The algebraic groupoid structure of the universal Picard-Vessiot ring, differential operators and Jet spaces.

Laiachi El Kaoutit
(collaboration in part with José Gómez-Torrecillas)

Universidad de Granada. Spain.
kaoutit@ugr.es

Interfaces of Noncommutative Geometry with the Representation Theory of Hopf Algebras and Artin Algebras.
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Galois-Grothendieck and Tannaka-Krein Reconstruction Theories.

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`Alg-Geometric problem`. Let $\mathcal{X}$ be a ringed space, and consider its category of *differential operators sheaves*. Can we characterize the subcategory of locally free quasi-coherent sheaves on $\mathcal{X}$ with differential operators structure sheaf?

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Solution of the reconstruction type problems in this context

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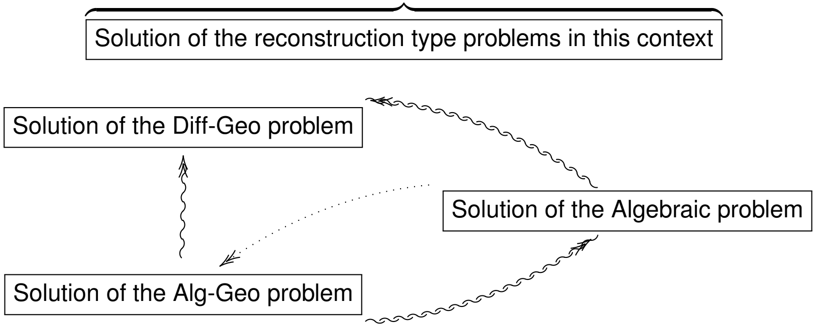

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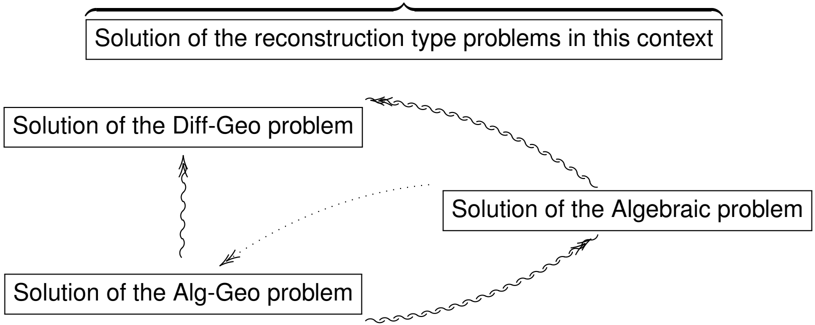

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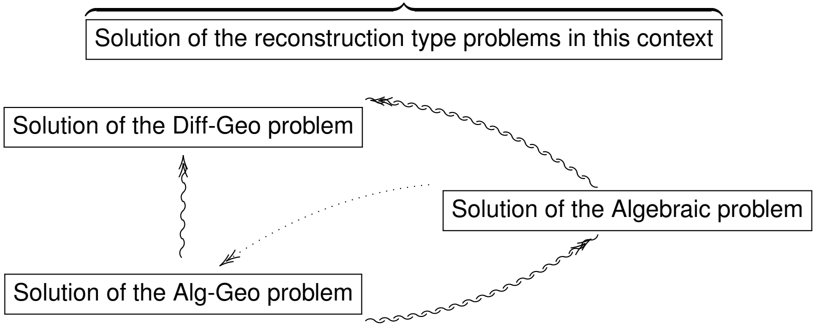

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It seem that in general this is a difficult problem which probably may have a solution in the future. So for the moment may be we should first treat some very simplest example.

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$$\mathbf{Diff}(\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}) = \Delta(\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}) = \mathbb{C}[X][Y, \partial/\partial X],$$

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$$\mathcal{A}_{\mathcal{U}} = \{M \in \mathrm{Mod}_{\mathcal{U}} |\ M \text{ is a free } A\text{-module of finite rank}\}$$

together with the *fiber functor* $\mathscr{O} : \mathcal{A}_{\mathcal{U}} \to add(A)$ to the category of locally free quasi coherent sheaves.

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Each object $M \in \mathcal{A}_{\mathcal{U}}$, is then a *differential module* and one can associated to it a *linear differential matrix equation*. In this way $\mathcal{A}_{\mathcal{U}}$ is nothing but the set of all linear differential matrix equations over the affine line $\mathbb{A}^1_{\mathbb{C}}$.

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The *'local' (or classical) differential Galois theory* says that there is a monoidal equivalence of categories

$$\{\{M\}\} \simeq \mathbf{Rep}_0(\mathcal{G}_M),$$

between the closed monoidal full subcategory of $\mathcal{A}_{\mathcal{U}}$ 'sub-generated' by M, and a full subcategory of the category of finite rank representations of $\mathcal{G}_M$.

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(iii) Is there some relation between the 'local' Galois groups $\mathcal{G}_M$ and the groupoid $Spec(\mathcal{V})$?

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$$T_{PQ} = \mathrm{Hom}_{\mathcal{A}}(P, Q),\ \ T_P = T_{PP},\ \ P^* = \mathrm{Hom}_{-A}(P, A),\ \ \text{for every } P, Q \in \mathcal{A}.$$

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The infinite comatrix A-coring associated to the fiber functor $\omega : \mathcal{A} \to add(A_A)$, is the universal object:

$$\mathscr{R}(\mathcal{A}) = \int^{P \in \mathcal{A}} P^* \otimes_{\mathbb{K}} P = \frac{\left(\bigoplus_{P \in \mathcal{A}} P^* \otimes_{T_P} P \right)}{\mathfrak{J}_{\mathcal{A}}},$$

where $\mathfrak{J}_{\mathcal{A}}$ is the $\mathbb{K}$-submodule generated by the set

$$\left\{ q^* \otimes_{T_Q} tp - q^* t \otimes_{T_P} p : \; q^* \in Q^*, \, p \in P, \, t \in T_{PQ}, \, P, Q \in \mathcal{A} \right\}.$$

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There is an A-corings morphism known as *the canonical map*:

$$\mathrm{can}_{\mathcal{A}^{\mathfrak{C}}} : \mathscr{R}(\mathcal{A}^{\mathfrak{C}}) \longrightarrow \mathfrak{C}, \quad \left(\overline{p^* \otimes_{T_P} p} \longmapsto (p^* \otimes_A \mathfrak{C}) \circ \varrho_P(p) \right).$$

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We will use the following terminology. An A-coring $\mathfrak{C}$ is said to be $\mathcal{A}^{\mathfrak{C}}$-*Galois coring* provided that $\mathrm{can}_{\mathcal{A}^{\mathfrak{C}}}$ is an isomorphism of A-corings. This is the case, for instance when $\mathrm{Comod}_{\mathfrak{C}}$ is an abelian category having $\mathcal{A}^{\mathfrak{C}}$ as a set of generators, and this is exactly the situation in the coalgebra case (i.e. when $A = \mathbb{K}$).

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The fiber functor is then the restriction of the forgetful functor $\eta_* : \mathrm{Mod}_{\mathcal{U}} \to \mathrm{Mod}_A$, that is, $\eta_* : \mathcal{A}_{\mathcal{U}} \to add(A_A)$. The associate A-coring $\mathscr{R}(\mathcal{A}_{\mathcal{U}})$ is simply denoted by $\mathcal{U}^{\circ}$, and referred to as *the finite dual of the A-ring* $\mathcal{U}$.

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The fiber functor is then the restriction of the forgetful functor $\eta_* : \mathrm{Mod}_{\mathcal{U}} \to \mathrm{Mod}_A$, that is, $\eta_* : \mathcal{A}_{\mathcal{U}} \to add(A_A)$. The associate A-coring $\mathscr{R}(\mathcal{A}_{\mathcal{U}})$ is simply denoted by $\mathcal{U}^{\circ}$, and referred to as *the finite dual of the A-ring $\mathcal{U}$*.

In case where $A = \mathbb{K}$ is assumed to be a field and $\mathcal{U}$ is a $\mathbb{K}$-algebra. Then $\mathcal{U}^{\circ}$ is the maximal coalgebra contained in the linear dual $\mathcal{U}^* = \mathrm{Hom}_{\mathbb{K}}(\mathcal{U}, \mathbb{K})$. It is well know in this case that $\mathcal{U}^{\circ}$ is a Hopf $\mathbb{K}$-algebra whenever $\mathcal{U}$ it is.

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The finite dual of $\mathbb{K}$-algebra is in fact *the topological dual of* $\mathcal{U}$ with respect to the linear topology defined by the set of ideals of finite co-dimension, and where $\mathbb{K}$ is endowed with the discrete topology. For instance, if $\mathcal{U}$ is the first Weyl algebra, then it finite dual as $\mathbb{C}$-algebra is 0.

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Bialgebroids. Let $\mathcal{U}$ be an A^{e}-ring ($A^{\mathrm{e}} = A \otimes_{\mathbb{K}} A^{op}$ is the enveloping algebra of A) via the ring extension $\eta : A^{\mathrm{e}} \to \mathcal{U}$ whose *source* and *target* maps are resp. denoted by

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The category $\mathrm{Mod}_{\mathcal{U}}$ is left closed with left inner hom-functors

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This is equivalent to say that the map defined by

$$\mathcal{T} : \mathcal{U} \otimes_{A^{op}} \mathcal{U} \to \mathcal{U} \diamond \mathcal{U}, \text{ sending } u \otimes_{A^{op}} v \mapsto uv_1 \otimes_A v_2,$$

is an isomorphism of right $\mathcal{U}$-modules.

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- Universal enveloping of Lie algebroids: $(A, \mathcal{U})$ is a (left or right) co-commutative Hopf algebroid with $\mathsf{s} = \mathsf{t}$, and $A \nsubseteq \mathscr{Z}(\mathcal{U})$ is not in the centre of $\mathcal{U}$.

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Following Rinehart, the pair (A, L) is called *Lie-Rinehart algebra* with *anchor* map ω, provided

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\begin{aligned}
(aX)(b) &= a(X(b)) \qquad X \in L,\ a, b \in A, \\
[X, aY] &= a[X, Y] + X(a)Y \qquad X, Y \in L, a \in A.
\end{aligned}
$$

where $X(a)$ stands for $\omega(X)(a)$.

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(ii) The pair $(A, \mathrm{Der}_{\mathbb{K}}(A))$ admits trivially a structure of (transitive) Lie-Rinehart algebra.

(iiii) A *Lie algebroid* is a vector bundle $\mathcal{E} \to \mathcal{M}$ over a smooth manifold, together with a map $\omega : \mathcal{E} \to T\mathcal{M}$ of vector bundles and Lie structure $[-, -]$ on the vector space $\Gamma(\mathcal{E})$ of global smooth sections of $\mathcal{E}$, such that the induced map $\Gamma(\omega) : \Gamma(\mathcal{E}) \to \Gamma(T\mathcal{M})$ is a Lie algebra homomorphism, and for all $X, Y \in \Gamma(\mathcal{E})$ and any $f \in \mathcal{C}^{\infty}(\mathcal{M})$ one has

$$[X, fY] \,=\, f[X, Y] + \Gamma(\omega)(X)(f)Y.$$

Then the pair $(\mathcal{C}^{\infty}(\mathcal{M}), \Gamma(\mathcal{E}))$ is obviously a Lie-Rinehart algebra.

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Associated to any Lie-Rinehart algebra (A, L), there is a universal object denoted by $(A, \mathcal{V}L)$ which is constructed as follows.

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Let $U(L)$ be the universal enveloping algebra of the Lie algebra L, and take the factor A-algebra of $A \otimes_\iota U(L)$:

$$\Pi : A \otimes_\iota U(L) \longrightarrow \mathcal{V}L := \frac{A \otimes_\iota U(L)}{\mathcal{J}_L}, \;\; \mathcal{J}_L := \langle a \otimes_\iota X - 1 \otimes_\iota aX \rangle_{a \in A,\, X \in L},$$

$A \otimes_\iota U(L) := A \otimes_{\mathbb{K}} U(L)$ denotes the twisted A-algebra defined by the twisting map: $\iota : U(L) \otimes_{\mathbb{K}} A \longrightarrow A \otimes_{\mathbb{K}} U(L)$ which sends

$$X \otimes a \longmapsto a \otimes X + X(a) \otimes 1, \quad \text{and} \quad 1 \otimes a \longmapsto a \otimes 1.$$

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If we take $A = \mathbb{C}[X]$, and consider the Lie-Rinehart algebra $(A, \mathrm{Der}_{\mathbb{C}}(A))$. Then it is easily checked that

$$\mathcal{V}\mathrm{Der}_{\mathbb{C}}(A) \;=\; \mathcal{U} = \mathbb{C}[X][Y, \partial/\partial X],$$

the first Weyl algebra.

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(iii) *Over a commutative base ring and under certain assumptions, there is a duality between the category of right co-commutative Hopf algebroids, and the category of commutative Hopf algebroids.*

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Corollary

The pair $(A, \mathcal{U}^\circ)$ is a commutative Hopf algebroid or equivalently $(Spec(\mathcal{U}^\circ), Spec(A))$ is an affine algebraic groupoid. In particular, $\mathcal{U}^\circ = \mathcal{V}$ is the universal Picard-Vessiot ring of A.

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Let $A = \mathbb{C}[X]$ and $\mathcal{U} = A[Y, \partial/\partial X]$ its differential operator algebra. Then the commutative Hopf algebroid $(A, \mathcal{U}^\circ)$ is a Galois A-coring. In particular the category of differential modules $\mathcal{A}_{\mathcal{U}}$ is isomorphic to the right Cauchy category $\mathcal{A}^{\mathcal{U}^\circ}$ of $\mathcal{U}^\circ$.

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$$A[X_{ij}, det_X^{-1}]/I_M \cong \mathcal{U}^\circ/\mathcal{J}_M := \mathcal{U}_M^\circ$$

is an isomorphism of differential Hopf A-algebras. Moreover, the associated affine algebraic group $\mathcal{G}_M$ is homeomorphic to a closed subset of the topological space Spec$(\mathcal{U}^\circ)$.

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(ii) *There is an open cover*

$$Spec(\mathcal{U}^\circ) = \bigcup_{M \in \mathcal{A}_\mathcal{U}} (Spec(\mathcal{U}^\circ) \setminus Spec(\mathcal{U}_M^\circ)).$$

Differential operators and Jet spaces.

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The space of *differential operators of order* k, $k \geq 0$ is defined by

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There is a filtrated system inside $\mathrm{Hom}_{\mathbb{K}}(P, Q)$

$$\mathrm{Diff}_0(P, Q) \subseteq \mathrm{Diff}_1(P, Q) \subseteq \cdots \subseteq \mathrm{Diff}_k(P, Q) \subseteq \mathrm{Diff}_{k+1}(P, Q) \subseteq \cdots$$

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Differential operators and Jet spaces.

`Diff-Operators.` Let A be a commutative $\mathbb{K}$-algebra and P, Q are A-modules. For any linear map $f \in \mathrm{Hom}_{\mathbb{K}}(P, Q)$ and element $a \in A$, we set

$$\delta_a(f) \ = \ fa - af \ \in \mathrm{Hom}_{\mathbb{K}}(P, Q), \text{ sending } p \mapsto f(ap) - af(p).$$

The space of *differential operators of order* k, $k \geq 0$ is defined by

$$\mathrm{Diff}_k(P, Q) \ = \ \left\{ f \in \mathrm{Hom}_{\mathbb{K}}(P, Q) | \ \delta_{a_o} \circ \cdots \circ \delta_{a_k}(f) = 0, \ \forall \, a_0, \cdots, a_k \in A \right\}$$

There is a filtrated system inside $\mathrm{Hom}_{\mathbb{K}}(P, Q)$

$$\mathrm{Diff}_0(P, Q) \subseteq \mathrm{Diff}_1(P, Q) \subseteq \cdots \subseteq \mathrm{Diff}_k(P, Q) \subseteq \mathrm{Diff}_{k+1}(P, Q) \subseteq \cdots$$

where $\mathrm{Diff}_0(P, Q) \ = \ \mathrm{Hom}_A(P, Q)$.

The composition of linear maps induces a *filtered* $\mathbb{K}$*-algebra* structure on the space $\cup_{k \geq 0} \mathrm{Diff}_k(P, P)$. In particular, we denote

$$\mathbf{Diff}(A) := \cup_{k \geq 0} \mathrm{Diff}_k(A, A)$$

and refer to as *the differential operators ring of A*.

Differential operators and Jet spaces.

Example

Let $\mathcal{M}$ be a smooth manifolds and set $A = \mathcal{C}^\infty(\mathcal{M})$. The graded algebra $\mathrm{gr}(\mathbf{Diff}(A))$ is isomorphic to the subalgebra of the algebra $\mathcal{C}^\infty(T^*\mathcal{M})$ consisting of functions whose restriction ot the fibers $T^*_z\mathcal{M}$ of the cotangent bundle are polinomials.

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Jet spaces. Let P be as before, for any $k \geq 0$, we denote by

$$\mu_k(P) := span_{\mathbb{K}}\{\delta^{a_0} \circ \cdots \circ \delta^{a_k}(a \otimes_{\mathbb{K}} p),\ a_0, \cdots, a_k \in A\}$$

where $\delta^r(a \otimes p) = a \otimes rp - ar \otimes p$, $r \in A$. The quotient A-bimodule

$$j_k : P \to A \otimes_{\mathbb{K}} P \to \mathcal{J}^k(P) := \frac{A \otimes_{\mathbb{K}} P}{\mu_k(P)},$$

is called the *k-Jet space of P*.

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Example

Let $\mathcal{M}$ be a smooth manifolds and set $A = \mathcal{C}^\infty(\mathcal{M})$. Assume that $P = \Gamma(\pi)$ the global smooth sections of some smooth vector bundle $\pi : \mathcal{E} \to \mathcal{M}$. Then there is a isomorphism of A-modules

$$\mathcal{J}^k(P) \;\cong\; \Gamma(J^k(\pi)) \text{ where } J^k(\pi) \text{ is the } k\text{-Jet bundle of } \pi.$$

Differential operators and Jet spaces.

`Duality between Diff-Operators and Jets.`

Differential operators and Jet spaces.

Duality between Diff-Operators and Jets. For any pair of A-modules P, Q and every $k \geq 0$, we have a commutative diagram

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\mathrm{Hom}_{\mathbb{K}}(P, Q) & \xrightarrow{\cong} & \mathrm{Hom}_{A-}(A \otimes_{\mathbb{K}} P, Q) \\
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Thus the functor $\mathrm{Diff}_k(P, -)$ is represented by $\mathcal{J}^k(P)$.

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Since $\{\mathcal{J}^k(A)\}_{k\geq 0}$ is an inverse system whose structural maps are $\nu_{l,k} : \mathcal{J}^l(A) \to \mathcal{J}^k(A)$, $l \leq k$, with universal equalities $\nu_{l,k} \circ j_l = j_k$. We can consider the *the infinite Jet space* (or *the prolongation Jet*)

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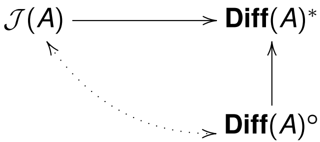
Using the above isomorphisms, we get

$$\mathbf{Diff}(A)^* \cong \varprojlim \mathrm{Hom}_{-A}(\mathbf{Diff}_k(A), A) \cong \varprojlim \left({}^*\mathcal{J}^k(A)\right)^*$$

Differential operators and Jet spaces.

Therefore, there is a diagram of linear maps



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Differential operators and Jet spaces.

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(i) Which structure then it have the prolongation Jet $\mathcal{J}(A)$?

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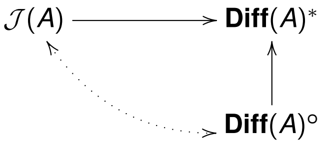

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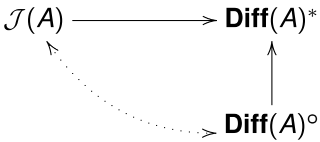

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Before answering to these questions, let see an alternative description of $\mathcal{J}(A)$. Let us denote by $\mathcal{K} := Ker(A \otimes_{\mathbb{K}} A \to A)$, we are considering $A \otimes_{\mathbb{K}} A$ as an augmented algebra over A. In fact $(A, A \otimes_{\mathbb{K}} A)$ is a commutative Hopf algebroid.

Differential operators and Jet spaces.

So we can consider the $\mathcal{K}$*-adic topology on* $A \otimes_{\mathbb{K}} A$, and so its completion $\widehat{A \otimes_{\mathbb{K}} A} = \varprojlim(\frac{A\otimes_{\mathbb{K}} A}{\mathcal{K}^n})$. This is a *complete Hopf algebroids*, in the sense of Quillen.

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For any commutative Hopf algebroid $(A, \mathcal{V})$, there is a commutative diagram

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For $A = \mathbb{C}[X]$, is there an isomorphism $\widehat{\mathbf{Diff}(A)}^{\circ} \cong \mathcal{J}(A)$ of complete Hopf algebroids?

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