Morita Theory for Commutative Hopf Algebroids, and Hovey-Strickland's Conjecture.

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- Lie groupoids with arrows the set of isomorphisms classes of (left) principal bundles (also know as *Hilsum-Skandalis* maps) form a category *MGPD* called a *Morita category of Lie groupoids*.
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- The trivial bundle construction defines a functor $\mathscr{P} : Gpd \to MGPD$ from the category of Lie groupoids to the Morita category of Lie groupoids. This functor sends weak equivalences to isomorphisms.
- The pair $(\mathscr{P}, MGPD)$ forms a universal solution: Any functor $\mathscr{F} : Gpd \to \mathcal{C}$ which sends weak equivalences to isomorphisms, factors uniquely through $\mathscr{P}$.

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A *commutative Hopf algebroid* is a *co-groupoid* kind of object in the category $\mathscr{A}lg_{\Bbbk}$. Specifically, this is a pair of commutative $\Bbbk$-algebras $(A, \mathcal{H})$ such that for any object C in $\mathscr{A}lg_{\Bbbk}$, we have in a functorial way, a groupoid structure

$$\mathscr{H}(C): \qquad \mathscr{A}lg_{\Bbbk}(\mathcal{H}, C) \underset{t^*}{\overset{s^*}{\rightrightarrows}} \!\!\xleftarrow{\varepsilon^*}\! \mathscr{A}lg_{\Bbbk}(A, C).$$

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Equivalently, there are morphisms of $\mathbb{k}$-algebras:

$$A \overset{s}{\underset{t}{\rightrightarrows}} \xleftarrow{\varepsilon} \mathcal{H} \qquad {}_s\mathcal{H}_t \xrightarrow{\Delta} {}_s\mathcal{H}_t \otimes_A {}_s\mathcal{H}_t, \qquad {}_s\mathcal{H}_t \xrightarrow{\mathscr{S}} {}_t\mathcal{H}_s.$$

source, target and identity arrow — composition — inverse arrow

satisfying the corresponding compatibility axioms of (a local) groupoid: co-associativity, co-unitary and idempotency properties.

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The category of right comodules $\mathrm{Comod}_{\mathcal{H}}$ is a symmetric monoidal category. The category of left $\mathcal{H}$-comodules is analogously defined and it is isomorphic via the antipode to the category of right comodules. Bicomodules are left and right comodules with colinear coactions.

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We say that $(A, \mathcal{H})$ is a *flat Hopf alegebroid* when ${}_{\mathrm{s}}\mathcal{H}$ (or $\mathcal{H}_{\mathrm{t}}$) is a flat A-module. In this case, both extensions s and t are faithfully flat. The category $\mathrm{Comod}_{\mathcal{H}}$ is a Grothendieck category if and only if $(A, \mathcal{H})$ is a flat Hopf algebroid and the forgetful functor $\mathscr{O}_{\mathcal{H}} : \mathrm{Comod}_{\mathcal{H}} \to \mathrm{Mod}_A$ is exact.

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Two flat Hopf algebroids are said to be *Morita equivalent* if their categories of (right) comodules are equivalent as symmetric monoidal categories.

Weak equivalences between Hopf algebroids.

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A *morphism of Hopf algebroids*: $\boldsymbol{\phi} : (A, \mathcal{H}) \to (B, \mathcal{K})$ is a pair of $\mathbb{k}$-algebra maps $\boldsymbol{\phi} = (\phi_0, \phi_1)$:

$$\phi_0 : A \to B, \qquad \phi_1 : \mathcal{H} \to \mathcal{K}$$

such that, for any object C in $\mathscr{A}lg_{\mathbb{k}}$, the associated pair of maps

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establishes, in a natural way, a morphism of groupoids, that is, a functor between the underlying categories.

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This is by definition a monoidal symmetric functor.

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Two Hopf algebroids $(A, \mathcal{H})$ and $(B, \mathcal{K})$ are said to be *weakly equivalent* if there is a third Hopf algebroid $(C, \mathcal{J})$ with diagram

$$\begin{array}{ccc} & (C, \mathcal{J}) & \\ \nearrow & & \nwarrow \\ (A, \mathcal{H}) & & (B, \mathcal{K}) \end{array}$$

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$$(A, \mathcal{H}) \longrightarrow (C, \mathcal{J}) \longleftarrow (B, \mathcal{K})$$

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Obviously two flat Hopf algebroid which are weakly equivalent, they are Morita equivalent. The converse was conjectured as follows by Mark Hovey and Neil Strickland

Now, if Φ_* is an equivalence of categories, then the counit $\Phi_*\Phi^*N \to N$ must be an isomorphism for all Σ-comodules N. In particular, $\Phi_*\Phi^*\Sigma \to \Sigma$ must be an isomorphism. But

$$\Phi_*\Phi^*\Sigma \cong B \otimes_A \Phi^*(\Sigma \otimes_B B) \cong B \otimes_A \Gamma \otimes_A B,$$

completing the proof. □

For rings R and S, we can have equivalences of categories between R-modules and S-modules that are not induced by maps $R \to S$; this is, of course, the content of Morita theory. However, two commutative rings are Morita equivalent if and only if they are isomorphic. We view our Hopf algebroids as fundamentally commutative objects, so we do not expect any non-trivial Morita theory.

Conjecture 6.3. *Suppose (A, Γ) and (B, Σ) are flat Hopf algebroids such that the category of Γ-comodules is equivalent to the category of Σ-comodules. Then (A, Γ) and (B, Σ) are connected by a chain of weak equivalences.*

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$$A \xrightarrow{\alpha} P \xleftarrow{\beta} B$$

is a diagram of $\Bbbk$-algebras, and P is an $(\mathcal{H},\mathcal{K})$-bicomodule algebra, that is, P is left $\mathcal{H}$-comodule algebra via α and a right $\mathcal{K}$-comodule algebra via β, such that

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- the canonical map

$$\mathrm{can}_{\mathcal{H},P} : P \otimes_B P \longrightarrow \mathcal{H} \otimes_A P, \quad \left(p \otimes_B q \longmapsto p_{(-1)} \otimes_A p_{(0)}q\right)$$

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Right principal bundles are similarly defined. A *principal bi-bundles* is simoultaniously a left and right principal bundle.

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- Given a morphism of flat Hopf algebroids $\boldsymbol{\phi} : (A, \mathcal{H}) \to (B, \mathcal{K})$, set $P := \mathcal{H} \otimes_{A\ \phi_0} B$ with the $\Bbbk$-algebra extensions

$$\alpha : A \to P,\ \alpha(a) = \mathsf{s}(a) \otimes_A 1_B, \quad \beta : B \to P,\ \beta(b) = 1_{\mathcal{H}} \otimes_A b.$$

Then (P, α, β) is a left principal $(\mathcal{H}, \mathcal{K})$-bundle, called *the trivial bundle* as it is the pull-back of the unit bundle $\boldsymbol{\phi}^*(\mathscr{U}(\mathcal{H}))$.

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Left principal $(\mathcal{H}, \mathcal{K})$-bundles form a category $PB^{\ell}(\mathcal{H}, \mathcal{K})$ where each morphism is an isomorphism, i.e. a groupoid. The cotensor product of left principal bundles, is again a left principal bundle. Thus left principal bundles over flat Hopf algebroids form a *bicategory* (in fact *bigroupoid*) denoted by PB^{ℓ}, and there is a 2-functor:

$$\mathscr{P} : HAlgd \longrightarrow PB^{\ell},$$

where *HAlgd* is the 2-category of flat Hopf algebroids.

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$$(A, \mathcal{H}) \xrightarrow{\boldsymbol{\alpha}=(\alpha, \alpha_1)} (P, \mathcal{H} \ltimes P \rtimes \mathcal{K}) \xleftarrow{\boldsymbol{\beta}=(\beta, \beta_1)} (B, \mathcal{K})$$

of Hopf algebroids, where α_1 and β_1 are the obvious maps. This Hopf algebroid is called the *two-sided translation Hopf algebroid*.

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- if (P,α,β) is a principal $(\mathcal{H},\mathcal{K})$-bibundle, then both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are weak equivalences. Therefore, $(A,\mathcal{H})$ and $(B,\mathcal{K})$ are weakly equivalent.

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The converse can be seen as follows: Take two weakly equivalent flat Hopf algebroids:

$$(A,\mathcal{H}) \xrightarrow{\varphi} (C,\mathcal{J}) \xleftarrow{\omega} (B,\mathcal{K})$$

Consider the trivial bundles $\varphi^*(\mathscr{U}(\mathcal{H}))$ and $\omega^*(\mathscr{U}(\mathcal{K}))$. These are principal bibundles, as φ and ω are weak equivalences, so that their cotensor product $P := \omega^*(\mathscr{U}(\mathcal{K}))\,\square_{\mathcal{J}}\,\varphi^*(\mathscr{U}(\mathcal{H}))$ is again a principal $(\mathcal{H},\mathcal{K})$-bibundle.

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Consider the trivial bundles $\varphi^*(\mathscr{U}(\mathcal{H}))$ and $\omega^*(\mathscr{U}(\mathcal{K}))$. These are principal bibundles, as φ and ω are weak equivalences, so that their cotensor product $P := \omega^*(\mathscr{U}(\mathcal{K}))\,\square_{\mathcal{J}}\,\varphi^*(\mathscr{U}(\mathcal{H}))$ is again a principal $(\mathcal{H},\mathcal{K})$-bibundle.

In this way this diagram is completed to the following one

$$\begin{array}{ccccc} & & (P, \mathcal{H} \ltimes P \rtimes \mathcal{K}) & & \\ & \nearrow^{\alpha} & & {}^{\beta}\nwarrow & \\ (A,\mathcal{H}) & & & & (B,\mathcal{K}) \\ & \searrow_{\varphi} & & {}_{\omega}\swarrow & \\ & & (C,\mathcal{J}) & & \end{array}$$

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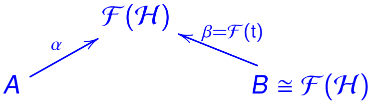
Conversely:

Let $\mathcal{F} : \mathrm{Comod}_{\mathcal{H}} \to \mathrm{Comod}_{\mathcal{K}}$ be a symmetric monoidal equivalence of categories, and consider $\mathcal{F}(\mathcal{H})$ as a $\Bbbk$-algebra with the canonical algebra extensions



Then the three-tuple $(\mathcal{F}(\mathcal{H}), \alpha, \beta)$ is a principal $(\mathcal{H}, \mathcal{K})$-bibundle.

The first main result.

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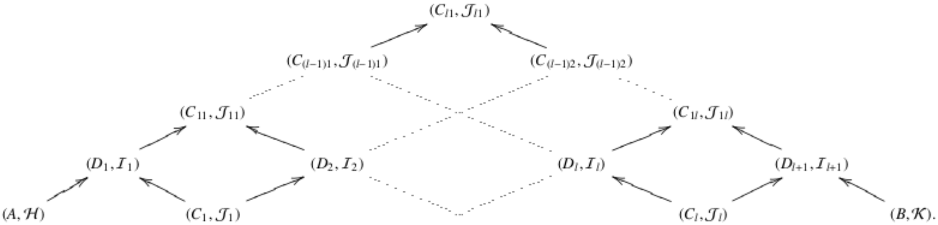
Theorem

Let $(A, \mathcal{H})$ and $(B, \mathcal{K})$ be two flat Hopf algebroids. The following are equivalent:

(1) *$(A, \mathcal{H})$ and $(B, \mathcal{K})$ are Morita equivalent.*

(2) *There is a principal bibundle connecting $(A, \mathcal{H})$ and $(B, \mathcal{K})$.*

(3) *$(A, \mathcal{H})$ and $(B, \mathcal{K})$ are weakly equivalent.*



$(C_{l1}, \mathcal{J}_{l1})$
$(C_{(l-1)1}, \mathcal{J}_{(l-1)1})$
$(C_{(l-1)2}, \mathcal{J}_{(l-1)2})$
$(C_{11}, \mathcal{J}_{11})$
$(C_{1l}, \mathcal{J}_{1l})$
$(D_1, \mathcal{I}_1)$
$(D_2, \mathcal{I}_2)$
$(D_l, \mathcal{I}_l)$
$(D_{l+1}, \mathcal{I}_{l+1})$
$(A, \mathcal{H})$
$(C_1, \mathcal{J}_1)$
$(C_l, \mathcal{J}_l)$
$(B, \mathcal{K})$.

The second main result.

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The 2-functor $\mathscr{P} : \mathit{HAlgd} \longrightarrow \mathrm{PB}^{\ell\, cop}$ from the 2-category of flat Hopf algebroids to the conjugate of PB^{ℓ}, sends any 1-cell $\phi : (A, \mathcal{H}) \to (B, \mathcal{K})$ to its associated trivial left principal bundle $\mathscr{P}(\phi) = \mathcal{H} \otimes_{\phi} B$.

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A 1-cell ϕ in *HAlgd* is a weak equivalence if and only if $\mathscr{P}(\phi)$ is an invertible 1-cell in $\mathsf{PB}^{\ell\, cop}$, i.e., *is part of an internal equivalence*.

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