Categories of comodules and chain complexes of modules.

Laiachi El Kaoutit
Joint work with A. Ardizzoni and C. Menini.

Universidad de Granada. Spain.
kaoutit@ugr.es

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Motivations overviews.

In 1990, D. Tambara, proved that if A is a finite dimensional algebra over field $\mathbb{k}$, then the functor

$$A \otimes_{\mathbb{k}} -\colon \mathbb{k}\texttt{-algebras} \longrightarrow \mathbb{k}\texttt{-algebras}$$

has a left adjoint

$$\mathcal{A}(A, -)\colon \mathbb{k}\texttt{-algebras} \longrightarrow \mathbb{k}\texttt{-algebras}.$$

The algebra $\mathcal{A}(A, A)$ has a natural structure of bialgebra called *coendomorphism bialgebra* and coacts universally on the algebra A. It turns out that, *if $\dim_{\mathbb{k}}(A) > 1$, then the category of right $\mathcal{A}(A, A)$-comodules is equivalent as a monoidal category to the category of chain complexes of $\mathbb{k}$-modules* .

This extended Manin's works on quadratic bialgebras, and those of Pareigis on certain Hopf algebras with generators and relations.

Tambara's proof relies on the use of a slightly variant of the equivalence between simplicial $\mathbb{k}$-vector spaces and chain complexes of $\mathbb{k}$-vector spaces, provided by Dold-Kan's normalization functor, as well as the Amitsur cosimplical vector space $Q_\bullet$ given by $Q_0 = \mathbb{k}$, $Q_1 = A$ and $Q_n = K^{\otimes_A (n-1)}$ for $n \geq 2$, where K is the kernel of the multiplication of A.

Question: It is possible to obtain an analogue of the above equivalence, when the base field $\Bbbk$ is replaced by any commutative algebra, or more generally any noncommutative base ring?

- As we will see, the answer is yes, and instead of coendomorphism bialgebra we will obtain a *coendomorphism left bialgebroid*.

Question: Why this should be interesting?

- This will provide a new examples of left bialgebroid defined over noncommutative ring by means of generators and relations.
- A monoidal equivalence between categories of chain complexes of (left) modules and comodules, allows one freely to transfer at least the model structure of chain complexes, as was described in Hovey's book, to left comodules. This in fact suggests that certain categories of comodules (instead of chain complexes) could be endowed within a (monoidal) model structure.

These indeed are our main motivations for further investigating the relationship between categories of chain complexes of modules and left comodules over bialgebroids.

A part from the previous motivations, that might be not so convincing answers, the origin of dealing with Hopf algebroids or bialgebroid with a noncommutative base ring, comes from noncommutative algebraic geometry.

Thus, if we want to have an analogue of the following correspondences in the context of noncommutative algebraic geometry:

$$\begin{tikzcd}
\text{Commutative rings} \arrow[r, leftrightsquigarrow, "\substack{\text{Abstract algebraic}\\ \text{geometry}}"] & \text{Affine schemes} \\
\substack{\text{Commutative}\\ \text{Hopf algebras}} \arrow[r, leftrightsquigarrow] & \substack{\text{Affine algebraic}\\ \text{groups}} \\
\substack{\text{Commutative}\\ \text{Hopf algebroids}} \arrow[r, leftrightsquigarrow] & \substack{\text{Groupoids in affine}\\ \text{schemes}}
\end{tikzcd}$$

then (left) bialgebroids or (left) Hopf algebroids with a noncommutative base rings should corresponds to some kind of ”groupoid” in affine schemes in noncommutative algebraic geometry.

Thus we need to complete the following correspondences:

$$\begin{array}{ccc} & \text{Noncommutative} & \\ & \text{algebraic geometry} & \\ \text{Noncommutative rings} & \longleftrightsquigarrow & \text{Affine schemes} \\ \text{Nonommutative Hopf algebras} & \longleftrightsquigarrow & \text{Affine algebraic groups} \\ \text{(left) Hopf algebroids} & \longleftrightsquigarrow & \text{(left)Groupoids in affine schemes} \end{array}$$

Apart from the above explanations and motivations, the construction of the coendomorphism bialgebroids performed hereby, uses in fact a noncommutative version of a well know techniques and methods from abstract algebraic geometry. Namely, the noncommutative version of Tannaka-Krein reconstruction.

Methodology.

In the non commutative setting, one basically starts with a small $\Bbbk$-linear monoidal category $(\mathcal{A}, \otimes, \mathbf{1})$ and a faithful monoidal functor [3] from $\mathcal{A}$ to the category of R-bimodules, $\chi : \mathcal{A} \to {}_R\mathrm{Mod}_R$ (the fiber functor), valued in the category finitely generated and projective left R-modules (i.e. locally free sheaves of finite rank).

There are several objects under consideration:

$$\Sigma(\chi) = \bigoplus_{\mathfrak{p} \in \mathcal{A}} \chi(\mathfrak{p}), \quad {}^{\vee}\Sigma(\chi) = \bigoplus_{\mathfrak{p} \in \mathcal{A}} {}^{*}\chi(\mathfrak{p}), \qquad \mathcal{G}(\mathcal{A}) = \bigoplus_{\mathfrak{p},\, \mathfrak{p}' \in \mathcal{A}} \mathrm{Hom}_{\mathcal{A}^o}\left(\mathfrak{p},\, \mathfrak{p}'\right).$$

Here the second is the right R-module direct sum of the left duals while the third is Gabriel's ring with enough orthogonal idempotents, attached to the opposite category. Using the canonical actions, we consider $\mathcal{L}(\chi) := \Sigma(\chi) \otimes_{\mathcal{G}(A)} {}^{\vee}\Sigma(\chi)$ as an R^{e}-bimodule, where $R^{\mathrm{e}} := R \otimes_{\Bbbk} R^{o}$ denotes the enveloping ring.

[3]Our setting requires an isomorphism only at the level of unit. That is, $R \cong \omega(\mathbf{1})$, while $\chi(- \otimes -) \to \chi(-) \otimes_R \chi(-)$ is not necessarily a natural isomorphism.

A well known argument in small additive categories says that the object $\mathcal{L}(\chi)$ solves the following universal problems in R-bimodules

$$\begin{aligned} \mathrm{Nat}\Big(\chi,\, -\otimes_R \chi\Big) &\cong \mathrm{Hom}_{R-R}\Big(\mathcal{L}(\chi),\, -\Big), \\ \mathrm{Nat}\Big(\chi\otimes_R \chi,\, -\otimes_R (\chi\otimes_R \chi)\Big) &\cong \mathrm{Hom}_{R-R}\Big(\mathcal{L}(\chi)\otimes_{R^e} \mathcal{L}(\chi),\, -\Big), \end{aligned}$$

where the R-bimodule structures of $\mathcal{L}(\chi)$ have been chosen properly. It is indeed this solution which allows us to construct a left R-bialgebroid (or a Hopf bialgebroid if desired). Of course there is an obvious (monoidal) functor connecting left unital $\mathcal{G}(\mathcal{A})$-modules and left $\mathcal{L}(\chi)$-comodule, namely

$$\Sigma(\chi)\otimes_{\mathcal{G}(\mathcal{A})} - : {}_{\mathcal{G}(\mathcal{A})}\mathsf{Mod} \longrightarrow {}_{\mathcal{L}(\chi)}\mathsf{Comod}.$$

In the case when each of the left R-modules $\chi(\mathfrak{p})$ is endowed with additional structure of left $\mathfrak{C}$-comodule for some R-coring $\mathfrak{C}$ (or certain given left R-bialgebroid), there is a map of R-corings, known as a *canonical map*,

$$\mathsf{can}_{\mathcal{G}(\mathcal{A})} : \mathcal{L}(\chi) \longrightarrow \mathfrak{C}$$

defined by using the left $\mathfrak{C}$-coaction of the $\chi(\mathfrak{p})$'s.

The associated coinduction functor leads to the following composition of functors

$$ {}_{\mathcal{G}(\mathcal{A})}\mathrm{Mod} \xrightarrow{\Sigma(\chi)\otimes_{\mathcal{G}(\mathcal{A})}-} {}_{\mathcal{L}(\chi)}\mathrm{Comod} \xrightarrow{(-)\mathrm{can}_{\mathcal{G}(\mathcal{A})}} {}_{\mathfrak{C}}\mathrm{Comod}. $$

Indeed this is the conceptual framework that allows us to compare certain categories of $\mathbb{k}$-linear functors with the categories of comodules over some corings (or left bialgebroids).

For instance, take $R = \mathbb{k}$ to be a field and A a finite dimensional $\mathbb{k}$-algebra. Consider the cochain complex $Q_\bullet$ mentioned above ▸ 1 and the monoidal $\mathbb{k}$-linear category $\mathbb{k}(\mathbb{N})$ generated by the natural number $\mathbb{N}$.

There is a fiber functor $\chi : \mathbb{k}(\mathbb{N}) \to \mathrm{Mod}_{\mathbb{k}}$ defined by $\chi(n) = Q_n$ on objects and sending the morphism $n \mapsto n+1$ to the differential $\partial : Q_n \to Q_{n+1}$, for every $n \in \mathbb{N}$. Using the previous arguments and notations, we then arrive to the following composition of functors

$$ Ch_+(\mathbb{k}) \xrightarrow[\cong]{\mathcal{O}} {}_{\mathcal{G}(\mathbb{k}(\mathbb{N}))}\mathrm{Mod} \xrightarrow{Q\otimes_{\mathcal{G}(\mathbb{k}(\mathbb{N}))}-} {}_{\mathcal{L}(\chi)}\mathrm{Comod} \xrightarrow{(-)\mathrm{can}_{\mathcal{G}(\mathbb{k}(\mathbb{N}))}} {}_{\mathcal{L}(A)}\mathrm{Comod} $$

where $\mathcal{O}$ is the canonical equivalence between chain complexes of $\mathbb{k}$-vector spaces and left unital $\mathcal{G}(\mathbb{k}(\mathbb{N}))$-modules, and $\mathcal{L}(A)$ is the coendomorphism bialgebra of A considered by Tambara.

The Sweedler-Takeuchi's products $- \times_R -$.

Fix a commutative ring $\Bbbk$ with 1, and $\Bbbk$-algebra R. We use the notation r^o, for $r \in R$, to denote the elements of the opposite ring R^o. By $R^{\mathrm{e}} := R \otimes R^o$ we denote the enveloping ring of R.

Let M be an R^{e}-bimodule, the underlying $\Bbbk$-module admits the following bimodules structures:

$$\begin{tikzcd}
 & & & {}_{R^{\mathrm{e}}}M_{R^{\mathrm{e}}} \arrow[dlll, squiggly, "(-)^o"'] \arrow[dll, squiggly] \arrow[ddl, squiggly, "(-)^l"] \arrow[ddr, squiggly] \arrow[drr, squiggly] \arrow[drrr, squiggly] & & & \\
{}_{1\otimes R^o}M_{1\otimes R^o} & {}_{R\otimes 1^o}M_{R\otimes 1^o} & & & & {}_{R\otimes 1^o}M_{1\otimes R^o} & {}_{1\otimes R^o}M_{R\otimes 1^o} \\
 & & {}_{R^{\mathrm{e}}}M & & M_{R^{\mathrm{e}}} & &
\end{tikzcd}$$

All these underlying structures are in a natural way functorial.

Another R^{e}-bimodule derived from a given R^{e}-bimodule M which will be used in the sequel is $M^{\dagger}$. The underlying $\Bbbk$-module of $M^{\dagger}$ is M, and an element $m \in M$ is denoted by $m^{\dagger}$ when it is viewed in $M^{\dagger}$. The R^{e}-biaction on $M^{\dagger}$ is given by

$$(p \otimes q^o)\, m^{\dagger}\, (r \otimes s^o) \;=\; \Big((p \otimes r^o)\, m\, (q \otimes s^o) \Big)^{\dagger},$$

for every $m^{\dagger} \in M^{\dagger}$, $p, r \in R$ and $q^o, s^o \in R^o$.
Here also we have a functor

$$(-)^{\dagger} : {}_{R^{\mathrm{e}}}\mathrm{Mod}_{R^{\mathrm{e}}} \to {}_{R^{\mathrm{e}}}\mathrm{Mod}_{R^{\mathrm{e}}}$$

which is an idempotent faithful functor, in the sense that, we have

$${}_{R^{\mathrm{e}}}(M^{\dagger})^{\dagger}{}_{R^{\mathrm{e}}} \;=\; {}_{R^{\mathrm{e}}}M_{R^{\mathrm{e}}}$$

and

$$\mathrm{Hom}_{R^{\mathrm{e}}-R^{\mathrm{e}}}\left(M^{\dagger},\, U^{\dagger}\right) \;=\; \mathrm{Hom}_{R^{\mathrm{e}}-R^{\mathrm{e}}}\left(M,\, U\right),$$

for every pair of R^{e}-bimodules U and M. This new R^{e}-bimodule leads, up to isomorphisms, to the already existing bimodules structures.

Now, let N be another R-bimodule, and consider the tensor product $M^o \otimes_R N$. The additive $\Bbbk$-submodule of invariant elements

$$(M^o \otimes_R N)^R = \left\{ \sum_i m_i^o \otimes_R n_i \middle| \sum_i r m_i^o \otimes_R n_i = \sum_i m_i^o \otimes_R n_i r, \text{ for all } r \in R \right\}$$

admits a structure of an R-bimodule given by the following actions:

$$\begin{aligned} r \rightharpoonup \left(\sum_i m_i^o \otimes_R n_i \right) &= \sum_i \Big((r \otimes 1^o)\, m_i \Big)^o \otimes_R n_i, \\ \left(\sum_i m_i^o \otimes_R n_i \right) \leftharpoonup s &= \sum_i \Big(m_i\, (s \otimes 1^o) \Big)^o \otimes_R n_i, \end{aligned}$$

where $\sum_i m_i^o \otimes_R n_i \in M^o \otimes_R N$ and $r, s \in R$.

In this way, to each R-bimodule N one can associate to it two functors:

$${}_{R^e}\mathrm{Mod}_{R^e} \xrightarrow{\left((-)^o \otimes_R N \right)^R} {}_R\mathrm{Mod}_R,$$

$${}_R\mathrm{Mod}_R \xrightarrow{\left(- \otimes^* N \right)^\dagger} {}_{R^e}\mathrm{Mod}_{R^e},$$

Lemma

Let N be an R-bimodule such that ${}_R N$ is finitely generated and projective module with left dual basis $\{(e_j, {}^*e_j)\}_{1 \leq j \leq m} \subset N \times {}^*N$. There is a natural isomorphism

$$\begin{array}{ccc}
\mathrm{Hom}_{R-R}\left(X, (M^o \otimes_R N)^R\right) & \longrightarrow & \mathrm{Hom}_{R^e-R^e}\left((X \otimes {}^*N)^\dagger, M\right) \\
\sigma & \longmapsto & \left[(x \otimes \varphi)^\dagger \longmapsto \left((M^o \otimes_R \varphi) \circ \sigma(x)\right)\right] \\
\left[x \longmapsto \sum_j \alpha\left(\left(x \otimes {}^*e_j\right)^\dagger\right)^o \otimes_R e_j\right] & \longleftarrow & \alpha
\end{array}$$

for every R-bimodule X and R^e-bimodule M. That is, the functor $(- \otimes {}^*N)^\dagger$ is left adjoint to the functor $((-)^o \otimes_R N)^R$.

Few words on the bi-functor $- \times_R -$.
As we have seen before there is a bi-functor

$$- \times_R - := \Big((-)^o \otimes_R - \Big)^R : {}_{R^e}\mathrm{Mod}_{R^e} \times {}_R\mathrm{Mod}_R \longrightarrow {}_R\mathrm{Mod}_R.$$

This is a Sweedler-Takeuchi's product of bimodules.

Notation:

$$\sum_i m_i \times_R n_i := \sum_i m_i^o \otimes_R n_i \in M \times_R N.$$

If N is an R^e-bimodule, then there are several structures of R-bimodules on N over which one can construct $M \times_R N$. Here we define $M \times_R N$ by using the R-bimodule ${}_{R \otimes 1^o} N_{R \otimes 1^o}$. In this way, $M \times_R N$ admits a structure of R^e-bimodule given by the following rule

$$(r \otimes s^o) \left(\sum_i m_i \times_R n_i \right) (p \otimes q^o) \\ = \sum_i \Big((r \otimes 1^o)\, m_i\, (s \otimes 1^o) \Big) \times_R \Big((1 \otimes p^o)\, n_i\, (1 \otimes q^o) \Big),$$

for every elements $\sum_i m_i \times_R n_i \in M \times_R N$ and $r, s, p, q \in R$.

Whence the R^e-biaction on $(M \times_R N)^\dagger$ is given by the formula:

$$(r \otimes s^o) \left(\sum_i m_i \times_R n_i \right)^\dagger (p \otimes q^o)$$
$$= \left(\sum_i \Big((r \otimes 1^o)\, m_i\, (p \otimes 1^o) \Big) \times_R \Big((1 \otimes s^o)\, n_i\, (1 \otimes q^o) \Big) \right)^\dagger .$$

In this way, the functor $- \times_R -$ is restricted to the category ${}_{R^e}\mathrm{Mod}_{R^e} \times {}_{R^e}\mathrm{Mod}_{R^e}$. Thus, we set

$$\begin{tikzcd}
{}_{R^e}\mathrm{Mod}_{R^e} \times {}_{R^e}\mathrm{Mod}_{R^e} \arrow[d, "\left((-)^o \underset{R}{\otimes} {}_{R\otimes 1^o}(-)_{R\otimes 1^o}\right)^R"'] \arrow[drr, dashed, "- \times_R -"] & & \\
{}_{R^e}\mathrm{Mod}_{R^e} \arrow[rr, "(-)^\dagger"'] & & {}_{R^e}\mathrm{Mod}_{R^e}
\end{tikzcd}$$

There is a natural transformation $(- \times_R -) \to (-)^l \otimes_R (-)^l$.

As one can notice there are different ways of constructing the product $- \times_R -$, apriory they will not necessary lead to the same functor (up to natural isomorphism).

Coendomorphism bialgebroids, examples.

Let A be an R-ring, and consider the functor

$$- \times_R A : {}_{R^e}\mathrm{Mod}_{R^e} \to {}_R\mathrm{Mod}_R.$$

For every pair of R^e-bimodules M and N, we have a well defined and R-bilinear maps:

$$\begin{array}{ccc} (M \times_R A) \otimes_R (N \times_R A) & \xrightarrow{\Phi^2_{(M,\,N)}} & (M \otimes_{R^e} N) \times_R A, \\ (m \times_R a) \otimes_R (n \times_R a') & \longmapsto & (m \otimes_{R^e} n) \times_R aa' \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{\Phi^0} & R^e \times_R A \\ r & \longmapsto & (r \otimes 1^o) \times_R 1_A \end{array}$$

where $\Phi^2_{(-,-)}$ is obviously a natural transformation. It turns out that $- \times_R A : {}_{R^e}\mathrm{Mod}_{R^e} \to {}_R\mathrm{Mod}_R$ is a monoidal functor.

Assume now that A is finitely generated and projective as left R-module, and fix a left dual basis $\{({}^*e_j, e_j)\}_{1 \leq j \leq n} \subset {}^*A \times A$. By the previous Lemma 1

$$\mathcal{R} = - \times_R A : {}_{R^e}\mathrm{Mod}_{R^e} \longrightarrow {}_R\mathrm{Mod}_R$$

is a right adjoint to the functor

$$\mathcal{L} = (- \otimes {}^*A)^\dagger : {}_R\mathrm{Mod}_R \longrightarrow {}_{R^e}\mathrm{Mod}_{R^e}.$$

By monoidal categories arguments, the adjunction $\mathcal{L} \dashv \mathcal{R}$ is restricted to the categories of ring extension. That is, we have a lifted adjunction

$$\begin{tikzcd}
R - \texttt{Rings} \arrow[r, shift left, "\mathcal{L}"] \arrow[d] & R^{\mathrm{e}} - \texttt{Rings} \arrow[l, shift left, "\mathcal{R}"] \arrow[d] \\
{}_R\mathrm{Mod}_R \arrow[r, shift left, "\mathcal{L}"] & {}_{R^{\mathrm{e}}}\mathrm{Mod}_{R^{\mathrm{e}}} \arrow[l, shift left, "\mathcal{R}"]
\end{tikzcd}$$

For a given R-ring C, the R^{e}-ring $\mathcal{L}(C)$ is defined by the quotient algebra

$$\mathcal{L}(C) \;=\; \mathcal{T}_{R^{\mathrm{e}}}\Big((C \otimes {}^*A)\Big)/\mathcal{I}_{\mathcal{L}(C)}$$

where $\mathcal{T}_{R^{\mathrm{e}}}\Big((C \otimes {}^*A)\Big)$ is the tensor algebra of the R^{e}-bimodule $(C \otimes {}^*A)$ (in fact $(C \otimes {}^*A)^{\dagger}$) and wherein $\mathcal{I}_{\mathcal{L}(C)}$ is the two-sided ideal generated by the set

$$\left\{ \sum_i \Big((c \otimes e_i\varphi) \otimes_{R^{\mathrm{e}}} (c' \otimes {}^*e_i)\Big) - (cc' \otimes \varphi);\; 1_R \otimes \varphi(1_A)^o - (1_C \otimes \varphi) \right\}$$

where $c,\ c' \in\ C$ and $\varphi \in {}^*A$

Now we take the image of A, that is $\mathcal{L}(A)$. We can show using the above adjunction that $\mathcal{L}(A)$ is left R-bialgebroid:

Proposition

Let A be an R-ring which is finitely generated and projective as left R-module with dual basis $\{({}^*e_i, e_i)\}_i$. Then $\mathcal{L}(A)$ is an $\times_R$-bialgebra with structure maps

$$\Delta : \mathcal{L}(A) \longrightarrow \mathcal{L}(A) \times_R \mathcal{L}(A), \left(\pi_A(a \otimes \varphi) \mapsto \sum_j \pi_A(a \otimes {}^*e_j) \times_R \pi_A(e_j \otimes \varphi) \right)$$

$$\varepsilon : \mathcal{L}(A) \longrightarrow \mathrm{End}_{\Bbbk}(R), \left(\pi_A(a \otimes \varphi) \longmapsto \left[r \mapsto \varphi(ar) \right] \right)$$

Moreover, A is a left $\mathcal{L}(A)$-comodule R-ring with a structure map the unit of the adjunction $\mathcal{L} \dashv \mathcal{R}$ at A:

$$\eta_A : A \longrightarrow \mathcal{L}(A) \times_R A, \quad \left(a \longmapsto \sum_i \pi_A(a \otimes {}^*e_i) \times_R e_i \right)$$

Where we have used the notation

$$\pi_A : A \otimes {}^*A \longrightarrow \mathcal{L}(A) = \mathcal{T}_{R^e}\Big((A \otimes {}^*A) \Big) / \mathcal{I}_{\mathcal{L}(A)}$$

Example

Assume that $A = R^n$, the obvious R-ring attached to the free R-module of rank n. So $\mathcal{L}(A)$ is a left R-bialgebroid generated as a ring by R^{e} and a set of R^{e}-invariant elements $\{x_{ij}\}_{1 \leq i, j \leq n}$ with relation

$$\begin{aligned} x_{ii}^2 &= x_{ii}, \quad \text{for all } i = 1, 2, \cdots, n. \\ x_{ji}\, x_{ki} &= 0, \quad \text{for all } j \neq k, \text{ and } i, j, k = 1, 2, \cdots, n. \\ \sum_{i=1}^{n} x_{ij} &= 1, \quad \text{for all } j = 1, 2, \cdots, n. \end{aligned}$$

The comultiplication and counit are given by

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes_R x_{kj}, \qquad \varepsilon(x_{ij}) = \delta_{ij}, \text{(Kronecker delta)}$$

for all $i, j = 1, 2, \cdots, n$.

Example

Let A be the trivial crossed product of R by a cyclic group of order 2, then $\mathcal{L}(A)$ is a left R-bialgebroid generated as an R^{e}-ring by two R^{e}-invariant elements x, y subject to the relations

$$xy + yx = 0 \quad \text{and} \quad 1 = x^2 + y^2.$$

The comultiplication and counit of the underlying R-coring structure are given by

$$\Delta(x) = x \otimes_R 1 + y \otimes_R x, \qquad \Delta(y) = y \otimes_R y, \quad \varepsilon(x) = 0, \quad \varepsilon(y) = 1.$$

For a cyclic group of order $n > 2$, the relations are quite complicated.

Example (Quaternion coendomorphism bialgebra)

Assume that $R = \Bbbk$ is a field with characteristic not equal to 2. Let A be the Hamilton quaternion $\Bbbk$-algebra associated to the pair $(-1,-1)$. That is, $A = \Bbbk \oplus \Bbbk\mathfrak{i} \oplus \Bbbk\mathfrak{j} \oplus \Bbbk\mathfrak{ij}$ with relation $\mathfrak{i}^2 = -1 = \mathfrak{j}^2$ and $\mathfrak{ij} = -\mathfrak{ji}$. The *quaternion coendomorphism bialgebra* $\mathcal{L}(A)$ is a $\Bbbk$-bialgebra, which is generated as an $\Bbbk$-algebra by elements $\{x_k, y_k, z_k, u_k\}_{1\leq k\leq 3}$ subject to the relations

$$\begin{aligned} 1 + x_k^2 &= y_k^2 + z_k^2 + u_k^2, \quad \text{for all } k = 1,2,3, \\ x_1x_2 + x_2x_1 &= y_2y_1 + y_1y_2 + u_2u_1 + u_1u_2 + z_2z_1 + z_1z_2, \end{aligned}$$

$$\begin{aligned} x_1y_1 &= -y_1x_1 + z_1u_1 - u_1z_1, & u_1y_1 &= y_1u_1 + z_1x_1 + x_1z_1, \\ z_1y_1 &= y_1z_1 - x_1u_1 - u_1x_1, & x_3 &= x_1x_2 - y_1y_2 - z_1z_2 - u_1u_2, \\ y_3 &= x_1y_2 + y_1x_2 - z_1u_2 + u_1z_2, & z_3 &= x_1z_2 + y_1u_2 + z_1x_2 - u_1y_2, \\ u_3 &= x_1u_2 - y_1z_2 + z_1y_2 + u_1x_2, & u_2x_1 &= -x_2u_1 - y_2z_1 - z_2y_1, \\ u_2z_1 &= -x_2y_1 - y_2x_1 - z_2u_1 + y_3, & u_2y_1 &= x_2z_1 + y_2u_1 + z_2x_1 + z_3, \\ x_2y_2 &= -y_2x_2 + z_2u_2 - u_2z_2, & x_2z_2 &= -y_2u_2 - z_2x_2 + u_2y_2, \\ x_2u_2 &= -y_2z_2 - z_2y_2 - u_2x_2. \end{aligned}$$

We did not succeed to simplify these relations!

Example

Let $A = R \oplus Rt$ be the trivial generalized R-ring i.e. the R-ring which is free as left R-module with basis $1 = (1, 0)$ and $\mathfrak{t} = (0, t)$ such that $\mathfrak{t}^2 = 0$. $\mathcal{L}(A)$ is left R-bialgebroid generated by the image of R^{e} and two R^{e}-invariant elements $\{x, y\}$ subject to the relations

$$xy + yx = 0, \quad \text{and} \quad x^2 = 0.$$

The comultiplication and counit of it underlying R-coring are given by

$$\begin{aligned} \Delta(x) &= x \otimes_R 1 + y \otimes_R x, \quad \varepsilon(x) = 0 \\ \Delta(y) &= y \otimes_R y, \quad \varepsilon(y) = 1. \end{aligned}$$

A is a left $\mathcal{L}(A)$-comodule ring with coaction: $\lambda : A \to \mathcal{L}(A) \otimes_R A$ sending

$$\lambda(1_A) = 1_{\mathcal{L}(A)} \otimes_R 1_A, \quad \lambda(\mathfrak{t}) = x \otimes_R 1_A + y \otimes_R \mathfrak{t}.$$

Comatrix (left) bialgebroids.

The general noncommutative Tannaka reconstruction.

Let $\mathcal{A}$ be a small full sub-category of an additive category. Following Gabriel, we can associate to $\mathcal{A}$ the ring with enough orthogonal idempotents $S = \oplus_{\mathfrak{p},\, \mathfrak{p}' \in \mathcal{A}} \mathrm{Hom}_{\mathcal{A}^o}(\mathfrak{p},\, \mathfrak{p}')$, where $\mathcal{A}^o$ is the opposite category of $\mathcal{A}$; ${}_S\mathrm{Mod}$ denotes the category of left unital S-module.

Let us denote by $add({}_RR)$ the full sub-category of ${}_R\mathrm{Mod}$ consisting of all finitely generated and projective left R-modules. Let

$$\chi : \mathcal{A} \longrightarrow add({}_RR)$$

be a faithful functor, refereed to as *fiber functor*. We denote by $\mathfrak{p}^\chi$ the image of $\mathfrak{p} \in \mathcal{A}$ under χ or by $\mathfrak{p}$ itself. Consider the left R-module direct sum of the $\mathfrak{p}$'s: $\Sigma = \oplus_{\mathfrak{p} \in \mathcal{A}} \mathfrak{p}$ (i.e. $\Sigma = \oplus_{\mathfrak{p} \in \mathcal{A}} \mathfrak{p}^\chi$) and the right R-module direct sum of their duals: ${}^\vee\Sigma = \oplus_{\mathfrak{p} \in \mathcal{A}} {}^*\mathfrak{p}$.

It is clear that ${}^\vee\Sigma$ is a left unital S-module while Σ is a right unital S-module. In this way Σ becomes an (R, S)-bimodule and ${}^\vee\Sigma$ an (S, R)-bimodule.

Then $\Sigma \otimes_S {}^{\vee}\Sigma$ is now an R-bimodule whose elements are described as a finite sum of diagonal ones, i.e. of the form $\iota_{\mathfrak{p}}(u_{\mathfrak{p}}) \otimes_S \iota_{^*\mathfrak{p}}(\varphi_{\mathfrak{p}})$ where $(u_{\mathfrak{p}}, \varphi_{\mathfrak{p}}) \in \mathfrak{p}^{\chi} \times ({}^*\mathfrak{p}^{\chi})$ and ι_{-} are the canonical injections in ${}^{\vee}\Sigma$ and Σ. From now on, we will use $u_{\mathfrak{p}} \otimes_S \varphi_{\mathfrak{p}}$ instate of $\iota_{\mathfrak{p}}(u_{\mathfrak{p}}) \otimes_S \iota_{^*\mathfrak{p}}(\varphi_{\mathfrak{p}})$ to denote a generic element of $\Sigma \otimes_S {}^{\vee}\Sigma$.

This bimodule admits a structure of an R-coring given by the following comultiplication

$$\begin{aligned} \Delta : \Sigma \otimes_S {}^{\vee}\Sigma &\longrightarrow (\Sigma \otimes_S {}^{\vee}\Sigma) \otimes_R (\Sigma \otimes_S {}^{\vee}\Sigma) \\ u_{\mathfrak{p}} \otimes_S \varphi_{\mathfrak{p}} &\longmapsto \textstyle\sum_i u_{\mathfrak{p}} \otimes_S {}^*u_{\mathfrak{p},\, i} \otimes_R u_{\mathfrak{p},\, i} \otimes_S \varphi_{\mathfrak{p}}, \end{aligned}$$

where, for a fixed $\mathfrak{p} \in \mathcal{A}$, the finite set $\{(u_{\mathfrak{p},\, i}, {}^*u_{\mathfrak{p},\, i})\}_i \subset \mathfrak{p} \times {}^*\mathfrak{p}$ is a left dual basis of the left R-module $\mathfrak{p}$; counit is given by evaluations.

With this structure $\Sigma \otimes_S {}^{\vee}\Sigma$ is refereed to as the *infinite comatrix coring* associated to the small category $\mathcal{A}$ and the fiber functor χ. It turns out that each of the left R-modules $\mathfrak{p}^{\chi}$ is actually a left $(\Sigma \otimes_S {}^{\vee}\Sigma)$-comodule with coaction, using the above notation is given by

$$\widetilde{\lambda}_{\mathfrak{p}} : \mathfrak{p} \longrightarrow \Sigma \otimes_S {}^{\vee}\Sigma \otimes_R \mathfrak{p}, \quad \left(u \longmapsto \sum_i u \otimes_S {}^*u_{\mathfrak{p},\, i} \otimes_R\, u_{\mathfrak{p},\, i} \right).$$

Another description of the infinite comatrices is given by the following isomorphism of R-bimodules

$$\Sigma \otimes_S {}^{\vee}\Sigma \;\cong\; \frac{\bigoplus_{\mathfrak{p} \in \mathcal{A}} \mathfrak{p} \otimes_{T_{\mathfrak{p}}} {}^{*}\mathfrak{p}}{\left\langle u\mathfrak{t} \otimes_{T_{\mathfrak{q}}} \varphi \,-\, u \otimes_{T_{\mathfrak{p}}} \mathfrak{t}\varphi \right\rangle_{\{u \in \mathfrak{p},\; \varphi \in {}^{*}\mathfrak{q},\; \mathfrak{t} \in T_{\mathfrak{q},\mathfrak{p}}\}}}$$

where $T_{\mathfrak{p}} := \mathrm{End}_{\mathcal{A}^o}(\mathfrak{p})$ and $T_{\mathfrak{p},\,\mathfrak{q}} := \mathrm{Hom}_{\mathcal{A}^o}\left(\mathfrak{p},\, \mathfrak{q}\right)$, for every objects $\mathfrak{p}, \mathfrak{q}$ in $\mathcal{A}$.

Theorem (Generalized Descent. (Gómez-Torrecillas, speaker 2004))

The following statements are equivalent.

(i) $(\Sigma \otimes_S {}^{\vee}\Sigma)_R$ *is flat,* $\{\mathfrak{p}^{\chi}\}_{\mathfrak{p} \in \mathcal{A}}$ *is a set of projective small generators for* ${}_{\Sigma \otimes_S {}^{\vee}\Sigma}\mathrm{Comod}$*;*

(ii) $(\Sigma \otimes_S {}^{\vee}\Sigma)_R$ *is flat and* $\Sigma \otimes_S - : {}_S\mathrm{Mod} \to {}_{\Sigma \otimes_S {}^{\vee}\Sigma}\mathrm{Comod}$ *is an equivalence of a categories;*

(iii) Σ_S *is faithfully flat unital module.*

Application to the cochain complex $Q_\bullet$.

Coming back to our initial situation: That is, $R \to A$ is a $\Bbbk$-algebra extension such that ${}_RA$ is finitely generated and projective module with a fixed basis $\{e_i, {}^*e_i\}_{1 \leq i \leq n}$. Let $Q_\bullet$ be the corresponding cochain complex of finitely generated and projective left R-modules

$$Q_0 = R, \; Q_1 = A, \; Q_2 = K, \cdots\cdots, Q_n = K \otimes_A \cdots \otimes_A K, \; n \geq 3.$$

Let us consider the $\Bbbk$-linear category $\Bbbk(\mathbb{N})$ whose objects are the natural numbers $\mathbb{N}$, and homomorphisms sets are defined by

$$\mathrm{Hom}_{\Bbbk(\mathbb{N})}\Big(n,\, m\Big) \;=\; \begin{cases} 0, \text{ if } \; m \notin \{n, n+1\} \\ \Bbbk.1_n, \text{ if } \; n = m \\ \Bbbk.\jmath_n^{n+1}, \text{ if } \; m = n+1. \end{cases}$$

The last two terms are free $\Bbbk$-modules of rank one. The induced ring with enough orthogonal idempotents is the free $\Bbbk$-module $B = \Bbbk^{(\mathbb{N})} \oplus \Bbbk^{(\mathbb{N})}$ generated by the set $\{\mathfrak{h}_n, \mathfrak{v}_n\}_{n \in \mathbb{N}}$, where $\mathfrak{h}_n$ and $\mathfrak{v}_n$ corresponds to 1_n and j_n^{n+1} respectively, subject to the following relations:

$$\begin{aligned} \mathfrak{h}_n\mathfrak{h}_m &= \delta_{n,m}\mathfrak{h}_n, \;\forall m, n \in \mathbb{N} \qquad \text{(Kronecker delta)} \\ \mathfrak{v}_n\mathfrak{v}_m &= \mathfrak{v}_m\mathfrak{v}_n \;=\; 0, \;\forall m, n \in \mathbb{N} \\ \mathfrak{v}_n\mathfrak{h}_{n+1} &= \mathfrak{v}_n \;=\; \mathfrak{h}_n\mathfrak{v}_n, \;\forall m, n \in \mathbb{N}. \end{aligned}$$

In other words B is the submodule of the ring of $(\mathbb{N} \times \mathbb{N})$-matrices over $\Bbbk$ of the form

$$\begin{pmatrix} \Bbbk & \Bbbk & 0 & 0 & & & \\ 0 & \Bbbk & \Bbbk & 0 & & & \\ 0 & 0 & \Bbbk & \Bbbk & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & \Bbbk & \Bbbk & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

consisting of matrices with only possibly two non-zero entries in each row: (i, i) and $(i, i+1)$.

It is clear that the category of unital left B-modules is isomorphic to the category $Ch_+(\Bbbk)$ of chain complexes of $\Bbbk$-modules. Precisely, this isomorphism functor $\mathcal{O}$ sends every chain complex $(V_\bullet, \partial^V)$ to its associated differential graded $\Bbbk$-module $\mathcal{O}(V_\bullet) = \oplus_{n \geq 0} V_n$ with the following left B-action

$$\mathfrak{h}_n . \sum_{n \geq 0} v_i = v_n, \quad \text{and } \mathfrak{v}_n . \sum_{n \geq 0} v_i = \partial^V(v_{n+1})$$

and acts in the obvious way on morphisms of chain complexes. The inverse functor is clear.

The convolution product on the left dual chain complex of $Q_\bullet$ is given as follows: For every $\varphi \in {}^*Q_n$ and $\psi \in {}^*Q_m$ with $n, m \geq 1$, we have a left R-linear map

$$\begin{aligned} \varphi \star \psi : Q_{n+m} &\longrightarrow R \\ x \otimes_A \partial(a) \otimes_A y &\longmapsto \varphi\Big(x\psi(ay)\Big) - \varphi\Big(xa\psi(y)\Big), \end{aligned}$$

where $x \in Q_n$, $y \in Q_m$, and $a \in A$. The convolution product with zero degree element is just the left and right R-actions of ${}^*Q_n$, for every $n \geq 1$, namely

$$\begin{aligned} r \star \varphi : Q_n &\longrightarrow R \\ x &\longmapsto \varphi(x\, r), \end{aligned} \qquad \begin{aligned} \varphi \star s : Q_n &\longrightarrow R \\ x &\longmapsto \varphi(x)\, s, \end{aligned}$$

for every elements $r, s \in R$ and $\varphi \in {}^*Q_n$.

Now we can consider the comatrix R-coring

$$Q \otimes_B {}^{\vee}Q, \text{ and the canonical map } \mathrm{can}_B : Q \otimes_B {}^{\vee}Q \longrightarrow \mathcal{L}(A),$$

associated to the fiber functor $\chi : \mathbb{k}(\mathbb{N}) \to add({}_R R)$ which sends $n \mapsto Q_n$. Here $\mathcal{L}(A)$ is the coendomorphism left R-bialgebroid of A. There is a natural transformation $(\chi \otimes_R \chi) \to \mathcal{L}(\chi) \otimes_R (\chi \otimes_R \chi)$ given by: $\widetilde{\lambda}_{n,m} : Q_n \otimes_R Q_m \to (Q \otimes_B {}^{\vee}Q) \otimes_R (Q_n \otimes_R Q_m)$

$$u_n \otimes_R u_m \longmapsto$$

$$\sum_{\alpha,\, \beta} \Big[(u_n \otimes_A u_m) \otimes_B ({}^*\omega_{n,\,\alpha} \star \partial^* \omega_{m,\,\beta}) + (u_n \otimes_A \partial u_m) \otimes_B ({}^*\omega_{n,\,\alpha} \star {}^*\omega_{m,\,\beta}) \Big] \otimes_R \Big(\omega_{n,\,\alpha} \otimes_R \omega_{m\,\beta} \Big)$$

for every $n, m \geq 1$, and by $\widetilde{\lambda}_{0,\,n} = \widetilde{\lambda}_{n,\,0} : Q_n \to (Q \otimes_B {}^{\vee}Q) \otimes_R Q_n$,

$$u_n \longmapsto \sum_{\alpha} (u_n \otimes_B {}^*\omega_{n,\,\alpha}) \otimes_R \omega_{n,\,\alpha},$$

where $\{(\omega_{n,\,\alpha},\, {}^*\omega_{n,\,\alpha})\}$ is a dual basis for ${}_R Q_n$, $n \geq 1$.

Using the previous natural transformation, we can extract from its using the bijection 1 , a unital multiplication $\mathcal{L}(A) \otimes_{R^e} \mathcal{L}(A) \to \mathcal{L}(A)$. We then arrive to the left R-bialgebroid structure of $Q \otimes_B {}^{\vee}Q$ (in fact the R^e-bimodule structure is that of $(Q \otimes_B {}^{\vee}Q)^{\dagger}$).

- The unit is:

$$R^e \to Q \otimes_B {}^{\vee}Q, \quad \text{sending } r \otimes s^o \mapsto (r \otimes_B s)$$

- The multiplication is defined by the formulae:

$$\begin{aligned}(u_n \otimes_B \varphi_n) \, . \, (u_m \otimes_B \varphi_m) \;=\; & \Big((u_n \otimes_A \partial u_m) \otimes_B (\varphi_n \star \varphi_m) \Big) \\ & + \Big((u_n \otimes_A u_m) \otimes_B (\varphi_n \star \partial \varphi_m) \Big)\end{aligned}$$

and

$$\begin{aligned}(u_n \otimes_B \varphi_n) \, . \, (r \otimes_B s) \;&=\; (u_n r \otimes_B s \varphi_n), \\ (r \otimes_B s) \, . \, (u_n \otimes_B \varphi_n) \;&=\; (r u_n \otimes_B \varphi_n s), \quad \forall r, s \in R.\end{aligned}$$

for every pair of generic elements $(u_n \otimes_B \varphi_n)$ and $(u_m \otimes_B \varphi_m)$ of $Q \otimes_B {}^{\vee}Q$ with $n, m > 0$,

Description of the main results.

Theorem

The R-coring $Q \otimes_B {}^{\vee}Q$ admits a structure of left R-bialgebroids, such that the canonical (or Galois map) $\mathrm{can}_B : Q \otimes_B {}^{\vee}Q \to \mathcal{L}(A)$ is an isomorphism of left R-bialgebroids.

Combining this Theorem with the generalized descent theorem ▸ 1 , we obtain

Theorem (**A**)

Let $R \to A$ be a $\Bbbk$-algebra map with A finitely generated and projective as left R-module. Consider the associated left R-bialgebroid $\mathcal{L}(A)$ and the cochain complex $Q_{\bullet}$ with its canonical right unital B-action and left $\mathcal{L}(A)$-coaction, where $B = \Bbbk^{(\mathbb{N})} \oplus \Bbbk^{(\mathbb{N})}$. Then the following statements are equivalent

1. *The right R-module ${}_{1 \otimes_{\Bbbk} R^o}\mathcal{L}(A)$ is flat and the functor $Q \otimes_B - : {}_B\mathrm{Mod} \longrightarrow {}_{\mathcal{L}(A)}\mathrm{Comod}$ is an equivalence of monoidal categories;*
2. *Q_B is a faithfully flat unital module.*

Corollary

The category $ch_+(\Bbbk)$ of chain complexes of $\Bbbk$-modules is monoidally equivalent to the category ${}_{\mathcal{L}(A)}\mathrm{Comod}$ of left comodules, if and only if Q_B is faithfully flat unital module.

Our next aim is to extended this Corollary to the category $ch_+(R)$ of chain complexes of left R-modules. So, consider the map of rings with same set of orthogonal idempotents:

$$B = \Bbbk^{(\mathbb{N})} \oplus \Bbbk^{(\mathbb{N})} \longrightarrow R^{(\mathbb{N})} \oplus R^{(\mathbb{N})} := C,$$

coming from the unit $\Bbbk \to R$. This enables us to consider the usual adjunction between the scalars-restriction functor and the tensor product functor

Let $\mathcal{J}$ be the left ideal of $\mathcal{L}(A)$ generated by the following set of elements

$$\Big\{\pi(ar \otimes \varphi) - \pi(a \otimes r\varphi)\Big\}_{a \in A,\, \varphi \in {}^*A,\, r \in R}.$$

Then $\mathcal{J}$ is a coideal of the underlying R-coring $\mathcal{L}(A)^l$.

Consider the quotient R-coring and the canonical surjection:

$$\overline{\pi} : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)/\mathcal{J} := \overline{\mathcal{L}}.$$

• In the commutative case the coideal $\mathcal{J}$ is in fact a two-sided ideal, and the quotient R-coring inherits a structure of bialgebroid from $\mathcal{L}(A)$. Thus, in this case, there are many ways by which one can compare the category of left comodules and category of chain complexes.

We can show that we have a morphism of rings $R \rightarrow \mathrm{End}_{\overline{\mathcal{L}}}(Q_n)$, for every $n \geq 0$. This leads to a faithful functor from the category $R(\mathbb{N})$ to the category of $(\overline{\mathcal{L}}, R)$-bicomodules (here R is considered as a trivial R-coring)

$$\chi' : R(\mathbb{N}) \rightarrow {}_{\overline{\mathcal{L}}}\mathrm{Comod}_R.$$

The composition of χ' with the forgetful functor gives rise then to a fiber functor $\omega : R(\mathbb{N}) \rightarrow {}_R\mathrm{Mod}_R$ whose image is in $add({}_R R)$.

Therefore, we can apply the Tannaka reconstructions. Thus, we have an infinite comatrix R-coring $Q \otimes_C {}^{\vee}Q$ together with a canonical map $\mathrm{can}_C : Q \otimes_C {}^{\vee}Q \longrightarrow \overline{\mathcal{L}}$ sending

$$\mathrm{can}_C(u_n \otimes_C \varphi_n) \;=\; \sum_{i_0, \cdots, i_{n-1}} \overline{\pi}\left(\pi(a_0 \otimes {}^*e_{i_0}) \cdots \pi(a_{n-1} \otimes {}^*e_{i_{n-1}})\right) \varphi\left(e_{i_0}\partial e_{i_1} \otimes_A \cdots \otimes_A \partial e_{i_{n-1}}\right).$$

for element $u_n \in Q_n$ of the form $u_n = a_0 \partial a_1 \otimes_A \cdots \otimes_A \partial a_{n-1}$, and $\varphi_n \in Q_n^*$.

Clearly we have a surjective map $\vartheta : Q \otimes_B {}^{\vee}Q \to Q \otimes_C {}^{\vee}Q$, as well as the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\vartheta) \arrow[r] \arrow[d, dashed] & Q \otimes_B {}^{\vee}Q \arrow[r, "\vartheta"] \arrow[d, "\mathrm{can}_B"] & Q \otimes_C {}^{\vee}Q \arrow[r] \arrow[d, "\mathrm{can}_C"] & 0 \\
0 \arrow[r] & \mathcal{J} \arrow[r] & \mathcal{L} \arrow[r, "\overline{\pi}"] & \overline{\mathcal{L}} \arrow[r] & 0
\end{tikzcd}$$

In this diagram, we have the following equality $\mathrm{can}_B(\mathrm{Ker}(\vartheta)) = \mathfrak{J}$. In particular, the map can_C is an isomorphism of R-corings. Thus the following Theorem also follows from [2] .

Theorem (**B**)

Let $R \to A$ be a $\Bbbk$-algebra map with A finitely generated and projective as left R-module. Consider $\mathfrak{L}(A)$ the associated left R-bialgebroid and $\mathfrak{J}$ the coideal of $\mathfrak{L}(A)$ generated by the set of elements $\{1_{\mathfrak{L}(A)}(r \otimes 1^o - 1 \otimes r^o)\}_{r \in R}$; denote by $\overline{\mathfrak{L}(A)} = \mathfrak{L}(A)/\mathfrak{J}$ the corresponding quotient R-coring. Consider the cochain complex $Q_\bullet$ with its structures of right unital C-module and left $\overline{\mathfrak{L}(A)}$-comodule. Then the following statements are equivalent

1. *The right R-module ${}_{1\otimes_\Bbbk R^o}\overline{\mathfrak{L}(A)}$ is flat and the functor $Q \otimes_C - : {}_C\mathrm{Mod} \longrightarrow {}_{\overline{\mathfrak{L}(A)}}\mathrm{Comod}$ is an equivalence of categories;*
2. *Q_C is a faithfully flat unital module.*

Therefore, we obtain that

Corollary

The category $ch_+(R)$ of chain complexes of left R-modules is equivalent to the category of left $\overline{\mathcal{L}}$-comodules, if and only if Q_C is faithfully flat module.

Notice that, if Q_C is faithfully flat, then the inverse functor of

$$Q \otimes_C (-) : {}_C\mathrm{Mod} \to {}_{\overline{\mathcal{L}}}\mathrm{Comod}$$

is given by the cotensor product

$${}^{\vee}Q \square_{\overline{\mathcal{L}}} (-) : {}_{\overline{\mathcal{L}}}\mathrm{Comod} \to {}_C\mathrm{Mod}.$$

Here the structure of $(C, \overline{\mathcal{L}})$-bicomodule of ${}^{\vee}Q$ is deduced from that of Q using the fact that each of the Q_n's is finitely generated and projective left R-module. The same argument runs for the inverse of the functor $Q \otimes_B - : {}_B\mathrm{Mod} \to {}_{\mathcal{L}}\mathrm{Comod}$.

Conditions under which Q_C is faithfully flat.

As was seen before a necessary condition for establishing an equivalence of categories of left comodules and chain complexes, is the faithfully flatness of the unital right module Q. The proof of this fact is actually the most difficult task in this theory. Here are some conditions under which this property is satisfies.

Theorem (C)

Let $R \to A$ be a $\Bbbk$-algebra map with A finitely generated and projective as left R-module. Assume further that A_R is finitely generated and projective, and the cochain complex $Q_\bullet$ is exact and splits, in the sense that, for every $m \geq 1$,

$$Q_m = \partial Q_{m-1} \oplus \overline{Q}_m = \mathrm{Ker}(\partial) \oplus \overline{Q}_m$$

as right R-modules, for some right R-module $\overline{Q}_m$. Then Q_C is a flat module. Furthermore, if $\Bbbk$ is a field and R is a division $\Bbbk$-algebra, then Q_C is faithfully flat.

The main example.

Consider the Example . It is easily seen that the kernel of the multiplication of A, i.e. $K = \mathrm{Ker}(A \otimes_R A \to A)$ is free as a left and right R-module with basis $\{\partial \mathfrak{t}, \mathfrak{t}\partial \mathfrak{t}\}$. In fact K is a free A-module with rank one and basis $\partial \mathfrak{t}$. We summarize the properties of the cochain complex $Q_\bullet$, as follows.

Proposition

The cochain complex $Q_\bullet$ associated to the trivial generalized ring $A = R \oplus Rt$, fulfils the following properties:

(i) *For every $m \geq 2$, Q_m is free as a left and right R-module with rank two, and its basis (on both sides) is given by the set $\left\{ \mathfrak{t}\partial\mathfrak{t} \otimes_A \cdots \otimes_A \partial\mathfrak{t},\ \partial\mathfrak{t} \otimes_A \cdots \otimes_A \partial\mathfrak{t} \right\}$.*

(ii) *Q is a homotopically trivial complex.*

(iii) *Q_C is faithfully flat module.*

In particular $ch_+(R)$ is equivalent to the category of comodules $_{\mathcal{L}(A)}\mathrm{Comod}$, and the equivalence is monoidal whenever R is a commutative ring.

teşekkür ederim !