings of the International Conference on Mathematics and its Applications, ICMA 2004, pp. 135–160. Kuwait University, Kuwait (2005)
6. Caenepeel, S., Militaru, G., Zhu, S.: Frobenius and Separable Functors for Generalized Module Categories and Nonlinear Equations, Lecture Notes in Mathematics, vol. 1787. Springer, New York (2002)
7. El Kaoutit, L., Gómez-Torrecillas, J.: Comatrix corings: Galois coring, Descent theory, and a structure theorem for cosemisimple corings. Math. Z. **244**, 887–906 (2003)

8. El Kaoutit, L., Gómez-Torrecillas, J.: Infinite comatrix corings. Internat. Math. Res. Notices **39**, 2017–2037 (2004)
9. El Kaoutit, L., Gómez-Torrecillas, J.: Morita duality for corings over Quasi-Frobenius rings. In: Caenepeel, S., Van Oystaeyen, F. (eds.) Hopf Algebras in Non-commutative Geometry and Physics, Lecture Notes in Pure and Applied Mathematics, vol. 239, pp. 137–154. Marcel Dekker, New York (2005)
10. El Kaoutit, L., Gómez-Torrecillas J.: Corings with exact rational functors and injective objects. In: Brzeziński, T., Gómez-Pardo, J.L., Shestakov, I., Smith, P.F. (eds.) Modules and Comodules, Trends in Mathematics, vol. XII, pp. 185–201. Birkhäuser, Boston (2008)
11. El Kaoutit, L., Gómez-Torrecillas, J., Lobillo, F.J.: Semisimple corings. Algebra Colloq. **11**(4), 427–442 (2004)
12. Gómez-Torrecillas, J.: Coalgebras and comodules over commutative ring. Rev. Roumaine Math. Pures Appl. **43**(5–6), 591–603 (1998)
13. Gómez-Torrecillas, J., Năstăsescu, C.: Quasi-co-Frobenius coalgebras. J. Algebra **174**, 909–923 (1995)
14. Gómez-Torrecillas, J., Vercruysse, J.: Comatrix corings and Galois comodules over firm rings. Algebr. Represent. Theory **10**, 271–306 (2007)
15. Harada, M.: Perfect categories I. Osaka J. Math. **10**, 329–341 (1973)
16. Harada, M.: Perfect categories II. Osaka J. Math. **10**, 343–355 (1973)
17. Lin, I.-P.: Semiperfect coalgebras. J. Algebra **49**, 357–373 (1977)
18. Mares, E.A.: Semi-perfect modules. Math. Z. **82**, 347–360 (1963)
19. Stenström, B.: Rings of Quotients. Springer, Berlin (1975)
20. Sweedler, M.E.: The predual theorem to the Jacobson-Bourbaki theorem. Trans. Amer. Math. Soc. **213**, 391–406 (1975)
21. Zimmermann-Huisgen, B.: Pure submodules of direct products of free modules. Math. Ann. **224**, 233–245 (1976)