The Physics of Electroweak Symmetry Breaking

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Chapter 1

Introduction

Our goal in this course in to discuss our current understanding of electroweak symmetry breaking (EWSB), including the lack of experimental evindence for the precise mechanism that triggers it. We will also investigate different alternative realizations, starting with the Standard Model (SM) and moving on to different models beyond the SM (BSM), their strengths and weaknesses and the current constraints on them. The ultimate goal is to develop the knowledge and tools to properly interpret the results that the LHC will provide us with in order to fully understand the process of EWSB.

These notes are meant as a guide to the different topics we will cover in the course but are not going to be comprehensive. We will try to always give references to the literature and text books to complement the discussion here.

Chapter 2

Symmetries in Physics and their Fate

Literature: The topics in this Chapter are covered in many textbooks. I have mainly followed Pokorski [1], Donoghue, Golowich and Holstein [2], Georgi [3] and Peskin and Schroeder [4]. Particular topics are also covered in lecture notes or review articles that will be cited correspondingly. Finally, I have used extensively Contino's notes (`http://indico.phys.ucl.ac.be/conferenceDisplay.py?confId=148`).

2.1 Classical Equations of Motion

Our main tool in this course will be quantum field theory and its building block, the action,

$$S[\phi] = \int d^4x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x)), \tag{2.1}$$

where $\mathcal{L}$ is the Lagrangian density (from now on denoted simply as the Lagrangian), that we assume depends only on fields and their first derivatives (and we have not made explicit possible extra space-time, internal or flavor indices in the fields). The classical equations of motion for the fields are obtained from Hamilton's principle of stationary action, for arbitrary variations of the fields (but with fixed "boundary conditions"). That is,

$$\delta S = 0, \tag{2.2}$$

for arbitrary variations of the fields,

$$\phi(x) \to \phi(x) + \delta\phi(x), \tag{2.3}$$

with $\delta\phi(x) = 0$ for the boundary of integration Ω (which can include spatial infinity). Computing explicitly the variation of the action we obtain

$$\delta S = \int_\Omega d^4x \, \left[\frac{\delta \mathcal{L}}{\delta \phi} \delta\phi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta \partial_\mu \phi \right] = \int_\Omega d^4x \, \left\{ \left[\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \right] \delta\phi + \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta\phi \right] \right\}, \tag{2.4}$$

where we have used that the coordinates x do not change in the variation,

$$\delta\partial_\mu\phi = \partial_\mu\delta\phi,$$

and integration by parts. The second term in square brackets results in a surface term that vanishes because the variation is zero at the boundary. The action will be stationary for arbitrary variations if the classical equations of motion are satisfied

$$\boxed{\frac{\delta\mathcal{L}}{\delta\phi_i^\nu} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i^\nu} = 0.} \tag{2.5}$$

We have explicitly written the corresponding equations for an arbitrary number of flavors (denoted by i) of vector fields (with an explicit space-time index ν) as an example. Note that any two Lagrangians that differ by a total detivative of an arbitrary function of the fields,

$$\mathcal{L}' = \mathcal{L} + \partial_\mu V^\mu(\phi),$$

give exactly the same equations of motion, due to de vanishing of the variation of the fields at the boundary.

2.2 Continuous Symmetries and Conservation Laws

A symmetry of the action is some (infinitesimal) change of the fields, $\delta\phi$, such that the action remains invariant,

$$S[\phi + \delta\phi] = S[\phi], \tag{2.6}$$

or in terms of the Lagrangian,

$$\mathcal{L} + \delta\mathcal{L} \equiv \mathcal{L}(\phi + \delta\phi, \partial_\mu\phi + \delta\partial_\mu\phi) = \mathcal{L}(\phi, \partial_\mu\phi) + \partial_\mu V^\mu(\phi, \partial_\mu\phi, \delta\phi). \tag{2.7}$$

Here we don't assume that the variation of the fields vanishes at the boundary but that the fields and their derivatives vanish fast enough at the boundaries to make the boundary term vanish.

Symmetries have important consequences. They imply conserved currents and definite statements about the spectrum of the theory. The first consequence is given by **Noether's theorem**, which states that for any invariance of the action under a *continuous* transformation of the fields, there exists a classical charge Q which is time-independent ($\dot{Q} = 0$) and is associated with a conserved current, $\partial_\mu J^\mu$. The theorem includes both internal and space-time symmetries. For a continuous internal symmetry, it is just a simple consequence of the classical equations of motion. If we impose the equations of motion on (2.4), we obtain

$$\delta\mathcal{L} = \big[\mathrm{EoM}\big]\delta\phi + \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi\right] = \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi\right] = \partial_\mu V^\mu. \tag{2.8}$$

Thus, there is a current associated to the symmetry,

$$J^\mu \equiv \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta\phi - V^\mu, \tag{2.9}$$

that is conserved,

$$\partial_\mu J^\mu = 0, \tag{2.10}$$

and an associated charge

$$Q(t) \equiv \int d^3x \, J_0(t, x), \tag{2.11}$$

which is a constant of motion,

$$\dot{Q} = \int d^3x \, \partial_0 J_0 = - \int d^3x \, \partial_i J^i = 0,$$

provided the currents fall off sufficiently rapidly at the space boundary Ω.

Let us consider the case that the symmetry is not only a symmetry of the action but also of the Lagrangia, $V^\mu = 0$. Furthermore, let's assume the symmetry is a linear unitary transformation of the fields,

$$\delta\phi = i\epsilon_a T^a \phi, \tag{2.12}$$

where ϵ_a are a set of infinitesimal parameters (that for the moment will be considered independent of space-time, *i.e.* we will be considering a **global symmetry**) and T^a, $a = 1, \ldots, m$ are a set of $N \times N$ hermitian matrices acting on flavor space (ϕ is a vector in such space), which satisfy the Lie algebra of the corresponding symmetry group,

$$[T^a, T^b] = i f_{abc} T^c, \qquad \mathrm{Tr} T^a T^b \propto \delta_{ab}. \tag{2.13}$$

Applying our general expression of Noether's current to this particular case for $V^\mu = 0$ and using that the ϵ_a are arbitrary parameters, we get the conserved currents (with standard normalization)

$$\boxed{J^\mu_a = -i \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} T^a \phi, \quad a = 1, \ldots, m.} \tag{2.14}$$

Our derivation of Noether's theorem also tells us what the divergence of the current is in the case that the transformation is *not* a symmetry of the Lagrangian,

$$\partial_\mu J^\mu = \delta \mathcal{L}. \tag{2.15}$$

Before going into some examples of conserved currents in physical systems, it is important to emphasize that the charges associated to these currents,

$$Q_a(t) \equiv \int d^3x \, J^0_a(t, x), \tag{2.16}$$

even if they are not conserved (because the transformation is not a good symmetry of the action or lagrangian), they are still the generators of the transformation (2.12). First, they obviously satisfy the same commutation relations as the matrices T^a. Second, they actually generate the field transformations (2.12) through Poisson brackets. Let us introduce the conjugate momenta for the fields $\phi(x)$ (that we assume different from zero)

$$\Pi(t,x) = \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi(t,x))}. \tag{2.17}$$

The Hamiltonian of the system reads

$$H = \int d^3x\,[\Pi \partial_0 \phi - \mathcal{L}], \tag{2.18}$$

and the Poisson bracket of two functionals $F_{1,2}$ of the fields Π and ϕ is defined as

$$\{F_1(t,x), F_2(t,x)\} = \int d^3z \left[\frac{\delta F_1(t,x)}{\delta \phi(t,z)} \frac{\delta F_2(t,y)}{\delta \Pi(t,z)} - \frac{\delta F_1(t,x)}{\delta \Pi(t,z)} \frac{\delta F_2(t,y)}{\delta \phi(t,z)} \right]. \tag{2.19}$$

In particular we have

$$\{\Pi(t,x), \phi(t,y)\} = -\delta(x-y), \tag{2.20}$$

$$\{\Pi(t,x), \Pi(t,y)\} = \{\phi(t,x), \phi(t,y)\} = 0. \tag{2.21}$$

The Noether charge can be written as

$$Q^a(t) = \int d^3x\, \Pi(t,x)(-iT^a)\phi(t,x). \tag{2.22}$$

Computing its Poisson bracket with ϕ we get

$$\boxed{\{Q^a(t), \phi(t,\mathbf{x})\} = (iT^a)\phi(t,\mathbf{x}),} \tag{2.23}$$

and thus, Q^a generates the transformation on the fields (independently of whether it is conserved or not).

If the transformation does not only affect the fields but also the coordinates, Noether's theorem has to be generalized. We will show here the result (see Pokorski for a full discussion). Assuming

$$x'^\mu = x^\mu + \delta x^\mu = x^\mu + \epsilon^\mu(x), \tag{2.24}$$

and

$$\phi'(x') = \exp[-iT(\epsilon)]\phi(x), \tag{2.25}$$

so that for infinitesimal transformations we have

$$\phi'(x') = \phi'(x) + \delta x^\mu \partial_\mu \phi(x) + \ldots, \tag{2.26}$$

$$\partial'_\mu \phi'(x') = \partial_\mu \phi'(x) + \delta x^\nu \partial_\mu \partial_\nu \phi(x) + \ldots\ . \tag{2.27}$$

A sufficient condition for the transformation to be a symmetry of the action is

$$\mathcal{L}(\phi'(x'), \partial'\phi'(x'))d^4x' = [\mathcal{L}(\phi(x), \partial\phi(x)) - \partial_\mu V^\mu(\phi(x))]d^4x. \tag{2.28}$$

It can then be seen that

$$\partial_\mu J^\mu \equiv \partial_\mu \left\{ \left[\mathcal{L} g^\mu_\rho - \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\rho \phi \right] \delta x^\rho + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta\phi + V^\mu \right\} = 0. \tag{2.29}$$

Note: All we have said in this section is strictly true for Classical Field Theory. What happens when one goes to QFT? The effects of symmetries generalize quite straightforwardly, with some subtleties. For one, the fields are no longer classical fields but quantum operators, and as such, transform as

$$\phi_i(x) \to U\phi_i(x)U^{-1} = [\exp(-i\theta^a T^a)]_{ij}\phi_j(x). \tag{2.30}$$

Also, Poisson brackets go to standard (anti)commutators,

$$[Q^a, \phi] = -T^a\phi \tag{2.31}$$

i.e. $\{,\} \to -i[,]$. One particle states in QFT corresponding to the quantum excitations of a particular field are constructed from the creation operators of the corresponding field acting on the vacuum,

$$a^\dagger_\psi(k)|0\rangle = |k_\psi\rangle, \tag{2.32}$$

and therefore how the spectrum behaves under the symmetry depends on how the vacuum does. Finally, QFT involves, at the quantum level, singularities that have to be regularized. It is sometimes the case that no regularization is possible that it is compatible with the classical symmetry. The system is then said to be anomalous under the symmetry (it is symmetric at the classical level but not at the quantum level). In the path integral formulation of QFT the system is anomalous when the integral measure is not invariant under the symmetry.

2.3 Examples

A set of nice examples are given by Georgi [3], involving both scalars and fermions. We encourage the interested student to study them in detail. Here we just mention what the examples are:

2.3.1 Scalars

Consider a set of N real scalar fields. The Kinetic Lagrangian reads,

$$\mathcal{L}_K = \frac{1}{2}\partial_\mu\phi^T S\partial^\mu\phi, \tag{2.33}$$

with S a symmetric, positive definite matrix. We can then define a linear transformation on the fields $\phi \to L\phi$, with L satisfying $L^T SL = 1$, (L is given by an orthogonal matrix that diagonalizes S times a rescaling of the diagonal entries) so that the kinetic Lagrangian now reads,

$$\mathcal{L}_K = \frac{1}{2}\partial_\mu \phi^T \partial^\mu \phi. \tag{2.34}$$

This Lagrangian has an obvious $SO(N)$ symmetry. Masses and other couplings can break this symmetry as discussed in Georgi's book.

2.3.2 Fermions

Consider now a set of N massless, free, spin-1/2, Dirac fermions. The Lagrangian reads,

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi, \tag{2.35}$$

which has a $U(N)$ symmetry. This is not the largest symmetry of the kinetic Lagrangian, though. Writing the fermions in their chirality components we have

$$\mathcal{L} = i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R, \tag{2.36}$$

making a larger, chiral, $U(N)_L \times U(N)_R$ symmetry apparent. A Dirac mass term, mixing left and right chiralities, will break this chiral symmetry. If a subgroup of n masses are degenerate, the corresponding vector subgroup, $SU(n)_V$ of equal left and right rotations in the subspace of degenerate fermions will survive.

2.4 Symmetries and Degeneracy

One important result in quantum mechanics is that states that are related by a symmetry of the system have to be degenerate. This is a simple consequence of the fact that the generator of the symmetry, is the conserved Noether's charge, Q_a. Being conserved, it commutes with the Hamiltonian and therefore leads to degenerate states. Let us consider two eigenstates of the Hamiltonian related by a symmetry, $|\psi\rangle = Q_a|\chi\rangle$. Then we have,

$$E_\psi|\psi\rangle = H|\psi\rangle = HQ_a|\chi\rangle = Q_a H|\chi\rangle = E_\chi Q_a|\chi\rangle = E_\chi|\psi\rangle, \tag{2.37}$$

and therefore $E_\psi = E_\chi$. Physical states are grouped in representations of the symmetry, all with the same energy (mass). This result -states related by a symmetry are degenerate- is equaly valid in quantum field theory. However, in QFT, the one-particle states associated to two fields which are related by a symmetry do not need to be degenerate. The reason is that in QFT, the one-particle states are created from the vacuum, and for the two physical states to be related, not only the fields have to be related by the symmetry but the vacuum has to be *invariant* under the symmetry. Let's see why in a bit more detail. Let's consider the two fields $\phi_1(x)$ and $\phi_2(x)$, corresponding to particles of type "1" and "2", respectively, and that are related by the action of some symmetry,

$$\phi_1(x) = i[Q, \phi_2(x)], \tag{2.38}$$

with Q a hermitian operator generating the symmetry. The same is then true for the corresponding creation and annihilation operators,

$$a_1 = i[Q, a_2]. \tag{2.39}$$

The particle states are then related as follows,

$$|1\rangle = a_1^\dagger|0\rangle = i[Q, a_2^\dagger]|0\rangle = iQa_2^\dagger|0\rangle - ia_2^\dagger Q|0\rangle = iQ|2\rangle - ia_2^\dagger Q|0\rangle. \tag{2.40}$$

Thus, the particle states satisfy $|1\rangle = iQ|2\rangle$ only if the vacuum is invariant $Q|0\rangle = 0$. If the vacuum is invariant, then the physical particle states are related by the symmetry when the corresponding fields are and have therefore to be degenerate. If the vacuum is not invariant (we then say that the symmetry is spontaneously broken), then it is not true that particle states whose corresponding fields are related by the symmetry have to be degenerate.

2.5 Spontaneous Breaking of a Global Symmetry

System with spontaneously broken symmetries, *i.e.* those for which the Lagrangian is invariant under the symmetry but the vacuum is not, are interesting and relevant to the real world. We have seen in the previous section that a spontaneously broken symmetry no longer implies degeneracy of the physical states. Another important result in spontaneously broken symmetries is **Goldstone's Theorem**, which states that for every spontaneously broken continuous global symmetry, there is a massless scalar boson with the same quantum numbers of the broken generator. Before going into the details of the theorem, we will discuss a very simple example.

2.5.1 Spontaneously Broken $U(1)$ Symmetry

Consider a complex scalar with Lagrangian,

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - V(|\phi|) = \partial_\mu\phi^*\partial^\mu\phi - [\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2]. \tag{2.41}$$

This system has a $U(1)$ global symmetry $\phi \to e^{i\alpha}\phi$. Assuming that a perturbative expansion exists, the vacuum of the system will be close to the classical vacuum, given by the constant field configuration that minimizes the potential. If $\mu^2 > 0$, then the minimum of the potential is $\langle\phi\rangle = 0$ (recall λ has to be positive for the Hamiltonian to be bounded from below) and the vacuum is also invariant under the symmetry. If on the other hand we have $\mu^2 < 0$, then the minimum of the potential happens for any field configuration such that

$$|\phi|^2 = -\frac{\mu^2}{2\lambda} \equiv v^2 > 0. \tag{2.42}$$

One important result in quantum field theory (in infinite space) is that the vacuum state for a theory that satisfies the standard QFT properties (including the clustering principle) can be only one point in the manifold of degenerate minima of the potential. In our case, that means that we have to choose one particular phase for the true vacuum of our system. In particular, we can choose the vacuum to be real,

$$\langle \phi \rangle = v, \tag{2.43}$$

and define a new field, $\phi' = \phi - v$, which has zero vev (and whose quantum excitations are therefore orthogonal to the vacuum state). In terms of this new field, we have, defining the real and imaginary parts of $\phi' = (\phi_1 + i\phi_2)/\sqrt{2}$, the following Lagrangian (up to an irrelevant constant term)

$$\mathcal{L} = \frac{1}{2}\Big[\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2\Big] - \frac{1}{2}4\lambda v^2 \phi_1^2 - \sqrt{2}\lambda v \phi_1(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2, \tag{2.44}$$

which is the Lagrangian of two real scalars, one with positive mass squared, $m_1^2 = 4\lambda v^2$, and one massless, with triple and quartic interactions among them. Alternatively, instead of using the real and imaginary parts of ϕ' as the relevant degrees of freedom, one can use the modulus and the phase,

$$\phi = \left(v + \frac{\rho}{\sqrt{2}}\right) e^{i\eta/(\sqrt{2}v)}, \tag{2.45}$$

with $\langle \rho \rangle = \langle \eta \rangle = 0$. We then have

$$\phi^* \phi = (v + \rho/\sqrt{2})^2, \tag{2.46}$$

and therefore the potential is clearly independent of η. The whole Lagrangian reads, for these fields,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \rho \partial^\mu \rho + \frac{1}{2}\left(1 + \frac{\rho}{\sqrt{2}v}\right)^2 \partial_\mu \eta \partial^\mu \eta - \frac{1}{2}4\lambda v^2 \rho^2 - \sqrt{2}\lambda v \rho^3 - \frac{\lambda}{4}\rho^4. \tag{2.47}$$

In particular we now have that the massless scalar, η, has disappeared from the potential and it only couples derivatively.

We will see in the next sections and examples that this particular choice of variables generalizes to arbitrary symmetry breaking patterns and has several advantages.

Note: For a continuous symmetry, we have to have a manifold of degenerate minima of the potential, since given any minimum, the manifold in the field space obtained by continuously transforming with the symmetry the original minimum gives the same value of the potential (since it is invariant under the symmetry). This also ensures that the choice of one particular vacuum, enforced by the properties of QFT in infinite space, is physically equivalent to any other choice, since they are related by a symmetry of the Lagrangian.

2.5.2 Goldstone's Theorem

In the previous example we have seen how the spontaneous breaking of the global $U(1)$ symmetry resulted in a massless scalar that, with the right choice of variables, coupled only derivatively. This result generalizes to an arbitrary pattern of spontaneous global symmetry breaking as we discuss now. Let us discuss it in the context of a system of (fundamental or composite) scalar fields with the following Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi), \tag{2.48}$$

where ϕ is some multiplet of spliness fields and $V(\phi)$ is invariant under some symmetry group action

$$\delta\phi = i\epsilon_a T^a \phi, \tag{2.49}$$

with T^a imaginary antisymmetry matrices (so that ϕ is hermitian). In perturbation theory, the vacuum, around we should perturb, is given by the minimum of the potential. Thus the fields ϕ will have a vev that minimize the potential $\langle\phi\rangle = \lambda$, satisfying

$$V_j(\lambda) = 0, \quad V_{jk}(\lambda) \geq 0. \tag{2.50}$$

We have used the notation

$$V_{j_1\cdots j_n} = \frac{\partial^n}{\partial\phi_{j_1}\ldots\phi_{j_n}} V(\phi).$$

Expanding around the minimum, we have, for the physical perturbations $\phi' = \phi - \lambda$,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi' \partial^\mu \phi' - V(\lambda) - \frac{1}{2}V_{ij}(\lambda)\phi'_i\phi'_j + \ldots \ . \tag{2.51}$$

Thus, the term proportional to $V_{ij}(\lambda)$ is the mass squared term for the physical fields, which is definite positive (no tachyons).

Now, this vacuum, λ, does not need to be invariant under the symmetry. In particular, the generators of the symmetry group G can be separated in generators, Y^i that annihilate the vacuuum and generators, $X^{\hat{a}}$ that do not:

$$(T^a) = (Y^i, X^{\hat{a}}), \quad Y^i\lambda = 0, \quad X^{\hat{a}}\lambda \neq 0, \tag{2.52}$$

where Y^i form a subgroup H of G. We then say that the symmetry is spontaneously broken from G to H. Since the potential is invariant under the symmetry, we have

$$0 = V(\phi + \delta\phi) - V(\phi) = iV_k(\phi)\epsilon_a (T^a)_{kl}\phi_l. \tag{2.53}$$

Differenciating this expression with respect to ϕ_j and using that ϵ_a is arbitrary, and evaluating the resulting quantity at $\phi = \lambda$ we get

$$0 = V_{jk}(\lambda)(T^a)_{kl}\lambda_l + V_k(\lambda)(T^a)_{kj} = V_{jk}(\lambda)(T^a)_{kl}\lambda_l = M^2 T^a \lambda, \tag{2.54}$$

where we have used that λ is a minimum of the potential and that the second derivative at λ gives the mass squared matrix for the physical fields. This equality is trivially satisfied for the generators of the unbroken group, as they annihilate the vacuum. The condition is however non-trivial for the generators of the coset space (the broken generators) implying that $X^{\hat{a}}\lambda$ is an eigenvector of M^2 with zero eigenvalue. This corresponds to a massless scalar field, given by

$$\phi^T X^{\hat{a}}\lambda. \tag{2.55}$$

Note that there is one massless scalar per broken generator. Also note that it has the same quantum numbers under the unbroken subgroup as the corresponding generator. Finally, note that we have chosen one set of λ that minimizes the potential. However, since the potential is invariant under the symmetry G, it is obvious that any $\lambda' = g\lambda$, with g an element of G is also a minimum of the potential. Since G is a continuous group, we have a continuous manifold of degenerate minima. Physically, only one of these minima can be used (in an infinite space), since going from one to another would require an infinite change in all space. However, all the different minima are physically equivalent and any choice we make can be converted in a different one by a simple relabeling of the generators, without altering the physics.

Note: Assume $[Y^i, Y^j] = ic^{ijk}Y^k + ic^{ij\hat{a}}X^{\hat{a}}$ and apply it to the vacuum (note that the commutator annihilates the vacuum)

$$0 = ic^{ijk}Y^k\lambda + ic^{ij\hat{a}}X^{\hat{a}}\lambda = ic^{ij\hat{a}}X^{\hat{a}}\lambda, \tag{2.56}$$

which implies that $c^{ij\hat{a}} = 0$ and H is a subgroup. This also implies (from the antisymmetry of the structure constants) that

$$[Y^i, X^{\hat{a}}] = ic^{i\hat{a}\hat{b}}X^{\hat{b}}, \tag{2.57}$$

and therefore $X^{\hat{a}}$ transform under some representation of the subgroup H. In general, the commutator of two broken generators does not live in H. However, if there is a parity operation P which leaves the Lie algebra of the group G invariant and is such that

$$PY^iP^{-1} = Y^i, \quad PX^{\hat{a}}P^{-1} = -X^{\hat{a}}, \tag{2.58}$$

then we obviously have

$$[X^{\hat{a}}, X^{\hat{b}}] = ic^{\hat{a}\hat{b}i}Y^i. \tag{2.59}$$

Our previous proof of Goldstone's theorem has been based on the analysis of the classical potential and therefore relies on perturbation theory. Goldstone's theorem is however more general and can be proven to all orders in perturbation theory [5]. The details can be found in the original article or in Pokorski's book. Here we will just sketch the proof following

Pokorski. The idea is to prove that the spontaneous breaking of a global continuous symmetry implies the existence of poles at $p^2 = 0$ in certain Green's functions and then that this poles imply the presence of massless scalar particles in the physical spectrum.

Let us consider the following Green's function

$$G^a_{\mu,k}(x-y) = \langle 0|Tj^a_\mu(x)\phi_k(y)|0\rangle, \tag{2.60}$$

where j^a_μ is the current corresponding to a generator Q^a of the symmetry G and ϕ_k belongs to an irreducible multiplet of real scalar fields. This Green's function satisfies a Ward identity that can be obtained by differentiating it being careful with the derivative of the θ functions involved in the time ordering,

$$\begin{aligned} \partial^\mu_{(x)} G^a_{\mu,k}(x-y) &= \delta(x^0-y^0)\langle 0|[j^a_0(x),\phi_k(y)]|0\rangle = \delta(x^0-y^0)\langle 0|-T^a_{kj}\phi_j(y)\delta(\mathbf{x}-\mathbf{y})|0\rangle \\ &= -\delta(x-y)T^a_{kj}\langle 0|\phi_j(y)|0\rangle = -\delta(x-y)T^a_{kj}\langle 0|\phi_j(0)|0\rangle. \end{aligned} \tag{2.61}$$

We have used the transformation properties of the field as generated by the Noether's current and translational invariance of the vacuum. This Ward identity, in the case of conserved currents, is valid both for bare and renormalized fields. The Fourier transform of the Ward identity reads

$$ip^\mu \tilde{G}^a_{\mu,k}(p) = T^a_{kj}\langle 0|\phi_j(0)|0\rangle, \tag{2.62}$$

where

$$G^a_{\mu,k}(x-y) = \int \frac{d^4p}{(2\pi)^4}\exp[-ip(x-y)]\tilde{G}^a_{\mu,k}(p). \tag{2.63}$$

Plugging the most general form of the (Fourier-transformed) Green's function as allowed by Lorentz invariance, $\tilde{G}^a_{\mu,k}(p) = p_\mu F^a_k(p^2)$, in the Ward's identity we get

$$F^a_k(p^2) = -\frac{i}{p^2}T^a_{kj}\langle 0|\phi_j(0)|0\rangle, \tag{2.64}$$

which implies that the Green's function corresponding to a generator that does not annihilate the vacuum,

$$T^a_{kj}\langle 0|\phi_j(0)|0\rangle \neq 0, \tag{2.65}$$

has a pole at $p^2 = 0$.

Let us now consider the following matrix element

$$\langle 0|j^a_\mu(x)|\pi^k(p)\rangle = if^a_k p_\mu e^{-ipx}, \tag{2.66}$$

where $|\pi^k(p)\rangle$ describes a particle of mass m_k which is a quantum of the field ϕ_k. The reduction formula relates this matrix element to the Green's function $\tilde{G}^s_{\mu,k}(p)$. Defining

$$G_{kk'} = \int \frac{d^4q}{(2\pi)^4}\frac{-\delta_{kk'}}{q^2-m_k^2+i\epsilon}e^{-iq(y-x)}, \tag{2.67}$$

$$\int d^4y G^{-1}(x-y)G(y-z) = \delta(x-z), \tag{2.68}$$

we get

$$\begin{aligned}
\langle 0|j^a_\mu(x)|\pi^k(p)\rangle &= \int d^4y d^4z e^{-ipz} G^a_{\mu,k'}(x-y) i G^{-1}_{kk'}(y-z) \\
&= \lim_{p^2\to m_k^2} e^{-ipx} \tilde{G}^a_{\mu,k'}(p) i \tilde{G}^{-1}_{kk'}(p) \\
&= -\lim_{p^2\to m_k^2} e^{-ipx} \tilde{G}^a_{\mu,k'}(p) i (p^2 - m_k^2).
\end{aligned} \tag{2.69}$$

Putting together (2.66) with this equation we get,

$$\lim_{p^2\to m_k^2} \tilde{G}^a_{\mu,k'}(p) i (p^2 - m_k^2) = -f^a_k p_\mu, \tag{2.70}$$

which implies $m_k = 0$, for those Green's functions that have a massless pole (*i.e.* those corresponding to generators that don't annihilate the vacuum), as we wanted to prove. This proof also gives us the value of f^a_k,

$$f^a_k = i T^a_{kj} \langle 0|\phi_j(0)|0\rangle. \tag{2.71}$$

Thus, there must be a massless boson, $|\Pi^a(p)\rangle = i|\pi^k(p)\rangle T^a_{kj}\langle 0|\phi_j(0)|0\rangle$, corresponding to each broken generator.

Note: Pokorski goes far beyond the discussion we have included here. Interesting topics related to spontaneously broken global symmetries include systems with spontaneous and explicit symmetry breaking and patterns of symmetry breaking.

2.6 Spontaneous Breaking of a Local Symmetry

In the previous section we discussed Goldstone's theorem, which implies the existence of a massless scalar for each spontaneously broken generator of a continuous global symmetry. What happens if the symmetry that is sponatenously broken is local? In particular the corresponding global symmetry given by keeping the gauge parameters constant is also a symmetry of the Lagrangian so Goldstone's theorem should also apply, right? The answer is actually no. The reason is that in our non-perturbative proof of the theorem, Lorentz invariance and a Hilbert space with positive norm were required. However, gauge theories do not satisfy both requirements simultaneously. In a covariant gauge, the theory contains states of negative norm whereas in a gauge in which the theory has only states of positive norm, it is not manifestly covariant. This means that in gauge theories, Goldstone's theorem does not hold and, as we will see, the Higgs mechanism operates.

2.6.1 An Abelian Example

Let us start with the simple example of a complex scalar with a *local* $U(1)$ symmetry.

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^* D_\mu\phi - \lambda(\phi^*\phi - v^2)^2, \tag{2.72}$$

where $D_\mu\phi = (\partial_\mu + igA_\mu)\phi$. This system is invariant under a local $U(1)$ transformation

$$\phi(x) \to e^{i\alpha(x)}\phi(x), \quad A_\mu(x) \to A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x). \tag{2.73}$$

If $v^2 > 0$, the minimum of the potential occurs for $|\phi|^2 = v^2$ and we have spontaneous breaking of the local symmetry. Expanding the physical fields around the true vacuum,

$$\phi(x) = v + \frac{1}{\sqrt{2}}(h(x) + i\pi(x)), \tag{2.74}$$

and using

$$D_\mu\phi = \frac{1}{\sqrt{2}}[\partial_\mu h - gA_\mu\pi] + \frac{i}{\sqrt{2}}[\partial_\mu\pi + gA(\sqrt{2}v + h)], \tag{2.75}$$

the Lagrangian reads,

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}[\partial_\mu h - gA_\mu\pi]^2 + \frac{1}{2}[\partial_\mu\pi + gA(\sqrt{2}v + h)]^2 \\
&\quad -\lambda\left[\sqrt{2}vh + \frac{h^2+\pi^2}{2}\right]^2 \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}(\partial_\mu\pi)^2 \\
&\quad +\frac{1}{2}2g^2v^2A_\mu A^\mu - \frac{1}{2}4\lambda v^2h^2 + \sqrt{2}gvA_\mu\partial^\mu\pi \\
&\quad +gA_\mu(h\partial^\mu\pi - \pi\partial^\mu h) + g^2A_\mu A^\mu(\sqrt{2}vh + h^2/2 + \pi^2/2) \\
&\quad -\frac{\lambda}{4}(h^2+\pi^2)^2 - \lambda\sqrt{2}vh(h^2+\pi^2).
\end{aligned} \tag{2.76}$$

Note that the kinetic mixing between A_μ and π makes it difficult to interpret the degrees of freedom in this Lagrangian. Up to this kinetic mixing, the Lagrangian seems to represent a *massive* gauge field, with mass $m_A = \sqrt{2}gv$, a massive scalar with mass $m_h = 2\sqrt{\lambda}v$, a massless scalar π and their interactions. Note that π, which is the field that with its kinetic mixing is creating the problem of interpretation, is what would correspond to the Goldstone boson if the symmetry was global. How can we get a proper identification of the relevant degrees of freedom? There are several things we can do but they all go through fixing the gauge in a way that the relevant degrees of freedom are manifest. The simplest possibility is to go to the so-called unitary gauge, in which the make a gauge transformation that locally removes the would-be Goldstone boson. In our particular example, we just need to do a gauge transformation that removes, locally, the phase of ϕ, so that ϕ remains real. This is easier to see in the polar coordinates (2.45). It is then clear that we can reach the unitary gauge by performing a gauge transformation with $\alpha(x) = -\eta(x)/(\sqrt{2}v)$, so that

$$\phi^U(x) = v + \rho(x)/\sqrt{2}, \quad A^U_\mu(x) = A_\mu + (1/\sqrt{2}vg)\partial_\mu\eta, \tag{2.77}$$

and the Lagrangian then reads,

$$\mathcal{L} = -\frac{1}{4}(F^U_{\mu\nu})^2 + \frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}2g^2(v+\rho/\sqrt{2})^2(A^U_\mu)^2 - \frac{1}{2}4\lambda v^2\rho^2 - \sqrt{2}\lambda v\rho^3 - \frac{\lambda}{4}\rho^4. \tag{2.78}$$

The Lagrangian has simplified enormously. Also, the particle content is apparent in the unitary gauge, in which there is no kinetic mixing between different fields. We have a massive gauge boson, a massive scalar and their interactions. Note also that the would-be Goldstone boson has completely disappeared from the spectrum. The number of degrees of freedom is however unchanged. Originally we had two real scalar degrees of freedom plus a massless gauge boson (another two degrees of freedom, the two transverse polarizations), in the Unitary gauge we still have four real degrees of freedom, one in ρ and three in the massive gauge boson (two transverse plus the longitudinal polarization). It is usually said that the would-be Goldstone boson is combined with the gauge boson to provide the longitudinal component (it is *eaten* by the gauge boson to gain its mass). Although in the unitary gauge the particle content is manifest (all dof are physical), other gauges are sometimes useful, in particular in explicit calculations (and also in the discussion of renormalizability of gauge theories). Let us go to a covariant gauge by means of the following gauge fixing term,

$$\mathcal{L}_{GF} = -(\partial_\mu A^\mu)^2/2a. \tag{2.79}$$

The quadratic part of the gauge-fixed Lagrangian then reads

$$\begin{aligned}\mathcal{L} + \mathcal{L}_{GF} &= -\frac{1}{2}A_\mu\left[-g^{\mu\nu}\partial^2 + \left(1-\frac{1}{a}\right)\partial^\mu\partial^\nu - m_A^2 g^{\mu\nu}\right]A_\nu \\ &\quad +\frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h^2 h^2 + \frac{1}{2}(\partial_\mu\pi)^2 + m_A A_\mu\partial^\mu\pi + \dots\ .\end{aligned} \tag{2.80}$$

Let us now consider the free particles (which are not necessarily the physical states) and include the interactions as a perturbation. In the Landau gauge ($a = 0$) we have (we do not care about h which does not affect the argument) a massless gauge boson with propagator

$$-iD_{\mu\nu} \equiv -\frac{i}{p^2+i\epsilon}(g_{\mu\nu} - p_\mu p_\nu/p^2) \equiv -\frac{i}{p^2+i\epsilon}\mathcal{P}_{\mu\nu}, \tag{2.81}$$

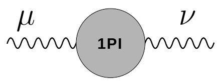
where in the second equality we have defined the transverse projector, and a massless would be Goldstone with propagator,

$$\frac{i}{p^2+i\epsilon}. \tag{2.82}$$

The full gauge boson propagator is then

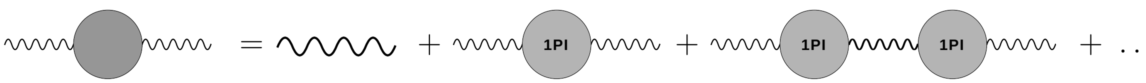
$$\Delta_{\mu\nu}(p) = -iD_{\mu\nu}\frac{1}{1-\Pi(p^2)}, \tag{2.83}$$

where $\Pi(p^2)$ is defined by the gauge-invariant 1PI vacuum polarization tensor (here, since we are considering some of the quadratic terms in the Lagrangian as a perturbation, by 1PI we really mean 1 gauge boson irreducible)



$$\equiv i\Pi^{\mu\nu}(p) = i\Pi(p^2)(p^2\eta^{\mu\nu} - p^\mu p^\nu) = i\Pi(p^2)p^2\mathcal{P}^{\mu\nu}.$$

The exact propagator is then given by the infinite series of diagrams shown



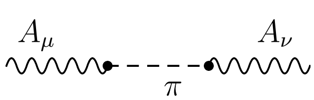
which results in the following form of the exact propagator (we do not write explicitly the factors of $i\epsilon$ in the intermediate calculations)

$$\begin{aligned}\Delta^{\mu\nu} &= -\frac{i}{p^2}\mathcal{P}^{\mu\nu} + \left(-\frac{i}{p^2}\mathcal{P}^{\mu\rho}\right) i\Pi(p^2)p^2\mathcal{P}_{\rho\sigma}\left(-\frac{i}{p^2}P^{\sigma\nu}\right) + \dots \\ &= -\frac{i}{p^2}\mathcal{P}^{\mu\nu}\Big[1 + \Pi(p^2) + \Pi(p^2)^2 + \dots\Big] = -iD^{\mu\nu}\frac{1}{1-\Pi(p^2)},\end{aligned} \tag{2.84}$$

where we have made use of the fact that $\mathcal{P}^{\mu\nu}\mathcal{P}_{\nu\sigma} = \mathcal{P}^\mu_\sigma$ (it is a projector) and have explicitly summed the geometric series. Now, to leading order in the interactions, we get a direct contribution to $\Pi_{\mu\nu}$ from the mass term,

$$i\Pi_{\mu\nu}(p^2)^m = im_A^2 g^{\mu\nu}, \tag{2.85}$$

plus a contribution from the exchange of a ϕ field



$$= (-m_A p_\mu)\,\tfrac{i}{p^2}\,(m_a p_\nu) = -im_A^2\tfrac{p_\mu p_\nu}{p^2}.$$

Thus, the total contribution is transverse as is should and we get

$$\Pi(p^2) = m_A^2/p^2, \tag{2.86}$$

so that the total propagator reads,

$$\Delta_{\mu\nu}(p) = -\frac{i}{p^2 - m_A^2}\mathcal{P}_{\mu\nu}, \tag{2.87}$$

which corresponds to the massive propagator in the Landau gauge. We see then how the exchange of the Goldstone boson provides the correct pole in the propagator as predicted by the gauge invariance of the Lagrangian.

Another set of gauges that are interesting are the so-called R_ξ or renormalizable gauges. They are obtained by means of the following gauge fixing term

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi m_A \pi)^2. \tag{2.88}$$

The quadratic part of the Lagrangian then reads

$$\begin{aligned}\mathcal{L} + \mathcal{L}_{GF} &= -\frac{1}{2}A_\mu \left[-g^{\mu\nu}\partial^2 + \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu - m_A^2 g^{\mu\nu}\right] A_\nu \\ &+\frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h^2 h^2 + \frac{1}{2}(\partial_\mu \pi)^2 - \frac{\xi}{2}m_A^2\pi^2 + \ldots \ .\end{aligned} \tag{2.89}$$

In particular note that the gauge fixing term removed the kinetic mixing between the gauge boson and the would-be Goldstone. In a general R_ξ gauge, the propagators for the different fields are

$$\begin{aligned}A_\mu &: \frac{-i}{p^2 - m_A^2 + i\epsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2 - \xi m_A^2}\right] \\ &= -i\frac{g_{\mu\nu} - p_\mu p_\nu/m_A^2}{p^2 - m_A^2 + i\epsilon} - i\frac{p_\mu p_\nu/m_A^2}{p^2 - \xi m_A^2 + i\epsilon},\end{aligned} \tag{2.90}$$

$$\pi : i/(p^2 - \xi m_A^2 + i\epsilon), \tag{2.91}$$

$$h : i/(p^2 - m_h^2 + i\epsilon). \tag{2.92}$$

The Landau gauge corresponds to $\xi = 0$ whereas the unitary gauge corresponds to $\xi \to \infty$. In the unitary gauge the corresponding Goldstone boson does not propagate and we recover only physical states. In a renormalizable gauge ($\xi \neq \infty$) we have written the gauge propagator as the propagator of a standard massive gauge boson (in unitary gauge) plus a term correponding to a scalar negative norm boson coupled to the source of the vector boson derivatively. Another gauge that is sometimes useful for calculations is the 't Hooft-Feynman gauge, $\xi = 1$, in which the gauge boson has a very simple propagator $\propto g^{\mu\nu}$ and the Goldstone boson has mass m_A.

2.6.2 Spontaneous Symmetry Breaking of non-Abelian Gauge Symmetries

Our previous analysis goes almost unchanged to the non-abelian case. In this section and the next we follow section 20.1 of [4], to which we refer the reader for further details.

We will consider a system of real scalars invariant under a local non-abelian symmetry with generators t^a,

$$\phi \to (1 + i\alpha^a(x)t^a)\phi = (1 - \alpha^a(x)T^a)\phi, \tag{2.93}$$

where in the second equality we have used the fact that the fields are real (so that the t^a are hermitian and pure imaginary) to write $t^a = iT^a$ with T^a real and antisymmetric. The covariant derivative reads,

$$D_\mu\phi = (\partial_\mu + gA_\mu^a T^a)\phi. \tag{2.94}$$

The scalar kinetic term reads

$$\frac{1}{2}(D_\mu\phi_i)^2 = \frac{1}{2}(\partial_\mu\phi_i)^2 + gA^a_\mu\partial^\mu\phi_i T^a_{ij}\phi_j + \frac{1}{2}g^2 A^a_\mu A^\mu_b (T^a\phi)_i(T^b\phi)_i. \tag{2.95}$$

Let us now assume that the scalar potential is such that some of the fields acquire a vacuum expectation vacuum (vev) that spontaneously breaks the symmetry,

$$\langle\phi_i\rangle = \lambda_i, \tag{2.96}$$

and expand around the correct vacuum $\phi \to \lambda + \phi$. The scalar kinetic term, expanded around the vacuum λ gives us the mass matrix for the gauge bosons,

$$\Delta\mathcal{L} = \frac{1}{2}m^2_{ab}A^a_\mu A^{b\mu} = \frac{1}{2}g^2(T^a\lambda)_i(T^b\lambda)_i A^a_\mu A^{b\mu}. \tag{2.97}$$

Note that this mass matrix is positive definite, since in any basis, the diagonal entries are strictly non-negative. Also note that, if for any a we have $T^a\lambda = 0$, then the corresponding gauge boson is massless and the symmetry along that generator is not broken. Also in the kinetic term there is a kinetic mixing between the Goldstone bosons (recall the Goldstone bosons were $\pi^a = \phi^T T^a\lambda$) and the gauge bosons corresponding to the broken generators,

$$\Delta\mathcal{L} = gA^a_\mu\partial^\mu\phi_i(T^a\lambda)_i. \tag{2.98}$$

Now we can repeat the calculation that we did for the abelian symmetry of the 1PI vacuum polarization tensor for the massless gauge bosons, including the mass and mixing with Goldstones as a perturbation, replacing (2.85) and (2.6.1) with the ones we have computed here. That gives, for the vacuum polarization tensor

$$\begin{aligned} i\Pi_{\mu\nu}(p) &= ig^2(T^a\lambda)_i(T^b\lambda)_i g_{\mu\nu} + \big(gp_\mu(T^a\lambda)_i\big)\frac{i}{p^2}\big(-gp_\nu(T^b\lambda)_i\big) \\ &= im^2_{ab}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right), \end{aligned} \tag{2.99}$$

which again is transverse as required by gauge invariance but gives a mass to the gauge bosons associated to the broken generators. A similar discussion about the gauge fixing as was done in the section of the abelian symmetry can be done here (note however that the quantization of non-abelian gauge symmetries, although also a resonably straight forward generalization of the case of abelian symmetries, involves new features that are worth studying, see Peskin and Schroeder's book for instance).

2.6.3 Formal Aspects of the Higgs Mechanism

All we have said in the last two sections about the Higgs mechanism relied on the existance of a set of scalars that acquired a vev. Spontaneous symmetry breaking of global and local symmetries is however more general and does not require the presence of fundamental scalars. In fact, no fundamental scalar degrees of freedom seem to play a role in known examples of global symmetry breaking, like superconductivity or chiral symmetry breaking in QCD. Of course, we know from Goldstone's theorem that some scalar degrees of freedom will be present in systems with spontaneous symmetry breaking, the Goldstone bosons, but these do not need to be fundamental scalars and can instead be composite objects made out of fermions or other species. In this section, we will discuss the Higgs mechanism in a more general set-up, in which we do not start with a set of fundamental scalars describing the process of spontaneous symmetry breaking. The final result will however be essentially unchanged with respect to the examples we saw in the last two sections. The presence of massless scalar degrees of freedom associated to the broken symmetries (the Goldstone bosons) will play a crucial role in giving mass to the broken gauge bosons in a gauge invariant way, *i.e.* keeping the vacuum polarization tensor properly transverse. Our starting point is a general system with a global symmetry G, represented by the Lagrangian $\mathcal{L}_0$. The corresponding Noether current,

$$J^a_\mu = -\frac{\delta \mathcal{L}_0}{\delta \partial^\mu \phi} T^a \phi, \tag{2.100}$$

(note that ϕ here represents all the degrees of freedom in the system that, let us stress it again, are not necessarily scalars) satisfies

$$\delta \mathcal{L}_0 = \partial_\mu J^{a\mu} = 0, \tag{2.101}$$

We can promote the global symmetry to a local one $\delta\phi = -\alpha^a(x) T^a \phi$ by coupling the system to a set of gauge bosons through the conserved Noether currents,

$$\mathcal{L} = \mathcal{L}_0 + g A^a_\mu J^{a\mu} + \mathcal{O}(A^2), \tag{2.102}$$

where we have not made explicit the terms of order higher than linear that are required to make the Lagrangian gauge invariant. The important feature is that, at the linear level, the coupling of the gauge bosons to our system is fixed by gauge invariance and it goes through the Noether currents of the global symmetry. Thus, in any computation that involves vertices with only one gauge boson, we ca use the Lagrangian above. These Noether currents are closely related to the corresponding Goldstone bosons. At low energies, Goldstone bosons are just infinitesimal rotations of the vacuum, $Q^a|0\rangle$, where Q^a is the charge associated to the Noether current. $J^{a\mu}$ has therefore the right quantum numbers to create or destroy a Goldstone boson from the vacuum. We can parametrize the matrix element of such process as

$$\langle 0|J^{a\mu}(x)|\pi_k(p)\rangle = -ip^\mu F^a_k e^{-ipx}, \tag{2.103}$$

where p is the on-shell momentum of the Goldstone boson and F^a_k is a matrix of contants that vanish for unbroken generators. From the conservation of the Noether current we have

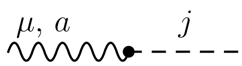
$$0 = \partial_\mu \langle 0|J^{a\mu}(x)|\pi_k(p)\rangle = -p^2 F^a_k e^{-ipx}, \tag{2.104}$$

which agrees with the fact that the Goldstone bosons are massless.

We can now see how the Higgs mechanism operates in our general system. By now we know that, in order to do that, we have to study the vacuum polarization tensor of the corresponding gauge bosons. The Ward identities associated to the gauge symmetry require it to be transverse, of it has to have the form

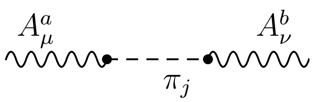
$$i\Pi_{\mu\nu}(p) = i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)(m_{ab}^2 + \mathcal{O}(p^2)). \tag{2.105}$$

Computing the non-singular term in our general theory is not easy but the singular part comes exclusively from the exchange of our massless Goldstones, whose coupling to the gauge bosons can be deduced from (2.102) and (2.103) and is given by



$$= gp^\mu F_j^a.$$

The Goldstone exchange contribution to the vacuum polarization tensor is therefore



$$= \left(-gp_\mu F_j^a\right)\tfrac{i}{p^2}\left(gp_\nu F_j^b\right) = -i\tfrac{p_\mu p_\nu}{p^2} g^2 F_j^a F_j^b.$$

Thus, we have the mass term for the gauge bosons,

$$m_{ab}^2 = g^2 F_j^a F_j^b. \tag{2.106}$$

Note that, given the generality of the system, we have not been able to compute the non-singular part of the vacuum polarization tensor. We have instead used gauge invariance to deduce its form (transverse) and computed the singular contribution generated by the exchange of the massless Goldstone. This process is completely general, and can be applied to any model with local spontaneous symmetry breaking.

2.7 The sigma model

Let us now discuss a simple model that will be useful for many aspects of the course. The σ model was introduced for the first time by Gell-Mann and Levy [6]. In the preparation of these notes, I have mainly used [1] and [2]. The linear σ model contains two Dirac fermions u, d that we will jointly denote by

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}$$

and four real scalars σ' and $\vec{\pi} = (\pi^1, \pi^2, \pi^3)$. The lagrangian is given by

$$\begin{aligned}\mathcal{L}_\sigma^{\text{lin}} &= \frac{1}{2}(\partial_\mu\sigma')^2 + \frac{1}{2}(\partial_\mu\pi^i)^2 - \frac{\mu^2}{2}(\sigma'^2 + \vec{\pi}^2) - \frac{\gamma}{4}(\sigma'^2 + \vec{\pi}^2)^2 \\ &+ \bar{\psi} i\partial\!\!\!/\psi + g\bar{\psi}(\sigma' + i\vec{\sigma}\cdot\vec{\pi}\gamma^5)\psi,\end{aligned} \tag{2.107}$$

where $\vec{\sigma}$ are the Pauli matrices. As we saw in the previous examples, the fermion kinetic term has an $SU(2)_L \times SU(2)_R$ global symmetry (we neglect for the moment the $U(1)$ factors, one of which corresponds to baryon number and the other one is anomalous). We will want to insist that the full lagrangian is invariant under this chiral symmetry. This requirement determines the transformation properties of the scalar fields. In particular, writing the fermions in their corresponding chiral components we get

$$\begin{aligned}\mathcal{L}_\sigma^{\rm lin} &= \frac{1}{4}\mathrm{Tr}(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma) - \frac{\mu^2}{4}\mathrm{Tr}(\Sigma^\dagger\Sigma) - \frac{\lambda}{16}[\mathrm{Tr}(\Sigma^\dagger\Sigma)]^2 \\ &+ \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R + g\bar{\psi}_L\Sigma\psi_R + g\bar{\psi}_R\Sigma^\dagger\psi_L, \end{aligned} \tag{2.108}$$

where we have defined the combination of scalars $\Sigma \equiv \sigma' + i\vec{\sigma}\cdot\vec{\pi}$, which satisfies

$$\sigma'^2 + \vec{\pi}^2 = \frac{1}{2}\mathrm{Tr}(\Sigma^\dagger\Sigma). \tag{2.109}$$

In order for the Lagrangian to be invariant under the chiral symmetry, Σ has to transform as a bidoublet,

$$\psi_{L,R} \to U_{L,R}\psi_{L,R}, \quad \Sigma \to U_L\Sigma U_R^\dagger, \tag{2.110}$$

with $U_{L,R}$ arbitrary $SU(2)$ matrices

$$U_{L,R} = \exp(-i\alpha^a_{L,R}\sigma^a/2). \tag{2.111}$$

Using the following properties of the Pauli matrices, $\mathrm{Tr}\,\sigma^i = 0$ and $\sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k + \delta^{ij}$, we have

$$\sigma' = \frac{1}{2}\mathrm{Tr}\Sigma, \quad \pi^k = -\frac{i}{2}\mathrm{Tr}[\sigma^k\Sigma].$$

The corresponding infinitesimal transformations for the scalars are then

$$\sigma' \to \sigma' + \frac{1}{2}(\vec{\alpha}_L - \vec{\alpha}_R)\cdot\vec{\pi}, \tag{2.112}$$

$$\pi^k \to \pi^k - \frac{1}{2}(\alpha^k_L - \alpha^k_R)\sigma' - \frac{1}{2}\epsilon^{klm}\pi^l(\alpha^m_L + \alpha^m_R). \tag{2.113}$$

In particular, under an isospin rotation ($\alpha_L = \alpha_R$) σ' and $\vec{\pi}$ transform, respectively, as a singlet and a triplet, whereas under an axial rotation ($\alpha_L = -\alpha_R$), all four fields mix,

$$\sigma' \to \sigma' + \vec{\alpha}_L\cdot\vec{\pi}, \tag{2.114}$$

$$\vec{\pi} \to \vec{\pi} - \sigma'\vec{\alpha}_L. \tag{2.115}$$

Noether's theorem tells us that the invariance of the Lagrangian implies the corresponding conserved currents. Using the general definition of Noether's current and the explicit expression for the variation of the different fields under a L and R rotation, we get

$$\begin{aligned} J^k_{L\mu} &= \bar{\psi}_L\gamma_\mu\frac{\sigma^k}{2}\psi_L - \frac{i}{8}\mathrm{Tr}[\sigma^k(\Sigma\partial_\mu\Sigma^\dagger - \partial_\mu\Sigma\Sigma^\dagger)] \\ &= \bar{\psi}_L\gamma_\mu\frac{\sigma^k}{2}\psi_L - \frac{1}{2}(\sigma'\partial_\mu\pi^k - \pi^k\partial_\mu\sigma') + \frac{1}{2}\epsilon^{klm}\pi^l\partial_\mu\pi^m, \end{aligned} \tag{2.116}$$

and

$$\begin{aligned} J^k_{R\mu} &= \bar\psi_R\gamma_\mu\frac{\sigma^k}{2}\psi_R + \frac{i}{8}\mathrm{Tr}[\sigma^k(\partial_\mu\Sigma^\dagger\Sigma - \Sigma^\dagger\partial_\mu\Sigma)] \\ &= \bar\psi_R\gamma_\mu\frac{\sigma^k}{2}\psi_R + \frac{1}{2}(\sigma'\partial_\mu\pi^k - \pi^k\partial_\mu\sigma') + \frac{1}{2}\epsilon^{klm}\pi^l\partial_\mu\pi^m. \end{aligned} \tag{2.117}$$

These currents can be combined into conserved vector currents,

$$V^k_\mu = J^k_{L\mu} + J^k_{R\mu} = \bar\psi\gamma_\mu\frac{\sigma^k}{2}\psi + \epsilon^{klm}\pi^l\partial_\mu\pi^m, \tag{2.118}$$

and axial-vector currents

$$A^k_\mu = J^k_{L\mu} - J^k_{R\mu} = \bar\psi\gamma_\mu\gamma_5\frac{\sigma^k}{2}\psi + \pi^k\partial_\mu\sigma' - \sigma'\partial_\mu\pi^k. \tag{2.119}$$

By now we know too well that the invariance of the Lagrangian is not necessarily the full story. Depending on the parameters of the scalar potential, the vacuum of the system might be non invariant under the chiral symmetry, which is then spontaneously broken. The scalar potential can be written, up to an irrelevant constant factor

$$V(\sigma',\pi^k) = \frac{\lambda}{4}\left(\sigma'^2 + \vec\pi^2 + \frac{\mu^2}{\lambda}\right)^2. \tag{2.120}$$

If $\mu^2 > 0$, then the minimum of the potential is for $\sigma' = \pi^k = 0$, and the vacuum is also invariant under the chiral symmetry. This means that our fields represent quantum excitations around the true vacuum and they correspond to degenerate multiplets (all four scalars have the same mass, similarly for the two fermions). If on the other hand we have $\mu^2 < 0$, then the minimum of the potential (and therefore the vacuum of the system in the classical approximation) satisfies

$$\sigma'^2 + \vec\pi^2 = -\frac{\mu^2}{\lambda} \equiv v^2. \tag{2.121}$$

The true vacuum of the system corresponds to a point in this manifold. Any choice is physically equivalent, since they are all related by the symmetry of the Lagrangian (different choices just amount to a renaming of the broken and unbroken symmetries) but the vacuum is one and only one of the states satisfying (2.121) and not a linear combination of different ones (in infinite space-time). One choice that we can always make is

$$\langle\sigma'\rangle = v, \quad \langle\pi^k\rangle = 0. \tag{2.122}$$

With this choice, the vacuum is still invariant under the isospin (vector) symmetry but not under the axial symmetry. Thus, this pattern of symmetry breaking (which in the current model is the only possible one) corresponds to $SU(2)_L \times SU(2)_R$ being spontaneously broken to $SU(2)_V$. Let us see what implications this pattern of symmetry breaking has for the spectrum.

In order to preserve orthogonality of the vacuum to one-particle states, we define the physical field,

$$\sigma = \sigma' - v, \tag{2.123}$$

so that $\langle\sigma\rangle = 0$. The Lagrangian now reads, up to an irrelevant constant,

$$\begin{aligned} \mathcal{L} &= \bar{\psi} i\partial\!\!\!/ \psi + gv\bar{\psi}\psi + \frac{1}{2}[(\partial_\mu\sigma)^2 - 2\lambda v^2\sigma^2] + \frac{1}{2}(\partial_\mu\vec{\pi})^2 \\ &+ g\bar{\psi}[\sigma + i\vec{\sigma}\cdot\vec{\pi}]\psi - \frac{\lambda}{4}(\sigma^2+\vec{\pi}^2)^2 - \lambda v\sigma(\sigma^2+\vec{\pi}^2), \end{aligned} \tag{2.124}$$

which describes a nucleon of mass $-gv$, a σ meson of mass $m_\sigma = \sqrt{2\lambda}v$ and an isospin triplet of massless pions, $\vec{\pi}$, and their interactions. We can also compute, in the tree approximation,

$$\langle 0|A^a_\mu(x)|\pi^b(q)\rangle = \langle 0|\sigma'|0\rangle\langle 0|\partial_\mu\pi^a(x)|\pi^b(q)\rangle = -\langle 0|\sigma'|0\rangle i q_\mu\delta^{ab}e^{-iqx} = i f_\pi q_\mu \delta^{ab} e^{-iqx}. \tag{2.125}$$

Thus, we have $f_\pi = -\langle 0|\sigma'|0\rangle = -v$. All other elements in $A^a_\mu(x)$ will give zero when applied over the vacuum.

Note that, as predicted by Goldstone's theorem, we have found three massless Goldstones, which correspond to the three broken generators. ($G = SU(2)_L \times SU(2)_R$ has six generators out of which the three in $H = SU(2)_V$ remain unbroken, leaving three broken generators.) Using the commutation relations of the generators

$$[T^i_{L,R}, T^j_{L,R}] = i\epsilon^{ijk}T^k_{L,R}, \quad [T^i_L, T^j_R] = 0, \tag{2.126}$$

we obtain

$$[T^i_{V,A}, T^j_{V,A}] = i\epsilon^{ijk}T^k_V, \quad [T^i_V, T^j_A] = i\epsilon^{ijk}T^k_A. \tag{2.127}$$

In particular, the last equation tells us that the broken generators transform as a triplet under the unbroken isospin group. The Goldstone bosons transforming as an isospin triplet therefore agrees also with the theorem.

What would happen if we include an explicit symmetry breaking term? Let us consider we add an isospin preserving but chiral breaking term like

$$\mathcal{L}' = -\epsilon\sigma' = -\frac{\epsilon}{2}\mathrm{Tr}\Sigma. \tag{2.128}$$

This has several implications. The axial-vector current is no longer conserved,

$$\partial^\mu A^a_\mu = \delta_{A^a}\mathcal{L}' = -\epsilon\delta_{A^a}\sigma' = -\epsilon\pi^a, \tag{2.129}$$

and the corresponding charge is time dependent,

$$\frac{dQ^a_A}{dt} = -\epsilon\int d^3x\, \pi^a. \tag{2.130}$$

Recall however that this charge still generates the corresponding transformations on the fields. Another implication is that the minimum of the potential is modified. We now have

$$\partial_{\sigma'} V = \sigma' \big[\mu^2 + \gamma(\sigma'^2 + \vec{\pi}^2)\big] + \epsilon, \tag{2.131}$$
$$\partial_{\pi^i} V = \pi^i \big[\mu^2 + \gamma(\sigma'^2 + \vec{\pi}^2)\big], \tag{2.132}$$

so that the degeneracy of the minimum in the potential is lifted and the vacuum is aligned with the perturbation,

$$\langle \pi^i \rangle = 0, \quad \langle \sigma' \rangle = v, \tag{2.133}$$

with $\mu^2 v + \gamma v^3 + \epsilon = 0$. Defining $\sigma = \sigma' - v$ we get a mass for the pions

$$m_\pi^2 = \mu^2 + \lambda v^2 = -\epsilon/v = \epsilon/f_\pi. \tag{2.134}$$

This gives us the divergence of the axial-vector current as

$$\partial^\mu A_\mu^a = -f_\pi m_\pi^2 \pi^a(x). \tag{2.135}$$

This equation is the basis of the PCAC (partially conserved axial vector current) approximation.

2.7.1 Different Representations of the Sigma Model

In the previous section we saw that the σ model, for $\mu^2 < 0$, represents massive nucleons and a scalar singlet plus a triplet of massless pions. If we are interested in energies much smaller than the scale of symmetry breaking, we should be able to integrate out the σ field and obtain an effective theory containing only the pions. However, it is not obvious how to do that. Taking the limit $v \to \infty$ does not seem to really decouple σ, since the coupling $\sigma\pi^2$ grows with v. The reason is that if we integrate out σ we are explicitly breaking the chiral symmetry. This is a typical situation in spontaneously broken continuous symmetries, in which the decoupling theorem does not apply. It turn out, however, that we can still write an effective Lagrangian only in terms of the pion fields that is invariant under the full chiral symmetry. This is at the expense of having a much more complicated, non-linear, transformation rule for the Goldstone bosons. This non-linear realization of the symmetry takes advantage of the fact that the Goldstone bosons are a parametrization of the manifold of degenerate "vacua" and therefore can be represented by the local orientation with respect to the vacuum. In the next section we will describe the general theory behind non-linear representations of spontaneously broken gauge symmetries. Here, we will just see it in the example of the σ model. Let us consider the following representation of the σ model,

$$\Sigma = \sigma' + i\vec{\sigma} \cdot \vec{\pi} = S'U, \quad U = \exp(i\vec{\sigma} \cdot \vec{\xi}/v). \tag{2.136}$$

The exponential can be computed using the properties of the Pauli matrices, which result in,

$$U = \cos\left(\frac{\xi}{v}\right) + i\frac{\vec{\sigma} \cdot \vec{\xi}}{\xi} \sin\left(\frac{\xi}{v}\right), \tag{2.137}$$

where $\xi = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$. Equivalently, we have

$$\sigma' = S' \cos\frac{\xi}{v} = S' + \ldots, \quad \pi^i = S' \frac{\xi^i}{\xi} \sin\frac{\xi}{v} = S' \frac{\xi^i}{v} + \ldots \, . \tag{2.138}$$

In particular note that $S'^2 = \sigma'^2 + \vec{\pi}^2 = \mathrm{Tr}(\Sigma^\dagger \Sigma)/2$ is invariant under the full chiral symmetry. Also we have, in the new coordinates, the following vacuum,

$$\langle S \rangle \equiv \langle S' - v \rangle = 0, \quad \langle \xi^i \rangle = 0. \tag{2.139}$$

Inserting this representation in the Lagrangian, we get

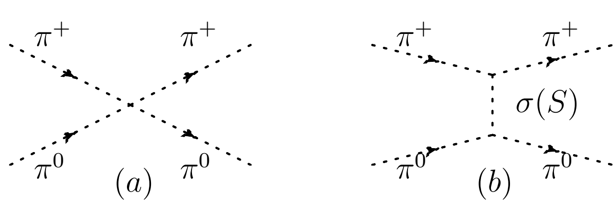
$$\begin{aligned} \mathcal{L} &= \bar{\psi} i \partial\!\!\!/ \psi + g(v+S)\big(\bar{\psi}_L U \psi_R + \bar{\psi}_R U^\dagger \psi_L\big) + \frac{1}{2}\Big[(\partial_\mu S)^2 - 2\lambda v^2 S^2\Big] \\ &+ \frac{(v+S)^2}{4} \mathrm{Tr}\big(\partial_\mu U \partial^u U^\dagger\big) - \lambda v S^3 - \frac{\lambda}{4} S^4. \end{aligned} \tag{2.140}$$

Note that this is just a redefinition of the fields, so the physics is completely unchanged. Also note that the new massless Goldstones ξ^i no longer appear in the potential but have only derivative (bosonic) interactions. Finally, the matrix U transforms under the chiral group the same way Σ does (since S' does not transform),

$$U \to L U R^\dagger. \tag{2.141}$$

We have said that the new representation is just a redefinition of fields and therefore does not change the physics. This is in fact a powerful theorem in field theory on representation independence first proved by R. Haag [7]. It states that if two fields are related nonlinearly, *e.g.* $\phi = \chi F(\chi)$ with $F(0) = 1$, then the same experimental observables result if one calculates with the field ϕ using $\mathcal{L}(\phi)$ or instead with χ using $\mathcal{L}\big(\chi F(\chi)\big)$. The proof consists basically of demonstrating that (i) two S-matrices are equivalent if they have the same single particle singularities, and (ii) since $F(0) = 1$, ϕ and χ have the same free field behaviour and single particle singularities.

Let us see in action Haag's theorem using the two different representations of the σ model. Let us compute scattering amplitudes of Goldstone bosons. In particular, we will compute the scattering $\pi^+\pi^0 \to \pi^+\pi^0$. The relevant diagrams are shown below.



1. Linear representation.

   The relevant part of the Lagrangian reads

$$\Delta\mathcal{L} = -\frac{\lambda}{4}(\vec{\pi}^2)^2 - \lambda v \sigma \vec{\pi}^2. \tag{2.142}$$

With this Lagrangian, the two diagrams in the figure contribute to the scattering amplitude

$$\mathcal{M}_{\pi^+\pi^0\to\pi^+\pi^0} = -2i\lambda + (-2i\lambda v)^2 \frac{i}{p^2 - m_\sigma^2} = -2i\lambda \left[1 + \frac{2\lambda v^2}{p^2 - 2\lambda v^2}\right] = \frac{ip^2}{v^2} + \dots \; . \tag{2.143}$$

Note that, although the couplings are non-derivative, the constant term at low energies cancels and we are left with an amplitude that vanishes in the low energy limit.

2. Non-linear representation.

   In this case we have the Lagrangian

$$\Delta\mathcal{L} = \frac{1}{4}(v+S)^2 \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \dots \; . \tag{2.144}$$

   The contribution proportional to diagram (b) in the figure is of order p^4 (two factors at each vertex) and can be neglected at low energies. The contribution of diagram (a) is of order p^2 as can be clearly seen by writing the relevant part of the Lagrangian in terms of the Goldstone fields $\vec{\xi}$. Using the explicit form of U in terms of the Goldstone fields, Eq. (2.137), and $\partial_\mu \xi = (\vec{\xi}\cdot\partial_\mu\vec{\xi})/\xi$, we get

$$\begin{aligned} \Delta\mathcal{L} &= \frac{1}{4}(v+S)^2 \frac{2}{v^2}\Big\{(\partial_\mu\xi)^2 + \frac{\sin^2(\xi/v)}{(\xi/v)^2}\left[(\partial_\mu\vec{\xi})^2 - (\partial_\mu\xi)^2\right]\Big\} \\ &= \frac{1}{6v^2}\left[(\vec{\xi}\cdot\partial_\mu\vec{\xi})^2 - \vec{\xi}^2(\partial_\mu\vec{\xi}\cdot\partial^\mu\vec{\xi})\right] + \frac{S}{v}(\partial_\mu\vec{\xi})^2 + \dots \;, \end{aligned} \tag{2.145}$$

   where we have also explicitly shown the coupling of the scalar to the Goldstone bosons to show that it will contribute at order p^4 in the amplitude and also for future use.

   Note: The same result could have been obtained from the expansion of the exponential and the commutation relations of the Pauli matrices. Care has to be taken when using this approach in including all relevant terms to this order (in particular there is a contribution to order ξ^4 that comes from the linear term in $\partial_\mu U$ times the triple Goldstone term in $\partial^\mu U^\dagger$ and viceversa).

   The corresponding amplitude reads

$$\mathcal{M}_{\pi^+\pi^0\to\pi^+\pi^0} = \frac{i(p'_+ - p_+)^2}{v^2} + \dots = \frac{ip^2}{v^2} + \dots \; . \tag{2.146}$$

As expected, we see that the amplitude for the Goldstone boson scattering is identical in both representations, although the final result can preceed through different diagrams in different representations. We have also seen that Goldstone boson scattering is proportional to p^2 and therefore vanishes at low energies. This is a completely general property of Goldstone bosons and is obvious in the non-linear representation, since their only couplings are derivative. Haag's theorem guarantees that we can use either (or any other one) representation for the sigma model and all the physical results that we compute will be the same. This is very useful, as it allows us to use whichever representation makes our calculations (or our understanding of the physics involved) easier. We have actually seen an example in the scattering amplitude we have computed. Obtaining the correct contribution at low energies $\propto p^2$ required a cancelation between different contributions in the linear realization whereas it was immediate in the non-linear representation. In this non-linear representation, we were able to compute the Goldstone boson scattering at low energies without the contribution of the field S, *i.e.*, we had a low energy effective Lagrangian for the Goldstone bosons that did not involve the scalar singlet S. The reason is that we have replaced the particle content in the linear representation, an isospin singlet σ plus a triplet π transforming joinly (linearly) as a bidoublet under the full chiral symmetry, with the one of the non-linear representation, a singlet (under the full chiral symmetry) S, plus three scalars $\vec{\xi}$ which transform non-linearly under the chiral symmetry, as defined in (2.141) (note that the quantity U transforms linearly under the chiral symmetry, but the fields $\vec{\xi}$ transform non-linearly). There is actually a subset of chiral transformations under which the fields $\vec{\xi}$ transform linearly, the vector (isospin) transformations. Indeed, if we take $L = R = 1 + i\vec{\alpha}\cdot\vec{\sigma} + \ldots$ we have

$$\begin{aligned} LUR^\dagger &= L\left[\sum_{n=0}^{\infty}\frac{1}{n!}(i\vec{\xi}\cdot\vec{\sigma}/v)^n\right]L^\dagger = \sum_{n=0}^{\infty}\frac{1}{n!}(i\xi^i/v\,L\sigma^i L^\dagger)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(i\xi^i/v\,R_{ij}\sigma^j)^n = \exp(i\vec{\xi'}\cdot\vec{\sigma}/v), \end{aligned} \tag{2.147}$$

where $\xi'^{\,j} = R_{ij}\xi^i = R^T_{ji}\xi^i$ and R is the generator of the linear representation of the broken generator. Thus, we have seen that the Goldstone bosons in the non-linear representation transform non-linearly under the chiral transformation, to compensate for the non-transformation properties of S but they transform linearly under the unbroken group (recall that σ was already a singlet under the isospin symmetry so nothing to compensate here). This is just one example of a very general result that we will discuss in the next lecture.

Let us summarize here what we have learned from the σ model. We have seen how the sigma model allows for a very simple description of the spontaneous breakdown of the chiral $SU(2)_L\times SU(2)_R$ symmetry down to its vectorial, isospin, $SU(2)_V$ subgroup. In the process fermions can acquire a mass (which is forbidden by the chiral symmetry, now spontaneously broken), one of the scalar degrees of freedom, a singlet under the vectorial symmetry, also acquires a mass. Finally there are three scalar degrees of freedom that remain massless. These are the Goldstone bosons predicted by Goldstone's theorem. They have the quantum numbers of the broken generators arise and how they do not mix with the fourth scalar degree of freedom under isospin transformations. Chiral rotations, however mix the four of them, making it difficult to obtain a low energy effective Lagrangian only in terms of the Goldstone fields without explicitly breaking the chiral symmetry. This difficulty was overcome by changing the linear representation to the non-linear one, in which we replace the four scalar degrees of freedom by a scalar that is a singlet under the full chiral symmetry plus a triplet of scalars that transform linearly (as a triplet) under the unbroken isospin symmetry but transform non-linearly under chiral rotations (to compensate for the non-transformation of the singlet). In this situation, we can actually decouple the singlet in a chirally invariant way. This can be done by simply freezing its value to the vev v. This way, we are left with the so-called **non-linear sigma model**, with Lagrangian,

$$\mathcal{L}_\sigma^{\rm non-lin} = \bar{\psi} i \partial\!\!\!/ \psi + gv\big(\bar{\psi}_L U \psi_R + \bar{\psi}_R U^\dagger \psi_L\big) + \frac{v^2}{4}\mathrm{Tr}\big(\partial_\mu U \partial^u U^\dagger\big). \qquad (2.148)$$

Haag's theorem ensures that the physical calculations performed with both representations are the same. Of course, in the non-linear sigma model, we have decoupled one of the particles and it can only be considered an effective theory that will eventually break down. One can consider the non-linear sigma model as the leading term in an expansion in momentum of the most general effective Lagrangian with chiral symmetries. This is the approach of chiral perturbation theory to low energy QCD.

Note: Note that we could have used the equivalence $SU(2)_L \times SU(2)_R \equiv SO(4)$ to write the non-linear representation in the equivalent form

$$\phi_0 = \begin{pmatrix} \pi^1(x) \\ \pi^2(x) \\ \pi^3(x) \\ \sigma(x) \end{pmatrix} = (v+S)\exp(i\xi^{\hat{a}} X^{\hat{a}}/v) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad (2.149)$$

where $X^{\hat{a}}$ are the generators of $SO(4)$ corresponding to the broken axial-vector transformations. Written this way, the S and $\vec{\xi}$ fields correspond exactly to the ones we used in the non-linear representation. In particular we can decouple the singlet by just fixing it to its vev to obtain the non-linear sigma model. This can be written in terms of the exponential of the pion fields,

$$\exp(i\xi^{\hat{a}} X^{\hat{a}}/v)\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}}, \qquad (2.150)$$

or equivalently, in terms of a set of real scalar fields transforming as the (4) representation of $SO(4)$, $\phi(x)$, subject to the chirally invariant constraint

$$\phi^2(x) = v^2. \qquad (2.151)$$

Again, both representations are completely equivalent, although one could be more useful than the other in particular situations.

2.8 Non-linear Realization of a Symmetry

Literature: Here we will follow the discussion in Pokorski and also the nice review by Feruglio [8]. I have also found useful the lectures on chiral symmetry by Peskin [9] and the discussion in Weinberg's second QFT book [10]. Obviously, the original papers by CCWZ [11, 12] are also worth reading.

We have seen in the previous sections how theories with spontaneously broken symmetries have less restrictions on the spectrum of the theory. Fields transforming under different representations of the unbroken symmetry do not need to have the same masses. It is then tempting to write our theory in terms of multiplets of the unbroken symmetry, instead of multiplets under the full symmetry. The fact that the symmetry is spontaneously broken, however, (*i.e.* the Lagrangian is still invariant under the symmetry) imposes some constraints on the couplings of the different fields. How can we then impose the right constraints from the symmetry but work with building blocks that have good representation properties only under the unbroken symmetry? The answer, in the case of the sigma model, was to work with the non-linear representation. In that case, we could work with building blocks that had good (linear) transformation properties under unbroken group, the scalar singlet S and the triplet of Goldstones $\vec{\xi}$. With these building blocks, we could construct a theory that was invariant under the full chiral symmetry, at the expense that the Goldstone fields transform non-linearly under the (broken) chiral symmetry. This procedure, can actually be generalized to an arbitrary symmetry breaking pattern. The trick that we mentioned but didn't fully exploit in the example of the sigma model is the fact that the Goldstone bosons can be considered a local rotation of the vacuum among the continuous set of degenerate vacua,

$$|\alpha\rangle = \exp(i\alpha^{\hat{a}} X^{\hat{a}})|0\rangle. \tag{2.152}$$

In the non-linear representation, a field is given by the singlet S times a set of angles $\xi^{\hat{a}}(x)$ specifying the vacuum orientation which sets the Goldstones locally to zero.

Let us see the general theory of non-linear realization of a broken symmetry. This was first developed by Coleman, Wess and Zumino and Callan, Coleman, Wess and Zumino (CCWZ) in [11, 12].

One can introduce the non-linear realization of a group on a manifold at a very formal level, which shows the generality of the procedure. Feruglio discusses these formal developments in his review and we will discuss them in the lectures. Here we will just summarize the main results, without further proof or discussion. The set up is a real manifold M (which in our field theoretic description will be a set of scalar fields) and a Lie group G acting on it. The manifold has a special point, the origin (for us it will be the vacuum), such that a subgroup H of G leaves the origin invariant. The first important result is Haag's theorem that we discussed in previous lectures. It describes "allowed" transformations as those that have Jacobian one at the origin. They give raise to identical S-matrix elements. The second result is that it is always possible to choose coordinates on M (scalar field representation) such that the action of H on M is linear. The corresponding coordinates are said to be in standard form. A final theorem states that any non-linear representation of the group G on M can be always brought to the standard form by means of an allowed coordinate transformation.

Let us now be a bit more explicit. We will consider a scalar theory and choose coordinates on the manifold of scalars that generate the non-linear realization of the group in its standard form. Let us consider a theory with some symmetry group G which is spontaneously broken to a subgroup H. Let G be a compact, connectetd, semi-simple Lie group. In some neighbourhood of the identity of G, we can write an element of G as,

$$g = \exp(i\xi^{\hat{a}} X^{\hat{a}}) \exp(iu^i Y^i), \tag{2.153}$$

where as in previous sections $X^{\hat{a}}$ and Y^i are, respectively, the broken and unbroken generators.

Note: The scalar fields $\xi(x)$ are coordinates of the manifold of left cosets G/H at each point of space-time. The set of group elements $l(\xi) = \exp(i\xi^{\hat{a}} X^{\hat{a}})$ parametrizes this manifold. The decomposition (2.153) means that once we have a parametrization $l(\xi)$, each group element g can be uniquely decomposed into a product $g = lh$, where $h \in H$. l is the representative member of the coset to which g belongs and h connects l to g within the coset.

Let $\Phi_0(x)$ be a field which transforms according to a linear representation of G. Let us write it as follows,

$$\Phi_0(x) = \exp(-i\xi^{\hat{a}} X^{\hat{a}})\Phi(x) \equiv U\Phi, \tag{2.154}$$

where $X^{\hat{a}}$ here denotes the matrix representation of the broken generators appropriate for $\Phi_0(x)$. Let us now perform an arbitrary G rotation of the field Φ_0,

$$\begin{aligned} \Phi_0'(x) &= \exp(-i\alpha_a G_a)\Phi_0(x) = \exp(-i\alpha_a G_a) \exp(-i\xi^{\hat{a}} X^{\hat{a}})\Phi(x) \\ &= \exp(-i\alpha_a' G_a)\Phi(x) = \exp(-i\xi'^{\hat{a}}(\xi,\alpha) X^{\hat{a}}) \exp(-iu^i(\xi,\alpha) Y^i)\Phi(x). \end{aligned} \tag{2.155}$$

In terms of the group elements, we have just used the property that the product of an element of the group g with another element of the group, in particular an element of the coset $l(\xi)$ is inside the group and therefore can be decomposed as the product of an element of the coset times an element of the unbroken subgroup, $gl(\xi) = g'(g,\xi) = l(\xi')h$, where $\xi' = \xi'(g,\xi)$ and $h = h(g,\xi)$. Thus, the field Φ_0 can be represented by the fields $\xi^{hata}(x)$ and $\Phi(x)$ for which a general *global* transformation of the group G is realized as a non-linear transformation of the fields $\xi^{\hat{a}}$ and a *local* transformation belonging to the unbroken subgroup H on the multiplet Φ,

$$\xi^{\hat{a}}(x) \rightarrow \xi'^{\hat{a}}(x) = \xi'^{\hat{a}}(g, \xi(x)), \tag{2.156}$$

$$\Phi(x) \rightarrow \exp(iu^i(x) Y^i)\Phi(x) = h(g,\xi(x))\Phi. \tag{2.157}$$

The transformation properties are defined by (2.155). We say that the fields ξ, Φ form a non-linear realization of the group G. What happens if the transformation is under the unbroken subgroup H? In that case the transformation of the Goldstone fields simplify, becoming linear,

$$\begin{aligned}\Phi_0'(x) &= \exp(-i\alpha^i Y^i)\exp(-i\xi^{\hat{a}}X^{\hat{a}})\Phi(x) = \exp(-i\alpha^i Y^i)\exp(-i\xi^{\hat{a}}X^{\hat{a}})\exp(i\alpha^i Y^i)\exp(-i\alpha^i Y^i)\Phi(x) \\ &= = \exp(-i\xi^{\hat{a}}R_{\hat{a}\hat{b}}X^{\hat{b}})\exp(-i\alpha^i Y^i)\Phi(x) = \exp(-i\xi'^{\hat{b}}X^{\hat{b}})\exp(-i\alpha^i Y^i)\Phi(x), \end{aligned} \tag{2.158}$$

where R_{ab} is the matrix representation of the linear transformation of the broken generators under the unbroken group,

$$\exp(-i\alpha^i Y^i)X^{\hat{a}}\exp(i\alpha^i Y^i) = R_{\hat{a}\hat{b}}X^{\hat{b}}. \tag{2.159}$$

We would like to use this non-linear realization of the group to construct G-invariant Lagrangians. The problem is that, despite the fact that the symmetry is global, the non-linear realization involves the Goldstone fields and therefore is space-time dependent. Thus, derivatives have to be transformed into covariant derivatives. Furthermore, the non-linearity of the transformation makes the transformation of $\partial_\mu\xi$ complicated. Instead of starting with this object, we will consider one that has simpler transformation properties,

$$e^{-i\xi\cdot X}\partial_\mu e^{i\xi\cdot X} = iD_\mu^{\hat{a}}X^{\hat{a}} + iE_\mu^i Y^i \equiv iD_\mu + iE_\mu, \tag{2.160}$$

where we have used that the object we are considering is a generator of the group and can therefore be expanded in generators. Let us consider now $\partial_\mu\Phi_0$,

$$\begin{aligned}\partial_\mu\Phi_0 &= \partial_\mu\Big[e^{i\xi\cdot X}\Phi\Big] = e^{i\xi\cdot X}e^{-i\xi\cdot X}\partial_\mu\Big[e^{i\xi\cdot X}\Phi\Big] \\ &= e^{i\xi\cdot X}\Big[iD_\mu^{\hat{a}}X^{\hat{a}} + iE_\mu^i Y^i\Phi + \partial_\mu\Phi\Big] \\ &= e^{i\xi\cdot X}\Big[iD_\mu^{\hat{a}}X^{\hat{a}}\Phi + (\partial_\mu + iE_\mu^i Y^i)\Phi\Big]. \end{aligned} \tag{2.161}$$

But $\partial_\mu\Phi_0$ transforms under a general global transformation g the same as Φ_0 does, thus, the term in brackets transforms under g as Φ does

$$\Big[iD_\mu^{\hat{a}}X^{\hat{a}}\Phi + (\partial_\mu + iE_\mu^i Y^i)\Phi\Big] \to h(g,\xi(x))\Big[iD_\mu^{\hat{a}}X^{\hat{a}}\Phi + (\partial_\mu + iE_\mu^i Y^i)\Phi\Big]. \tag{2.162}$$

Since different irrepresentations of H do not mix in Φ under a G transformation, we can set to zero all elements in Φ except those belonging to a single R_i. But then the two terms in brackets belong to different H representations in R. Thus, they must transform independently. On one hand we have

$$D_\mu^{\hat{a}}X^{\hat{a}}\Phi \to hD_\mu^{\hat{a}}X^{\hat{a}}\Phi = D_\mu'^{\hat{a}}X^{\hat{a}}\Phi' = hh^{-1}D_\mu'^{\hat{a}}X^{\hat{a}}h\Phi. \tag{2.163}$$

We therefore have

$$D_\mu' = hD_\mu h^{-1} = D_\mu^{\hat{b}}e^{iu\cdot Y}X_{\hat{b}}e^{-iu\cdot Y} = D_\mu^{\hat{b}}R_{\hat{b}\hat{a}}X_{\hat{a}}. \tag{2.164}$$

Similarly,

$$
\begin{aligned}
(\partial_\mu + iE_\mu \cdot Y)\Phi &\rightarrow h(\partial_\mu + iE_\mu)\Phi = (\partial_\mu + iE')\Phi' = (\partial_\mu + iE')h\Phi \\
&= h(\partial_\mu + ih^{-1}E'h + h^{-1}\partial_\mu h)\Phi. \qquad (2.165)
\end{aligned}
$$

Thus,

$$
E'_\mu = hE_\mu h^{-1} + i(\partial_\mu h)h^{-1} = hE_\mu h^{-1} - ih\partial_\mu h^{-1}. \qquad (2.166)
$$

We therefore have three basic ingredients, Φ, D_μ and E_μ on which a general g transformation acts as a local $h(g, \xi(x)) \in H$ transformation belonging to the unbroken subgroup. Thus, a general H invariant Lagrangian built out of functions of these building blocks is automatically G-invariant. Let us summarize here the transformations of these fields,

$$
\begin{aligned}
\Phi &\rightarrow h(g, \xi(x))\Phi, \qquad (2.167) \\
D_\mu &\rightarrow h(g, \xi(x))D_\mu h^{-1}(g, \xi(x)), \quad D_\mu^{\hat{a}} \rightarrow R^T_{\hat{a}\hat{b}} D_\mu^{\hat{b}}, \qquad (2.168) \\
E_\mu &\rightarrow h(g, \xi(x))E_\mu h^{-1}(g, \xi(x)) - ih(g, \xi(x))\partial_\mu h^{-1}(g, \xi(x)), \qquad (2.169)
\end{aligned}
$$

where R is the matrix representation of H under which the broken generators transform. In particular, the quantity $\mathcal{E}_\mu \equiv \partial_\mu + iE_\mu$ acts as an H-covariant derivative,

$$
\mathcal{E}_\mu \Phi \rightarrow h(g, \xi(x))\mathcal{E}_\mu \Phi. \qquad (2.170)
$$

With these building blocks we can construct in a very simple way G-invariant Lagrangians out of multiplets of the unbroken group. What happens if the symmetry is (partially) gauged? Essentially the same formalism can be used, with the replacement of the usual derivative with the gauge covariant derivative

$$
\partial_\mu \rightarrow \partial_\mu + iA_\mu = \partial_\mu + iA_\mu^a G^a.
$$

We define now,

$$
e^{-i\xi\cdot X}(\partial_\mu + iA_\mu)e^{i\xi\cdot X} = i\bar{D}_\mu^{\hat{a}} X^{\hat{a}} + i\bar{E}_\mu^i Y^i = i\bar{D}_\mu + i\bar{E}_\mu, \qquad (2.171)
$$

where now $\bar{D}_\mu = \bar{D}_\mu(\xi, A)$, $\bar{E}_\mu = \bar{E}_\mu(\xi, A)$.

$$
\begin{aligned}
(\partial_\mu + iA_\mu)\Phi_0 &= (\partial_\mu + iA_\mu)e^{i\xi\cdot X}\Phi = e^{i\xi\cdot X}(\partial_\mu + e^{-i\xi\cdot X}\partial_\mu e^{i\xi\cdot X} + ie^{-i\xi\cdot X}A_\mu e^{i\xi\cdot X})\Phi \\
&= e^{i\xi\cdot X}\Big\{\partial_\mu + e^{-i\xi\cdot X}\big[\partial_\mu + iA_\mu\big]e^{i\xi\cdot X}\Big\}\Phi \\
&= e^{i\xi\cdot X}\Big\{i\bar{D}_\mu + (\partial_\mu + i\bar{E}_\mu)\Big\}\Phi. \qquad (2.172)
\end{aligned}
$$

But since under a local G transformation, $(\partial_\mu + iA_\mu)\Phi_0$ transforms as Φ_0 does, that means that $[i\bar{D}_\mu + (\partial_\mu + i\bar{E}_\mu)]\Phi$ transforms the same way Φ does. Therefore, under *local* G transformations we have,

$$
\begin{aligned}
\Phi &\rightarrow h(g(x), \xi(x))\Phi, \qquad (2.173) \\
\bar{D}_\mu &\rightarrow h(g(x), \xi(x))\bar{D}_\mu h^{-1}(g(x), \xi(x)), \qquad (2.174) \\
\bar{\mathcal{E}}_\mu \Phi &\equiv (\partial_\mu + i\bar{E}_\mu)\Phi \rightarrow h(g(x), \xi(x))\bar{\mathcal{E}}_\mu \Phi. \qquad (2.175)
\end{aligned}
$$

Thus, we can build an invariant Lagrangian under local G tranformations by constructing, out of Φ, $\bar{D}_\mu$ and $\bar{\epsilon}_m u$ a Lagrangian invariant under local H transformations.

Chapter 3

Electroweak Symmetry in the Standard Model

Literature: The Standard Model is covered in many reviews and textbooks. Here we are going to take a slightly different approach than just introducing the SM but instead we will describe the experimentally confirmed part of the SM and argue that an EWSB sector is needed. We will then describe the EWSB sector in the SM, including its sucesses and problems. This will motivate alternative EWSB sectors that we will consider in the rest of the course. In general, I have used Contino's notes (`http://indico.phys.ucl.ac.be/conferenceDisplay.py?confId=148`) as a rough guide. Some other references I found particularly useful during the preparation of these notes are the books by Peskin and Schroeder [4]; Pokorski [1] and Donoghue, Golowich and Holdstein [2]. Very interesting as an overview of electroweak symmetry breaking in and beyond the Standard Model are the lectures by Grojean [13]. Also very interesting are the lectures by Veltman on the Higgs system [14].

3.1 Electroweak Symmetry and its Breaking

Most of the Standard Model (SM) of particle physics has been tested to extremely good accuracy up to energies of the order of the electroweak symmetry breaking (EWSB) scale $v \sim 174$ GeV. It is based on a Yang-Mills theory with an $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group spontaneously broken to $SU(3)_C \times U(1)_Q$ and three families of fermions with the quantum numbers displayed in Table 3.1, in which we have also displayed the chirality of each fermion (we use $\psi_{L,R} = \mathcal{P}_{L,R}\psi$ with $\mathrm{P}_{L,R} \equiv (1 \mp \gamma^5)/2$ the chirality projectors). Neutrino masses also require the introduction of RH neutrinos, which are singlets under the SM gauge symmetry. In these lectures we are interested in the mechanism of EWSB and will disregard neutrino masses altogether. The particular mechanism of EWSB has however escaped experimental confirmation so far. The SM solution for EWSB, despite its

Table 3.1: Quantum numbers for a family of SM fermions. The hypercharge assignment is arbitrary, fixed by convention.

$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	u_R	d_R	$l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	e_R
$(3,2)_{1/6}$	$(3,1)_{2/3}$	$(3,1)_{-1/3}$	$(1,2)_{-1/2}$	$(1,1)_{-1}$

simplicity, is not fully satisfactory. Assuming that naturalness is a good guiding principle, new physics is required at around the TeV scale either to replace the Higgs as the mediator for EWSB or to make it more natural, in a way that we will discuss in these lectures. We will then dedicate this lecture to study some of the features of the mechanism of EWSB in the SM, its successes and weaknesses, as a motivation for the need of physics beyond the SM (BSM) and a guide for what makes a model BSM interesting, relevant, and phenomenologically viable.

We are therefore going to describe the Lagrangian of the SM, as it has been experimentally confirmed so far. We will start with the kinetic Lagrangian, which reads

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W^{i\,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R, \tag{3.1}$$

where $\psi_{L,R}$ run over all the SM fermions of the appropriate chirality. Following standard notation (see for instance [4]) we have defined the covariant derivative by

$$D_\mu = \partial_\mu - i g_a T^a A^a_\mu = \partial_\mu - i[g_s t^a G^a_\mu + g T^i_L W^i_\mu + g' Y B_\mu], \tag{3.2}$$

where we have denoted T^a and A^a_μ the general list of generators and gauge bosons and t^a and T^i_L are the $SU(3)_C$ and $SU(2)_L$ generators in the corresponding representation of the field the covariant derivative is acting on, normalized to $\text{Tr}[T^\alpha T^\beta] = \delta^{\alpha\beta}/2$ for the fundamental representation (*i.e.* they are $t^a = \lambda^a/2$, $T^i_L = \sigma^i/2$ for a color triplet and weak doublet, with λ and σ the Gell-mann and Pauli matrices, respectively). The gauge boson field tensors are defined in general by

$$[D_\mu, D_\nu] = -i g_a F^a_{\mu\nu} T^a \Rightarrow F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_a f^{abc} A^b_\mu A^c_\nu. \tag{3.3}$$

In particular, for our choice of gauge groups, we have

$$G^a_{\mu\nu} \equiv \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu, \tag{3.4}$$
$$W^i_{\mu\nu} \equiv \partial_\mu W^i_\nu - \partial_\nu G^j_\mu + g \epsilon^{ijk} W^j_\mu W^k_\nu, \tag{3.5}$$
$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{3.6}$$

As advertised, this Lagrangian has a local $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry. Defining

$U(x) \equiv \exp[iT^a\alpha^a(x)]$, with T^a running over all 12 (=8+3+1) generators, we have

$$\begin{aligned}
\psi(x) &\rightarrow U(x)\psi(x) \approx (1 + iT^a\alpha^a(x) + \ldots)\psi(x), &(3.7)\\
A_\mu(x) &\equiv A^a_\mu(x)T^a \rightarrow U\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^\dagger \approx A_\mu + i[\alpha, A_\mu] + \frac{1}{g}\partial_\mu\alpha, &(3.8)\\
D_\mu\psi(x) &\rightarrow U(D_\mu\psi). &(3.9)
\end{aligned}$$

Apart from this local symmetry, we know from our examples in previous lectures that the kinetic term for the fermions has an extra $U(3)^5$ global symmetry (in our previous language this would be $U(3)^u_L \times U(3)^u_R \times U(3)^d_L \times U(3)^d_R$ for the quark system with the restriction that u_L and d_L are tied together by the $SU(2)_L$ symmetry and therefore there is only one "Left" rotation in the quark sector. Similarly, for the lepton sector we have a common left rotation for the three l_L and a right rotation for the e_R).

One important property of the SM gauge symmetry is that it is (its electroweak part) **chiral**, by which we mean that left-handed fermions transform in different representations of the gauge group than right-handed fermions. In particular, a mass term for the fermions, which mixes left and right components, is not gauge invariant and is therefore forbidden *unless the symmetry is spontaneously broken.* Obviously, the gauge invariance also forbids a mass term for the gauge bosons which, unless the symmetry is spontanously broken, should all be massless.

Note: Chirality of the electroweak symmetry implies other constraints, namely, the possible presence of anomalies. This constraints the quantum numbers of the chiral fields. The fermion content of the SM is anomaly free.

Experimentally, we observe however that fermions and some of the gauge bosons have finite masses. This means that the gauge symmetry must be spontaneously broken and experimental data indicate that it is broken down to QCD, $SU(3)_C$, and electromagnetism, $U(1)_{\rm em}$. These two symmetries are vector-like, which means that all fermions have left and right components transforming under the same representation. Thus, Dirac masses are allowed by the unbroken gauge invariance. Similarly, the gauge bosons corresponding to the broken generators, will get masses through the Higgs mechanism.

Traditionally, one would write here the gauge invariant Higgs Lagrangian, that in the SM is responsible for spontaneously breaking the electroweak symmetry. Instead, we will follow our *experimental guiding principle* and write down the mass Lagrangian for the fermions and gauge bosons.

$$\mathcal{L}_{\rm mass} = \mathcal{L}^g_{\rm mass} + \mathcal{L}^f_{\rm mass}. \tag{3.10}$$

For the electroweak gauge bosons, we have,

$$\mathcal{L}^g_{\rm mass} = \frac{1}{2}m^2_W[(W^1_\mu)^2 + (W^2_\mu)^2] + \frac{1}{2}m^2_Z(c_W W^3_\mu - s_W B_\mu)^2, \tag{3.11}$$

where c_W, s_W are the cosine and sine of the Weinberg angle that define which combination of W^3 and B becomes the massive Z. This Lagrangian gives mass to $W^{1,2}_\mu$ and a linear combination of W^3_μ and B_μ, leaving massless the orthogonal combination $s_W W^3_\mu + c_W B_\mu$. Performing the following redefinition of fields,

$$W^\pm_\mu = \frac{W^1_\mu \mp i W^2_\mu}{\sqrt{2}}, \quad \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c_W & -s_W \\ s_W & c_W \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \Leftrightarrow \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} c_W & s_W \\ -s_W & c_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}, \tag{3.12}$$

where $W^\pm_\mu$ have electric charged ± 1, Z_μ has charge 0 and A_μ is the massless photon, we have

$$\mathcal{L}^g_{\text{mass}} = m^2_W W^+_\mu W^{-\,\mu} + \frac{1}{2} m^2_Z (Z_\mu)^2. \tag{3.13}$$

This mass Lagrangian breaks (we assume the breaking is spontaneous) the electroweak symmetry $SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$

$$i(D_\mu - \partial_\mu) = g T^3_L W^3_\mu + g' Y B_\mu + \ldots = (g s_W T^3_L + g' c_W Y) A_\mu + (g c_W T^3_L - g' s_W Y) Z_\mu + \ldots = eQA_\mu + \ldots, \tag{3.14}$$

which implies,

$$eQ = g s_W \left(T^3_L + \frac{g'}{g} \frac{c_W}{s_W} Y \right) \to g s_W (T^3_L + Y), \tag{3.15}$$

where we have use the fact that the normalization of the $U(1)_Y$ factor is not fixed to make $Y(g' c_W)/(g s_W) \to Y$, (this is actually the convention used in Table 3.1), or equivalently

$$g' = g c_W / s_W \Rightarrow \begin{cases} s_W = g' / \sqrt{g^2 + g'^2}, \\ c_W = g / \sqrt{g^2 + g'^2}, \end{cases} \tag{3.16}$$

Thus, we have

$$e = g s_W = g' c_W, \quad Q = T^3_L + Y. \tag{3.17}$$

The mass lagrangian for the fermions reads (we neglect neutrino masses here),

$$\mathcal{L}^f_{\text{mass}} = -\sum_{ij=1}^{3} \left\{ \bar{u}^i_L m^u_{ij} u^j_R + \bar{d}^i_L m^d_{ij} d^j_R + \bar{e}^i_L m^e_{ij} e^j_R + \text{h.c.} \right\}. \tag{3.18}$$

Using the global symmetries of the kinetic Lagrangian, we can put the mass matrices in the following form

$$m^u_{ij} = V^\dagger_{ij} m^u_i, \quad m^d_{ij} = m^d_i \delta_{ij}, \quad m^e_{ij} = m^e_i \delta_{ij}, \tag{3.19}$$

with V_{ij} the unitary 3×3 CKM matrix (alternatively, one could put the up quark sector mass diagonal and the down quark sector one diagonal up to a left rotation given by the CKM matrix) and m^u_i, m^d_i, m^e_i the diagonal masses of the charge 2/3 quarks, charge $-1/3$ quarks and charged leptons, respectively. Note that the mass terms we have written for the fermions are compatible with the unbroken electromagnetic symmetry (but not with the original electroweak symmetry). Also, these mass terms break the global symmetries down to $U(1)_B$ and $U(1)_{L_i}$.

The total Lagrangian, after EWSB, reads (we do not display the QCD part from now on, since it does not play any role in EWSB),

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}W^+_{\mu\nu}W^{-\,\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + m_W^2 W^+_\mu W^{-\,\mu} + \frac{1}{2}m_Z^2(Z_\mu)^2 \\ &+ \sum\left[\bar{\psi}_L i\not{D}\psi_L + \bar{\psi}_R i\not{D}\psi_R\right] - \left[\bar{u}^i_L V^\dagger_{ij} m^u_j u^j_R + \bar{d}^i_L m^d_i d^i_R + \bar{e}^i_L m^e_i e^i_R + \text{h.c.}\right],\end{aligned} \quad (3.20)$$

where the (electroweak part of the) covariant derivative now reads

$$i(D_\mu - \partial_\mu) = \frac{g}{\sqrt{2}}(T^+W^+_\mu + T^-W^-_\mu) + \frac{g}{c_W}[T^3_L - s_W^2 Q]Z_\mu + eQA_\mu, \quad (3.21)$$

and we have defined $T^\pm = T^1 \pm iT^2 = (\sigma^1 \pm i\sigma^2)/2$.

3.2 The Need for an Electroweak Symmetry Breaking Sector

In the previous section we have described the Lagrangian of the spontaneously broken $SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$ model that very successfully reproduces experimental data up to the electroweak scale. From previous lectures we know that in order for a local symmetry to be broken in a spontaneous way (so that the Lagrangian is still invariant under the symmetry), the Higgs mechanism has to be in action. For that, we just need the would-be Goldstone bosons that will combine with the broken gauge bosons to give them a mass in a gauge invariant way (*i.e.* resulting in Ward identity preserving, transverse vacuum polarization tensors). The question now is, can we do with just these (three) would-be Goldstone bosons and write the Lagrangian using a non-linear representation? The answer is, of course, yes, we can, but not for long. It turns out that the theory we just wrote does not preserve some of the fundamental features of realistic quantum field theories at high energies. In particular, the theory as written, violates unitarity at high energies. One can describe the physics at the EWSB scale by means of the so-called electroweak chiral Lagrangian, that uses the non-linear representation, keeping only the Goldstone bosons that are required to give masses to the W and the Z. We will do so by using the formalism we developed in the previous lectures. We will start by parametrizing the generators of the group in terms unbroken generators,

$$Q = T^3_L + Y,$$

and generators of the coset, for which we choose,

$$T^i_L.$$

Let us now start with a singlet under the unbroken group,

$$\Phi(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.22)$$

In order for this to be a singlet under the unbroken subgroup, it has to have zero electric charge and therefore hypercharge $Y = Q - T_L^3 = 1/2$. This field can be promoted to a full representation of the electroweak group by multiplying it with the exponential of the Goldstone bosons,

$$\Phi_0 = \exp[i\vec{\pi}\cdot\vec{\sigma}/(\sqrt{2}v)]\Phi_0, \tag{3.23}$$

transforms as a $(1,2)_{1/2}$ under the full gauge symmetry,

$$\Phi_0 \to e^{i\vec{\alpha}_L\cdot\vec{\sigma}/2}e^{i\alpha_Y/2}\Phi_0. \tag{3.24}$$

We prefer to work with the Goldstone fields in terms of a 2×2 matrix by constructing another doublet out of the conjugate of Φ_0,

$$\tilde{\Phi}_0 = i\sigma^2\Phi_0^* \sim (1,2)_{-1/2}, \tag{3.25}$$

and constructing the matrix

$$\Sigma = (\tilde{\Phi}_0\,\Phi_0) = \exp(i\vec{\pi}\cdot\vec{\sigma}/(\sqrt{2}v)). \tag{3.26}$$

The $SU(2)_L \times U(1)_Y$ symmetry can be seen acting on Σ as

$$\Sigma \to e^{iw_L^i\sigma^i/2}\Sigma e^{-iw_Y\sigma^3/2}, \tag{3.27}$$

(with fermions we need to also gauge $U(1)_{B-L}$ in this minimal setup). Thus, we can write a fully gauge invariant Lagrangian involving only the massive gauge bosons (plus the photon). The building blocks of this theory are

- The field strengths: $\hat{W}_{\mu\nu} \equiv W^i_{\mu\nu}\sigma^i/2, B_{\mu\nu}$.
- The Lorentz vector $V_\mu \equiv (D_\mu\Sigma)\Sigma^\dagger$, involving the covariant derivative of Σ,

$$D_\mu\Sigma \equiv \partial_\mu\Sigma - i[\frac{g}{2}W^I_\mu\sigma^I\Sigma - \frac{g'}{2}\Sigma\sigma^3 B_\mu]. \tag{3.28}$$

- The combination of $\Sigma-$fields: $T \equiv \Sigma\sigma^3\Sigma^\dagger$.

Except for $B_{\mu\nu}$, which is invariant under $SU(2)_L \times U(1)_Y$, the transformation of all the other combinations of fields, denoted collectively by Φ, under the full $SU(2)_L \times U(1)_Y$ symmetry is

$$\Phi \to e^{iw_L^I\sigma^I/2}\Phi e^{-iw_L^I\sigma^I/2}. \tag{3.29}$$

Doing an expansion in momenta and keeping up to second order we can write the Chiral EW Lagrangian

$$\begin{aligned}\mathcal{L}_{EWCh} &= -\frac{1}{2}\mathrm{Tr}(\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{v^2}{2}\mathrm{Tr}[(D_\mu\Sigma)^\dagger D^\mu\Sigma] \\ &+ a_0\frac{v^2}{4}[\mathrm{Tr}(TV_\mu)]^2 + a_1 i\frac{gg'}{2}B_{\mu\nu}\mathrm{Tr}(T\hat{W}_{\mu\nu}) + a_8\frac{g^2}{4}[\mathrm{Tr}(T\hat{W}_{\mu\nu})]^2 + \ldots\ .\end{aligned} \tag{3.30}$$

The coefficients of the first three terms are chosen so that the kinetic terms of the gauge and Goldstone bosons are canonically normalized. Let us expand the kinetic term of the Goldstones. First, we expand $D_\mu\Sigma$ in powers of the Goldstone fields,

$$\begin{aligned} D_\mu\Sigma &= i\left[\frac{\partial_\mu\pi}{\sqrt{2}v} - \frac{g}{2}W_\mu + \frac{g'}{2}\sigma^3 B_\mu\right] + \ldots \\ &= i\left[\sum_{b=1}^{2}\left(\frac{\partial_\mu\pi^b}{\sqrt{2}v} - \frac{g}{2}W_\mu^b\right)\sigma^b + \left(\frac{\partial_\mu\pi^3}{\sqrt{2}v} - \frac{g}{2}W_\mu^3 + \frac{g'}{2}B_\mu\right)\sigma^3\right] + \ldots, \end{aligned} \tag{3.31}$$

where the dots denote terms with more than one field. The kinetic term for the Goldstones can then be written

$$\mathcal{L}_\Sigma = \frac{v^2}{2}\mathrm{Tr}[(D_\mu\Sigma)^\dagger D^\mu\Sigma] = |\partial_\mu\pi^+ - \frac{g}{\sqrt{2}}vW_\mu^+|^2 + \frac{1}{2}(\partial_\mu\pi^3 - \frac{g}{\sqrt{2}c_W}vZ_\mu)^2. \tag{3.32}$$

In the last equality we have used the definition of the Z boson and c_W that we introduced in the previous section. Also, we have defined $\pi^\pm = (\pi^1 \mp i\pi^2)/\sqrt{2}$. The masses of the gauge bosons are aparent in the unitary gauge ($\Sigma = 1_{2\times 2}$, $\pi^i = 0$)

$$\mathcal{L} = \frac{1}{2}g^2v^2W_\mu^+W^{-\,\mu} + \frac{1}{4}v^2\sqrt{g^2+g'^2}Z_\mu Z^\mu = m_W^2 W_\mu^+W^{-\,\mu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu. \tag{3.33}$$

Thus we get

$$\left.\begin{array}{l} m_W = g\frac{v}{\sqrt{2}} \\ m_Z = \sqrt{g^2+g'^2}\frac{v}{\sqrt{2}} \end{array}\right\} \Rightarrow \rho \equiv \frac{m_W^2}{m_Z^2 c_W^2} = 1. \tag{3.34}$$

This relation between the W and Z masses is not coincidence. There is a symmetry that guarantees this result. The importance of this relation, as parametrized by ρ (or through its close cousin the T parameter $\hat{T} \sim \rho - 1$), is that it has been measured to with a per mille accuracy, finding very good agreement with the SM prediction.

Instead of the unitary gauge, one can fix the gauge to an R_ξ gauge by adding the following gauge-fixing term

$$\mathcal{L}_{\mathrm{G.F.}} = -\frac{1}{2\xi}\left(\partial^\mu W_\mu^i + \xi\frac{gv}{\sqrt{2}}\pi^i\right)^2 - \frac{1}{2\xi}(\partial^\mu B_\mu - \xi\frac{g'v}{\sqrt{2}}\pi^3)^2. \tag{3.35}$$

In this family of gauges, the gauge-Goldstone boson kinetic mixing is cancelled. The gauge boson propagators read, for the $W^\pm$,

$$\frac{-i}{k^2-m_i^2}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2-\xi m_i^2}\right], \tag{3.36}$$

where $m_i = m_W, m_Z, m_A$ for the W, Z and A, respectively. The Goldstones, on the other hand, have propagators,

$$\frac{i}{k^2-\xi m_i^2}, \tag{3.37}$$

where $m_i = m_W, m_Z$ where corresponds.

3.2.1 Custodial symmetry

Let us discuss what the symmetry reason behind the relation $\rho = 1$ is. The pattern of symmetry breaking described by Σ has a larger *global* symmetry than $SU(2) \times U(1)$. The $U(1)_Y$ symmetry acts on the right of Σ with the σ^3 Pauli matrix. We can extend this $U(1)$ symmetry to a full $SU(2)_R$ (global) symmetry under which the kinetic term $\mathcal{L}_\Sigma$ is still invariant. The vacuum is given by $\Sigma = 1_{2\times 2}$, which is invariant under the vector $SU(2)_{L+R}$ symmetry (equal left and right rotations). Now, note that the left rotation actually coincides with the gauge $SU(2)_L$ symmetry, under which the three gauge bosons W^I_μ transform as a triplet (the adjoing of $SU(2)_L$). The remaining symmetry after EWSB is the simultaneous rotation from the left and the right with the same matrix, but in particular there is an arbitrary left rotation that remains unbroken. Thus, W^I_μ transform as a triplet under the remaining custodial symmetry and therefore they must receive the same contribution to their mass from EWSB. This global symmetry pattern $SU(2)_L \times SU(2)_R \to SU(2)_{L+R}$ has been dubbed custodial symmetry, as it protects the ρ (or T) parameter. Indeed, the presence of a symmetry under which the three $SU(2)_L$ gauge bosons transform as a triplet, together with unbroken $U(1)_{\text{em}}$ gauge invariance, ensure $\rho = 1$.

Note: Let us show that custodial symmetry plus unbroken $U(1)_{\text{em}}$ imply $\rho = 1$. First write the Z and the photon as arbitrary combinations of W^3 and B,

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} W^3 \\ B \end{pmatrix}, \tag{3.38}$$

where $c \equiv \cos\theta$, $s \equiv \sin\theta$ with θ an arbitrary angle, to be determined by $Q = T^3 + Y$. The coupling of the photon is then

$$gT^3 W^3 + g'YB = (sgT^3 + cg'Y)A + \ldots = eQA + \ldots, \tag{3.39}$$

from which we obtain $c = g/\sqrt{g^2 + g'^2}$, $s = g'/\sqrt{g^2 + g'^2}$. Now allowing for an arbitrary mass matrix in the neutral sector we get

$$\begin{aligned} \mathcal{L}^0_m &= \frac{v^2}{4} \begin{pmatrix} W^3 & B \end{pmatrix} \begin{pmatrix} g^2 & a \\ a & b \end{pmatrix} \begin{pmatrix} W^3 \\ B \end{pmatrix} \\ &= \begin{pmatrix} Z & A \end{pmatrix} \begin{pmatrix} c^2 g^2 - 2sca + s^2 b & sc(g^2 - b) + (c^2 - s^2)a \\ sc(g^2 - b) + (c^2 - s^2)a & s^2 g^2 + 2sca + s^2 b \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix}. \end{aligned} \tag{3.40}$$

Requiring the photon mass and mixing with the Z to vanish determines a and b and therefore the Z mass, which is given by

$$\frac{1}{2} m_Z^2 = \frac{v^2}{4}(g^2 + g'^2) = \frac{1}{2} \frac{m_W^2}{c_W^2}, \tag{3.41}$$

and therefore $\rho = 1$.

Note however that the custodial symmetry is a good symmetry of $\mathcal{L}_\Sigma$ but not of the full electroweak chiral Lagrangian. In particular $T = \Sigma\sigma^3\Sigma^\dagger$ is not invariant under the full $SU(2)_R$ global symmetry and therefore operators involving it can give a direct contribution to the ρ parameter. Actually, the operator with coefficient a_0 does induce a correction to it.

Which sectors of the SM break the custodial symmetry? One we have already seen, the hypercharge. That means that at loop level, there will be extra corrections to ρ proportional to g'. Another sector that breaks the custodial symmetry is the fermionic sector, through the Yukawa couplings. These can be written as

$$\mathcal{L} = \begin{pmatrix} \bar{t}_L & \bar{b}_L \end{pmatrix} \Sigma \begin{pmatrix} \lambda_t t_R \\ \lambda_b b_R \end{pmatrix}. \tag{3.42}$$

We have written the couplings in a way that makes it transparent the breaking of the custodial symmetry. It is the difference between the top and bottom Yukawa couplings that break the global $SU(2)_R$ symmetry, and therefore the custodial symmetry. This means that, due to the large difference between the top and bottom masses, one can expect a relatively large contribution from this sector to ρ at one loop. This contribution is in fact an interesting example of non-decoupling of heavy physics, in which the conditions of the Appelquist-Carazzone theorem are not satisfied. The reason is that in the heavy top limit, the large mass comes from a dimensionless coupling that becomes large, thus hitting strong coupling. Furthermore, the resulting low energy theory is non-renormalizable and therefore non-vanishing effects in the large top mass limit can be expected.

We will compute the leading contribution to ρ in the heavy top limit as an exercise.

An important lesson to learn from this discussion, and from the top contribution is that tree level violations of the custodial symmetry by new physics will be in general very strongly constrained and therefore custodial symmetry should be a natural ingredient in BSM model building. Also, non-decoupling physics that violates the custodial symmetry, can have non-negligible effects on ρ at the loop level and should be considered in models of physics BSM.

3.2.2 Unitarity Violation

The model as we have described it so far, can be used to describe the physics up to the EW scale that has been experimentally tested. It is however not consistent when extrapolated at higher energies (another question that we will address later is whether the description of experimental data in terms of this model is accurate or not). The reason is that the scattering of longitudinal gauge bosons (or equivalently of the Goldstone bosons that are eaten by them to become massive) grows with the energy. However, cross sections cannot be arbitrary large, as they are constrained by unitarity. The growth with energy of longitudinal gauge boson scattering can be easily understood by noting that a massive spin-1 particle with momentum k^μ can be represented by

$$A^\mu = \epsilon^\mu \exp(ik_\nu x^\nu), \tag{3.43}$$

where the polarization vector satisfies $\epsilon_\mu \epsilon^\mu = -1$ and $k_\mu \epsilon^\mu = 0$. Thus, if we assume the vector is moving in the $\hat{z}$ direction ($k^\mu = (E, 0, 0, k)$, with $k^\mu k_\mu = E^2 - k^2 = M^2$), the three polarizations can be written as

$$\epsilon_+^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0), \quad \epsilon_-^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0), \tag{3.44}$$

for the LH and RH *transverse* polarizations and

$$\epsilon_L^\mu = \left(\frac{k}{M}, 0, 0, \frac{E}{M}\right) = \frac{k^\mu}{M} + \mathcal{O}\left(\frac{M}{E}\right), \tag{3.45}$$

for the *longitudinal* one.

Exercise: Confirm the form of the longitudinal polarization. Hint: Start in the rest frame and perform a boost to with momentum k along the $\hat{z}$ axis. Compute the boost of the polarization vector along that direction $\epsilon = (0, 0, 0, 1)$. Using the on-shell condition, check that at high energies, the longitudinal polarization aligns with the momentum.

Thus, we see that at high energies, the longitudinal polarizations aligns with the momentum of the gauge boson. This causes a growth in the scattering amplitude that is incompatible with unitarity. Let us first try to understand the result and then compute it using a neat trick to simplify the calculation. Let us assume we are in the unitary gauge, in which Σ is equal to the identity, and let us consider the scattering of longitudinal W bosons,

$$W_L^+ W_L^- \to W_L^+ W_L^-, \tag{3.46}$$

which proceeds through the three diagrams shown in Fig. 3.1. The naive energy dependence

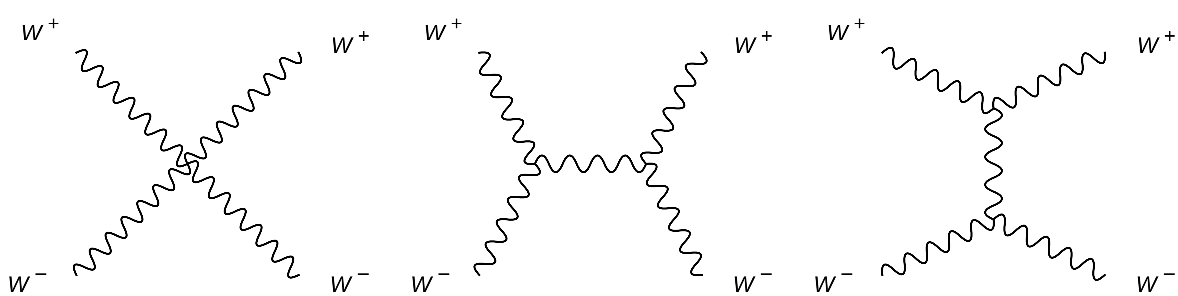


Figure 3.1: Diagrams contributing to longitudinal W scattering. In the second and third diagrams, either the Z or the γ are exchanged in the s and t channels, respectively.

of the amplitude can be easily estimated by recalling that $\epsilon_L \sim E$, the trilinear couplings are proportional to momentum (and therefore can pick up a power of the energy), and the propagators go like $\sim (1 + E^2)/E^2$. Thus, the contact interaction diagram has potentially

$\mathcal{O}(E^4)$ contributions whereas the two diagramas with exchange of Z and A are potentially $\mathcal{O}(E^6)$. It turns out that the terms $\sim E^6$ coming from the $p^\mu p^\nu/m^2$ terms in the propagators cancel out on each diagram, making each individual diagram of order $\sim E^4$. If the couplings come from an $F^2_{\mu\nu}$ term, these leading order $\sim E^4$ cancels among the three diagrams. There is however a residual $\mathcal{O}(E^2)$ term that does not cancel. This term will violate unitarity at high energies. The result of the full scattering amplitude is in fact

$$\mathcal{M}(W_L^+W_L^- \to W_L^+W_L^-) = \frac{g^2}{4m_W^2}(s+t). \tag{3.47}$$

Instead of doing the calculation for the gauge bosons, we will make use of the Equivalence Theorem that states that amplitudes involving longitudinally polarized gauge bosons are equal to amplitudes with the corresponding Goldstone bosons as external particles up to corrections of order m^2/E^2, with m the mass of the gauge boson and E the energy involved in the scattering. At leading order in m_W^2/E^2, we can therefore replace the external $W^\pm$ with the corresponding Goldstone bosons $\pi^\pm$. We will furthermore work in the t'Hooft-Feynman gauge ($\xi = 1$), in which the gauge boson propagators have the simple form,

$$\frac{-ig^{\mu\nu}}{p^2-m^2}, \tag{3.48}$$

and therefore they go like $\sim 1/E^2$ at large energies. The counting now goes as follows, gauge couplings of two Goldstone bosons are momentum dependent and therefore go like $\sim E$ whereas a Goldstone boson quartic coupling has two powers of momentum and therefore goes like $\sim E^2$. Thus, the diagrams with an internal gauge boson propagator go like $\sim E^0$ whereas the contact interaction grows with energy like $\sim E^2$. Thus, this is the only diagram we have to compute. The quartic interaction of the Goldstone bosons is given by the corresponding term in $\mathcal{L}_\Sigma$. We have actually computed it in the study of the sigma model getting,

$$\mathcal{L}_\Sigma = \frac{1}{12v^2}\left[(\pi^i\partial_\mu\pi^i)^2 - \pi^i\pi^i(\partial_\mu\pi^j\partial^\mu\pi^j)\right] + \ldots = \frac{g^2}{24m_W^2}(\pi^-\partial_\mu\pi^+ - \pi^+\partial_\mu\pi^-)^2 + \ldots . \tag{3.49}$$

Using this expression, we obtain for the amplitude,

$$\mathcal{M}(\pi^+\pi^- \to \pi^+\pi^-) = \frac{g^2}{4m_W^2}(s+t)\left[1 + \mathcal{O}\left(\frac{m_W^2}{E^2}\right)\right]. \tag{3.50}$$

Let us now be more quantitative on the violation of unitarity. In order to do that, we expand the amplitude in partial waves,

$$\mathcal{M} = 16\pi\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)a_l, \tag{3.51}$$

where P_l are the Legendre polynomials, that satisfy the orthonormality condition

$$\int_{-1}^{1} dx\, P_l(x)P_{l'}(x) = \frac{2}{(2l+1)}\delta_{ll'}. \tag{3.52}$$

The first few polynomials are

$$P_0(x) = 1, \tag{3.53}$$
$$P_1(x) = x, \tag{3.54}$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1). \tag{3.55}$$

They all satisfy $P_l(1) = 1$. Using the fact that for a $2 \to 2$ process, the cross section is given by $\mathrm{d}\sigma/\mathrm{d}\Omega = |\mathcal{M}|^2/(64\pi^2 s)$, with $\mathrm{d}\Omega = 2\pi \mathrm{d}\cos\theta$, we get

$$\begin{aligned}\sigma &= \frac{8\pi}{s}\sum_{l,l'=0}^{\infty}(2l+1)(2l'+1)a_l a_{l'}^* \int_{-1}^{1} dx\, P_l(x)P_{l'}(x) \\ &= \frac{16\pi}{s}\sum_{l=0}^{\infty}(2l+1)|a_l|^2. \end{aligned} \tag{3.56}$$

The optical theorem relates the total cross section with the imaginary part of the forward amplitude,

$$\sigma = \frac{1}{s}\mathrm{Im}\mathcal{M}(\theta = 0) = \frac{16\pi}{s}\sum_{l=0}^{\infty}(2l+1)\mathrm{Im}(a_l) = \frac{16\pi}{s}\sum_{l=0}^{\infty}(2l+1)|a_l|^2. \tag{3.57}$$

In practice, this has to be satisfied for each partial wave. Thus, we have

$$|a_l|^2 = \mathrm{Im}(a_l) \Rightarrow \mathrm{Re}(a_l)^2 + \left(\mathrm{Im}(a_l) - \frac{1}{2}\right)^2 = \frac{1}{4}, \tag{3.58}$$

which is a circle in the complex plane of radius 1/2, centered at $i/2$. Therefore we have $|\mathrm{Re}(a_l)| < 1/2$. [1] This has to be applied to the partial wave amplitudes, that read,

$$a_l = \frac{1}{32\pi}\int_{-1}^{1} d\cos\theta\, P_l(\cos\theta)\mathcal{M} = \frac{1}{16\pi(s-4m_W^2)}\int_{-s+4m_W^2}^{0} dt\, P_l\left(1 + \frac{2t}{s-4m_W^2}\right)\mathcal{M}, \tag{3.59}$$

where in the second equality we have used properties of a $2 \to 2$ scattering process of four particles with the same mass.

Note: Recall that, for a $2 \to 2$ process of four particles with the same mass we have $s = 4E^2$, $t = -2p^2(1-\cos\theta)$, where E, p and θ are the energy, modulus of trimomentum and scattering angle of the collision (thus $m^2 = E^2 - p^2$), we have

$$\int_{-1}^{1} d\cos\theta = \frac{2}{s-4m^2}\int_{-s+4m^2}^{0} dt, \tag{3.60}$$

and $\cos\theta = 1 + 2t/(s-4m^2)$.

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[1] This is an all order in perturbation theory statement, imaginary parts of the scattering amplitude are generated at the next order in perturbation theory.

Inserting the expression we got for the amplitude, we have, for the first few partial waves

$$\begin{aligned} a_0 &= \frac{g^2}{128\pi}\frac{s}{m_W^2}\left(1+\mathcal{O}\frac{m_W^2}{s}\right), \\ a_1 &= \frac{g^2}{384\pi}\frac{s}{m_W^2}\left(1+\mathcal{O}\frac{m_W^2}{s}\right). \end{aligned} \tag{3.61}$$

Requiring $\mathrm{Re}(a_0) \leq 1/2$ we obtain that unitarity is violated at a scale

$$\Lambda_{\mathrm{unit.}} = \sqrt{s_{\max}} = \frac{8\sqrt{\pi}}{g} m_W \approx 22 m_W \approx 1.7 \text{ TeV}. \tag{3.62}$$

A more detailed calculation, involving several coupled channels, gives a bound

$$\Lambda_{\mathrm{unit.}} \lesssim 700\,\mathrm{GeV}. \tag{3.63}$$

It should be noted however that these bounds are approximate bounds on the perturbative violation of unitarity. They are not strict, as non-perturbative effects will not turn on abruptly. Nevertheless they are a good indicator that something has to happen at energies $\lesssim$ 1 TeV to unitarize longitudinal gauge boson scattering. We do not know yet what this is but it is clear that the quantum field theory that is represented by the electroweak chiral Lagrangian, necessarily breaks down and has to be replaced by something else at such energies. This is the main motivation for the LHC that will extensively explore the multi-TeV region.

Thus, we see that although current experimental data can be explained in terms of just the observed particles (including the longitudinal components of the gauge bosons or, equivalently, the would-be Goldstone bosons), unitarity requires an electroweak symmetry breaking sector, that unitarizes longitudinal gauge boson scattering. In the rest of the course we will study in some detail different alternatives for this EWSB sector, the experimental constraints on them and their advantages and disadvantages. We start with the SM solution in the next section.

3.3 The SM EWSB sector: unitarity restoration with one scalar

3.3.1 Higgs unitarization of longitudinal gauge boson scattering

The SM simply consists on linearizing the non-linear chiral electroweak Lagrangian. We know how to do that linearization from our analysis of the sigma model, we just need to define the linearly transforming field,

$$\phi = (v + \frac{h}{\sqrt{2}})\Sigma\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} i\pi^+ \\ v + \frac{h - i\pi^3}{\sqrt{2}}\end{pmatrix} \equiv \begin{pmatrix}\phi^+\\ \phi_0\end{pmatrix} \sim (1,2)_{1/2}, \tag{3.64}$$

where in the last equality we have denoted the quantum numbers of the linear multiplet. Note that in the linear representation we have included a fourth scalar, the Higgs boson, that completes the linear multiplet. We can construct, out of the complex conjugate another doublet with opposite hypercharge,

$$\tilde{\phi} \equiv i\sigma^2\phi^* = \begin{pmatrix} \phi_0^* \\ -\phi^- \end{pmatrix} \sim (1,2)_{-1/2}. \tag{3.65}$$

The mass terms for the gauge bosons and fermions are derived in the SM from the gauge invariant scalar Lagrangian,

$$\mathcal{L}_{SM} = \mathcal{L}_{\rm kin} + |D_\mu\phi|^2 - \lambda\Big[(\phi^\dagger\phi) - v^2\Big]^2 - \Big[\bar{q}^i_L\tilde{\phi}V^\dagger_{ij}\lambda^u_j u^j_R + \bar{q}^i_L\phi\lambda^i_d d^i_R + \bar{l}^i_L\phi\lambda^i_e e^i_R + \text{h.c.}\Big]. \tag{3.66}$$

The Higgs does not really play any role in the mass generation for the gauge bosons or fermions (as we discussed in the previous sections, the Goldstones are enough to do that), however it does play an important role in making the model consistent at high energies. In particular, the scalar kinetic term includes couplings between the Higgs field h and the gauge bosons and therefore it contributes to longitudinal gauge boson scattering. These contributions are represented in Fig. 3.2. The gauge couplings to the Higgs are not

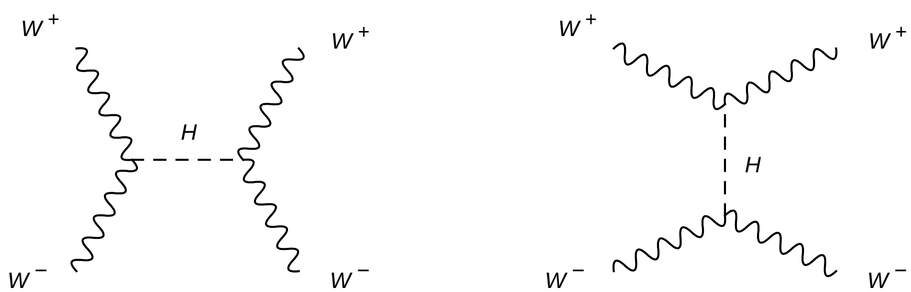


Figure 3.2: Diagrams contributing to longitudinal W scattering from the exchange of the Higgs in the s and t channels, respectively.

momentum dependent but are proportional to the corresponding gauge boson mass. Thus, we get a contribution that goes lie $\sim E^2$ at large energies ($\sim E^4$ from the longitudinal polarizations times $\sim E^{-2}$ from the propagator). Assuming $2m_W^2 \ll s, t$ and replacing $\epsilon^\mu_L \approx k^\mu/m$, the full amplitude reads,

$$\mathcal{M} = \frac{(igm_W^2)^2}{4m_W^4}\left[\frac{s^2}{s-m_H^2} + \frac{t^2}{t-m_H^2}\right] = -\frac{g^2}{4m_W^2}\left[\frac{s^2}{s-m_H^2} + \frac{1}{t-m_H^2}\right]. \tag{3.67}$$

Adding this contribution to the one we obtained for the higgless scattering, we get

$$\begin{aligned} \mathcal{M}(W_L^+W_L^- \to W_L^+W_L^-) &= \frac{g^2}{4m_W^2}\left[s + t - \frac{s^2}{s-m_H^2} - \frac{t^2}{t-m_H^2}\right] \\ &= -\frac{g^2}{4m_W^2}m_H^2\left[\frac{s}{s-m_H^2} + \frac{t}{t-m_H^2}\right]. \end{aligned} \tag{3.68}$$

In particular, the full amplitude no longer grows at high energies. The contribution of order $\sim E^2$ cancels exactly between the gauge and Higgs terms, thus the Higgs unitarizes exactly longitudinal gauge boson scattering. Of course it can only unitarize it at energies of the order of the Higgs mass itself. Unitarity therefore still puts an upper bound on the Higgs boson mass. In particular, inserting the amplitude we have obtained in the s-wave partial amplitude and taking the large s limit, we get

$$a_0 \sim \frac{g^2}{32\pi}\frac{m_H^2}{m_W^2}. \tag{3.69}$$

Requiring Re $a_0 \leq 1/2$ we obtain

$$m_H \leq \frac{4\sqrt{\pi}}{g} m_W \approx 11 m_W \approx 880 \text{ GeV}. \tag{3.70}$$

3.3.2 Constraints on the Higgs mass

As we have seen in the previous section, although the Higgs mass is a free parameter in the SM,

$$m_H^2 = 4\lambda v^2, \tag{3.71}$$

it is constrained by unitarity. In fact there are other constraints on the Higgs mass that we will discuss in this section.

Strong coupling

A simple constraint on the Higgs mass is simply strong coupling. The heavier the Higgs is, the larger λ has to be and therefore the scalar self-interactions become stronger. This is typical of any spontaneous symmetry breaking sector. The closer to the unitarity cut-off the scalar sector is, the more strongly coupled it has to be, since perturbative unitarity is saturated and beyond that point, unitarity has to be satisfied by higher order terms. Also, a heavier Higgs has larger and larger decay width,

$$\Gamma(h \to W^+W^-) \approx \frac{1}{32\pi^2}\frac{m_h^3}{v^2}. \tag{3.72}$$

For $m_h \gtrsim 1$ TeV, the width becomes of the same order as the mass itself and the very notion of the Higgs as a particle does not make much sense.

Triviality

At the quantum level, the coefficients of the Higgs potential

$$V(h) = -\frac{1}{2}\mu^2 h^2 + \frac{1}{4}\lambda h^4 \tag{3.73}$$

change with energy. The RGE for the Higgs quartic reads, at one loop,

$$16\pi^2 \frac{\mathrm{d}\lambda}{\mathrm{d}\ln Q} = 24\lambda^2 - (3g'^2 + 9g^2 - 12\lambda_t^2)\lambda + \frac{3}{8}g'^4 + \frac{3}{4}g'^2 g^2 + \frac{9}{8}g^4 - 6\lambda_t^4 + \dots \,. \tag{3.74}$$

In the limit of large Higgs mass (large λ), the first term dominates, making the Higgs mass a growing function of Q. The solution to the λ dominated RGE is

$$\lambda(Q) = \frac{m_H^2}{4v^2 - (3/2\pi^2) m_H^2 \ln(Q/\sqrt{2}v)}, \tag{3.75}$$

which presents a Landau pole at

$$Q = \sqrt{2}v \exp \frac{8\pi^2 v^2}{3m_H^2}. \tag{3.76}$$

Some form of new physics has to enter at energies smaller than this value in order to prevent the theory to explode. This gives a cut-off scale for the SM,

$$\Lambda_{\text{triviality}} = \sqrt{2}v \exp \frac{8\pi v^2}{3m_H^2}. \tag{3.77}$$

Equivalently, for a fixed value of the SM cut-off, this gives an upper bound on the Higgs mass. In particular, we cannot take the limit $\Lambda \to \infty$, since that leads to $\lambda = 0$ (trivial theory) for which no EWSB can occur.

Stability

Let us now consider the small Higgs mass limit. In that case it is not the quartic itself but the top Yukawa that dominates the RGE for λ. The negative sign in that case makes λ a decreasing function of Q. The energy dependence of the top Yukawa coupling is given by the RGE

$$16\pi^2 \frac{\mathrm{d}\lambda_t}{\mathrm{d}\ln Q} = \frac{9}{2}\lambda_t^3 + \dots \,. \tag{3.78}$$

The solution of the system of two RGEs gives,

$$\lambda_t^2(Q) = \frac{\lambda_{t,0}^2}{1 - (9/16\pi^2)\lambda_{t,0}^2 \ln(Q/Q_0)}, \tag{3.79}$$

$$\lambda(Q) = \lambda_0 - \frac{(3/8\pi^2)\lambda_{t,0}^4 \ln(Q/Q_0)}{1 - (9/16\pi^2)\lambda_{t,0}^2 \ln(Q/Q_0)}. \tag{3.80}$$

At large energies the top Yukawa drives λ to negative values. Again, some new physics must enter before $\lambda = 0$ to avoid the potential to be unbounded from below (in practice one could allow the potential to be unbounded from below provided we live in a long enough lived local minimum of the potential). This implies another cut-off for the SM,

$$\Lambda_{\text{stab.}} = \sqrt{2}v \exp \frac{4\pi^2 m_H^2}{6\lambda_t^4 v^2}. \tag{3.81}$$

Now for a fixed value of the cut-off, the Higgs mass cannot be smaller than some value as given by the previous equation.

These bounds have been summarized in Fig. 3.3 (taken from [15])

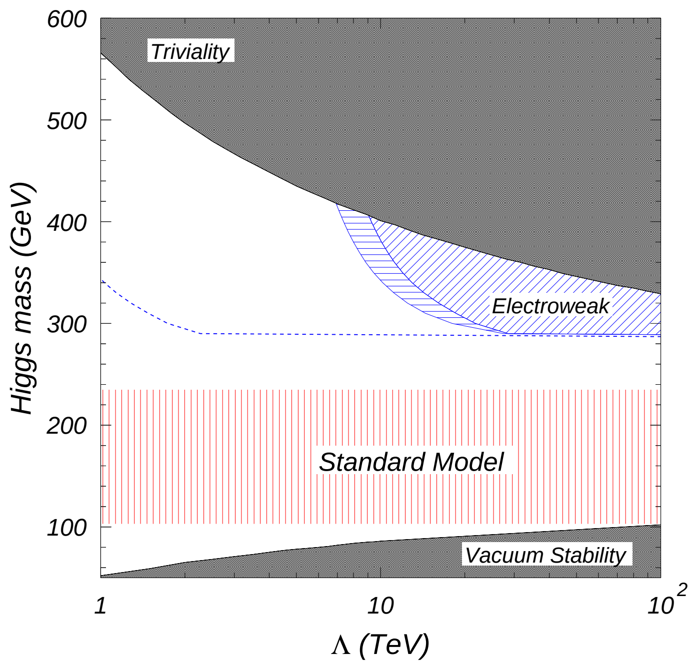


Figure 3.3: Bounds on the Higgs mass from triviality and stability. Figure taken from [15].

In the next few lectures, we will see that the presence of these cut-offs in the SM makes the SM realization of EWSB quite unnatural. This unnaturalness problem of the SM will motivate us to continue our quest in the search of alternative realizations of EWSB. Another motivation to study other realizations of EWSB beyond the SM is the fact that there is no dynamical mechanism for EWSB in the SM. The EW symmetry is broken because we put, by hand, a negative mass squared in the Higgs potential. Before jumping into other models of EWSB we will however review the amazing consistency of the SM with current experimental data and we will unveil the second crucial role the Higgs boson plays: consistency with electroweak precision tests. Any of the models of new physics that we will study along the course, apart from improving on the unnaturalness problem of the SM and, if possible, provide a dynamical mechanism for EWSB, they should also fulfill the important roles that the Higgs plays in the SM, restoration of unitarity and compatibility with EWPT.

3.4 The hierarchy problem

The hierarchy problem is based on the observation that *theories with light fundamental scalars are not natural*. It is usually stated as "Quantum corrections to the Higgs mass are quadratically divergent", or in equations,

$$\delta m_H^2 = \left[\frac{1}{4}(9g^2 + 2g'^2) - 6y_t^2 + 6\lambda\right]\frac{\Lambda^2}{32\pi^2}, \tag{3.82}$$

where y_t is the top Yukawa coupling (we have neglected the contribution from lighter fermions) and we have assumed a common cut-off Λ to regulate the momentum integrals. Thus, the natural value of the Higgs mass is the cut-off of the theory. In order for the EWSB scale v to be much lower than the cut-off we need a delicate cancellation between this quantum corrections and the bare parameters of the model. It is natural to assume that at least the Planck mass, which is the scale at which gravity becomes strong and quantum gravity effects are relevant, is a cut-off of the SM as we know it. If there is nothing but the SM between the scale of EWSB and the Planck mass, the bare parameters of the Higgs potential have to be adjusted to cancel the quantum corrections to one part in $\sim 10^{15}$! This is definitely not a satisfactory feature of the mechanism of EWSB in the SM.

We would like to stress that the hierarchy problem is a real problem that can be stated without any mention to cut-off or regularization dependence. One could naively argue that the quadratic divergence is an artificial result or the regularization used and that one can simply avoid it (and therefore prove that it is a bogus problem) by going to dimensional regularization, in no power divergences can appear. The precise statement of the hierarchy problem is that the mass of a fundamental scalar is *quadratically sensitive to high energy thresholds* and therefore, it is unnatural for a scalar to be much lighter than any other scale in the model. This is a statement about renormalized quantities and has nothing to do with the regularization method used. This feature is actually an example of a more general property, first discussed by 't Hooft, which says that a small parameter is technically natural if the system acquires a larger symmetry when such parameter is set to zero. An example is a massive fermion

$$\mathcal{L} = \bar{\psi} i \not{D} \psi + m\bar{\psi}\psi = \bar{\psi}_L i\not{D}\psi_L + \bar{\psi}_R i\not{D}\psi_R + m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L). \tag{3.83}$$

This theory has a global $U(1)$ symmetry $\psi \to e^{i\alpha}\psi$. However, if we take the mass to zero, there is a larger, chiral symmetry $\psi \to e^{i\gamma^5\alpha}\psi$. Thus, the fermion mass can be naturally much lighter than other scales in the theory. Technically what happens is that corrections to the fermion mass, that violate the chiral symmetry, have to proceed through a coupling that breaks the chiral symmetry. But the fermion mass itself is the only parameter that breaks the symmetry. Thus, quantum corrections to the fermion mass are proportional to the fermion mass itself and remain small if we started with a small tree level value. Another example is gauge boson masses, that have an associated gauge symmetry when the mass is taken to zero. There is however in general no symmetry associated to a massless scalar with non-derivative couplings and therefore light scalars are unnatural.

Let us study these features in a toy model. The model under consideration is a Yukawa type theory with two scalars and a fermion. One of the scalars is massless at tree level, while the other fields will have arbitrary masses. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(\partial_\mu \Phi)^2 + \bar{\psi} i \partial\!\!\!/ \psi - \frac{1}{2} m_\phi^2 \phi^2 - \frac{1}{2} m_\Phi^2 \Phi^2 - m_\psi \bar{\psi}\psi - \frac{1}{4}\lambda \phi^2 \Phi^2 - y_\phi \phi \bar{\psi}\psi - y_\Phi \Phi \bar{\psi}\psi. \tag{3.84}$$

In principle we could have written more terms in the Lagrangian but we just want to see how a heavy threshold affects differently scalar and fermion masses. Doing the calculation in dimensional regularization and the $\overline{MS}$ renormalization scheme we obtain at one loop, for the scalar mass,

$$\delta m_\phi^2 = \frac{y_\phi^2}{4\pi^2} m_\psi^2 \Big[1 - 2 \ln \frac{m_\psi^2}{\mu^2} + \mathcal{O}(m_\phi^2/m_\psi^2)\Big] - \frac{\lambda}{32\pi^2} m_\Phi^2 \Big[1 - \ln \frac{m_\Phi^2}{\mu^2}\Big]. \tag{3.85}$$

The correction to the fermion self-energy gives a fermion mass

$$\delta m_\psi = m_\psi \Big[\frac{5}{4} - \frac{3}{2} \ln \frac{m_\Phi^2}{\mu^2} + \mathcal{O}(m_\psi^2/m_\Phi^2)\Big] + (\Phi \to \phi). \tag{3.86}$$

Exercise: Compute the corrections in Eqs.(3.85) and (3.86). Hint: Compute the corresponding one loop self-energies in dimensional regularization; perform the Feynman parameter integrals; evaluate the self-energies at $p^2 = m^2$ and expand the result in inverse powers of large masses. The scalar self-energy has two types of contributions, one with the other scalar and one with the fermion running in the loop; the fermion has only one type of contribution.

We see that, if ψ or Φ are heavy, the ϕ mass is quadratically sensitive to these large scales, *even if we set $m_\phi = 0$ at tree level.* The fermion on the other hand, even if the scalars are extremely heavy, is only logarithmically sensitive to the heavy scale. Furthermore, the corrections to the fermion mass are proportional to the fermion mass itself (multiplicative renormalization) and therefore if we start with a light fermion, it will remain light after radiative corrections have been taken into account. Thus, fermion masses are stable against radiative corrections whereas scalar masses are unstable, quadratically sensitive to higher energy thresholds.

There is a related difference between scalars versus fermions or gauge bosons, which has to do with the number of degrees of freedom for massless versus massive particles with or without spin. This is summarized in Table 3.2 in which we see that massive fermions or gauge bosons have more degrees of freedom that the corresponding massless fields. These extra degrees of freedom have to come from somewhere to give mass to the corresponding field. This feature reflects the enhanced symmetry in the massless case. Scalars on the other hand have the same number of degrees of freedom independently of their mass. Thus, it is not much different to have massless or massive scalar, a footprint of their instability against radiative corrections.

Table 3.2: Number of degrees of freedom for massive versus massless particles of different spins and associated symmetry for the massless case

spin	massive	massless	symmetry
0	1	1	-
$\frac{1}{2}$	4	2	chiral
1	3	2	gauge

The hierarchy problem is a serious issue. One could try to argue that the SM being a renormalizable theory, it does not need to have a UV cut-off. It is not easy to reconcile that with the existance of the Planck mass, at which gravity becomes strongly coupled but even without resorting to gravity, we have seen that essentially for any given Higgs mass there is a UV cut-off due to triviality of stability. Thus, unless we are willing to assume a extremelly fine-tuned realization of EWSB, we should expect some new physics not far from the EW scale stabilizing the Higgs mass (or replacing the Higgs altogether). Before considering this possibility, it is useful to analyze the degree of compatibility of the SM with current experimental data. This will make explicit the second crucial role that the Higgs boson plays in the SM, namely compatibility with EWPT, and also with show how models of new physics can be constrained by current experimental data.

3.5 The Standard Model as an Effective Theory

In the previous lecture we saw that the EWSB sector of the SM is not fully satisfactory at the theoretical level and that one can confidently expect new physics to show up around the TeV scale to stabilize the EWSB scale. However, it is also true that, as a description of current experimental data, the SM is extremelly successful. In this lecture we will discuss how successful it is in a quantitative way and, at the same time, we will discuss how to test models of new physics against current experimental data. This is of course not a course on precision tests of the SM and therefore we will not go into the details of all observables that have been experimentally tested. We will be content with having a clear idea of which are the most constrained observables and a simple way of estimating bounds on the scale of new physics from electroweak precision tests (EWPT).

3.5.1 Precision tests of the Standard Model

The main experimental tests of the SM come from low energy (mainly leptons off nucleons) scattering data, precision CP, P or flavour violating experiments, e^+e^- scattering at and around the Z pole (LEP and SLC), e^+e^- scattering above the Z pole, up to energies ~ 200 GeV (LEP2) and Tevatron data ($p\bar{p}$ collider at 2 TeV center of mass energy).

The gauge sector of the SM has three independent parameters, g, g' and v. Including the scalar sector, we only have one further parameter, the quartic coupling λ, that can be exchanged by the Higgs mass m_H. Finally, adding the fermion sector, we have a large number of new parameters, the Yukawa couplings. However, only the top Yukawa coupling is reasonably large, all the other ones (or the relevant combinations appearing in the CKM matrix) are small as corresponds to their small masses and mixing angles. Thus, excluding flavour physics and very small parameters, the SM has five independent relevant parameters,

$$g, g', v, y_t, \lambda, \tag{3.87}$$

in terms of which, we can obtain definite predictions in the SM for any observable. It is customary to exchange these parameters with the observables that have been experimentally measured with the best precision, usually,

$$M_Z, G_F, \alpha_{em}, m_t, m_H, \tag{3.88}$$

which are respectively the Z mass, the Fermi Constant as measured in muon decay, the electromagnetic coupling constant, the top mass and the Higgs mass. Of course the Higgs mass has not been measured yet, but the fact that we have not found it yet means that it is heavy, which in turns implies that λ is large. Thus, we cannot neglect it as it can give important corrections to EW observables at the quantum level (this is true thanks to the extreme precision of the experimental data). We can then express, in the context of the SM, all other observables in terms of these five and compare these predictions with experimental data. The SM predictions have been computed to at least one loop (and in some observables to more than two loops) and a comparison with experimental data results in an excellent fit (see for instance [16]). The result of the fit can be summarized in the following points.

- The observables at the Z pole are measured typically with $\lesssim 10^{-3}$ relative precision. Thus, the quantum structure of the SM ($\sim 1/16\pi^2 \sim 10^{-2}$) is fully probed by these experimental data.

- Some of the observables that show some discrepancies are the total hadronic cross section at the Z pole, σ_h (pull=+2), and the forward-backwards asymmetry of $e^+e^- \to b\bar{b}$ at the Z pole, A^b_{FB} (pull=-2.7). This latter observable is somewhat problematic because the other observable that is also sensitive to the b couplings to the Z, $R_b \equiv \Gamma(b\bar{b})/\Gamma(\text{had})$, is in very good agreement with the SM prediction (pull=+0.8).

- The dependence of some observables on m_t is quadratic, therefore the top mass is strongly constrained indirectly by EWPT and the result agrees very well with direct measurements at Tevatron (although the latest measurements of m_t and m_W at the Tevatron start to show some tension). The dependence on the Higgs mass is only logarithmic, thus the sensitivity much weaker. Nevertheless, experimental data is so precise that a bound on the SM Higgs mass can be put

$$46\,\text{GeV} \le m_H \le 154\,\text{GeV}, \quad \text{(from EWPT at 90\% C.L. \textbf{in the SM})}. \tag{3.89}$$

  It is important however to emphasize that this bound only applies to the SM Higgs. One could have a heavier Higgs whose effects on EWP observables are compensated by some new physics.

Note: Alternatively this constraint can be seen as the **second crucial role the Higgs plays in the SM**, making it compatible with EWPT. The Higgs not only mediates longitudinal gauge boson gauge boson scattering to render it unitary but it also mediates quantum corrections to EW observables that are compatible with experiment is the Higgs is light enough (and no other physics beyond the SM is present).

- LEP2 data is far less precise ($\sim 10^{-2}$ relative precision) than the Z pole data but this is compensated by the gain in energy, $\sqrt{s} \le 209$ GeV and LEP2 data is very important as a test of new physics.

Another important piece of information is the unsuccessful Higgs search at LEP, that has put a limit on a SM Higgs mass [17]

$$m_H \ge 114.4\,\text{GeV} \quad \text{(for a SM Higgs at 95\% C.L. from direct searches)}. \tag{3.90}$$

Non-SM Higgses can be lighter than that limit, see [18] for a recent review, although there is a model-independent limit $m_H \ge 82$ GeV.

3.5.2 Constraints on new physics

Having seen how well the SM compares with experimental data, the next step is to learn how to perform a similar comparison of models of new physics. Since the SM agrees so well with experimental data, it is natural to assume that the new physics has a high enough typical scale for its effects to be a small correction to SM physics. In principle, one can take their favourite model of new physics, with their new parameters beyond the ones of the SM, and compute all precision observables in terms of these parameters. A fit to the EWP observables will then decide whether the particular model is excluded or compatible with current data. A more interesting way of doing this study, however, is to use Effective Lagrangians (EL) as a tool to parametrize physics beyond the SM. In this way, we can do the analysis for a completely general extension of the SM (with some mild assumptions) and perform the comparison with experiment. This will put bounds on the coefficients of the different operators appearing in the EL. We then only need to compute the values of such coefficients in our particular model and will automatically know the constraints on our model.

Effective theories are based on the idea that, at some particular energy, at which we are doing an observation (experiment), the details at much higher energies (shorter distances) are irrelevant for the description of the observation. Or said in a different way, when doing experiment with a typical wave-length, the details of the universe at much shorter distances are averaged out and can be simply parametrized by a number of unknown coefficients.

In our particular case, the idea is that, provided we pick up the right degrees of freedom and symmetries relevant at the energies at which we are doing experiment, we can parametrize the unknown short-distance physics by an infinite expansion of operators built with the chosen degrees of freedom and preserving the relevant symmetries,

$$\mathcal{L}_{\text{eff}} = \sum_d \mathcal{L}_d = \sum_d \sum_i \alpha_i^{(d)} \mathcal{O}_i^{(d)}, \tag{3.91}$$

where $\mathcal{L}_d$ is the sum of all operators with mass dimension d, $[\mathcal{O}_i^{(d)}] = d$. In D space-time dimensions the Lagrangian has mass dimension D. Thus, the coefficients of the expansion have $[\alpha^{(d)}] = D - d$. In particular, for $D = 4$, we have that the coefficients have mass dimension $[\alpha^{(d)}] = 4 - d$, which is negative for all operators of dimension higher than four.

Note: The mass dimension of the Lagrangian in a $D-$dimensional space time is D, since the action $S = \int d^D x \mathcal{L}$ is dimensionless. In an field theory in D dimensions, the mass dimension of the different fields can be found by looking at their kinetic terms, taking into account that a derivative has mass dimension 1 for arbitrary D. Thus, scalars and gauge bosons have mass dimension $[\phi] = [A_M] = D/2 - 1$ whereas fermions have mass dimension $[\psi] = (D-1)/2$. In particular, taking $D = 4$ we recover the usual mass dimension 1 for bosons and 3/2 for fermions. The mass dimension for the rest of the parameters in the Lagrangian can be obtained as outlined above.

This EL will parametrize physics up to some scale Λ at which a new threshold appears and our model has to be completed by a more fundamental theory that incorporates new states. Although it looks hopeless to try to do any physics with such a Lagrangian, that has an infinite number of terms, the reality is luckly quite the opposite. Recall that the coefficient of dimension higher than D have negative mass dimension and are therefore suppressed by the cut-off scale Λ. For instance in four dimensions we have

$$\alpha^{(d>4)} = \frac{a^{(d)}}{\Lambda^{d-4}}, \tag{3.92}$$

with $a^{(d)}$ a dimensionless constant which is expected to vary from $\sim 4\pi$ for strongly coupled theories to ~ 1 for weakly coupled theories that couple at tree level with the SM or even

$\sim 1/16\pi^2$ if the interaction only proceeds through quantum effects. Thus operators of higher dimensions will be suppressed by increasing powers of E/Λ, where E is the energy at which we are doing experiment. The crucial point is that if we are interested in describing the world around us, we are nevertheless limited by experimental precision and therefore we do not need theoretical predictions with infinite precision. Thus, as long as we only need some finite precision in our calculations, we can keep a finite number of operators in the EL, and throw away all operators with a dimension higher than the critical one to obtain the required precision. This allows us to cut the sum in d in Eq. (3.91). The number of possible operators of some dimension d, built with a finite number of fields is finite, although it can be very large if the number of fields or the dimension are large. Another important property is the fact that operators that are related by the classical equations of motion are redundant [19]. This means that, two operators that are related by the classical equation of motion give the same physics, even including quantum effects. Thus, we can reduce the sum over i in Eq. (3.91) by eliminating those operators that are redundant by the classical equations of motion.

Note that an EL is non-renormalizable as we have coefficients with negative mass dimensions (precisely the virture of EL at low energies). This means that an infinite number of counterterms will be required to renormalize the theory. This is obviously expected from the fact that our theory breaks down at high energies and furthermore already includes an infinite number of possible operators allowed by the symmetries (and therefore has an infinite number of counterterms). Our saviour is again the finite precision that we require on our predictions. Assuming that we only need a finite precision, we only need to keep a finite number of operators and higher order counterterms are irrelevant for the experimental predictions.

Given the agreement between the SM and experimental data, a very reasonable choice of relevant degrees of freedom and symmetries are those of the SM. The choice is whether we keep the Higgs boson or we work in the non-linear sigma model representation of the SM (known as the electroweak chiral Lagrangian). In both cases it is easy to compute the first few terms of the EL. For instance, assuming the SM with the Higgs boson in the linear representation, there is only one allowed operator of dimension five, assuming lepton number violation [20, 21, 22]

$$\mathcal{L}_5 = \frac{a^{(5)}}{\Lambda}\epsilon_{ij}\epsilon_{kl}\bar{l}^{ci}_R\phi^j\phi^l l^k_L + \text{h.c.}, \tag{3.93}$$

which gives a Majorana mass to neutrinos. Given the smallness of such masses, it is natural to assume approximate lepton and baryon number conservation as did W. Buchmüller and D. Wyler when they classified all possible non-redundant operators of dimension six [23]. The total number of such operators, up to flavour indices, is 81. One can then take these 81 operators and study their effects on EWPT. Such an exercise has been recently performed by several groups [24, 25, 26] (in the latter reference they make special emphasis in determining what the minimal set of operators is constrained by EWPT).

In this lecture we will be a bit less ambitious and will consider a simplifying assumption in order to have a smaller set of parameters. We follow [27] (although we will use a slightly different normalization of the fields) in assuming that new physics is **universal** in the following sense. We assume that there are some gauge boson fields $\bar{W}^I_\mu$ and $\bar{B}_\mu$ to which the light fermions couple as they do to the SM gauge bosons,

$$\mathcal{L} = -g\bar{W}^I_\mu J^\mu_I - g'\bar{B}_\mu J^\mu_Y + \dots\ . \tag{3.94}$$

We have defined the fermion currents as

$$J^\mu_I = \sum_f \psi^f_L \frac{\sigma^I}{2}\gamma^\mu \psi^f_L, \tag{3.95}$$

$$J^\mu_Y = \sum_f \psi^f Y_f \gamma^\mu \psi^f, \tag{3.96}$$

where the sum runs over all light fermions in the SM. Note that these interpolating gauge fields are not necessarily the SM gauge bosons but can have a component of new physics. The important feature, that is the very definition of these interpolating fields (and of the universality of new physics) is that the only gauge interactions (apart from QCD of the light fermions is the one given in Eq. (3.94). We will also assume that the threshold of new physics is far enough above the relevant energies that we can safely expand in powers of energy (or momentum). Finally we will just assume unbroken QED (and in particular electric charge conservation). With these assumptions all effects of new physics relevant for EWPT can be encoded in the self-energies of the interpolating fields. In the spirit of effective theories we can split these self-energies in two parts, one local tree level correction from new physics and another that contains all loop corrections from the SM fields. It is the first part that we are considering here (all effects of the SM, including quantum corrections, will be taken into account in the second calculable part). Furthermore, we can keep only the terms proportional to $\eta^{\mu\nu}$ in the two point function, as the terms proportional to the external momenta $p^\mu p^\nu$ will vanish or be negligible when contracted with conserved currents or currents built with light fermions. Thus, we can parametrize the most general, $U(1)_Q$ gauge invariant, universal Lagrangian by (we do not explicitly write the vector indices that are contracted with $\eta^{\mu\nu}$)

$$\mathcal{L} = -W^+\Pi_{+-}(p^2)W^- - \frac{1}{2}W^3\Pi_{33}(p^2)W^3 - W^3\Pi_{3B}(p^2)B - \frac{1}{2}B\Pi_{BB}(p^2)B. \tag{3.97}$$

Furthermore, the assumption that new physics occurs at a high scale allows us to expand the two point functions in a momentum expansion,

$$\Pi(p^2) = \Pi(0) + p^2\Pi'(0) + \frac{1}{2}(p^2)^2\Pi''(0) + \dots, \tag{3.98}$$

where we have kept only operators of dimension six or lower (recall that the self-energies have mass dimension 2). Thus, assuming we can keep only up to dimension six operators, we have $3 \times 4 = 12$ independent coefficients (the two point function and the first two derivatives at $p^2 = 0$ for each of the four combinations). Not all of those are however independent. First, three of these coefficients can be removed by canonically normalizing the fields. This corresponds to the determination of g, g' and v in the SM,

$$\begin{aligned} \Pi'_{+-}(0) &= \Pi'_{BB}(0) = 1, \\ \Pi_{+-}(0) &= -m_W^2 = -(80.425 \text{ GeV})^2. \end{aligned} \tag{3.99}$$

The remaining 9 parameters are not yet fully independent. The reason is that we have not yet required that $U(1)_Q$ is unbroken (other than electric charge conservation). Imposing conservation of the $U(1)$ group generated by $Q = T_3 + Y$ we obtain the following two consistency conditions

$$\begin{aligned} &{g'}^2\Pi_{33} + g^2\Pi_{BB} + 2gg'\Pi_{3B} = 0, \\ &g\Pi_{BB} + g'\Pi_{30} = 0. \end{aligned} \tag{3.100}$$

Exercise: Prove the two consistency conditions in Eq. (3.100). Hint: Assume the photon and the Z are an arbitrary unitary rotation of W^3 and B; determine the rotation angle in terms of g and g' by requiring that the photon couples to Q; then impose that the photon is massless and does not mix with the Z.

We are therefore left with $7 = 12 - 3 - 2$ coefficients that parametrize any new universal physics beyond the SM. A smart choice of these seven independent parameters is given in Table 3.3.

Table 3.3: Coefficients of the most general Lagrangian of new universal physics BSM. The expressions below are also valid when the normalization conditions, Eq. (3.99), have not been imposed.

Coefficients	Dimension-6 operator	$SU(2)_C$	$SU(2)_L$
$\hat{S} = \frac{g}{g'}\frac{\Pi'_{3B}(0)}{\Pi'_{+-}(0)}$	$(\phi^\dagger\sigma^I\phi)W^I_{\mu\nu}B^{\mu\nu}$	$+$	$-$
$\hat{T} = \frac{\Pi_{33}(0)-\Pi_{+-}(0)}{-\Pi_{+-}(0)}$	$\vert\phi^\dagger D_\mu\phi\vert^2$	$-$	$-$
$\hat{U} = \frac{\Pi'_{+-}(0)-\Pi'_{33}(0)}{\Pi'_{+-}(0)}$	Dim. 8	$-$	$-$
$V = \frac{-\Pi_{+-}(0)}{2}\left(\frac{\Pi''_{33}(0)-\Pi''_{+-}(0)}{\Pi'_{+-}(0)}\right)$	Dim. 10	$-$	$-$
$X = \frac{-\Pi_{+-}(0)}{2\Pi'_{+-}(0)}\frac{\Pi''_{3B}(0)}{\sqrt{\Pi'_{+-}(0)\Pi'_{BB}(0)}}$	Dim. 8	$+$	$-$
$Y = \frac{-\Pi_{+-}(0)}{2\Pi'_{+-}(0)}\frac{\Pi''_{BB}(0)}{\Pi'_{BB}(0)}$	$(\partial_\rho B_{\mu\nu})^2$	$+$	$+$
$W = \frac{-\Pi_{+-}(0)}{2\Pi'_{+-}(0)}\frac{\Pi''_{33}(0)}{\Pi'_{+-}(0)}$	$(D_\rho W^I_{\mu\nu})^2$	$+$	$+$

These coefficients are related to the Peskin-Takeuchi S, T, U paremeters [28] by

$$S = 4s_W^2 \hat{S}/\alpha_{em}, \quad T = \hat{T}/\alpha_{em}, \quad U = -4s_W^2 \hat{U}/\alpha_{em}. \tag{3.101}$$

We have also shown in the table which $SU(2)_L \times U(1)_Y-$invariant dimension 6 operator (assuming a fundamental Higgs) generates the corresponding operator and whether they preserve or violate custodial symmetry and $SU(2)_L$ symmetry. From the dimension of the different operators we see that there is a hierarchy between the different coefficients,

$$\hat{U} \sim \frac{m_W^2}{\Lambda^2}\hat{T}, \quad V \sim \frac{m_W^4}{\Lambda^4}\hat{T}, \quad X \sim \frac{m_W^2}{\Lambda^2}\hat{S}, \tag{3.102}$$

where we have assumed that operators preserving/breaking the same groups of symmetries are generated at a similar scale. Thus, in models with new universal physics, there are four oblique parameters that fully parametrize corrections to EWPT,

$$\hat{S}, \hat{T}, W, Y. \tag{3.103}$$

In particular, we have that, for universal physics, $\hat{T}$ is related to the ρ parameter,

$$\rho = 1 + \hat{T}, \quad \text{Universal physics.} \tag{3.104}$$

This could be expected from the fact that $\hat{T}$ is the only of the four parameters that violates custodial symmetry. A fit to these four coefficients was performed in [27] with the result shown in Table 3.4. The main conclusion is that these four coefficients are constrained to be $\lesssim 10^{-3}$. It is interesting to note that LEP data alone does not allow to constraint independently the four parameters, but only three combinations of them. It is LEP2 (less precise but higher energies) data that allows for an independent determination of the four parameters. Also note that the result of the fit depends on the Higgs mass. The reason is the logarithmic dependence of the different coefficients (mainly $\hat{S}$ and $\hat{T}$) on the Higgs mass (it is only the SM part that depends on the Higgs mass, but upon comparison with experiment, the constraint on new physics is modified accordingly). It is interesting to

Table 3.4: Fit to universal corrections to the SM

Fit	$10^3\hat{S}$	$10^3\hat{T}$	$10^3 W$	$10^3 Y$
115 GeV Higgs	0.0 ± 1.3	0.1 ± 0.9	0.1 ± 1.2	-0.4 ± 0.8
800 GeV Higgs	-0.9 ± 1.3	2.1 ± 1.0	0.0 ± 1.2	-0.2 ± 0.8

compute the leading dependence on m_t and m_H of the relevant self-energies that affect our four parameters. The leading top mass dependence is quadratic, and represents a nice example of non-applicability of the decoupling theorem. The dependence on the Higgs mass is only logarithmic as implied by the screening theorem [29],

$$\hat{S} = \frac{G_F m_W^2}{12\sqrt{2}\pi^2} \ln\left(\frac{m_h^2}{m_{h\,\mathrm{ref}}^2}\right) + \ldots, \quad \hat{T} = -\frac{3G_F m_W^2}{4\sqrt{2}\pi^2}\frac{g'^2}{g^2} \ln\left(\frac{m_h^2}{m_{h\,\mathrm{ref}}^2}\right) + \ldots\,. \tag{3.105}$$

Let us start with the m_t dependence of the ρ parameter. This correction has nothing to do with the gauge symmetry, but rather with the violation of custodial symmetry due to the difference between the top and bottom Yukawa couplings. In particular, it is due to the large value of the top Yukawa (and the small value of the bottom one). We can therefore do the calculation in the limit of large top mass. In that limit, the top does not decouple, because we are taking a dimensionless coupling very large (and the resulting effective theory is non-renormalizable). In this limit, it is the coupling to the Goldstone bosons that grows with the top Yukawa and therefore we can do the calculation in the gauge-less limit, given by the Lagrangian,

$$\mathcal{L}_{\text{gauge-less}} = \bar{\psi} i \not{D} \psi + |D_\mu \phi|^2 - V(\phi) - (\lambda_t q_L \tilde{\phi} t_R + \text{h.c.}), \tag{3.106}$$

where the gauge fields are not dynamical but just considered external classical sources. Recall Eq. (3.32) for the kinetic term of the scalar, including the Goldstone bosons,

$$\mathcal{L}_{\text{Kin}}(\pi_i) = Z_2^{(+)} |\partial_\mu \pi^+ - \frac{g}{\sqrt{2}} v W_\mu^+|^2 + \frac{1}{2} Z_2^{(3)} |\partial_\mu \pi^3 - \frac{g}{\sqrt{2} c_W} v Z_\mu|^2, \tag{3.107}$$

where we have now included arbitrary wave-function renormalization constants $Z_2^{(+)}$ and $Z_2^{(3)}$, for π^+ and π^3, respectively, which will be 1 at tree level but will receive corrections at higher orders in perturbation theory, while preserving the same covariant structure of the kinetic terms. Expanding the squares in (3.107) we get an all order result, in the gauge-less limit for the ρ parameter,

$$\rho = \frac{m_W^2}{m_Z^2 c_W^2} = \frac{Z_2^{(+)}}{Z_2^{(3)}}, \tag{3.108}$$

in terms of the ratio of the wave-function renormalization of the two eaten Goldstones. Thus, we only need to compute the wave-function renormalization of the Goldstone bosons which, to leading order, is given by the p^2 term in the expansion around $p^0 = 0$ (we are neglecting masses of order m_W against m_t) of the diagrams shown in Fig. 3.4. The

Figure 3.4: Goldstone boson self energies

self-energy for the charged Goldstone bosons reads, in dimensional regularization,

$$\begin{aligned}\mathcal{M}_+^2(p^2) &= \frac{3}{16\pi^2}\Big\{(\lambda_t^2+\lambda_b^2)\Big[-a(m_b)-a(m_t)+(p^2-m_b^2-m_t^2)B_0(p;m_b,m_t)\Big] \\ &\qquad + 4\lambda_t\lambda_b m_t m_b B_0(p;m_b,m_t)\Big\} \qquad (3.109)\\ &= \frac{3}{16\pi^2 v^2}m_t^2(p^2-m_t^2)B_0(p;0,m_t)+\ldots = \frac{3}{16\pi^2 v^2}m_t^2\left(\Delta+\frac{1}{2}-\ln\frac{m_t^2}{\mu^2}\right)p^2+\ldots\,,\end{aligned}$$

where we have used the corresponding Passarino-Veltman functions, which are described at the end of this lecture. In the second equality we have neglected terms proportional to m_b whereas in the last one we have expanded in powers of p^2 around $p^2=0$ and retained the term proportional to p^2, which is the one that will determine the wave-function renormalization. We have defined $\Delta\equiv 1/\epsilon-\gamma_E+\ln 4\pi$. The self-energy of the neutral Goldstone is

$$\begin{aligned}\mathcal{M}_3^2(p^2) &= -\frac{3}{16\pi^2}\lambda_t^2\Big[2a(m_t)-p^2B_0(p;m_t,m_t)\Big]+(m_t,\lambda_t\to m_b,\lambda_b)\\ &= \frac{3}{16\pi^2 v^2}m_t^2 p^2 B_0(p;m_t,m_t)+\ldots = \frac{3}{16\pi^2 v^2}m_t^2 p^2\left(\Delta-\ln\frac{m_t^2}{\mu^2}\right)+\ldots \qquad (3.110)\end{aligned}$$

Again in the first equality we have neglected terms proportional to the bottom mass (and those that do not contribute to the wave-fucntion renormalization of the Goldstone boson) and in the second one we have kept the term that renormalizes the wave function. Thus, we have for the ρ parameter

$$\rho=\frac{Z_2^{(+)}}{Z_2^{(3)}}\approx\frac{1+\frac{3}{16\pi^2}\lambda_t^2\left(\Delta+\frac{1}{2}-\ln\frac{m_t^2}{\mu^2}\right)}{1+\frac{3}{16\pi^2}\lambda_t^2\left(\Delta-\ln\frac{m_t^2}{\mu^2}\right)}\approx 1+\frac{3}{32\pi^2}\lambda_t^2=1+\frac{3}{8}\frac{G_F}{\sqrt{2}}\frac{m_t^2}{\pi^2}. \qquad (3.111)$$

Using similar methods it can be shown that another observable receiving quadratic m_t corrections is the coupling of the LH bottom to the Z. If we parametrize this coupling by

$$-\frac{g}{2c_W}[1-\frac{2}{3}s_W^2+\delta g_b]\bar{b}_L\gamma^\mu b_L Z_\mu, \qquad (3.112)$$

the leading correction to δg_b is (see for instance Pokorski's text book [1]),

$$\delta g_b=-\frac{G_F m_t^2}{4\pi^2\sqrt{2}}. \qquad (3.113)$$

Similarly, the dependence on the Higgs mass, can be obtained by computing the W^3-B self-energy, with a charged Goldstone running inside the loop for the $\hat{S}$ parameter and the self-energy of the charged Goldstones with a hypercharge gauge boson (and a charged Goldstone) running in the loop for the $\hat{T}$ parameter (note that we need a hypercharge gauge boson running in the loop as this is the only source of custodial violation in the gauge sector). For details of these calculations, see for instance [30].

3.5.3 Contribution of dimension 6 operators

Let us compute the contribution to the four relevant oblique parameters in the case of universal physics from the corresponding dimension 6 operators.

- $\hat{T}$

$$\begin{aligned}\mathcal{L}_{\hat{T}} &= \frac{\alpha_{\hat{T}}}{\Lambda^2}|\phi^\dagger D_\mu \phi|^2 = \frac{\alpha_{\hat{T}}}{\Lambda^2}\left| v^2 \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{g}{\sqrt{2}} W_\mu^+ \\ -\frac{g}{2} W_\mu^3 + \frac{g'}{2} B_\mu \end{pmatrix} \right|^2 \\ &= \frac{\alpha_{\hat{T}}}{\Lambda^2}\frac{v^4}{4}(gW_\mu^3 - g'B_\mu)^2 = \frac{\alpha_{\hat{T}}}{\Lambda^2}\frac{v^4}{4}[g^2 W_\mu^3 W_3^\mu + g'^2 B_\mu B^\mu - 2gg' W_\mu^3 B^\mu]\end{aligned} \tag{3.114}$$

Thus, we have

$$\hat{T} = \frac{\Pi_{33} - \Pi_{+-}}{m_W^2} = -\frac{\alpha_{\hat{T}}}{2}\frac{v^4 g^2}{\Lambda^2 m_W^2} = -\alpha_{\hat{T}}\frac{v^2}{\Lambda^2}. \tag{3.115}$$

- $\hat{S}$

$$\begin{aligned}\mathcal{L}_{\hat{S}} &= \frac{\alpha_{\hat{S}}}{\Lambda^2}\phi^\dagger \sigma^I \phi W_{\mu\nu}^I B^{\mu\nu} = -\alpha_{\hat{S}}\frac{v^2}{\Lambda^2} W_{\mu\nu}^3 B^{\mu\nu} \\ &= -2\alpha_{\hat{S}}\frac{v^2}{\Lambda^2}\partial_\mu W_\nu^3(\partial^\mu B^\nu - \partial^\nu B^\mu) = -2\alpha_{\hat{S}}\frac{v^2}{\Lambda^2} W_\nu^3 p^2 \eta^{\mu\nu} B_\mu + \dots\end{aligned} \tag{3.116}$$

Thus

$$\hat{S} = \frac{g}{g'}\Pi'_{3B} = 2\frac{g}{g'}\alpha_{\hat{S}}\frac{v^2}{\Lambda^2}, \tag{3.117}$$

- Y

$$\begin{aligned}\mathcal{L}_Y &= \frac{\alpha_Y}{\Lambda^2}(\partial_\rho B_{\mu\nu})^2 = -2\frac{\alpha_Y}{\Lambda^2}\partial_\mu B_\nu \partial^2 B^{\mu\nu} \\ &= 2\frac{\alpha_Y}{\Lambda^2}(p^2)^2 B_\nu \eta^{\mu\nu} B_\mu + \dots = -\frac{1}{2}\frac{(p^2)^2}{2}\left(-8\frac{\alpha_Y}{\Lambda^2}\right) B_\mu B^\mu + \dots\end{aligned} \tag{3.118}$$

Thus,

$$Y = \frac{m_W^2}{2}\Pi''_{BB} = -4\alpha_Y\frac{m_W^2}{\Lambda^2}. \tag{3.119}$$

- W (similar to Y)

$$\begin{aligned}\mathcal{L}_W &= \frac{\alpha_W}{\Lambda^2}(D_\rho W_{\mu\nu}^I)^2 = -2\frac{\alpha_W}{\Lambda^2} D_\mu W_\nu^I D^2 W^{I\,\mu\nu} + \dots \\ &= 2\frac{\alpha_W}{\Lambda^2}(p^2)^2 W_\nu^I \eta^{\mu\nu} W_\mu^I + \dots = -\frac{1}{2}\frac{(p^2)^2}{2}\left(-8\frac{\alpha_W}{\Lambda^2}\right) W_\mu^I W^{I\,\mu} + \dots\end{aligned} \tag{3.120}$$

Thus,

$$W = \frac{m_W^2}{2}\Pi''_{33} = -4\alpha_W\frac{m_W^2}{\Lambda^2}. \tag{3.121}$$

3.5.4 A Simple Example

As a simple example of universal physics let us consider the effect of a heavy copy of the hypercharge gauge boson,

$$\mathcal{L} = \mathcal{L}_{\rm SM} + \mathcal{L}_{B'}, \qquad (3.122)$$

where the relevant parts of the two Lagrangians can be schematically written as

$$\begin{aligned} \mathcal{L}_{\rm SM} &= -\frac{1}{2}W_3^\mu(p^2 - m_W^2)W_{3\,\mu} - t_0 m_W^2 W_3^\mu B_\mu \\ &- \frac{1}{2}B^\mu(p^2 - t_0^2 m_W^2)B_\mu + gJ_L^{3\,\mu}W_{3\,\mu} + g' J_Y^\mu B_\mu, \qquad (3.123) \\ \mathcal{L}_{B'} &= -\frac{1}{2}B'^{\,\mu}(p^2 - M^2)B'_\mu + g' J_Y^\mu B'_\mu. \qquad (3.124) \end{aligned}$$

we have defined $t_0 = s_0/c_0 = g'/g$ and the standard fermion currents $J_Y^\mu = \sum_\psi \bar{\psi} Y \gamma^\mu \psi$, $J_L^{i\,\mu} = \sum_\psi \bar{\psi} T_L^i \gamma^\mu \psi$. Let us proceed now along the standard route by integrating out the heavy boson. This can be done at tree level by introducing the solution of the classical equation of motion back in the Lagrangian and expanding in inverse powers of the heavy mass. The classical equation of motion for B' reads

$$(M^2 - p^2)B'_\mu + g' J_\mu^Y = 0 \Rightarrow B' = -g' \frac{J_Y^\mu}{M^2} + \mathcal{O}\left(\frac{p^2}{M^2}\right). \qquad (3.125)$$

Inserting it back in the Lagrangian and retaining only up to dimension 6 operators we get

$$\mathcal{L}_{B'} = -\frac{1}{2}g'^{\,2}\frac{J_\mu^Y(p^2 - M^2)J_Y^\mu}{M^4} - g' J_Y^\mu g' \frac{J_\mu^Y}{M^2} + \ldots = -\frac{1}{2}g'^{\,2}\frac{J_\mu^Y J_Y^\mu}{M^2} + \ldots \,. \qquad (3.126)$$

Thus, integrating out B' leads to four-fermion interactions. This four fermion interaction, however can be written in terms of purely oblique corrections. To see that, we can make use of the classical equations of motion for B,

$$g' J_Y^\mu = (p^2 - t_0^2 m_W^2)B^\mu + t_0 m_W^2 W_3^\mu. \qquad (3.127)$$

Using this, we can write the contact interaction as

$$-\frac{1}{2}g'^{\,2}\frac{J_Y^2}{M^2} = -\frac{1}{2}B\frac{(p^2 - t_0^2 m_W^2)^2}{M^2}B - \frac{1}{2}W_3\frac{t_0^2 m_W^4}{M^2}W_3 - t_0 m_W^2 W_3 \frac{(p^2 - t_0^2 m_W^2)}{M^2}B, \quad (3.128)$$

Adding this term to the SM Lagrangian we get

$$\begin{aligned} \mathcal{L}_6 &= -\frac{1}{2}W_3^\mu\Big[p^2 - m_W^2\big(1 - t_0^2\frac{m_W^2}{M^2}\big)\Big]W_{3\,\mu} - t_0 m_W^2 W_3^\mu\Big[\frac{p^2}{M^2} + 1 - t_0^2\frac{m_W^2}{M^2}\Big]B_\mu \\ &-\frac{1}{2}B^\mu\Big[p^2\big(1 - 2t_0^2\frac{m_W^2}{M^2}\big) - t_0^2 m_W^2\big(1 - t_0^2\frac{m_W^2}{M^2}\big) + \frac{p^4}{M^2}\Big]B_\mu \\ &+gJ_L^{3\,\mu}W_{3\,\mu} + g' J_Y^\mu B_\mu + \mathcal{O}(p^6) + \mathcal{O}(1/M^4). \qquad (3.129) \end{aligned}$$

The mass Lagrangian for the neutral gauge bosons reads,

$$\mathcal{L}_6^{\text{mass}} = \left(1 - t_0^2 \frac{m_W^2}{M^2}\right) \mathcal{L}_{SM}^{\text{mass}}, \tag{3.130}$$

which guarantees that it is diagonalized in exactly the same way as in the SM,

$$Z = c_0 W_3 - s_0 B, \quad A = s_0 W_3 + c_0 B, \tag{3.131}$$

with a massless photon and a *modified Z mass*,

$$m_Z^2 = \left(1 - t_0^2 \frac{m_W^2}{M^2}\right) m_{Z\,SM}^2 = \left(1 - t_0^2 \frac{m_W^2}{M^2}\right) \frac{m_W^2}{c_0^2}. \tag{3.132}$$

Since the rotation is the same and the couplings to fermions are also the same, the photon (the massless combination) couples to $Q = T_L^3 + Y$ as it should. In particular the W mass is not modified, so we have

$$\rho = 1 + \alpha T = \frac{m_W^2}{m_Z^2 c_0^2} = 1 + t_0^2 \frac{m_W^2}{M^2} + \ldots\,, \tag{3.133}$$

which is a direct contribution to the T parameter.

Note that with the use of the equation of motion for B we have been able to write the original effective Lagrangian with four-fermion interactions in a purely oblique form, *i.e.* corrections only to the gauge boson self-energies. The reason is that the model of new physics we have considered fall into the category of universal new physics. This is easy to see by noting that the fermion gauge couplings can be written in the form,

$$\Delta\mathcal{L} = g J_L^{3\,\mu} \bar{W}_{3\,\mu} + g' J_Y^\mu \bar{B}_\mu, \tag{3.134}$$

where we have defined the interpolating fields $\bar{W}_3 \equiv W_3$, $\bar{B} \equiv B + B'$. Following the recipe for models with universal new physics, we know that all the relevant corrections come then from the self energies of these interpolating fields. Thus, we need to compute the effective Lagrangian for $\bar{W}_3$ and $\bar{B}$, instead of the one for the original fields. In fact this is the Lagrangian we have computed (after the use of equation of motions for B to tranform the four-fermion interactions into purely oblique corrections). A simpler way to obtain the same lagrangian, in the case of universal physics, is to integrate out not the massive field but the combination of the massive field and the SM one that doesn't couple to fermions (this is usually called the holographic approached, after its use in models with extra dimensions with a holographic motivation). Let's do it that way. The full Lagrangian for the neutral gauge bosons can be written, in obvious notation,

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2} W_3 \Pi_{33} W_3 - W_3 \Pi_{3B} B - \frac{1}{2} B \Pi_{BB} B - \frac{1}{2} B' \Pi_{B'B'} B' + g J_3 W_3 + g J_y (B + B') \\ &= -\frac{1}{2} W_3 \Pi_{33} W_3 + g J_3 W_3 + g J_y B_+ - \frac{1}{2} W_3 \Pi_{3B} B_+ - \frac{1}{8} B_+ (\quad + \quad) B_+ \\ &\quad -\frac{1}{8} B_- (\quad + \quad) B_- - \frac{1}{4} B_+ (\quad - \quad) B_- - \frac{1}{2} W_3 \Pi_{3B} B_-,\end{aligned} \tag{3.135}$$

where we have defined $B_\pm \equiv B \pm B'$ and $(\ \pm\ ) \equiv \Pi_{BB} \pm \Pi_{B'B'}$. The classical EoM for B_- can be written as

$$B_- = -\frac{(\ -\ )B_+ + 2\Pi_{3B} W_3}{(\ +\ )}. \tag{3.136}$$

Inserting it back in the Lagrangian we get

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}W_3\Pi_{33}W_3 - W_3\Pi_{3B}B - \frac{1}{2}B\Pi_{BB}B - \frac{1}{2}B'\Pi_{B'B'}B' + gJ_3W_3 + gJ_y(B+B') \\ &= -\frac{1}{2}W_3\Pi_{33}W_3 + gJ_3W_3 + gJ_yB_+ - \frac{1}{2}W_3\Pi_{3B}B_+ - \frac{1}{8}B_+(\ +\ )B_+ \\ &\quad \frac{1}{8}\frac{[(\ -\ )B_+ + 2\Pi_{3B}W_3]^2}{(\ +\ )}.\end{aligned} \tag{3.137}$$

This effective Lagrangian, when expanded in powers of momenta, gives exactly Eq. (3.129). Such Lagrangian can be written in the standard form of universal new physics, Eq. (3.98) with

$$\Pi_{33} = -m_W^2\left(1 - t_0^2\frac{m_W^2}{M^2}\right) + p^2 + \ldots, \tag{3.138}$$

$$\Pi_{\bar{B}\bar{B}} = -m_W^2 t_0^2\left(1 - t_0^2\frac{m_W^2}{M^2}\right) + p^2\left(1 - 2t_0^2\frac{m_W^2}{M^2}\right) + \frac{p^4}{2}\frac{2}{M^2} + \ldots, \tag{3.139}$$

$$\Pi_{3\bar{B}} = m_W^2 t_0\left(1 - t_0^2\frac{m_W^2}{M^2}\right) + p^2 t_0\frac{m_W^2}{M^2} + \ldots\ . \tag{3.140}$$

We have not modified the charged current sector, so the two normalization conditions in Eq. (3.99) involving the charged sector are automatically satisfied. The one involving $\bar{B}$ is not, however and in order to get $\Pi'_{\bar{B}\bar{B}}(0) = 1$ we need to perform the following renormalization of the $\bar{B}$ field,

$$\bar{B} \to N_B\bar{B} \equiv \left(1 + t_0^2\frac{m_W^2}{M^2}\right)\bar{B}. \tag{3.141}$$

In particular this means that $g' \to N_B g'$, $\Pi_{\bar{B}\bar{B}} \to N_B^2\Pi_{\bar{B}\bar{B}}$ and $\Pi_{3\bar{B}} \to N_B\Pi_{3\bar{B}}$ (but note that in our expressions we still have $t_0 = g'/g$). In order to avoid confusion, we will write explicitly the factors of N_B and the two point functions refer to the ones we have written. With these new redefined two point functions, it is trivial to check that Eqs. (3.100), which states the preservation of QED, are satisfied. Also, we can now compute all the relevant oblique parameters. In particular we have,

$$\hat{T} = \frac{\Pi_{33}(0) - \Pi_{+-}(0)}{m_W^2} = \frac{\Pi_{33}(0)}{m_W^2} + 1 = -\left(1 - t_0^2\frac{m_W^2}{M^2}\right) + 1 = t_0^2\frac{m_W^2}{M^2}, \tag{3.142}$$

$$\hat{S} = \frac{g}{N_B g'}N_B\Pi'_{3\bar{B}}(0) = \frac{g}{g'}\Pi'_{3\bar{B}}(0) = \frac{g}{g'}t_0\frac{m_W^2}{M^2} = \frac{m_W^2}{M^2}, \tag{3.143}$$

$$Y = \frac{m_W^2}{2}N_B^2\Pi''_{\bar{B}\bar{B}}(0) = \frac{m_W^2}{2}\left(1 + 2t_0^2\frac{m_W^2}{M^2}\right)\frac{2}{M^2} = \frac{m_W^2}{M^2}, \tag{3.144}$$

$$W = \frac{m_W^2}{2}\Pi''_{33}(0) = 0. \tag{3.145}$$

Note that, in this particular case and at the order we are working, the canonical normalization of $\bar{B}$ didn't actually have any effect. Also note recall $\hat{T} = \alpha T$ so indeed we reproduce our previous calculation of the T parameter.

3.5.5 Summary

We have seen in this section that the SM is a very good description of the available experimental data. We have discussed how to use EWPT to constraint models of new physics and seen that in general, very small deviations are allowed. This has lead to the so called little hierarchy problem. New physics is required by naturalness to be below ~ 1 TeV in order to stabilize the EWSB scale, however, EWPT push the scale of new physics closer to the multi-TeV scaler. This means that, either the new physics has some special properties that somehow hides their effects at low energies while still being able to stabilize the EWSB scale or new fine-tuning is reintroduced in the theory. Successful models of new physics typically have such property of hiding large effects. Custodially preserving new physics is a very simple example. Theories with a discrete symmetry that forbids tree-level interations between new physics and the SM particles (like R parity in SUSY or T parity in Little Higgs models) prevents large tree level corrections while still allowing for a resolution of the hierarchy problem -which is occurs at the quantum level-. Nevertheless, most of models of new physics, still suffer from some amount of fine-tuning, which is typically of the order of $\sim$ few % (thus the name little hierarchy, as opposed to the $\sim 10^{-15}$ fine-tuning in the SM).

In the following lectures we will overview some ideas on how to solve the hierarchy problem and discuss their successes and weaknesses.

Note: Some properties of Passarino-Veltman functions. We follow the discussion in [31]. The one and two point PV functions can be defined as:

$$\frac{i}{16\pi^2}A(m_1) = \mu^{2\epsilon}\int \frac{d^nk}{(2\pi)^n}\frac{1}{N_1}, \tag{3.146}$$

$$\frac{i}{16\pi^2}[B_0, B^\mu, B^{\mu\nu}](12) = \mu^{2\epsilon}\int \frac{d^nk}{(2\pi)^n}\frac{[1, k^\mu, k^\mu k^\nu]}{N_1 N_2}, \tag{3.147}$$

where $n = 4 - 2\epsilon$ and

$$N_1 = k^2 - m_1^2 + \mathrm{i}\epsilon, \tag{3.148}$$

$$N_2 = (k + p_1)^2 - m_2^2 + \mathrm{i}\epsilon. \tag{3.149}$$

The exact result for A and B_0 read, using $\Delta \equiv \frac{1}{\epsilon} - \gamma_E + \ln(4\pi)$,

$$A(m) = m^2[\Delta + 1 - \ln(m^2/\mu^2)], \tag{3.150}$$

and

$$\begin{aligned} B_0(q^2; m_1, m_2) &= \Delta - \ln\frac{m_1 m_2}{\mu^2} + 2 \\ &+ \frac{1}{q^2}\Big[(m_2^2 - m_1^2)\ln\frac{m_1}{m_2} \\ &+ \sqrt{\lambda(q^2 + i\epsilon, m_1^2, m_2^2)}\mathrm{ArcCosh}[\frac{m_1^2 + m_2^2 - q^2 - i\epsilon}{2m_1 m_2}]\Big], \end{aligned} \tag{3.151}$$

where we have used the definitions,

$$\lambda(a, b, c) \equiv a^2 + b^2 + c^2 - 2ab - 2ac - 2bc, \tag{3.152}$$

and

$$\mathrm{ArcCosh}(z) = \ln(z + \sqrt{z^2 - 1}). \tag{3.153}$$

We can also write the higher order two point functions in terms of these ones. For instance

$$B^\mu = \frac{p_1^\mu}{2p^2}\left[A(m_1) - A(m_2) + (m_2^2 - m_1^2 - p^2)B_0(p^2; m_1, m_2)\right]. \tag{3.154}$$

Some important expansions of B_0 are

$$B_0(m^2, m, M) = 1 - \ln\frac{M^2}{\mu^2} + \left(\frac{1}{2} - \ln\frac{M^2}{m^2}\right)\frac{m^2}{M^2} + \mathcal{O}\left(\frac{m^4}{M^4}\right) \tag{3.155}$$

$$B_0(m^2, M, M) = -\ln\frac{M^2}{\mu^2} + \frac{m^2}{6M^2} + \mathcal{O}\left(\frac{m^4}{M^4}\right). \tag{3.156}$$

Chapter 4

Electroweak Symmetry Breaking in Supersymmetric Models

Note: Supersymmetric theories are very well covered in many textbooks and review articles. We will follow in the lectures mainly Martin's Supersymmetry Primer [32]. In fact, we are not going to produce lecture notes for the supersymmetry lectures. Instead, we will mention what is being explained in each lecture and where it can be found in Martin's review. (The references will be to v5 of the review.)

4.1 Motivation for Supersymmetry

Where: Section 1 of the primer

Contrary to scalars with non-derivative interactions, fermion masses are protected by chiral symmetry so that light fermions are natural. One possible way of solving the hierarchy problem is to use a symmetry that relates scalars to fermions, so that the chiral protection is extended to the Higgs boson. Such a possibility is actually highly non-trivial and the resulting symmetry, called supersymmetry, is very constrained in models with a non-trivial S-matrix. Once we have a symmetry that relates scalars to fermions, or more generally, bosons to fermions. It is easy to see how loop contributions to the Higgs mass receive contributions from both particles and their superpartners and the two contributions can exactly cancel. This is a good motivation to investigate further the implications of supersymmetry.

4.2 Formalism of Supersymmetry: Building Supersymmetric Lagrangians

After a review of two component Weyl spinor notation for fermions in four dimensions (section 2 of the primer) we go on to build supersymmetric Lagrangians.

4.2.1 The Simplest Supersymmetric Lagrangian: Non-interacting Wess-Zumino Model

Where: Section 3.1 of the primer.

We start with a Weyl fermion and a massless complex scalar (2 degrees of freedom each), assume the simplest possible supersymmetric transformation for the scalar

$$\delta\phi = \epsilon\psi, \quad \delta\phi^* = \epsilon^\dagger\psi^\dagger, \tag{4.1}$$

with ϵ an infinitesimal symmetry parameter, which is a spinor with mass dimension $-1/2$, and guess what the supersymmetry transformation of the fermion should be so that the Lagrangian is invariant up to a total derivative.

We then have to check that the supersymmetry algebra closes, which means that the commutator of two succesive supersymmetric transformations is a symmetry of the Lagrangian. It turn out that the commutator is proportional to the four-momentum up to the equations of motion for the fermion. That means that the algebra closes on-shell but not off-shell (which makes sense because off-shell, the fermion has four degrees of freedom and not the two of an off-shell complex scalar). This is fixed by introducing a non-propagating auxiliary (F) field (a complex scalar -2 dof- with mass dimension -2) whose supersymmetric transformation is proportional to the equations of motion of the fermion. This way the supersymmetry algebra also closes off-shell. This forms the chiral supermultiplet, that contains a massless scalar and a massless fermion.

4.2.2 Non-gauge Interactions of Chiral Multiplets

Where: Section 3.2 of the primer.

The most general renormalizable Lagrangian involving chiral supermultiplets that is invariant under the supersymmetry transformation (up to total derivatives) can be obtained in terms of a holomorphic function of the scalar components (non-auxiliary) of the chiral supermultiplets involved. By holomorphic we mean that it is a function of ϕ_i and not of ϕ^{*i}. This function is called the superpotential,

$$W = L^i\phi_i\frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k. \tag{4.2}$$

The superpotential has mass dimension 3 and determines the interactions as follows (after integrating out the auxiliary fields)

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}\left(W^{ij}\psi_i\psi_j + W^*_{ij}\psi^{\dagger\, i}\psi^{\dagger\, j}\right) - W^i W^*_i, \tag{4.3}$$

where

$$W_{ij\ldots k} \equiv \frac{\delta W}{\delta\phi_i\delta\phi_j\ldots\delta\phi_k}. \tag{4.4}$$

In particular, this determines (part of) the scalar potential.

4.2.3 Lagrangians for Gauge Multiplets

Where: Sections 3.3 and 3.4 of the primer.

A supersymmetric Lagrangian for gauge bosons contains a number of gauge bosons transforming in the adjoint of some Lie group, a similar number of fermions also transforming in the adjoint (2 dof for each type of field per element of the adjoint) plus the same number of real scalar auxiliary fields to close the supersymmetry algebra off-shell. Once we introduce the auxiliary fields and write reasonable supersymmetric transformations fo the different fields (taking into account reality of the adjoint representation and that gauge invariance has to be preserved) it is easy to check the invariance of the kinetic Lagrangian for the different fields (that involves covariant derivatives, both in the Lagrangian and in the supersymmetric transformations and therefore includes gauge couplings) and that the supersymmetry algebra closes.

In order to get gauge interactions for chiral supermultiplets, we transform normal derivatives into covariant derivatives (both in the Lagrangian and the supersymmetry transformations). This is not enough however to obtain a supersymmetric Lagrangian. The reason is that this only couples the gauge boson members of the vector multiplet with the chiral multiplets but supersymmetry relates the gauge boson to the gauginos and the D^a auxiliary fields. Thus, we need to include direct couplings between those and the fermions and sfermions of the chiral supermultiplet. There are only three gauge invariant possible couplings, whose coefficients are fixed by the requirement of a supersymmetry invariant Lagrangian. In the process, an extra term in the supersymmetric transformation of the F auxiliary fields of the chiral supermultiplets involving the gauginos is required.

With that we have all renormalizable interactions compatible with gauge invariance and supersymmetry. Integrating out the D^a auxiliary fields we get a new $D-$term contribution to the scalar potential which is a quartic term with gauge couplings as coupling constant.

4.2.4 The hierarchy problem in SUSY

Let us see how supersymmetry solves the hierarchy problem. We consider as an example the top mass contribution to the Higgs mass squared. Supersymmetry requires a superpartner for each SM particle.

$$q_L \leftrightarrow \tilde{q}_L = (\tilde{t}_L, \tilde{b}_L), \quad t_R \leftrightarrow \tilde{t}_R. \tag{4.5}$$

Let us consider the following interaction Lagrangian

$$\mathcal{L}_{\text{int}} = -(\lambda_F \bar{t}_R \tilde{\phi}^\dagger q_L + \text{h.c.}) + \lambda_L |\tilde{\phi}^\dagger \tilde{q}_L|^2 + \lambda_R |\tilde{t}_R \phi|^2. \tag{4.6}$$

To this Lagrangian we could add (and we will in the next section) other terms that are gauge invariant but supersymmetry breaking

$$\mathcal{L}_{\text{soft}} = \lambda_{LR}(\tilde{t}_R \tilde{q}_L^i \epsilon^{ij} \phi^j + \text{h.c.}) + \tilde{q}_L^\dagger \tilde{q}_L m_L^2 + \tilde{t}_R^\dagger \tilde{t}_R m_R^2. \tag{4.7}$$

Let us assume for simplicity that $\lambda_{LR} = 0$. The one loop correction to the Higgs mass is given by

$$\begin{aligned} \delta m_H^2 &= -\frac{|\lambda_F|^2 2N_c}{16\pi^2}[\Lambda^2 - 6m_F^2 \log\frac{\Lambda}{m_F} + 2m_F^2] \\ &\quad - \sum_{s=L,R}\left\{\frac{\lambda_s N_c}{16\pi^2}[-\Lambda^2 + 2m_s^2 \log\frac{\Lambda}{m_s}] + \frac{(\lambda_s v)^2 N_c}{16\pi^2}[-1 + 2\log\frac{\Lambda}{m_s}]\right\}, \end{aligned} \tag{4.8}$$

where $m_F \equiv \lambda_F v/\sqrt{2}$. If we have $\lambda_L = \lambda_R = |\lambda_F|^2$, as required by SUSY, we get an exact cancellation of quadratic divergencies,

$$\begin{aligned} \delta m_H^2 &= -\frac{2N_c|\lambda_F|^2}{16\pi^2}\Big\{(m_L^2 + m_R^2 - 2m_F^2)\log\frac{\Lambda}{m_L} \\ &\qquad - 6m_F^2 \log\frac{m_L}{m_R} + (m_R^2 + 2m_F^2)\log\frac{m_L}{m_R}\Big\}. \end{aligned} \tag{4.9}$$

In fact, if SUSY was exactly unbroken ($\mathcal{L}_{\text{soft}} = 0$), then we have $m_L = m_R = m_F$ and the whole contribution cancels exactly

$$\delta m_H^2 = 0 \text{ (unbroken SUSY)}. \tag{4.10}$$

However $m_L = m_R = m_F$ is not allowed phenomenologically but we have seen that we can add SUSY breaking terms that do not spoil the cancellation of quadratic divergencies (soft SUSY breaking terms). If $m_F \ll m_L \sim m_R$ and defining $m_{\tilde{t}}^2 \equiv (m_L^2 + m_R^2)/2$, we get

$$\delta m_H^2 \approx -\frac{2N_c}{16\pi^2}|\lambda_t|^2 m_{\tilde{t}}^2 \log\frac{\Lambda^2}{m_{\tilde{t}}^2}, \tag{4.11}$$

so that we see that physically, the quadratic divergence is cut-off by the stop mass.

4.2.5 Soft Supersymmetry Breaking Interactions

Where: Section 4 of the primer

Supersymmetry requires degeneracy between members of a supermultiplet. However, we have not observed experimentally the superpartners of the SM particles. This means that supersymmetry must be spontaneously broken. We can proceed to study models of spontaneous supersymmetry breaking or we can parametrize all possible supersymmetry breaking terms that do not spoil the cancellation of quadratic contributions to the Higgs potential. These are the so-called soft supersymmetry breaking terms, and correspond to operators with mass dimension smaller than four (i.e. involving dimensionful coupling constants). The soft supersymmetry breaking Lagrangian of a general theory has the form

$$\mathcal{L}_{\text{soft}} = -\left(\frac{1}{2}M_a\lambda^a\lambda^a + \frac{1}{6}a^{ijk}\phi_i\phi_j\phi_k + \frac{1}{2}b^{ij}\phi_i\phi_j + t^i\phi_j\right) + \text{h.c.} - (m^2)^i_j\phi^{*\,j}\phi_i, \tag{4.12}$$

which are enough to give masses to all superpartners. One operator that is formally soft but that might give rise to quadratic divergencies in the Higgs mass is

$$-\frac{1}{2}c_i^{jk}\phi^{*\,i}\phi_j\phi_k + \text{h.c.}, \tag{4.13}$$

although in models of spontaneous supersymmetry breaking it is typically negligible small.

4.2.6 The minimal supersymmetric standard model

In the SM we have just one scalar doublet ϕ, that gives masses to both $T^3 = \pm 1/2$ components. This is done by coupling one of the fields to ϕ and one to ϕ^*. In SUSY however, non-gauge couplings come from the superpotential, which is a holomorphic function. That means that we can couple the fields to either ϕ or ϕ^*, but not to both. We thus need two scalar doublets in the minimal supersymmetric extension of the SM,

$$\phi_u = \begin{pmatrix} H_u^0 \\ H_u^- \end{pmatrix} \sim (2)_{-1/2}, \quad \phi_d = \begin{pmatrix} H_d^+ \\ H_d^0 \end{pmatrix} \sim (2)_{+1/2}. \tag{4.14}$$

This constraint can be also understood from the fact that scalars in SUSY bring along the corresponding superpartners, which are Weyl fermions and therefore contribute to anomalies. Adding two doublets with opposite hypercharges provides the simplest way of cancelling such anomalies. Apart from this subtlety, the MSSM corresponds to the obvious supersymmetryzation of the SM.

The MSSM superpotential reads

$$W_{MSSM} = -\bar{u}y_u QH_u - \bar{d}y_d QH_d - \bar{e}y_e LH_d + \mu H_u H_d. \tag{4.15}$$

Note in particular that there is not $HH\tilde{f}$ term allowed by gauge invariance and therefore not quartic Higgs coupling is generated from the superpotential.

The soft supersymmetry breaking Lagrangian in the MSSM reads

$$\begin{aligned}
\mathcal{L}_{\text{soft}}^{\text{MSSM}} &= -\frac{1}{2}(M_3\tilde{g}\tilde{g} + M_2\tilde{W}\tilde{W} + M_1\tilde{B}\tilde{B} + \text{h.c.}) \\
&\quad -(\tilde{\bar{u}}a_u\tilde{Q}H_u + \tilde{\bar{d}}a_d\tilde{Q}H_d - \tilde{\bar{e}}a_e\tilde{Q}H_d + \text{h.c.}) \\
&\quad -\tilde{Q}^\dagger m_Q^2\tilde{Q} - \tilde{L}^\dagger m_L^2\tilde{L} - \tilde{\bar{u}}m_{\bar{u}}^2\tilde{\bar{u}}^\dagger - \tilde{\bar{d}}m_{\bar{d}}^2\tilde{\bar{d}}^\dagger - \tilde{\bar{e}}m_{\bar{e}}^2\tilde{\bar{e}}^\dagger \\
&\quad -m_{H_u}^2H_u^*H_u - m_{H_d}^2H_d^*H_d - (bH_uH_d + \text{h.c.}).
\end{aligned} \tag{4.16}$$

Let us look more in detail at the Higgs potential. We have two scalar (complex) doublets, 8 dof in total. We will break the $SU(2)_L \times U(1)_Y$ gauge symmetry to $U(1)_Q$, eating three Goldstone bosons in the way, thus we are left with 5 real physical scalars in the spectrum (from the Higgs sector). The Higgs potential can be written as

$$\begin{aligned}
V &= (|\mu|^2 + m_{H_u}^2)(|H_u^0|^2 + |H_u^+|^2) + (|\mu|^2 + m_{H_d}^2)(|H_d^0|^2 + |H_d^-|^2) \\
&+ [b(H_u^+H_d^- - H_u^0H_d^0) + \text{h.c.}] + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 + |H_u^+|^2 - |H_d^0|^2 - |H_d^-|^2)^2 \\
&+ \frac{1}{2}g^2|H_u^+H_d^{0\,*} + H_u^0H_d^{-\,*}|^2.
\end{aligned} \tag{4.17}$$

Using $SU(2)_L$ rotations we can always fix $\langle H_u^+\rangle = 0$. We then have

$$\begin{aligned}
0 &= \frac{\partial V}{\partial H_d^+} = (|\mu|^2 + M_{H_d}^2)H_d^- + \frac{1}{4}(g^2 + g'^2)(H_u^0|^2 - |H_d^0|^2 - |H_d^-|^2)(-H_d^-) + \frac{g^2}{2}|H_u^0|^2H_d^- \\
&= H_d^-[(|\mu|^2 + M_{H_d}^2) - \frac{1}{4}(g^2 + g'^2)(H_u^0|^2 - |H_d^0|^2 - |H_d^-|^2) + \frac{g^2}{2}|H_u^0|^2],
\end{aligned} \tag{4.18}$$

so that $\langle H_d^-\rangle = 0$ is a minimum of the potential for which QED remains unbroken. Using these two assumptions we are left with the following potential

$$V = (|\mu|^2 + m_{H_u}^2)|H_u^0|^2 + (|\mu|^2 + m_{H_d}^2)|H_d^0|^2 - [bH_u^0H_d^0 + \text{h.c.}] + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 - |H_d^0|^2)^2. \tag{4.19}$$

The only complex parameter is b, but we can make it real by absorbing its phase in H_u^0 or H_d^0. Thus, we can take b real and positive. But then we have

$$0 = \frac{\partial V}{\partial H_u^0} = [|\mu|^2 + m_{H_u}^2 + \frac{g^2 + g'^2}{4}(|H_u^0|^2 - |H_d^0|^2)]H_u^{0\,*} - bH_d^0, \tag{4.20}$$

which implies that the phases of $H_{u,d}^0$ are opposite and can therefore be eliminated by a $U(1)_Y$ transformation (recall the two Higgses have opposite hypercharges). Thus we can take

$$\langle H_{u,d}^0\rangle = v_{u,d}, \tag{4.21}$$

real and positive. This means that the Higgs potential is CP conserving and therefore we can classify the 5 physical scalars according to their CP properties.

The positive quartic coupling ensures stability except for the D-flat direction $v_u = v_d$. Positivity of the potential along that flat direction requires

$$2b < 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2. \tag{4.22}$$

Similarly, requiring that a combination of H_u^0 and H_d^0 gets a negative mass squared at $H_u^0 = H_d^0 = 0$ implies

$$b^2 > (|\mu|^2 + m_{H_u}^2)(|\mu|^2 + m_{H_d}^2). \tag{4.23}$$

In particular, for $m_{H_u} = m_{H_d}$ the two conditions are incompatible. A particular case is a supersymmetric theory (no SUSY breaking) for with both vanish. Thus, if SUSY is unbroken, EWSB does not occur.

Let us denote

$$\tan\beta \equiv \frac{v_u}{v_d}. \tag{4.24}$$

We have

$$v_u^2 + v_d^2 = v^2 = (174 \text{ GeV})^2 = \frac{2m_Z^2}{g^2 + g'^2}. \tag{4.25}$$

$\tan\beta$ is currently not fixed by experiment. The different parameters are related by the condition of minimum of the potential

$$\partial_{H_u^0} V = 0 \quad \Rightarrow \quad m_{H_u}^2 + |\mu|^2 - b\cot\beta - \frac{m_Z^2}{2}\cos 2\beta = 0, \tag{4.26}$$

$$\partial_{H_d^0} V = 0 \quad \Rightarrow \quad m_{H_d}^2 + |\mu|^2 - b\tan\beta + \frac{m_Z^2}{2}\cos 2\beta = 0. \tag{4.27}$$

In particular we have

$$\sin 2\beta = \frac{2b}{m_{H_u}^2 + m_{H_d}^2 + 2|\mu|^2}, \tag{4.28}$$

and

$$m_Z^2 = \frac{|m_{H_d}^2 - m_{H_u}^2|}{\sqrt{1 - \sin^2 2\beta}} - m_{H_u}^2 - m_{H_d}^2 - 2|\mu|^2, \tag{4.29}$$

which exemplify the so-called μ problem, which stands for the fact that all these quantities, $|\mu|^2, m_{H_{u,d}}^2, b$ seem to have to be of the same order but the former is SUSY preserving whereas the latter three are SUSY breaking and therefore they are expected to have very different origins.

We can write the two scalar doublets in terms of mass and CP eigenstates

$$\begin{pmatrix} H_u^0 \\ H_d^0 \end{pmatrix} = \begin{pmatrix} v_u \\ v_d \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} c_\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix} \begin{pmatrix} h^0 \\ H^0 \end{pmatrix} + \frac{i}{\sqrt{2}} \begin{pmatrix} s_\beta & c_\beta \\ -c_\beta & s_\beta \end{pmatrix} \begin{pmatrix} G^0 \\ A^0 \end{pmatrix}, \tag{4.30}$$

and

$$\begin{pmatrix} H_u^+ \\ H_d^{-\,*} \end{pmatrix} = \begin{pmatrix} s_\beta & c_\beta \\ -c_\beta & s_\beta \end{pmatrix} \begin{pmatrix} G^+ \\ H^+ \end{pmatrix}, \tag{4.31}$$

where $G^{\pm,0}$ are the eaten Goldstones, h^0 and H^0 are CP even neutral scalars, A^0 is a CP odd neutral scalar and H^+ is a CP even charge (charge +1) scalar. The resulting mass eigenstates are

$$\begin{aligned} m_{G^{\pm,0}} &= 0, && (4.32)\\ m^2_{A^0} &= \frac{2b}{\sin 2\beta} = 2|\mu|^2 + m^2_{H_u} + m^2_{H_d}, && (4.33)\\ m^2_{h^0,H^0} &= \frac{1}{2}(m^2_{A^0} + m^2_Z \mp \sqrt{(m^2_{A^0} - m^2_Z)^2 + 4m^2_Z m^2_{A^0}\sin^2 2\beta}), && (4.34)\\ m^2_{H^\pm} &= m^2_{A^0} + m^2_W, && (4.35) \end{aligned}$$

and the rotation angles are related by

$$\frac{\sin 2\alpha}{\sin 2\beta} = -\frac{m^2_{H_0} + m^2_{h^0}}{m^2_{H_0} - m^2_{h^0}}, \quad \frac{\tan 2\alpha}{\tan 2\beta} = \frac{m^2_{A^0} + m^2_Z}{m^2_{A^0} - m^2_Z}. \tag{4.36}$$

From these equations we see that $m_{A^0,H^0,H^\pm}$ can be arbitrarily large, as they grow with $b/\sin 2\beta$. m_{h^0} on the other hand is bounded from above

$$m_{h^0} \leq m_Z|\cos 2\beta|. \tag{4.37}$$

This is a tree level result but already shows one of the problems with supersymmetric theories (or at least minimal ones). The fact that LEP didn't find the Higgs is, in most or parameter space, incompatible with this inequality. loop contributions modify this constraint, making it still compatible with LEP bounds, if only at the prize of some amount of fine-tuning.

Let us consider for simplicity the decoupling limit $m_{A^0} \gg m_Z$. In that case we also have $m_{H^0,H^\pm} gg m_Z$. In that limit the only light scalar is h^0, which behaves like the SM Higgs. We thus obtain the following contribution from the top/stop loops

$$m^2_{h^0} \approx m^2_Z \cos^2 2\beta + \frac{N_c}{4\pi}\frac{m_t^4}{v^2}\log\frac{m^2_{\tilde{t}}}{m^2_t}, \tag{4.38}$$

where we have neglected for simplicity the mixing between $\tilde{t}_{L,R}$ and have taken $m_{\tilde{t}_L} = m_{\tilde{t}_R} = m_{\tilde{t}}$. In this expression we have assumed EWSB (so that the physical Higgs mass squared is determined by v, $\tan\beta$ and the quartic coupling). In the best case scenario we get

$$m_{h^2} \lesssim 135 \text{ GeV}, \tag{4.39}$$

assuming $m_{\tilde{t}} \lesssim$ TeV. Even extending the spectrum, as long as the new particles are below around one TeV and couplings remain perturbative all the way to the unification scale, it is difficult to get beyond ~ 150 GeV. The reason we have required the stop mass to be below 1 TeV is that the improvement in the Higgs mass bound can only be obtained at the expense of fine-tuning. We do not see it in the expression of the Higgs mass because in a way we have cheated when writing it. In fact, from our discussion of the Hierarchy problem we know that, at energies below the scale of the superpartners we just have the SM and we know in the SM the Higgs mass squared is quadratically sensitive to the UV physics. Where has this quadratic sensitivity gone? The reason it didn't show up explicitly is that we *assumed* EWSB, and therefore the quadratically divergent contribution to $m^2_{h^0}$ is fixed by v and $\tan\beta$ and the calculation we have shown is really due to the one loop contribution to the Higgs quartic coupling (which is not quadratically but just logarithmically divergent). The fine-tuning problem can be understood by computing the Higgs mass squared before EWSB. At tree level we have (in the decoupling limit we have $h^0 = \sqrt{2}(s_\beta H^0_u + c_\beta H^0_d)$),

$$\begin{aligned} V &= 2h_0^2\{(|\mu|^2 + m^2_{H_u})s^2_\beta + (|\mu|^2 + m^2_{H_d})c^2_\beta - 2bs_\beta c_\beta\} \\ &= 2h_0^2\{|\mu|^2(1 - s^2_{2\beta}) + m^2_{H_u}s^2_\beta(1 - 2c^2_\beta) + m^2_{H_d}c^2_\beta(1 - 2s^2_\beta)\} \\ &= 2h_0^2\{|\mu|^2 c^2_{2\beta} - m^2_{H_u}s^2_\beta c_{2\beta} + m^2_{H_d}c^2_\beta c_{2\beta}\} + \ldots, \end{aligned} \tag{4.40}$$

so that

$$m^2_{h^0} = 4\{|\mu|^2 c^2_{2\beta} - c_{2\beta}(m^2_{H_u}s^2_\beta + m^2_{H_d}c^2_\beta)\}. \tag{4.41}$$

The one loop contribution on the other hand is

$$\delta m^2_{h^0} \approx \frac{N_c}{4\pi}|\lambda_t|^2 m^2_{\tilde{t}} \log\frac{\Lambda^2}{m^2_{\tilde{t}}}, \tag{4.42}$$

which, as expected, is quadratically sensitive to the stop mass. Large physical m_{h^0} requires a large $m_{\tilde{t}}$ but a large stop mass gives a large loop correction to $\delta m^2_{h^0}$ that has to be cancelled by the tree level correction thus implying fine-tuning. As an example if we take

$$m_{\tilde{t}} \approx 700 \text{ GeV}, \quad \Lambda \approx 10^{15} \text{ GeV}, \quad \tan\beta \approx 20 - 30, \tag{4.43}$$

so that $m_{h^0} \approx 120$ GeV, we get

$$\delta m^2_{h^0} \approx 10^6 \Rightarrow \text{F.T.} \approx \left|\frac{\delta m^2_{h^0}}{m^2_{h^0}}\right| \approx \frac{10^6}{10^4} \approx 10^2 \Rightarrow\sim \%\text{F.T.}. \tag{4.44}$$

Which shows the fine-tuning problem in the MSSM. This fine-tuning problem can be improved with a smaller cut-off Λ and also by the adition of new fields that can give extra contributions to the Higgs potential (for instance in the nMSSM).

The MSSM, up to some technical problems, for which solutions have been proposed, and a mild fine-tuning, is a well defined and successful theory of EWSB up to very high energies. Two other interesting aspects of SUSY is that, when evolved towards high energies, the three gauge coupling constants seem to converge to much better precision than in the SM (gauge coupling unification), and the fact that EWSB can be generated radiatively, related to the large top Yukawa coupling.

One aparent drawback of the MSSM is that baryon and lepton numbers are no longer accidental symmetries, as we can write terms in the superpotential that violate them

$$W = \lambda_u \tilde{L}\tilde{Q}\tilde{\bar{D}} + \lambda_2 \tilde{\bar{U}}\tilde{\bar{D}}\tilde{\bar{D}} + \ldots. \tag{4.45}$$

This would induce fast proton decay, which is phenomenologically ruled out. However we can define a discrete parity, R parity, that is + on the SM particles and - on the supersymmetric partners. This ensures that proton decay operators are forbidden by the symmetry. It also ensures that there is no tree level coupling between SM particles and one superpartner. This is actually a great advantage, since it automatically cancels most of the tree level contributions of new particles to electroweak precision observables, thus allowing for very light superpartners without contradicting current experimental observables (flavor is a completely separate issue that requires some protection). Note that couplings "in pairs", which is what is needed for the one loop solution to the Hierarchy problem are still allowed by R parity and therefore the nice features of supersymmetric models are not spoiled by it.

Chapter 5

Unitarity Restoration by an Infinite Tower of Resonances: Technicolor

Literature: In the preparation of these lectures I have found useful the reviews by Farhi and Susskind [33] and the lectures by Peskin [34], Lane [35] and R. Contino
(`http://indico.phys.ucl.ac.be/conferenceDisplay.py?confId=148`)

5.1 Introduction

In previous lectures, we have seen that EWSB in the SM although quite minimal, is not completely satisfactory. There is no dynamical explanation for the origin of EWSB plus the actual mechanism suffers from the hierarchy problem. It is actually amusing to notice that, supporting the idea that the naturalness problem of the SM is a real problem, no fundamental scalars have so far been found in nature. In case you are wondering, this is not because no symmetry breaking process has been observed in nature. On the contrary, we have examples of symmetry breaking in which nature has chosen an explicit realization that does not suffer from the same problems of the SM. We will start our lecture with a discussion of low energy QCD and the spontaneous breaking of chiral symmetry. This spontaneous symmetry breaking does not suffer from any hierarchy problem. The scattering of Goldstone bosons is unitarized by an infinite tower of hadronic resonances (composite states) and not surprisingly, there is no fundamental scalars in the model. These ideas motivated a scaled up version of QCD as a natural explanation of EWSB. The idea behind technicolor is thus simple and beautiful, and it is reminiscent of a symmetry breaking mechanism that we have actually observed in nature. Nevertheless, particular realizations of this idea are not necessarily so nice, as they encounters some phenomenological problems that force us to complicate models quite a bit. Before getting to the actual discussion of technicolor ideas we will review what happens in the known example of symmetry breaking by strong interactions: chiral symmetry breaking in low energy QCD.

5.2 Low energy QCD and chiral symmetry breaking

5.2.1 The pattern of chiral symmetry breaking and restoration of unitarity

We have taken this discussion from [4] Let us consider QCD with two massless flavors (given the smallness of the u and d masses, this is not a bad approximation of the real world). The fermionic sector of the QCD Lagrangian reads,

$$\mathcal{L} = \bar{q} i \not{D} q = \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R, \tag{5.1}$$

where we have defined the doublet of fields $q_{L,R} = (u_{L,R}, d_{L,R})^T$ and D_μ is the QCD covariant derivative. This Lagrangian is invariant under the following global transformations

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \to U_L \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \to U_R \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \tag{5.2}$$

where $U_{L,R}$ are arbitrary 2×2 unitary matrices. Separating the $U(1)$ and the $SU(2)$ parts of the transformations, we get that the Lagrangian has, at the classical level an $SU(2)_L \times SU(2)_R \times U(1)_L \times U(1)_R$ symmetry with associated currents

$$j^\mu_{L,R} = \bar{q}_{L,R} \gamma^\mu q_{L,R}, \quad j^{\mu\, a}_{L,R} = \bar{q}_{L,R} \gamma^\mu \frac{\sigma^a}{2} q_{L,R}, \tag{5.3}$$

where σ^a are the Pauli matrices. The sum of left and right handed currents give (three times) the baryon number and the isospin currents,

$$j^\mu = \bar{q}\gamma^\mu q, \quad j^{\mu\, a} = \bar{q}\gamma^\mu \frac{\sigma^a}{2} q, \tag{5.4}$$

whereas the difference gives the axial currents

$$j^{\mu\, 5} = \bar{q}\gamma^\mu \gamma^5 q, \quad j^{\mu\, 5\, a} = \bar{q}\gamma^\mu \gamma^5 \frac{\sigma^a}{2} q. \tag{5.5}$$

The former of the two is anomalous and therefore not a real symmetry of the Lagrangian. $j^{\mu\, 5\, a}$ on the other hand is a real symmetry of the Lagrangian, that is however spontaneously broken by the QCD vacuum.

At low energies, QCD becomes strongly coupled and quark-anti quark pairs bound together and condense. The vacuum state with a quark anti-quark condensate is characterized by a non-zero vev for the scalar operator

$$\langle 0|\bar{q}q|0\rangle = \langle 0|\bar{q}_L q_R + \bar{q}_R q_L|0\rangle \neq 0, \tag{5.6}$$

which is then non-invariant under a chiral transformation $U_L \neq U_R$. Note however that the vectorial $SU(2)$, characterized by $U_L = U_R$ leaves the vacuum invariant. Thus, the axial symmetry is spontaneously broken by the QCD vacuum.

Note 1: The statement above is just a hypothesis, as due to strong coupling we cannot compute the true QCD vacuum. This pattern of chiral symmetry breaking is however well motivated phenomenologically. The reason is that the generators of the left and right transformations are related by parity

$$PQ^a_{L,R}P^{-1} = Q^a_{R,L},$$

which means that for each isospin multiplet with a fixed parity, we should find in nature a degenerate multiplet with opposite parity. The lack of such observational evidence strongly suggests spontaneous breaking of the chiral symmetry.

Note 2: Recall that we have already seen this pattern of symmetry breaking in two different situations, one is the sigma model (that was invented precisely as a model of chiral symmetry breaking in QCD -before QCD was invented-) and the other is the global custodial symmetry in the SM.

The corresponding Goldstone bosons associated to the broken symmetry are created from the vacuum by the axial currents,

$$\langle 0|j^{\mu\,5\,a}(x)|\pi^b(p)\rangle = -ip^\mu f_\pi \delta^{ab} e^{-ip\cdot x}, \tag{5.7}$$

where π^a are the Golstone bosons of the broken chiral symmetry. f_π is known as the pion decay constant and has dimension of mass. It is guaranteed to have the same value for the three Goldstone bosons due to the unbroken vectorial symmetry (isospin), under which the three Goldstone bosons transform as a triplet. There is actually an isospin triplet of mesons, the pions, which are much lighter than the rest of the hadronic resonances, $m_{\pi^\pm} = 139$ MeV, $m_{\pi^0} = 135$ MeV, $m_\rho = 770$ MeV, and are threfore naturally identified with the Goldstone bosons of the broken axial symmetry. This allows to measure f_π in pion decays, with the result $f_\pi \approx 93$ MeV. At energies much smaller than m_{had} we can integrate out the hadrons and we are left with just the low energy effective Lagrangian of the Goldstone bosons, the chiral Lagrangian,

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots, \tag{5.8}$$

with

$$\mathcal{L}^{(2)} = \frac{f_\pi^2}{4}\text{Tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma), \tag{5.9}$$

and

$$\mathcal{L}^{(4)} = \alpha_1[\text{Tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma)]^2, +\alpha_2 \text{Tr}(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma)\text{Tr}(\partial^\mu \Sigma^\dagger \partial^\nu \Sigma), \tag{5.10}$$

where we have defined the matrix of Goldstone bosons as we did in the SM

$$\Sigma = e^{i\vec{\pi}\cdot\vec{\sigma}/(\sqrt{2}f_\pi)}. \tag{5.11}$$

This chiral Lagrangian, represented by the non-linear sigma model, is identical in essence to the EW Chiral Lagrangian, that represents the SM without a Higgs. Thus, we know it will have the same kind of sick behaviour in the scattering amplitudes for Goldstone bosons at high energies. Indeed, the pion scattering amplitude grows with the energy as $(E/m_\rho)^2$, where m_ρ represents the cutoff of the non-linear sigma model, *i.e.* the mass of the first hadronic resonance. However, QCD is a renormalizable well-defined theory so something must restore unitarity in pions scattering. The SM solution is to unitarize this amplitude by the exchange of a fundamental scalar, the Higgs boson, and provided the Higgs mass is not too high, this occurs in a perturbative region. The SM represents a linear sigma model UV completion of the chiral non-linear sigma model. In the case of low energy QCD there is no fundamental scalar and the amplitude grows until it hits strong coupling. In that case higher order terms in the perturbative expansion become important and, eventually, one has to include all the terms or, equivalently, include explicitly the hadronic resonances. These hadronic resonances are the ones that, in QCD, restore unitarity of the Goldstone boson scattering. In particular, it is an experimental fact that the first resonances are responsible in the most part for such unitarization (vector meson dominance): it is the exchange of the ρ and the a_1 that mainly unitarize pion scattering. The interesting feature is that, because the main source of unitarization is different in each case, a partial wave decomposition of the amplitude gives a different answer for different channels, depending on which way the non-linear sigma model is completed in the UV. For instance, in the linear sigma model case, there will be a resonance in the partial wave with zero spin and isospin, whereas in the QCD case, the resonance appears in the spin 1, isospin 1 channel, corresponding to the vector resonance exchanged, the ρ. In fact, experimental data confirms the latter picture as shown in Fig. 5.1, in which we show the experimental data on the vector, isospin 1 partial amplitude in pion-pion scattering, together with the LO chiral prediction ($\mathcal{O}(E^2)$) in dashed line and the NLO one ($\mathcal{O}(E^4)$) in solid line. We

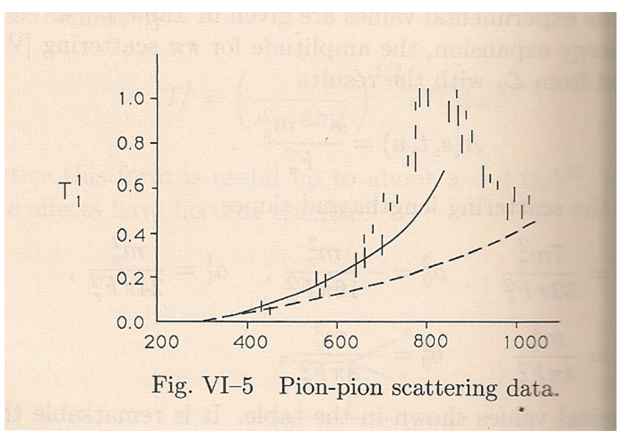


Figure 5.1: Experimental data on the vector, isospion 1 partial amplitude of pion pion scattering together with the LO (dashed) and NLO (solid) chiral predictions.

see how the LO chiral prediction does not show any effect of the resonance, whereas the NLO starts showing the effect of the ρ (an imaginary part of the amplitude is generated at one loop). Experimental data shows the actual resonance that would require all terms in the perturbative expansion to be reproduced. If the restoration of unitarity was due to a scalar (even if not composite, if the restoration of unitarity was done at weak coupling and there were no strongly coupled interactions) we would have found a similar behaviour in the T_0^0 partial amplitude instead.

Thus, the picture of low energy QCD, for two massless quarks, is that the original $SU(2)_L \times SU(2)_R \times U(1)_V$ is spontaneously broken to $SU(2)_V \times U(1)_V$, leading to three Goldstone bosons, the pions, associated to the spontaneous breaking of the axial symmetry. Goldstone boson scattering does not violate unitarity due to the exchange of resonances of the strongly coupled theory that completes in the UV the non-linear sigma model that represents the interactions of the pions. Note that the pions themselves are composite states of the strongly interacting theory (they are $\bar{q}q$ bound states).

5.2.2 Turning on weak interactions

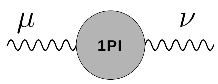
Let us now discuss what happens if one considers the EW gauge bosons coupled to this system (but assuming exact chiral symmetry, *i.e.* exactly massless u and d quarks). It turns out that the $SU(2)_L \times U(1)$ symmetry is actually a subgroup of the global symmetry of the original Lagrangian, corresponding to the generators of the global $SU(2)_L$ and $Y = T_3^R + \frac{B}{2}$, with B the baryon number (1/3 for quarks), which are weakly gauged (by which we mean that we assume the corresponding global symmetry is actually a local symmetry, with its associated gauge bosons, and small coupling constant). The QCD vacuum breaks the original symmetry down to $SU(2)_V \times U(1)_B$ and therefore the surviving *gauge* group is a $U(1)$ with generator $Q = T_3^L + T_3^R + B/2 = T_3^L + Y$, which is, precisely, electromagnetism.

Note: The gauge sector has actually nothing to do with the $U(1)_B$ group -it is not charged under it-. Thus, in what gauge bosons regards, the EWSB is fixed by the symmetry breaking pattern $SU(2)_L \times SU(2)_R \to SU(2)_V$, which guarantees that the $U(1)$ group generated by $T_L^3 + T_R^3 = Q$ (for the gauge bosons) is unbroken.

This spontaneous breaking of the $SU(2)_L \times U(1)$ symmetry means that the $W^\pm$ and the Z get a mass by eating the corresponding would-be Goldstone bosons, which are nothing but the pions of the chiral symmetry breaking. This is a particular case of the discussion we had at the beginning of the course when discussing the Higgs mechanism. In order to see this more explicitly, let us consider the corrections to the EW gauge boson propagators, which can be written, at tree level in the Landau ($\xi = 0$) gauge as (recall that, at this level, the EW gauge bosons are still massless),

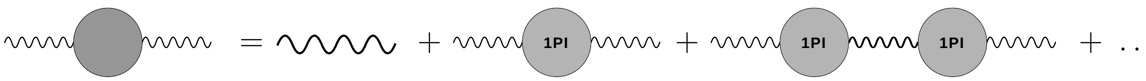
$$G^{\mu\nu}(p) = -\frac{i}{p^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \equiv -\frac{i}{p^2}\mathcal{P}^{\mu\nu}, \tag{5.12}$$

where in the second equality we have defined the transverse projector. Let us denote the 1PI two point function by



$$\equiv i\Pi^{\mu\nu}(p) = i\Pi(p^2)(p^2\eta^{\mu\nu} - p^\mu p^\nu) = i\Pi(p^2)p^2\mathcal{P}^{\mu\nu}.$$

The exact two point function is then given by the infinite series of diagrams shown



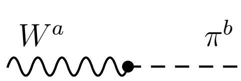
which results in the following form of the exact two-point function

$$\begin{aligned} G^{\mu\nu} &= -\frac{i}{p^2}\mathcal{P}^{\mu\nu} + \left(-\frac{i}{p^2}\mathcal{P}^{\mu\rho}\right) i\Pi(p^2)p^2\mathcal{P}_{\rho\sigma}\left(-\frac{i}{p^2}P^{\sigma\nu}\right) + \dots \\ &= -\frac{i}{p^2}\mathcal{P}^{\mu\nu}\Big[1 + \Pi(p^2) + \Pi(p^2)^2 + \dots\Big] = -\frac{i}{p^2}\mathcal{P}^{\mu\nu}\frac{1}{1-\Pi(p^2)}, \end{aligned} \tag{5.13}$$

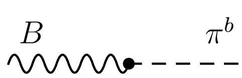
where we have made use of the fact that $\mathcal{P}^{\mu\nu}\mathcal{P}_{\nu\sigma} = \mathcal{P}^\mu_\sigma$ (it is a projector) and have explicitly summed the geometric series. It is then clear that the gauge boson can only acquire a mass if $\Pi(p^2)$ has a pole at $p^2 = 0$. This pole is actually present in the theory, and corresponds to the massless Goldstone boson. Thus, we expect the correction to the gauge boson mass to come from the exchange of the massless Golstone bosons in the gauge boson two point function. Let us compute such correction. The coupling of the EW gauge bosons to the corresponding currents is,

$$\mathcal{L} = -gW^a_\mu J_L^{\mu\,a} - g'B_\mu(J_R^{\mu\,3} + \frac{1}{6}J^\mu). \tag{5.14}$$

Using this coupling, together with the form of the pion matrix element, Eq. (5.7), we find the following expression for the annihilation of a pion into a gauge boson,



$$= ig\left(-\tfrac{1}{2}\right)(-if_\pi p^\mu\delta^{ab}) = -\tfrac{g}{2}f_\pi p^\mu \delta^{ab},$$



$$= ig'\left(\tfrac{1}{2}\right)(-if_\pi p^\mu\delta^{3b}) = \tfrac{g'}{2}f_\pi p^\mu \delta^{ab},$$

where the factors of $\pm 1/2$ come from the left and right chirality projectors in the decomposition of the gauge currents into the vector and axial vector currents. Using that the propagator of the Goldstone boson is $i\delta^{ab}/p^2$ we obtain, for the self-energies of the different gauge bosons

$$= \left(\tfrac{g}{2} f_\pi p_\mu\right) \tfrac{i}{p^2} \left(-\tfrac{g}{2} f_\pi p_\nu\right) \delta^{ab} = -i \tfrac{g^2}{4} f_\pi^2 \tfrac{p_\mu p_\nu}{p^2} \delta^{ab}$$

$$= \left(-\tfrac{g'}{2} f_\pi p_\mu\right) \tfrac{i}{p^2} \left(-\tfrac{g}{2} f_\pi p_\nu\right) \delta^{a3} = i \tfrac{gg'}{4} f_\pi^2 \tfrac{p_\mu p_\nu}{p^2} \delta^{a3}$$

$$= \left(-\tfrac{g'}{2} f_\pi p_\mu\right) \tfrac{i}{p^2} \left(\tfrac{g'}{2} f_\pi p_\nu\right) = -i \tfrac{g'^2}{4} f_\pi^2 \tfrac{p_\mu p_\nu}{p^2}$$

From the general form of the 1PI two point function, Eq. (5.13), we obtain (note that we have explicitly computed the term in $i\Pi_{\mu\nu} = i\Pi(p^2)(-p^\mu p^\nu + \ldots)$ and therefore have to take into account the minus sign),

$$\Pi_{ab}(p^2) = \frac{g^2}{4}\frac{f_\pi^2}{p^2}\delta^{ab} + \ldots, \tag{5.15}$$

$$\Pi_{aB}(p^2) = \Pi_{Ba}(p^2) = -\frac{gg'}{4}\frac{f_\pi^2}{p^2}\delta^{a3} + \ldots, \tag{5.16}$$

$$\Pi_{BB}(p^2) = \frac{g'^2}{4}\frac{f_\pi^2}{p^2} + \ldots, \tag{5.17}$$

where the dots represent other contributions that do not have poles at $p^2 = 0$. From this we see that the EW gauge bosons acquire a mass matrix of the form

$$\mathcal{L}_M = \frac{1}{2}\frac{f_\pi^2}{4}\begin{pmatrix} W^1 & W^2 & W^3 & B \end{pmatrix} \begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W^1 \\ W^2 \\ W^3 \\ B \end{pmatrix}. \tag{5.18}$$

This mass matrix results in the following masses for the gauge bosons,

$$m_W^2 = m_Z^2 c_W^2 = g^2\frac{f_\pi^2}{4}, \quad m_A^2 = 0. \tag{5.19}$$

Thus we see that the photon remains massless and the W and Z masses follow the by now well known relation $\rho = 1$. The reason for that is simply that QCD is custodially symmetric (the global symmetry pattern is the custodially preserving $SU(2)_L \times SU(2)_R \to SU(2)_{L+R}$). Thus, we see that QCD by itself spontaneously breaks the EW symmetry, maintaining the condition $\rho = 1$. Of course, this is not the full story for EWSB since, for instance, the W mass turns out to be

$$m_W(QCD) = g\frac{93\,\text{MeV}}{2} \approx 46\,\text{MeV}, \tag{5.20}$$

which is far from the experimentally measured $m_W \approx 80.4$ GeV. Also, the pions have been measured to be physical fields, and not the unphysical Goldstone bosons that are eaten by the gauge bosons to become massive.

Despite these drawbacks, QCD still represents, at the qualitative level, a successful realization of EWSB. The important question is, apart from the fact that this realization of symmetry breaking and unitarization of Goldstone boson scattering by resonances of a strongly coupled theory as opposed to a fundamental scalar, has been observed in nature, is it any better than the realization of EWSB in the SM through the Higgs? The answer is definitely yes, because the breaking of chiral symmetry by QCD does not suffer from the hierarchy problem. The reason is that QCD is an asymptotically free theory that condenses in the IR, generating the QCD scale, $\Lambda \sim 300$ MeV, by dimensional transmutation. The one loop running of the strong coupling is given by

$$\alpha_s(Q^2) = \frac{\alpha_s^0}{1 + b_0 \frac{\alpha_s^0}{4\pi} \ln(Q^2/Q_0^2)}, \tag{5.21}$$

where $b_0 = 11 - 2n_f/3$ and $\alpha_s^0 \equiv \alpha_s(Q_0^2)$. It is conventional to define Λ_{QCD} as the scale at which the strong coupling constant blows up,

$$1 + b_0 \frac{\alpha_s^0}{4\pi} \ln(\Lambda/Q_0^2) = 0, \tag{5.22}$$

in terms of which we can write the strong coupling constant as

$$\alpha_s(Q^2) = \frac{4\pi}{b_0 \ln(Q^2/\Lambda^2)}. \tag{5.23}$$

Let us assume now that there is a high energy threshold at which new particles are present. At these energies, the strong coupling constant is small and corrections to it remain in the perturbative regime. Thus, the threshold effects will have a small impact on the value of the strong coupling constant at such large energies Λ, $g_s^{(0)}$. Now the question is, given that threshold effect on $g_s^{(0)}$, do we expect the strong coupling scale Λ_{QCD} to be sensitive to this enormously high cutoff? The answer is no, we can actually naturally expect Λ_{QCD} to be exponentially smaller than Λ as can be seen by simple solving for Λ in terms of g_s^0 in the above equation. The result is

$$\Lambda_{QCD} = \Lambda \exp\left[-\frac{2\pi}{\alpha_s^0 b_0}\right] = \Lambda \exp\left[-\frac{8\pi}{g_s^{(0)\,2} b_0}\right]. \tag{5.24}$$

Thus, we see that, as long as g_s is not large in the UV, the QCD scale is preturbatively stable against UV thresholds. This is why it is natural to have $\Lambda_{QCD} \approx 300$ MeV despite the fact that we have observed other thresholds in nature, for instance the EWSB scale $v \sim 174$ GeV. For instance, a coupling constant of order $g_0 \sim 0.3$ at the Planck mass will generate a strong coupling scale $\Lambda_{QCD} \sim 1$ GeV. Note that this result does not mean that Λ_{QCD} is predicted to be the value we measured. What we have seen is that, provided the theory is asymptotically free, so that the coupling constant in the UV (where possible new thresholds will appear) is small, the scale of symmetry breaking generated by dimensional transmutation is naturally exponentially smaller than the cut-off of the theory, whether this is ~ 1 GeV, ~ 1 TeV or $\sim 10^{-5} GeV$ depends on the actual value of the cut-off and the coupling constant but the important feature is that a large hierarchy of scales is naturally predicted.

5.3 Technicolor

In this lecture we will discuss ideas motivated by the observed spontaneous breaking of the chiral symmetry by the QCD vacuum and its successful realization of EWSB at the qualitative level. The simplest idea is to consider a "scaled-up" version of QCD [36, 37] in which,

$$f_\pi = 93 \text{ MeV} \to F_\pi = \sqrt{2}v = 246 \text{ GeV}. \tag{5.25}$$

Of course, we do not have to have precisely three colors in this new version of QCD. Let us therefore consider an $SU(N_{TC})$ technicolor gauge group with a dynamically broken $SU(2)_L \times SU(2)_R \to SU(2)_{L+R}$ global symmetry. The scale of dynamical symmetry breaking will be assumed to be $\Lambda_{TC} \sim$ TeV and the three technipions will be the Goldstone bosons eaten by the W and the Z to get their masses. In order to be able to scale-up the results of QCD to technicolor with an arbitrary number of techni-colors (N_{TC}), we need to know how different quantities scale with N. This is not known in general but we have some very simple scaling properties in the large-N limit (see [38, 39]). Assuming $SU(N_{TC})$ is a confining theory for large N_{TC}, it is possible to write the n-point Green's functions of quark bilinears, in the large N_{TC} limit, as and infinite sum over stable intermediate meson states created out of the vacuum. In particular, for instance, the two point functions can be written as an infinite sum over stable resonances,

$$\begin{aligned} \langle T\{J_V^\mu J_V^\nu\}\rangle &= (q^2 g^{\mu\nu} - q^\mu q^\nu)\sum_n \frac{f_{\rho_n}^2}{q^2 - m_{\rho_n}^2}, && (5.26) \\ \langle T\{J_A^\mu J_A^\nu\}\rangle &= (q^2 g^{\mu\nu} - q^\mu q^\nu)\sum_n \Big[\frac{f_{a_n}^2}{q^2 - m_{a_n}^2} + \frac{1}{q^2} f_\pi^2\Big]. && (5.27) \end{aligned}$$

Note the massless pole corresponding to the Goldstone bosons in the axial currents. Using the scaling with large N_{TC} of the n-point bilinear functions, we can obtain the following scaling for different relevant masses and couplings,

$$\begin{aligned} m_\rho &\sim \text{constant}, && (5.28) \\ f_{\pi,\rho} &\sim \frac{\sqrt{N_{TC}}}{4\pi}, && (5.29) \\ g_{\pi\pi\rho} &\sim \frac{4\pi}{\sqrt{N_{TC}}}. && (5.30) \end{aligned}$$

In particular, both m_ρ and $f_\pi g_\rho$ are independent of N_{TC} in the large-N limit. This scaling properties assume that Λ_{QCD} is constant (which also implies m_ρ constant). However, we have seen that the electroweak gauge boson masses are proportional to f_π^2 rather than Λ_{QCD}^2. If we keep f_π fixed as we vary the number of technicolors, then $\Lambda \sim m_\rho \sim 1/\sqrt{N}$. In particular we have, for instance,

$$m_{\rho_{TC}} \sim \sqrt{\frac{3}{N_{TC}}}\frac{F_\pi}{f_\pi} m_\rho \approx \sqrt{\frac{3}{N_{TC}}}\frac{246}{0.093} 0.77 \approx 1.8 \text{ TeV (for } N_{TC} = 4). \tag{5.31}$$

Thus, smaller N_{TC} implies stronger coupling ($g_\rho \sim 1/\sqrt{N}$) and heavier ρ_{TC} (recall that this apparent dependence of $m_{\rho_{TC}}$ on N is due to the fact that we have fixed F_π).

Thus, the EW symmetry is broken by both the technicolor sector and QCD. The W and the Z eat a combination of π and π_{TC} to get a mass,

$$|\chi\rangle = \sin\alpha|\pi\rangle + \cos\alpha|\pi_{TC}\rangle, \tag{5.32}$$

where $v^2 = f_\pi^2 + F_\pi^2$ and $\tan\alpha = f_\pi/F_\pi \ll 1$. The longitudinal components of the EW gauge bosons are therefore mostly the technipions whereas the physical pions are mostly the QCD pions.

Note: This is easily seen by notizing that

$$\begin{aligned} \langle 0|J^{\mu\,5\,a}|\pi^b\rangle &= -ip^\mu f_\pi \delta^{ab}, && (5.33)\\ \langle 0|J^{\mu\,5\,a}|\pi^b_{TC}\rangle &= -ip^\mu F_\pi \delta^{ab}, && (5.34)\end{aligned}$$

and that

$$\langle 0|J^{\mu\,5\,a}|\text{physical pion}\rangle = 0. \tag{5.35}$$

If we have N_D technifermions that are doublets under $SU(2)_L$ and color singlets, the relation between F_π and m_W gets modified to

$$m_W = \frac{\sqrt{N_D}}{2} g F_\pi, \tag{5.36}$$

and therefore $F_\pi = 246\ \text{GeV}/\sqrt{N_D}$. Thus, in general, if we have a larger number of technicolors or of technidoublets, we have to replace the technipion decay constant with an effective one

$$F_\pi \to \sqrt{N_D N_{TC}/3}\, F_\pi, \tag{5.37}$$

so that its effect is effectively larger and therefore the actual value of F_π can be smaller. Note however that more technidoublets imply more Goldstone bosons. The new flavor symmetry must therefore be broken by gauge interactions so that these Goldstone bosons acquire masses.

5.3.1 The simplest example: minimal technicolor model of Weinberg and Susskind

Before going into the challenges that technicolor models have to overcome it is instructive to briefly introduce the simplest technicolor model introduced by Weinberg [36] and by Susskind [37]. The model has an $SU(N_T) \times SU(3) \times SU(2)_L \times U(1)_Y$ gauge symmetry. In addition to the SM fields, we add one flavor doublet of color singlet technifermions, (T, B).

They have quantum numbers,

$$Q_L^a = \begin{pmatrix} T \\ B \end{pmatrix}_L^a \sim (N_T, 1, 2, 0), \tag{5.38}$$

$$Q_R^a = \begin{pmatrix} T \\ B \end{pmatrix}_R^a \sim \begin{pmatrix} (N_T, 1, 1/2, 0) \\ (N_T, 1, -1/2, 0) \end{pmatrix}, \tag{5.39}$$

where $a = 1, \ldots, N_T$ and we have grouped the two RH techniquarks into a doublet to make explicit the global $SU(2)_L \times SU(2)_R \times [U(1)_A] \times U(1)_B$ (the axial abelian group is written in square brackets to remind the reader that it is anomalous and therefore not a true symmetry). This spectrum is anomaly free provided N_T is even. We assume this new technicolor group will become strongly coupled at a scale Λ_T at which techniquar condensates will form. Identically to what happens in low energy QCD, the fact that we get $\bar{Q}Q$ condensates and that LH and RH techniquarks transform differently under the electroweak group, will induce a sponatenous breaking of the electroweak symmetry. With just one set of doublets, we have exactly the same symmetry breaking pattern as in the chiral Lagrangian, three Goldstone bosons are generated, which are then eaten by the electroweak gauge bosons. The masses of the different resonances in the spectrum can be estimated by scaling with N_T (and N_D, the number of $SU(2)_L$ doublets if we include more than one) QCD results.

5.3.2 Phenomenological issues with technicolor and possible resolutions

This minimal version of technicolor, despite successfully describing EWSB in a natural way at the qualitative level has some phenomenological problems. We briefly summarize below the most important and the suggested solutions. An up to date discussion can be found in [40].

EWPT

Scaling up properties of QCD give estimations of the S parameter that are incompatible with experiment. Being a strongly coupled theory, we cannot compute the actual value of the S parameter in technicolor models. Peskin and Takeuchi used detailed properties of QCD, like vector meson dominance (saturation by first resonances) and large-N behaviour to get an estimation [28]

$$S = -4\pi \frac{d}{dq^2} \left(\Pi_{VV}(q^2) - \Pi_{AA}(q^2) \right) \Big|_{q^2=0} \approx 4\pi \left(1 + \frac{m_{\rho_T}^2}{m_{a_{1T}}^2} \right) \frac{F_\pi^2}{m_{\rho_T}^2} \approx 0.25 N_D \frac{N_{TC}}{3}, \tag{5.40}$$

which, taking into account the experimental limit $S \lesssim 0.3$ (assuming a similar contribution to T, otherwise the limit is more strict), seems to indicate that if technicolor models are just a scaled up version of QCD, they are most likely excluded experimentally. Alternatively, we can see the limit above as a bound on the technirho mass

$$m_{\rho_T} \gtrsim \sqrt{N_D} 1.8 \text{ TeV}, \tag{5.41}$$

which is difficult to make compatible with Eq. (5.31), which was computed for $N_D = 1$. Furthermore, even that estimation was difficult to make compatible with restoration of unitarity in longitudinal gauge boson scattering if ρ_T is the lightest mode responsible for such restoration.

A solution to this problem has necessarily to do with technicolor models that are *not* a scaled-up version of QCD. Walking technicolor, that we will discuss latter, is such an example.

Fermion masses and FCNC

So far we have not said anything about how the SM fermions should get a mass. In order for the techniquark condensates to give them a mass, we need fermions to couple to the technicolor currents. This is done in Extended Technicolor models, in which $SU(3)_C$, all flavor symmetries, and the technicolor group are assumed to be subgroups of an extended technicolor gauge group, which is broken at a scale Λ_{ETC},

$$SU(N_{ETC}) \stackrel{\Lambda_{ETC}}{\longrightarrow} SU(3)_C \times SU(N_{TC}). \tag{5.42}$$

Actually, the different flavor symmetries should be broken at different scales in order to give different masses to the SM fermions. The corresponding massive flavor gauge bosons will generate couplings between SM fermions and technifermions in the form of four-fermion operators. After integrating out the extended technicolor gauge bosons, we end up with an effective Lagrangian of the form

$$g_{ETC}^2 \left(\alpha_{ab} \frac{(\bar{T}\gamma_\mu t^a T)(\bar{T}\gamma^\mu t^b T)}{\Lambda_{ETC}^2}\Big|_{ETC} + \beta_{ab} \frac{(\bar{T}\gamma_\mu t^a q)(\bar{q}\gamma^\mu t^b T)}{\Lambda_{ETC}^2}\Big|_{ETC} + \gamma_{ab} \frac{(\bar{q}\gamma_\mu t^a q)(\bar{q}\gamma^\mu t^b q)}{\Lambda_{ETC}^2}\Big|_{ETC} \right), \tag{5.43}$$

where the subscript ETC indicates that the corresponding operators are generated at a scale Λ_{ETC}, $t^{a,b}$ denote de ETC generators and g_{ETC} is the ETC gauge coupling. The first type of terms will generate masses for the technipions (the goldstones that are not eaten by the electroweak gauge bosons), the second type will generate fermion masses and finally the third type can generate dangerous FCNC interactions. Let us look at the terms the SM generating fermion masses in a bit more detail, Fierz rearranging the corresponding operators, we get

$$\left(\frac{\beta g_{ETC}^2 \langle \bar{T}T \rangle|_{ETC}}{\Lambda_{ETC}^2} \right) \bar{q}q. \tag{5.44}$$

This gives a mass for the corresponding SM fermion, which runs logarithmically below Λ_{ETC}. In order to compute the SM fermion mass, we need to solve the Callan-Symanzik equation for the techniquark condensate,

$$\langle\bar{\Psi}\Psi\rangle_{ETC} = \langle\bar{\Psi}\Psi\rangle_{TC} \exp\left(\int_{\Lambda_{TC}}^{\Lambda_{ETC}} d\log\mu\,\gamma_m\right), \tag{5.45}$$

where the anomalous dimension for the operator $\bar{\Psi}\Psi$ is given in perturbation theory by

$$\gamma_m(\mu) = \frac{3C_2(R)}{2\pi}\alpha_{TC}(\mu) + \ldots, \tag{5.46}$$

with $C_2(R)$ the quadratic Casimir of the technifermions in the R representation of $SU(N_{TC})$. For the fundamental representation, we have $C_2(N_{TC}) = (N_{TC}^2 - 1)/2N_{TC}$. If technicolor behaves like QCD in the sense that the coupling constant decreases very fast with the energy, then the anomalous dimension can be taken essentially vanishing and we get the result for the techniquark condensate

$$\langle\bar{\Psi}\Psi\rangle_{ETC} \approx \langle\bar{\Psi}\Psi\rangle_{TC}. \tag{5.47}$$

Thus, the SM fermion mass, at the ETC scale, reads

$$m_q(ETC) \;\sim\; \frac{1}{\Lambda_{ETC}^2}\langle\bar{\Psi}\Psi\rangle_{ETC} \sim \frac{1}{\Lambda_{ETC}^2}\langle\bar{\Psi}\Psi\rangle_{TC} = \Lambda_{TC}\left(\frac{\Lambda_{TC}}{\Lambda_{ETC}}\right)^2 = \frac{4\pi F_\pi^3}{\Lambda_{ETC}^2}. \tag{5.48}$$

Where the last equality is an estimation based on the NJL model. We have also assumed a similar behaviour (that we will have to modify in more realistic set-ups) to QCD in the sense that the anomalous dimension becomes very small at higher energies. We therefore obtain the relation between the scale at which the corresponding flavor symmetry has to be broken and the generated fermion mass,

$$\Lambda_{ETC}(q) \approx \sqrt{\frac{4\pi 2^{3/2} v^3}{m_q N_D^{\frac{3}{2}}}} \approx \frac{13\ \text{TeV}}{N_D^{3/4}}\sqrt{\frac{1\ \text{GeV}}{m_q}}. \tag{5.49}$$

We have taken into account that the condensation scale is lower for a higher number of technidoublets N_D. We therefore see that fermion masses put an *upper* bound on the scale of ETC breaking. Furthermore, this upper bound is of the order of ($\sim$ 10 TeV for light GeV-ish quarks and seems hopelessly small ($\sim$ TeV) for the top. This is already a problem, since it is not obvious how to generate the top mass, but even a bigger problem is the fact that this same scale is the one that suppresses the dangerous FCNC interactions. For a realistic theory of flavor, we must have different SM fermions to couple to the same technifermion (or to two technifermions that are themselves related by the flavor symmetries). Thus, flavour violating four fermion interactions among the SM fermions, *i.e.* non-zero family dependent γ couplings, are generically expected. Note that even if these four-fermion interactions among SM fermions are flavor diagonal (but not proportional to the identity), they will induce FCNC when we rotate to the physical basis. For instance, one expects,

$$\mathcal{L} = \frac{g_{ETC}^2 V_{ds}^2}{\Lambda_{ETC}^2}(\bar{d}\Gamma^\mu s)\ (\bar{d}\Gamma'_\mu s) + \text{h.c.}, \tag{5.50}$$

where Γ_μ, Γ'_μ are either of $\gamma^\mu(1 \pm \gamma^5)/2$, and we have written the generated operator in terms of a dimensionless coefficient V_{ds} which depends on the particular flavor structure of the theory but is not expected to be much smaller than the product of the corresponding CKM elements. Using the experimental values of Δm_K and ϵ_K, we obtain the following approximate bounds on the scale of ETC,

$$\Lambda_{ETC} \gtrsim \begin{cases} 1300 \text{ TeV} \times g_{ETC}\sqrt{\mathrm{Re}(V_{ds}^2)}, \\ 16000 \text{ TeV} \times g_{ETC}\sqrt{\mathrm{Im}(V_{ds}^2)}. \end{cases} \tag{5.51}$$

This very stringent lower bound on the ETC scale is incompatible with our previous estimate of the upper bound value of Λ_{ETC} to give the SM fermions the right masses. Note however that we have made an important assumption, namely that the dynamics of TC is identical to that of QCD and in particular that the TC coupling is very small at large energies. This is clearly not necessarily the case in a general strongly coupled theory and we will see in the next section how departing from this assumption improves dramatically the behaviour of technicolor models.

Departing from QCD: Walking technicolor

All the main phenomenological problems with ETC came from the fact that we are considering a scaled-up version of QCD. In particular, we are assuming that the theory goes to its asymptotically freedom regime very fast above Λ_{TC}. This forced us to have quite low Λ_{ETC} scales in order to give the SM fermion the right masses. But these values generate FCNC that are in gross contradiction with observed experimental data. Also, the assumption of vector meson dominance and the relation between the first vector and axial-vector resonances in the technicolor spectrum (features just taken as they are in QCD) led to a too large contribution to the S parameter. What if ETC is actually not a scaled-up version

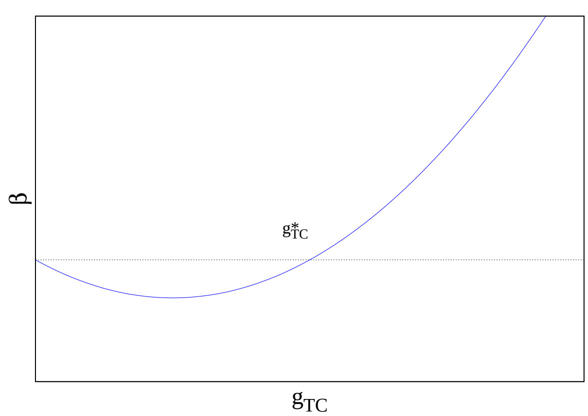


Figure 5.2: Strongly coupled IR fixed point. The theory remains nearly conformal until it condenses.

of QCD? What if, in particular, the technicolor coupling contant, instead of going to a very small value at energies just above Λ_{TC}, remains almost constant (it does not run but *walks*) all the way to Λ_{ETC}? Imagine that the theory has a non-perturbative IR fixed point at high energies as displayed in Fig. 5.2. If the theory is at its conformal point at $E \sim \Lambda_{ETC}$, it will behave like an approximately conformal theory for $\Lambda_{TC} \leq E \leq \Lambda_{ETC}$, spontaneously broken at Λ_{TC}. In that case, any operator is characterized by its *scaling* dimension at the fixed point, d_*, which, due to the strong coupling regime, can be significantly different from the classical dimension.

In particular, the solution of the Callan-Symanzik equation for the techniquark condensate, assuming a constant anomalous dimension, reads,

$$\langle \bar{\Psi}\Psi \rangle_{ETC} = \langle \bar{\Psi}\Psi \rangle_{TC} \left(\frac{\Lambda_{TC}}{\Lambda_{ETC}} \right)^{\gamma_m} . \tag{5.52}$$

The SM fermion mass at the ETC scale then reads,

$$m_q(ETC) \sim \frac{\langle \bar{\Psi}\Psi \rangle}{\Lambda_{ETC}^2} \sim \frac{\langle \bar{\Psi}\Psi \rangle_{TC}}{\Lambda_{ETC}^2} \left(\frac{\Lambda_{TC}}{\Lambda_{ETC}} \right)^{\gamma_m} \sim \Lambda_{TC} \left(\frac{\Lambda_{TC}}{\Lambda_{ETC}} \right)^{2+\gamma_m} . \tag{5.53}$$

Thus, we see that a large negative anomalous dimension can notably change the scaling behaviour, allowing for a higher value of Λ_{ETC} and therefore ameliorating the FCNC problem. In the extreme walking limit, in which $\alpha_{TC}(\mu)$ is constant, it is possible to obtain an approximate non-perturbative formula for the anomalous dimension,

$$\gamma_m(\mu) = -1 + \sqrt{1 - \alpha_{TC}(\mu)/\alpha_{TC}^*}, \tag{5.54}$$

where $\alpha_{TC}^* = \frac{\pi}{3C_2(R)}$. It is believed that $\gamma_m = -1$ is the signal of spontaneous chiral symmetry breaking. In extended technicolor, $\alpha_{TC}(\mu)$ is assumed to be near its critical value α_{TC}^* for $\Lambda_{TC} \lesssim \mu \lesssim \Lambda_{ETC}$, thus $\gamma_m \gtrsim -1$ and

$$m_q(ETC) \sim \Lambda_{TC} \frac{\Lambda_{TC}}{\Lambda_{ETC}}, \tag{5.55}$$

and can do with a safer value of the ETC scale to generate the fermion masses,

$$\Lambda_{ETC} \sim \frac{270 \text{ TeV}}{N_D^{3/4}} \sqrt{\frac{1 \text{ GeV}}{m_q}}. \tag{5.56}$$

Note that, contrary to the terms that give masses to the fermions, which in the extreme walking limit are suppressed by just one power of Λ_{ETC}, the FCNC operators, that involve four SM fermions, are still suppressed, even in walking technicolor models, by two powers of Λ_{ETC}. This means that compatibility between quark masses and FCNC and CP violation is much improved, although some tension is still present.

We have thus seen how conformal behavior at strong coupling, *walking technicolor*, allows to generate light fermion masses compatible with enough suppression of FCNC (or at least importantly improving the situation). There is also some evidence that **near conformal behavior can naturally reduce the contribution to the S parameter**. The main argument is that near conformal dynamics is usually accompanied by a nearly parity symmetric spectrum, *i.e.* degenerate vector and axial resonances. This degeneracy automatically minimizes the contribution to the S parameter.

Top quark mass and walking technicolor

One problem that not even near conformal ETC can solve is the generation of the top mass $m_t \approx 175$ GeV. The scale Λ_{ETC} needed to generate the top mass is of order $\Lambda_{ETC} \sim$ TeV for simple ETC and $\Lambda_{ETC} \sim 20$ TeV for walking technicolor. The former is obviously too low, and even the assumption of four-fermion interactions is no longer valid. The latter is somewhat higher, although it still seems to be difficult to make it compatible with FCNC constraints. Some attempts try to separate the ETC scale for different generations (Tumbling technicolor theories), thus improving the FCNC. However, it is not easy to generate the corresponding cascading of scales solely from gauge dynamics. Furthermore, even if we manage to separate the third generation from the other two, the bottom still comes together with the top and it is very difficult to get the hierarchy in masses between the top and the bottom without inducing large corrections to the $Zb\bar{b}$ coupling, which are strongly constrained experimentally.

We list now two of the suggested solutions to the top mass problem:

Topcolor-assisted technicolor One of the proposed solutions is called top color assisted technicolor. The basic idea is to assume two separated technicolor sectors, one that condenses at a scale $\sim$ TeV and couples preferentially to the top, inducing a top condensate that creates a composite state with the quantum numbers of the top. The second technicolor sector is a regular ETC that gives small masses to the SM fermions. In particular, the top quark is an admixture of a top that gets a small mass from the ETC sector and the composite with the quantum numbers of the top in the stronger technicolor sector. In this way the large top mass is naturally explained and does not introduce large corrections to the lighter sector, since it does not perturb strongly the ETC sector that gives mass to the lighter fermions (and that, with a bit of walking can safely avoid FCNC).

An alternative is that top condensation induces all the required EWSB, i.e. it is not the condensation of new technifermions that drives EWSB but a $\bar{t}t$ condensate. The problem is that, for the top to condense, one would need it to couple more strongly than its mass indicates. It would have to be $m_t \sim 600$ GeV for its "Yukawa coupling" to be strong enough to produce the condensation. Top see-saw models propose that the EWSB mass of the top is actually of that size ~ 600 GeV but it mixes with other heavier composites that make the physical mode lighter (in a small see-saw) to give the observed mass. From the low energy perspective, we can model this process through a 600 GeV top mass (from EWSB) that mixes with a $\sim$ TeV vector-like quark singlet. The lightest physical admixture of both quarks has a mass ~ 175 and is the top we measure.

It is not easy to make minimal top condensation models (with or without top see-saw) compatible with experimental observations, in particular when corrections to the $Zb\bar{b}$ coupling are considered.

Larger anomalous dimensions The limit $\gamma \lesssim 1$ for condensation is based on approximate models and numerical simulations but there is no formal proof of such bound.

Unitarity imposes a firm bound given by $\gamma \leq 2$. If one allows anomalous dimensions of order $\gamma \sim 1.5$, the top mass can be generated consistently with the bounds from FCNC and EWPT.

5.3.3 New Developments in Technicolor

The problems we have outlined in the previous sections made technicolor fade against its main competitor, SUSY, and in the last decade against new ideas in model building, some of which we will discuss in these lectures. The fact that SUSY was not found at LEP and the experimental bound on the Higgs mass starts to impose some amount of fine-tuning on minimal supersymmetric models, gave some new momentum to technicolor theories. It was also realized that some new ideas in model building could actually be used to understand better technicolor models. Thus, in the last few years, new more realistic technicolor models have emerged and their phenomenological implications will be studied at the LHC. Here we briefly describe two of these new routes in technicolor models.

Minimal walking models

The models we have described so far are based on $SU(N_{TC})$ with fermions transforming in the fundamental representation N_{TC}. The reason is that in the end, our only experience with these kind of models comes from QCD, for which that is the case. Furthermore, we typically need to use large-N_{TC} results, which made models less compatible with experimental data (for instance the S parameter naively grows with the number of technicolors). In recent years, an effort is being carried out to analyze the conformal window as a function of, not only the number of technicolor or techniflavors (number of weak doublets) but also as a function of the representation of the group. The results of these analyses seem to point to very minimal models, with a very small number of technicolors and techniflavors as being within the conformal window (thus leading to the required walking), resulting in the minimal naive contribution to the S parameter (defined as the one loop contribution of the technifermions assuming momentum independent constituent masses), and even allowing for the presence of realistic dark matter candidates. An interesting example of these new technicolor models is the *Ultra Minimal near Conformal Technicolor* model [41]. The model consists of an $SU(2)$ technicolor group with two Dirac flavors in the fundamental representation, the LH components transforming as a $SU(2)_L$ doublet with zero hypercharge, the RH components as $SU(2)_L$ singlets with hypercharge $\pm\frac{1}{2}$, respectively, plus two Weyl fermions, which are singlets under the electroweak symmetry but transform in the **adjoint representation** of the technicolor group. These new fermions transforming in a different representation of the group, bring the model into the conformal window, allowing for the model to walk. A detailed analysis of the symmetry breaking pattern shows an unbroken global technibaryon number. In some circumstances, the lightest technibaryon, which is stable due to technibaryon number conservation, is weakly interacting and electrically neutral and can therefore serve as a good dark matter candidate. It can be even possible to generate the dark matter relic density through a technibaryon number asymmetry, like it happens for the ordinary baryon density. In some models, the two can be related, which might explain the similarity of baryon to dark matter density today.

Unfortunately, some of the features of these models, like the existence of a conformal window for such small number of technifermions, can only be estimated due to the strong coupling. An important amount of resources is being currently used in studying these features in the lattice, which will provide a definite answer to some of these questions.

New tools for technicolor: extra dimensional models

As we said above, one of the main drawbacks of technicolor is the impossibility of precisely compute many of the phenomenological implications of the theory. That means that it is difficult to exclude these models with a high degree of confidence but it also means that it is extremely difficult to make reliable predictions. The situation changed in some way in the last decade when, due to the original work of Maldacena [42], it was realized that some strongly coupled gauge theories in d dimensions could be related to weakly coupled "dual" gravity theories in $d+1$ dimensions. The original idea, and the one that has the most "experimental" support (here by experimental we do not really mean particle physics experiments that theoretical tests that the duality has passed), is stated in a 10-dimensional string theory, compactified on $\Omega \times AdS_5$, where Ω is a 5-dimensional compact space and AdS_5 is 5-dimensional anti-deSitter space and its dual, that is a conformal theory (for instance $\mathcal{N}=4$ SYM for type II-B string theory with $\Omega = S^5$) defined on the boundary of AdS_5. The large-N limit of the conformal field theory corresponds to the weak coupling limit (the supergravity limit of the string theory) in the string side of the duality.

The AdS/CFT correspondence is therefore a duality between strongly coupled conformal theories and weakly coupled gravity theories defined in one more dimension. This sounds precisely like the kind of thing we need in order to study technicolor models. We want models that are near conformal and strongly coupled but we would like to be able to perform calculations with them. In order to do that, we only need to go to the dual weakly coupled models and make the corresponding calculations there. In fact, at that time, Randall and Sundrum (RS) had proposed a gravity model (which was soon extended to incorporate the full SM) in AdS_5 to solve the hierarchy problem in a purely geometric way [43]. It didn't take long to put these two ideas together and to conjecture a duality between RS models and strongly coupled conformal theories in four dimensions. This new conjecture rests on much less firm ground. It does not rely on string theory or highly supersymmetric gauge theories and it might be very well that we cannot compute which conformal theory our RS model is dual to. However, it is true that these 5D models share a lot of features in common with strongly coupled conformal theories and a qualitative AdS/CFT dictionary has been developed that seems to work pretty well. The main entries in the dictionary relevant for model building, are compiled in table 5.1, taken from [44]. Given these successes, even if we don't know which 4D CFT we are really studying, it is quite likely that there is one CFT that is (at least approximately) dual to our 5D models in AdS_5. We will in fact make more use of this duality in future lectures.

5D models of technicolor are defined on a slice of AdS_5, which has a metric, in confor-

Bulk of AdS	$\leftrightarrow$	CFT
Coordinate (z) along AdS	$\leftrightarrow$	Energy scale in CFT
Appearance of UV brane	$\leftrightarrow$	CFT has a cutoff
Appearance of IR brane	$\leftrightarrow$	conformal symmetry broken spontaneously by CFT
KK modes localized on IR brane	$\leftrightarrow$	composites of CFT
Modes on the UV brane	$\leftrightarrow$	Elementary fields coupled to CFT
Gauge fields in bulk	$\leftrightarrow$	CFT has a global symmetry
Bulk gauge symmetry broken on UV brane	$\leftrightarrow$	Global symmetry not gauged
Bulk gauge symmetry unbroken on UV brane	$\leftrightarrow$	Global symmetry weakly gauged
Higgs on IR brane	$\leftrightarrow$	CFT becoming strong produces composite Higgs
Bulk gauge symmetry broken on IR brane by BC's	$\leftrightarrow$	Strong dynamics that breaks CFT also breaks gauge symmetry

Table 5.1: Relevant rules for model building using the AdS/CFT correspondence

mally flat coordinates,

$$ds^2 = \frac{L_0^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2), \tag{5.57}$$

with $M_{Pl}^{-1} \approx L_0 \leq z \leq L_1 \approx \text{TeV}^{-1}$ the coordinate along the extra dimensions. This form of the metric shows the scale invariance of AdS, as it is invariant under $x, z \to \lambda x, \lambda z$, which hints at the conformal invariance of the dual theory (in fact the isometry group of AdS_5 is $SO(2,4)$, which corresponds to the conformal group in four dimensions). The conformal invariance is broken explicitly at the so called UV brane ($z = L_0$), which represents in the dual theory the coupling of external sources (the SM particles) to the CFT. It is also broken, although this time spontaneously due to condensation (in the dual theory), at the IR brane ($z = L_1$). Having obtained a near conformal theory we now need the chiral symmetry breaking $SU(2)_L \times SU(2)_R \to SU(2)_V$. The dictionary tells us that global symmetries in the CFT correspond to *bulk* gauge symmetries in the 5D side. Thus, we include an $SU(2)_L \times SU(2)_R \times U(1)_X$ bulk gauge symmetry. This will furthermore ensure the required global custodial symmetry that guarantees a small contribution to the T parameter.

Note: The original RS models, which did not have technicolor models as a goal, included a simpler $SU(2)_L \times U(1)_Y$ bulk gauge group. Not surprisingly, this lead to a very large contribution to the T parameter, which was subsequently fixed by enlarging the bulk gauge group, after the realization of the implications on the global symmetries in the dual picture.

The extra $U(1)_X$ group is introduced to obtain the correct hypercharge assignments. As we said, the IR brane represents the spontaneous breaking of conformal invariance due to condensation, thus we would like to have, at the IR brane, the bulk $SU(2)_L \times SU(2)_R$ symmetry been spontaneously broken to $SU(2)_V$. This can be done in the 5D model by means of a Higgs but that will actually leave a physical Higgs field in the spectrum, more along the lines of composite Higgs models that we will describe in the future. Another possibility, more in the spirit of technicolor models, is to break the symmetry by boundary conditions. This is a spontaneous breaking that does not leave any scalar in the low energy spectrum.

Note: It is not obvious that breaking gauge symmetries by boundary conditions is a spontaneous breaking. However, it was shown in [45] that consistent boundary conditions can be obtained from localized scalars acquiring a vev and taking this vev (and thus the corresponding physical Higgses) to infinity.

The UV brane, on the other hand, represents the explicit conformal breaking due to the coupling with the elementary, SM, fields. Thus, the bulk gauge symmetry, must be broken by boundary conditions to $SU(2)_L \times U(1)_Y$, where the hypercharge is a combination of the third component of $SU(2)_R$ and the $U(1)_X$. This model thus has all the right ingredients of a realistic technicolor model. Furthermore, in the limit that the 5D gauge coupling is small, we have a weakly coupled theory that will allow us to perform calculations. In this model, for instance, anomalous dimensions are related to the localization of the different fields in the extra dimension and can be very easily computed. This gives us the scaling of mass terms and different couplings that successfully reproduce the SM fermion masses. Furthermore, the same mechanism that suppressed the masses of the light fermions also suppresses FCNC for them. Although models that have been constructed still need some amount of cancellations to suppress enough the most dangerous FCNC processes, they are naturally close to current limits.

Apart from being able to compute the scaling of fermion masses and FCNC processes, we are also able to compute explicitly the restoration of unitarity in longitudinal gauge boson scattering for these models. It was shown in [45] that, as long as the breaking by boundary conditions is consistent (with the variation of the action in the bulk and branes), unitarity is exactly restored (in the sense that terms growing with energy like $\sim E^4$ and $\sim E^2$ vanish exactly) through the exchange of the whole tower of KK modes (resonances in the 4D dual) in the elastic channel. Still, the constant term can violate unitarity if the lightest KK mode is too heavy (similar to what happens in the SM with a too heavy Higgs). This imposes a bound on the lightest KK mode of the order of ~ 700 GeV. This bound is great for phenomenology, but it is clear that it might give rise to problems with EWPT and in particular with the S parameter. Before discussing that, it is worth mentioning that the fact that all KK modes contribute to the restoration of unitarity, means that inelastic channels will open at higher energies. In fact, it was shown that when inelastic channels are included, unitarity is violated at some high energy. This is due to the fact that 5D theories are intrinsically non-renormalizable and can only be considered effective theories below some cut-off. In fact, the unitarity bound from inelastic scattering approximately agrees with the naive cut-off of the 5D theory.

The fact that we need so light resonances to restore unitarity, means that more likely, we will have problems with EWPT. In fact, we also have that weak coupling in the 5D picture corresponds to large-N limit of the CFT, which goes in the direction of making the S parameter larger. The bound on the S parameter imposes an upper bound on N, which in turn imposes a lower bound on the 5D coupling. In fact, in minimal models, the two bounds are contradictory, and the models is excluded. Model building in the 5D side, however, allows us, at the expense of $\sim$ few % fine-tuning, to compensate the contribution to the S parameter with a vertex correction so that the total contribution to the "effective" S parameter cancels.

The heaviness of the top still poses a problem in these models although again model building techniques in 5D allow for Higgsless models that pass all EWPT constraints at tree level (with few per cent fine-tuning), have realistic fermion and gauge boson masses and preserve unitarity in longitudinal gauge boson scattering up to the cut-off of the theory. Loop corrections can be important in the case of the top but they can be argued to be of the same size as higher dimensional operators arising at the cut-off of the 5D theory. We are therefore in a situation in which we have gained *some* computational power, in the sense that we can compute our model to be realistic in the leading color approximation (leading corrections in $1/N$). Subleading color contributions (loop corrections on the 5D side) can be substantial in some cases but they are formally of the same order as uncalculable corrections coming from higher dimensional operators. This is somewhat uncomfortable but it is difficult to get further in calculability using 5D theories. Even more unsettling is the fact that although FCNC are well under control within the 5D, the same uncalculable higher dimensional operators, could contribute to light fermion FCNC well above current experimental data (this is forced upon in Higgsless models, whereas it can be avoided in other RS models, because of the requirement of cancellation of the S parameter with fermion vertex corrections).

More freedom, including some control over quantum corrections, can be obtained by means of the so called deconstruction of extra-dimensional models. This can be viewed as a coarse discretization of the extra dimension. The resulting theory is a weakly coupled 4D theory, that corresponds to the low energy effective theory of the extra dimensional one. It can be realized by means of a non-linear sigma model, in which case we will have to replace it with a UV completion at its cut-off, or by means of a linear sigma model, that has the ugly inconvenient of reintroducing the hierarchy problem.

In a way, both the 5D and the deconstructed Higgsless models sacrifice the solution of the big hierarchy problem (they could be considered duals to strongly coupled quasi-CFTs that solve the hierarchy problem but there are features of these 4D CFTs that we cannot compute in our intrinsically effective weakly coupled descriptions). This has become a bit of a trend in the last few years. I personally think that we should not forget the large hierarchy problem as the best motivation for physics beyond the SM but agree that, with the onset of the LHC in front of us, it is useful to prioritize our efforts. And possible solutions of the little hierarchy problem that can have phenomenological implications at the LHC seem like good places to start looking. A wealth of new models, including in a way the ones we have mentioned in this section, have been proposed in the last few years that solve the little hierarchy problem, allowing for a natural Higgs mass -or scale of electroweak symmetry breaking- with a cut-off that is compatible with current experimental constraints and at the same time have observable consequences at the LHC. If any of these models is confirmed, we might have to think about what UV completion takes over at the new cut-off of the theory, which typically is $\sim 1 - 10$ TeV.

Chapter 6

Composite Higgs Models

Literature: This section has been taken almost entirely from Roberto Contino's notes.

6.1 Introduction

Technicolor models are based on a beautiful, simple idea, which is actually realized in nature for another example of spontaneous symmetry breaking. Unfortunately, particular realizations of this simple idea clash with experimental observations and it is difficult to obtain realistic models that are not unnatural. Composite Higgs models [46, 47, 48] arose as an intermediate case between a fundamental scalar Higgs and technicolor models. In these models, the Higgs boson exists but it is a composite state of a strongly coupled new interaction. This allows these models to have some of the nice properties of technicolor models while avoiding some of the most constraining phenomenological problems:

- Being a bounds state, the Higgs is not sensitive to UV effects above the compositeness scale. This solves the hierarchy problem (instability of the Higgs mass against radiative corrections).
- We furthermore assume the Higgs is light as compare with the rest of the resonances because it is the Goldstone boson of a dynamically broken global symmetry of the strong sector.
- A light Higgs can (partially) restore unitarity in longitudinal gauge boson scattering, thus allowing for heavier vector resonances in the strong sector, compatible with EWPT. (The flavor problem will be discussed below).
- Composite Higgs models interpolate between technicolor and the SM with a fundamental Higgs. In principle, we can take the limit in which the vector resonances get heavier and heavier, but keeping the mass of the Goldstones at a few GeV scale. In that limit, the only effect of the strong sector is a scalar, with a natural cut-off, the

composite scale, that is very high. This is just the SM limit. Of course, in that case keeping the Higgs in the light spectrum requires fine-tuning. As we will see below, the Higgs mass is generated by radiative corrections but it is still proportional to the composite scale and keeping it much smaller than the compositeness scale requires to fine-tune this one loop correction.

6.2 Composite Higgs Models: General Structure

The starting point for composite Higgs models is very similar to technicolor models. We assume there is a strongly interacting sector with a global symmetry group G that is spontaneously broken, due to condensation, to a subgroup H. The SM group should be a subgroup of H. Furthermore, G/H should contain an $SU(2)_L$ doublet with hypercharge $\pm 1/2$. In that case we have:

- In the absence of the external gauging of G_{SM}, H is unbroken, and the Higgs is an exact Goldstone. This means that its potential vanishes at tree level (it can only have derivative couplings).
- The explicit breaking due to the gauging of G_{SM} will induce a potential for the Higgs (it is really a pseudo-Goldstone boson). The orientation of G_{SM} compared to H in the true vacuum (the misalignment of the vacuum) is a new parameter (with respect to technicolor theories), $\epsilon = v/F_\pi$. This parameter also parameterizes the interpolation between the SM ($\epsilon \ll 1$) and pure technicolor ($\epsilon \to 1$).

6.2.1 The minimal composite Higgs model

Let us consider the minimal composite Higgs model. We want H to include the electroweak group, so minimality requires it to *be* the electroweak group $H = SU(2)_L \times U(1)_Y$. We also want to have at least four Goldstone bosons that can act as the Higgs. The minimal choice would be a group with $8 = 4 + 4$ generators. Let us try with $G = SU(3)$. G has indeed H as a subgroup and the right number of generators so that there are 4 Goldstone bosons in the coset space. The next requirement is that the four Goldstones transform as the Higgs under H. We use the Gell-Mann matrices as the generators of $SU(3)$,

$$T^a = \frac{\lambda^a}{2}, \quad a = 1, \dots 8. \tag{6.1}$$

They satisfy the commutation relations, $[T^a, T^b] = i f_{abc} T^c$ with,

$$\begin{aligned} f_{123} &= 1, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}, \\ f_{147} &= f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}, \end{aligned} \tag{6.2}$$

all the others (that are not related by total antisymmetry) vanish. In particular, we have $(a, b, c = 1, 2, 3)$

$$[T^a, T^b] = i\epsilon^{abc}T^c, \quad [T^a, T^8] = 0, \tag{6.3}$$

so that T^a generate an $SU(2)$ subgroup and T^8 generates a $U(1)$ subgroup. The coset space is spanned by $T^{\hat{a}}$ with $\hat{a} = 4, 5, 6, 7$. Defining $T^+ \equiv T^4 - iT^5$ and $T^0 = T^6 - iT^7$, and grouping them in a two dimensional vector,

$$T_\phi = \begin{pmatrix} T^+ \\ T^0 \end{pmatrix}, \tag{6.4}$$

and using the commutation relations, we get,

$$[T^a, T_\phi] = -\frac{\sigma^a}{2}T_\phi, \quad [T^8, T_\phi] = -\frac{\sqrt{3}}{2}T_\phi. \tag{6.5}$$

Thus, defining $g_{SM} = g$ and $g'_{SM} = \sqrt{3}g$, we have that T_ϕ (and therefore the Goldstones associated to them) transforms as a doublet under $SU(2)$ with hypercharge 1/2, just like the SM Higgs. However, this has fixed the hypercharge coupling constant in terms of the $SU(3)$ coupling (as opposed to the one of a $U(1)$ group that does not come from a unified group). This fixed value is in disagreement with observation, as the sine of the Weinberg angle becomes,

$$s_W^2 = \frac{g'^2_{SM}}{g'^2_{SM} + g^2_{SM}} = \frac{3g^2_{SM}}{3g^2_{SM} + g^2_{SM}} = \frac{3}{4} \neq s^2_{W\,\mathrm{exp}} \approx 0.231. \tag{6.6}$$

This problem is easily fixed by starting with an extra $U(1)$ field under which the Higgs is not charged. A linear combination of the $U(1)$ that comes from the $SU(3)$ and the new one will give rise to the hypercharge, now with a free coupling. This minimal model has a phenomenological problem. The symmetry breaking pattern is not custodially symmetric and hadronic resonances will in general induce a too large T parameter in conflict with experimental data.

6.2.2 The minimal (custodial) composite Higgs model

Let us now construct a fully realistic model. The model will be minimal in the sense that the Goldstones in G/H still contain *only* the Higgs boson but now H is such that the global symmetry breaking pattern is custodial invariant. Custodial symmetry requires H to be at least $SU(2)_L \times SU(2) \sim SO(4)$ and the Higgs to transform as a bidoublete (equivalently as a fundamental of $SO(4)$). In order to get only 4 Goldstones we need G to have 10 generators ($SO(4)$ has six generators). As we will see, $SO(5)$ fulfills all the requirements. In order to generate a non-trivial Weinberg angle, we will add an extra $U(1)$ group so that in total we have $G = SO(5) \times U(1)_X$ and $H = SO(4) \times U(1)_X$. We have

$$\#\ \text{Goldstones} \ = \ \dim\ (SO(5)/SO(4)) = 4, \tag{6.7}$$

so that we have four real scalars as the only Golstones.

We will use the following basis of $SO(5)$ generators, in the fundamental representation,

$$\begin{aligned}
(T^a_L)_{i,j} &= -\frac{i}{2}\left[\frac{1}{2}\epsilon^{abc}(\delta^b_i\delta^c_j - \delta^b_j\delta^c_i) + (\delta^a_i\delta^4_j - \delta^a_j\delta^4_i)\right], \\
(T^a_R)_{i,j} &= -\frac{i}{2}\left[\frac{1}{2}\epsilon^{abc}(\delta^b_i\delta^c_j - \delta^b_j\delta^c_i) - (\delta^a_i\delta^4_j - \delta^a_j\delta^4_i)\right], \\
(T^{\hat{a}}_C)_{ij} &= -\frac{i}{\sqrt{2}}(\delta^{\hat{a}}_i\delta^5_j - \delta^{\hat{a}}_j\delta^5_i),
\end{aligned} \tag{6.8}$$

with $a = 1, 2, 3$, run over the $SU(2)_L$ and $SU(2)_R$ unbroken groups, $\hat{a} = 1, 2, 3, 4$ run over the coset space $SO(5)/SO(4)$ and $i, j = 1, \ldots, 5$. They are normalized to $\mathrm{Tr}T^AT^B = \delta^{AB}$. The corresponding commutation relations can be worked out to be,

$$[T^a_L, T^b_L] = i\epsilon^{abc}T^c_L, \quad [T^a_R, T^b_R] = i\epsilon^{abc}T^c_R, \quad [T^a_L, T^b_R] = 0, \tag{6.9}$$

$$[T^a_C, T^b_C] = \frac{i}{2}\epsilon^{abc}(T^c_L + T^c_R), \quad [T^a_C, T^4_C] = \frac{i}{2}(T^a_L - T^a_R), \tag{6.10}$$

$$[T^a_{L,R}, T^b_C] = \frac{i}{2}(\epsilon^{abc}T^c_C \pm \delta^{ab}T^4_C), \quad [T^a_{L,R}, T^4_C] = \mp\frac{i}{2}T^a_C. \tag{6.11}$$

In particular, it can be checked from these commutation relations that the generators in the coset space (and therefore the corresponding Goldstone bosons) transform as a $(2, 2)$ of $SU(2)_L \times SU(2)_R$ (or equivalently as a fundamental, (4) of $SO(4)$). Note that these are precisely the quantum numbers of the SM Higgs. The Goldstone bosons can be described in terms of a field Σ:

$$\Sigma = \Sigma_0 e^{\Pi/F_\pi}, \tag{6.12}$$

where $\Sigma_0 = (0, 0, 0, 0, 1)$, $\Pi = -iT^{\hat{a}}_C h^{\hat{a}}\sqrt{2}$. Expanding the exponential we get,

$$\Sigma = \frac{\sin h/F_\pi}{h}(h_1, h_2, h_3, h_4, h\cot h/F_\pi), \quad h = \sqrt{h^2_{\hat{a}}}. \tag{6.13}$$

The most general $SO(5) \times U(1)_X$ invariant Lagrangian, build out of the Goldstone bosons and the external (elementary) gauge fields (for the moment we consider the gauge fields as classical sources), at quadratic order in the gauge fields, is,

$$\mathcal{L}_\chi = \frac{1}{2}(P_T)_{\mu\nu}\left[\Pi^X_0(p^2)X_\mu X_\nu + \Pi_0(p^2)\mathrm{Tr}(A_\mu A_\nu) + \Pi_1(p^2)\Sigma A_\mu A_\nu \Sigma^T\right] + \ldots, \tag{6.14}$$

where $(P_T)_{\mu\nu} = \eta_{\mu\nu} - p_\mu p_\nu/p^2$ is the transverse projector so that terms with two gauge fields are the (quadratic) transverse part of $F_{\mu\nu}$, which transforms as $F_{\mu\nu} \to \Omega F_{\mu\nu}\Omega^{-1}$, with Ω an $SO(5)$ transformation. Also we have $\Sigma \to \Sigma\Omega^{-1}$, we have treated Σ as an external classical background (it has no momentum) and $\Pi_{0,1}(p^2)$ and $\Pi^X_0(p^2)$ are form factors that encode the dynamics of the strong sector (heavy resonances plus fluctuations of the Goldstones around the vacuum Σ).

We can get some information on the form factors by expanding around the $SO(4)$-preserving vacuum, Σ_0,

$$\mathcal{L} = \frac{1}{2}(P_T)_{\mu\nu}\Big[\Pi_0^X(p^2)X_\mu X_\nu + \Pi_0(p^2)A_\mu^a A_\nu^a + (\Pi_0 + \frac{1}{2}\Pi_1)A_\mu^{\hat{a}} A_\nu^{\hat{a}}\Big], \tag{6.15}$$

where A^a here represents the unbroken sources, $A^{\hat{a}}$ the broken ones, we have used that $(T\Sigma_0^T)_i = T_{i5}$ and $(T_L^a)_{i5} = (T_R^a)_{i5} = 0$, $(T_C^{\hat{a}})_{i5} = -\frac{i}{\sqrt{2}}\delta_i^{\hat{a}}$ and $(\Sigma_0 T)_i = T_{5i}$ with $(T_L^a)_{5i} = (T_R^a)_{5i} = 0$, $(T_C^{\hat{a}})_{5i} = \frac{i}{\sqrt{2}}\delta_i^{\hat{a}}$. We thus get

$$\boxed{\Pi_a \equiv \Pi_0, \quad \Pi_{\hat{a}} = \Pi_0 + \frac{1}{2}\Pi_1.} \tag{6.16}$$

Now, remember that we can only have a massless pole (due to the exchange of the Goldstone bosons) in the two point function of the broken generators. In particular, in the large N limit we have

$$\begin{aligned}
(P_T)_{\mu\nu}\Pi_a(p^2) &= \langle J_\mu^a J_\nu^a\rangle = (p^2\eta_{\mu\nu} - p_\mu p_\nu)\sum_n \frac{F_{\rho,n}^2}{p^2 - m_{\rho,n}^2} = (P_T)_{\mu\nu}\sum_n \frac{p^2 F_{\rho,n}^2}{p^2 - m_{\rho,n}^2}, \\
(P_T)_{\mu\nu}\Pi_{\hat{a}}(p^2) &= \langle J_\mu^{\hat{a}} J_\nu^{\hat{a}}\rangle = (p^2\eta_{\mu\nu} - p_\mu p_\nu)\Big[\frac{F_\pi^2}{2p^2} + \sum_n \frac{F_{a,n}^2}{p^2 - m_{a,n}^2}\Big] \\
&= (P_T)_{\mu\nu}\Big[\frac{F_\pi^2}{2} + \sum_n \frac{p^2 F_{a,n}^2}{p^2 - m_{a,n}^2}\Big].
\end{aligned} \tag{6.17}$$

Comparing the above expressions we get

$$\Pi_1(0) = F_\pi^2, \quad \Pi_0(0) = 0, \quad \Pi_0^X(0) = 0. \tag{6.18}$$

Now using $\Sigma = (\hat{h}_1 s_h, \hat{h}_2 s_h, \hat{h}_3 s_h, \hat{h}_4 s_h, c_h)$, with $\hat{h}_i \equiv h_i/h$, $s_h \equiv sin(h/F_\pi)$ and $c_h = \cos(h/F_\pi)$, and turning on only the $SU(2)_L \times U(1)_Y$ physical gauge fields, with $X_\mu = A_\mu^{3R} = B_\mu$ ($Y = T_3^R + X$), we get

$$\mathcal{L} = \frac{1}{2}(P_T)_{\mu\nu}\Big[(\Pi_0^X + \Pi_0 + \frac{s_h^2}{4}\Pi_1)B_\mu B_\nu + (\Pi_0 + \frac{s_h^2}{4}\Pi_1)L_\mu^a L_\nu^a + 2s_h^2\Pi_1 \hat{H}^\dagger T_a^L Y \hat{H} L_\mu^a B_\nu\Big], \tag{6.19}$$

where we have defined

$$\hat{H} \equiv (\hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{h}_4). \tag{6.20}$$

Note: The SM is embedded in the $SU(2)_L \times SU(2)_R \times U(1)_X$ as follows.

$$\mathcal{L} = -\frac{1}{4}\left[\frac{1}{g_L^2}L_{\mu\nu}^2 + \frac{1}{g_R^2}R_{\mu\nu}^2 + \frac{1}{g_X^2}X_{\mu\nu}^2\right] = -\frac{1}{4}\left[\frac{1}{g_L^2}L_{\mu\nu}^2 + \frac{1}{g_R^2}(R_{\mu\nu}^b)^2 + \frac{1}{g'^2}B_{\mu\nu}^2 + \frac{1}{g_{Z'}^2}Z'^2_{\mu\nu}\right], \tag{6.21}$$

where we have defined

$$R_\mu^3 = B_\mu + \frac{g_R^2}{g_R^2+g_X^2}Z'_\mu, \quad X_\mu = B_\mu - \frac{g_X^2}{g_R^2+g_X^2}Z'_\mu, \tag{6.22}$$

and the couplings and charges

$$g' = \frac{g_R g_X}{\sqrt{g_R^2+g_X^2}}, \quad g_{Z'} = \sqrt{g_R^2+g_X^2}, \tag{6.23}$$

and

$$Y = T_3^R + Q_X, \quad Q_{Z'} = \frac{g_R^2 T_3^R - g_X^2 Q_X}{g_R^2+g_X^2}. \tag{6.24}$$

With these redefinitions, the covariant derivative reads, in both basis,

$$iD_\mu = \not{L}^a T_a^L + \not{R}^a T_a^R + Q_X \not{X} + \ldots = \not{L}^a T_a^L + \not{R}^b T_b^R + Y\not{B} + Q_{Z'}\not{Z}' + \ldots . \tag{6.25}$$

Note that the relation between the two bases changes correpondingly if we use canonical normalization.

Expanding at order $\mathcal{O}(p^2)$ and setting $\hat{H} = (0,0,1,0)$, we get (using $\hat{H}^\dagger T_a^L Y \hat{H} L_\mu^a B_\nu \to -\frac{1}{4}L_\mu^3 B_\nu$)

$$\begin{aligned}\mathcal{L} &= (P_T)_{\mu\nu}\Big[\frac{1}{2}\frac{F_\pi^2 s_h^2}{4}(B_\mu B_\nu + L_\mu^3 L_\nu^3 - 2L_\mu^3 B_\nu) + \frac{F_\pi^2 s_h^2}{4}L_\mu^+ L_\nu^-\Big] \\ &+ \frac{1}{2}(P_T)_{\mu\nu}p^2\big[\Pi_0'(0)L_\mu^a L_\nu^a + (\Pi_0'(0) + \Pi_0^X{}'(0))B_\mu B_\nu - \frac{s_h^2}{2}\Pi_1'(0)L_\mu^3 B_\nu\big] + \ldots \end{aligned} \tag{6.26}$$

Note that so far we have not used canonically normalized gauge fields. The coupling constants are defined by canonically normalizing them,

$$\Pi_0'(0) = -\frac{1}{g^2}, \Pi_0'(0) + \Pi_0^X{}'(0) = -\frac{1}{g'^2}. \tag{6.27}$$

With canonical fields, we have

$$m_W^2 = g^2\frac{F_\pi^2 \sin^2(\langle h\rangle/F_\pi^2)}{4} = \frac{g^2 v^2}{4}, \tag{6.28}$$

where we have defined $v = 246$ GeV $= F_\pi \sin\frac{\langle h\rangle}{F_\pi}$. Thus, we have,

$$\epsilon = \frac{v}{F_\pi} = \sin\frac{\langle h\rangle}{F_\pi}. \tag{6.29}$$

In particular, note that the vev of the composite Higgs is not v but related to it through the equation above. If we expand around its vev, we can compute the couplings of the physical Higgs to the gauge bosons,

$$h^{\hat{a}} = \begin{pmatrix} 0 \\ 0 \\ \langle h \rangle + h \\ 0 \end{pmatrix}. \tag{6.30}$$

Expading to linear order in h, we have

$$F_\pi^2 s_h^2 = F_\pi^2 \left[s_{\langle h \rangle}^2 + 2 s_{\langle h \rangle} c_{\langle h \rangle} \frac{h}{F_\pi} + (c_{\langle h \rangle}^2 - s_{\langle h \rangle}^2) \frac{h^2}{F_\pi^2} \ldots \right] = v^2 + 2v\sqrt{1-\epsilon^2} h + (1-2\epsilon^2) h^2 + \ldots \; . \tag{6.31}$$

Thus, the couplings of the gauge bosons to the Higgs, are reduced with respect to the SM ones by

$$\boxed{\begin{array}{l} g_{VVh} = g_{VVh}^{SM} \, \sqrt{1-\epsilon^2}, \\ \\ g_{VVhh} = g_{VVhh}^{SM} \, (1-2\epsilon^2) \end{array}} \quad (V = W, Z). \tag{6.32}$$

This reduction in the Higgs couplings is a generic feature of composite Higgs models. The particular dependence on ϵ depends on the group structure but the reduction is generic. This can be understood by noting that the Goldstones can be parametrized as a belonging to a full representation of the group G, as we did in our case with Σ. However, only the part transforming under H will give masses to the gauge bosons (in our case $\Sigma_{1,\ldots,4}$). That part mixes with the part along the coset (Σ_5 in our case), which is a singlet under the SM. The fact that the Higgs lives partly in a singlet reduces its coupling to the SM fields accordingly to that mixing.

This reduced couplings have two obvious effects in the two important roles that the Higgs played in the SM, restoration of unitarity in longitudinal gauge boson scattering and contribution to EWPT to make the SM compatible with experimental data. Regarding the former, the amplitude for longitudinal charged gauge boson scattering, taking into account the shift in the Higgs couplings, reads,

$$\begin{aligned} \mathcal{M}(W^+W^- \to W^+W^-) &= \frac{g^2}{4m_W^2} \left[s - s\frac{s(1-\epsilon^2)}{s-m_h^2} + t - t\frac{t(1-\epsilon^2)}{t-m_h^2} \right] \\ &= \frac{g^2}{4m_W^2} \left[s\epsilon^2 - m_h^2(1-\epsilon^2)\frac{s}{s-m_h^2} + (s \to t) \right]. \end{aligned} \tag{6.33}$$

In particular, when $\epsilon \to 0$ we recover the SM result and the Higgs (which couples as the SM Higgs in that case) is fully responsible for the restoration of unitarity. The limit $\epsilon \to 1$, on the other hand, results in no contribution from the Higgs and restoration of unitarity has to be done by the tower of hadronic resonances (technicolor limit). For intermediate values $0 < \epsilon < 1$, there is still a term that grows with energy ($s\epsilon^2$) and therefore longitudinal gauge boson scattering violates unitarity (unless restored by the hadronic resonances) at a scale

$$\Lambda_{\text{unitarity}} = \frac{\Lambda_{\text{unitarity}}^{\text{No Higgs}}}{\epsilon}. \tag{6.34}$$

In order for the theory to make sense above that scale, we need the hadronic resonances to be lighter than that scale $m_\rho \leq \Lambda_{\text{unitarity}}$.

The implications of the reduced couplings for EWPT are also easy to estimate. The leading contribution to the S and T from diagrams including the Higgs, read, in the SM and in the heavy Higgs approximation,

$$S, T = a_{S,T} \log m_h + b_{S,T}, \tag{6.35}$$

where $a_{S,T}$ and $b_{S,T}$ are constants. This contribution can be seen as the Higgs mass cutting-off the UV sensitivity of loops involving gauge bosons. In the SM, the Higgs couplings are precisely the correct ones to exactly cancel off such UV dependence. In our case, the couplings are modified and they will not cancel exactly the UV dependence. In the same heavy Higgs limit, and calling Λ the UV cut-off of our effective description, we get

$$\begin{aligned} S, T &= a_{S,T}[(1-\epsilon^2)\log m_h + \epsilon^2 \log \Lambda] + b_{S,T} \\ &= a_{S,T}[\log m_h + \log(\Lambda/m_h)^{\epsilon^2}] + b_{S,T} = a_{S,T} \log\left[m_h \left(\frac{\Lambda}{m_h}\right)^{\epsilon^2}\right] + b_{S,T}, \end{aligned} \tag{6.36}$$

which simply amounts to using an effective, heavier, Higgs mass in EWPT

$$m_{\text{EWPT,eff}} = m_h \left(\frac{\Lambda}{m_h}\right)^{\epsilon^2}. \tag{6.37}$$

This implies a positive contribution to S and a negative contribution to T,

$$\Delta S = \frac{1}{12\pi}\log\left(\frac{m^2_{\text{EWPT,eff}}}{m^2_{\text{h,ref}}}\right), \quad \Delta T = -\frac{3}{16\pi c_W^2}\log\left(\frac{m^2_{\text{EWPT,eff}}}{m^2_{\text{h,ref}}}\right). \tag{6.38}$$

Of course, we get again the SM result $m_{\text{EWPT,eff}} = m_h$ for $\epsilon \to 0$ and the technicolor result $m_{\text{EWPT,eff}} = \Lambda$ for $\epsilon \to 1$.

6.2.3 EWSB in the minimal composite Higgs model

In this section we are going to study the one loop potential generated for the Higgs. This is given by the Coleman-Weinberg effective potential, which determines the correct vacuum in the presence of explicitly symmetry breaking terms. Before doing the calculation in the case of the minimal composite Higgs model, we will review a very illuminating example from QCD.

Electromagnetic contribution to the pion mass

The QCD example we would like to discuss as a useful motivation of the Higgs potential in our minimal composite Higgs model is the electromagnetic contribution to the pion mass. Let us consider again 2 quark QCD in the chiral limit (massless quarks). As we know well, this theory has a global $SU(2)_L \times SU(2)_R$ symmetry spontaneously broken by the QCD vacuum to $SU(2)_V$. The corresponding Goldstone bosons are the pions. In the chiral limit they are exactly massless. In fact, the have no potential in that limit as all their interactions have to be derivative. Gauging part of the global symmetry actually breaks it and a potential for the Goldstone bosons (and in particular a mass) is generated at the quantum level. In order to study such effect, we will turn on the electromagnetic interaction (we gauge the $U(1)_Q$ subgroup of H). In practice this means that we include an elementary photon, A_μ, external to the strongly coupled theory (QCD), see Fig. 6.1. We have not introduced at the moment the $W^\pm$ and Z bosons. Furthermore, $U(1)_Q$ is not

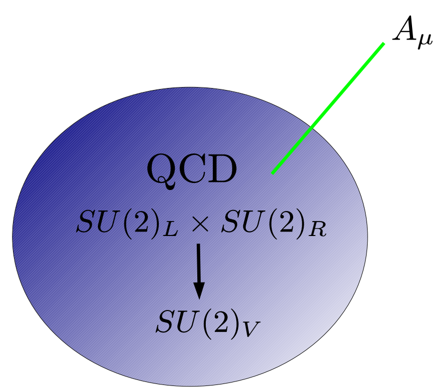


Figure 6.1: Elementary photon external to the strongly coupled theory (QCD). The pions are the Goldstone Bosons of the spontaneous symmetry breaking of the global symmetry.

broken by the QCD vacuum (it is embedded in the unbroken group, with $Q = T_3^L + T_3^R$, as far as the gauge bosons are concerned). This means that the photon does not eat any of the pions and they remain in the spectrum as massless scalars at tree level. The gauging of the electromagnetic group generates a potential for the (charged) pions at one loop. Our goal in this section is to compute such potential.

We first write the most general $SU(2)_L \times SU(2)_R$ invariant Lagrangian. We use the non-linear sigma model description of the Goldstone bosons, treating them as constant classical fields

$$\Sigma = e^{i\pi^a \sigma^a / f_\pi}, \quad \Sigma \to U_L \Sigma U_R^\dagger. \tag{6.39}$$

To make the analysis easier, we consider gauge bosons for the full $SU(2)_L \times SU(2)_R$ group. They are taken as external sources (we will switch off all of them except for the photon when time comes). At the quadratic level in those fields we have

$$\mathcal{L} = \frac{1}{2}(P_T)_{\mu\nu}\Big[\Pi_L(p^2)\mathrm{Tr}\big(L_\mu L_\nu\big) + \Pi_R(p^2)\mathrm{Tr}\big(R_\mu R_\nu\big) - \Pi_{LR}(p^2)\mathrm{Tr}\big(\Sigma^\dagger L_\mu \Sigma R_\nu\big)\Big], \quad (6.40)$$

$\Pi_L(p^2)$, $\Pi_R(p^2)$ and $\Pi_{LR}(p^2)$ are form factors that encode the effects of the exchange of the hadronic resonances. We can again get information from the form factors by going to the QCD vacuum $\Sigma = 1_{2\times 2}$. It is useful to rewrite the action in terms of vector (unbroken) and axial (broken) bosons,

$$V_\mu = \frac{R_\mu + L_\mu}{\sqrt{2}}, \quad A_\mu = \frac{R_\mu - L_\mu}{\sqrt{2}}. \quad (6.41)$$

We get

$$\mathcal{L} = \frac{1}{2}(P_T)_{\mu\nu}\Big[\frac{1}{2}\big(\Pi_L + \Pi_R + \Pi_{LR}\big)\mathrm{Tr}(V_\mu V_\nu) + \frac{1}{2}\big(\Pi_L + \Pi_R - \Pi_{LR}\big)\mathrm{Tr}(A_\mu A_\nu) + \frac{1}{2}\big(\Pi_R - \Pi_L\big)\mathrm{Tr}(V_\mu A_\nu + A_\mu V_\nu)\Big], \quad (6.42)$$

where

$$\Pi_{VV} = \frac{\Pi_L + \Pi_R - \Pi_{LR}}{2}, \quad (6.43)$$

$$\Pi_{AA} = \frac{\Pi_L + \Pi_R + \Pi_{LR}}{2}, \quad (6.44)$$

$$\Pi_{VA} = \frac{\Pi_R - \Pi_L}{2}. \quad (6.45)$$

Now we know that the only correlator that can have a massless pole (due to the exchange of the Goldstones) is $\langle A_\mu A_\nu \rangle$,

$$\langle A_\mu A_\nu \rangle = (p^2 \eta_{\mu\nu} - p_\mu p_\nu)\Big[\frac{f_\pi^2}{p^2} + \dots\Big]. \quad (6.46)$$

This implies

$$\Pi_{VV}(0) = \Pi_{VA}(0) = 0, \quad \Pi_{AA}(0) = f_\pi^2, \quad (6.47)$$

and therefore

$$\boxed{\Pi_L(0) = \Pi_R(0) = \frac{1}{2}\Pi_{LR}(0) = \frac{f_\pi^2}{2}.} \quad (6.48)$$

One other interesting implication is the fact that the left-right form factor

$$\Pi_{LR}(p^2) = \Pi_{AA}(p^2) - \Pi_{VV}(p^2), \quad (6.49)$$

is equal to the difference between the axial and vector correlators. In that sense it is an order parameter of chiral symmetry breaking, since it is zero or not depending on whether the chiral symmetry is (un)broken. Also, at energies much higher than Λ_{QCD} at which the breaking of the chiral symmetry is irrelevant, both correlators should be similar.

The effective Lagrangian that involves the photon and the pions is easy to obtain, by setting $L_\mu = R_\mu = Q v_\mu$, where v_μ represents the photon and

$$Q = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}. \tag{6.50}$$

We can also use the explicit form of $\Sigma = \cos(\pi/f_\pi) + i\hat{\pi}\cdot\vec{\sigma}\sin(\pi/f_\pi)$, with $\pi \equiv \sqrt{\pi^a\pi^a}$ and $\hat{\pi} \equiv \vec{\pi}/\pi$. The resulting effective Lagrangian reads,

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(P_T)_{\mu\nu}v_\mu v_\nu\Big[(\Pi_L(p^2)+\Pi_R(p^2))\mathrm{Tr}(Q^2) - \Pi_{LR}(p^2)\mathrm{Tr}(\Sigma^\dagger Q\Sigma Q)\Big] \\
&= \frac{1}{2}(P_T)_{\mu\nu}v_\mu v_\nu\Big[(\Pi_L(p^2)+\Pi_R(p^2)-\Pi_{LR}(p^2))\mathrm{Tr}(Q^2) \\
&\qquad + \Pi_{LR}(p^2)\frac{\sin^2(\pi/f_\pi)}{\pi^2}(\pi_1^2+\pi_2^2)\Big],
\end{aligned} \tag{6.51}$$

where we have used that $\mathrm{Tr}\,[\Sigma^\dagger Q\Sigma Q] = \mathrm{Tr}[Q^2] - \sin^2(\pi/f)(\pi_1^2+\pi_2^2)/\pi^2$. As we could have expected, we see that the photon does not induce any correction (as it doesn't couple to it directly) to the neutral pion but only to the charged ones. We can therefore set in the effective Lagrangian $\pi_3 = 0$ and therefore $\pi_1^2+\pi_2^2 = \pi^2$ to get

$$\mathcal{L} = \frac{1}{2}(P_T)_{\mu\nu}v^\mu v^\nu\Big[\Pi_\gamma(p^2) + \Pi_{LR}(p^2)\sin^2(\pi/f_\pi)\Big], \tag{6.52}$$

where we have defined $\Pi_\gamma(p^2) \equiv 2\mathrm{Tr}(Q^2)\Pi_{VV}(p^2)$.

We can now compute the one loop potential for the pions generated at one loop by photon exchange. This is given by the Coleman-Weinberg potential, that resums all mass insertions with zero external momentum at one loop (with the photon circulating in the loop). Introducing a generic ξ gauge,

$$\mathcal{L}_{GF} = -\frac{1}{2g^2\xi}(\partial_\mu v^\mu)^2,$$

we have for the propagator

$$i\Big[(P_T)_{\mu\nu}\frac{1}{\Pi_\gamma(p^2)} - \xi\frac{g^2}{p^2}(P_L)_{\mu\nu}\Big], \tag{6.53}$$

and for the "mass terms"

$$i\sin^2(\pi/f_\pi)\Pi_{LR}(p^2)(P_T)_{\mu\nu}. \tag{6.54}$$

The contribution of one loop diagrams with increasing number of mass insertions then gives

$$\begin{aligned}
V(\pi) &= 3i\int\frac{d^4p}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\Big[i^2\frac{\Pi_{LR}(p^2)}{\Pi_\gamma(p^2)}\sin^2(\pi/f_\pi)\Big]^n \\
&= \frac{3}{32\pi^2}\int_0^\infty dQ^2Q^2\log\Big[1+\frac{\Pi_{LR}(Q^2)}{\Pi_\gamma(Q^2)}\sin^2(\pi/f_\pi)\Big].
\end{aligned} \tag{6.55}$$

In the first line the 3 corresponds to the three polarization of the photon for a non-zero pion vev, the factor of $\frac{1}{2n}$ comes from symmetry factors in the diagrams whereas in the last equation we have resummed the series into a logarithm, Wick rotated to Euclidean momentum $Q^2 = -p^2$ and performed the angular integration $\int d\Omega = 2\pi^2$.

In order to continue with our calculation, we need to argue on the behaviour of the form factors at large Euclidean momenta and therefore on the convergence of the integral. Recall that Π_{LR} is an order parameter of chiral symmetry breaking and therefore we expect it go to go zero at large momenta. Thus, the integral is expected to be convergent and dominated by momenta smaller than the composite scale (this is what we mean by saying that pions are composite states and their masses are not UV sensitive). This statement can be made more precise by means of the operator product expansion (OPE) of the corresponding currents.

The time ordered product of two operators, $O_{1,2}(x)$, can be expressed as a sum of local operators times c-number coefficients that depend on the separation,

$$T\{O_1(x)O_2(0)\} = \sum_n c_{12}^n(x)O_n(0). \tag{6.56}$$

This equality is at the level of the operators, which imply the equality of any Green function made with them. The sum extends over all operators with the same quantum numbers of the product O_1O_2.

The OPE of the product two conserved currents, has the form, in Euclidean momentum space,

$$i\int d^4x\, e^{iq\cdot x}T\{J^\mu(x)J^\nu(0)\} = (q^2\eta^{\mu\nu} - q^\mu q^\nu)\sum_n c^{(n)}(q)O_n(0). \tag{6.57}$$

Purely based on dimensional analysis, it is clear that operators with larger mass dimension, are suppressed by a higher power of q^2 at large q,

$$c^{(n)}(q^2) \sim q^{-[O_n]}. \tag{6.58}$$

In QCD, the gauge invariant (color singlets) operators with dimension 6 or lower, spin zero (otherwise they cannot contribute to the vev without breaking Lorentz invariance) are

$$1 \text{ (Unity Operator)} \qquad (d=0) \tag{6.59}$$
$$O_M = \bar\psi M\psi \qquad (d=4) \tag{6.60}$$
$$O_G = G^a_{\mu\nu}G^{\mu\nu}_a \qquad (d=4) \tag{6.61}$$
$$O_\sigma = \bar\psi\sigma^{\mu\nu}T^a\tilde M\psi G^a_{\mu\nu} \qquad (d=6) \tag{6.62}$$
$$O_\Gamma = (\bar\psi\Gamma_1\psi)(\bar\psi\Gamma_2\psi) \qquad (d=6) \tag{6.63}$$
$$O_f = f^{abc}G^a_{\mu\nu}G^b_{\nu\gamma}G^c_{\gamma\mu} \qquad (d=6) \tag{6.64}$$

where M and $\tilde M$ are matrices in flavor space, whose elements are proportional to the quark masses and $\Gamma_{1,2}$ stand for some matrices acting on color, flavor and Lorentz indices. Out of these operators,

- The operators O_M and O_σ break explicitly the chiral symmetry and therefore vanish in the chiral limit (massless quarks).
- O_Γ is the only chiral invariant operator whose vev spontaneously breaks the chiral symmetry, distinguishing between the axial and vector currents.

As a consequence of the previous results, we have

$$\Pi_{LR}(q^2) = q^2 C_{O_\Gamma}(q^2)\langle O_\Gamma \rangle + \ldots = q^2 \left[\frac{\delta}{q^6} + \mathcal{O}(1/q^8) \right]. \tag{6.65}$$

The coefficient δ has been computed in perturbation theory by Shifman, Vainshtein and Zakharov [49],

$$\delta = +8\pi^2(\alpha_s/\pi + \mathcal{O}(\alpha_s^2))(\langle \bar{\psi}\psi \rangle)^2, \tag{6.66}$$

using the fact that $\langle O_\Gamma \rangle$ factorizes in $(\langle \bar{\psi}\psi \rangle)^2$ in the large-N limit. The OPE then guarantees the convergence of the integral, since

$$\Pi_\gamma(q^2) = -\frac{q^2}{e^2} + \mathcal{O}(q^0), \tag{6.67}$$

for large q^2. In fact, the integral can be reasonably well approximated by expanding the logarithm and keeping only the lowest term in Π_γ,

$$V(\pi) \approx \frac{3}{8\pi}\alpha \sin^2\left(\frac{\pi}{f_\pi}\right) \int_0^\infty dQ^2\, \Pi_{LR}(Q^2). \tag{6.68}$$

The integral can be now performed in the large-N limit as follows. We know the correlators in the large-N limit,

$$\Pi_{VV}(q^2) = q^2 \sum_n \frac{f_{\rho,n}^2}{q^2 - m_{\rho,n}^2}, \tag{6.69}$$

$$\Pi_{AA}(q^2) = q^2 \left[\frac{f_\pi^2}{q^2} + \sum_n \frac{f_{a,n}^2}{q^2 - m_{a,n}^2} \right]. \tag{6.70}$$

From the OPE we know the large q^2 behaviour of the LR correlator, which can be translated into sum rules for the decay constants and masses of the hadronic resonances in the large N limit

$$\Pi_{LR}(Q^2) \sim \frac{1}{Q^4} + \mathcal{O}(Q^{-6}), \quad \Rightarrow \begin{cases} \lim_{Q^2\to\infty} \Pi_{LR}(Q^2) = 0 \\ \lim_{Q^2\to\infty} Q^2\Pi_{LR}(Q^2) = 0 \end{cases} \tag{6.71}$$

If we insert in these equations the expression of the LR correlator in terms of the vector and axial ones in the large N limit, $\Pi_{LR} = \Pi_{AA} - \Pi_{VV}$, we get the first and second Weinberg sum rules,

$$\sum_n \left[f_{\rho,n}^2 - f_{a,n}^2 \right] = f_\pi^2, \tag{6.72}$$

$$\sum_n \left[f_{\rho,n}^2 m_{\rho,n}^2 - f_{a,n}^2 m_{a,n}^2 \right] = 0, \tag{6.73}$$

where we have used

$$\Pi_{LR} = \Pi_{AA} - \Pi_{VV} = f_\pi^2 + \sum_n \Big[f_{a,n}^2 \big(1 + \frac{m_{a,n}^2}{p^2} + \ldots\big) - f_{\rho,n}^2 \big(1 + \frac{m_{\rho,n}^2}{p^2} + \ldots\big)\Big]. \quad (6.74)$$

A further simplification comes from the assumption that the sum is dominated by the first vector and axial resonances, ρ, a_1. This so-called vector-meson dominance is experimentally confirmed in the case of QCD. In that case the first two Weinberg sum rules read

$$f_\rho^2 - f_{a_1}^2 = f_\pi^2, \quad f_\rho^2 m_\rho^2 - f_{a_1}^2 m_{a_1}^2 = 0, \quad (6.75)$$

which give

$$\boxed{f_\rho^2 = f_\pi^2 \frac{m_{a_1}^2}{m_{a_1}^2 - m_\rho^2}, \quad f_{a_1}^2 = f_\pi^2 \frac{m_\rho^2}{m_{a_1}^2 - m_\rho^2}.} \quad (6.76)$$

Thus, the Weinberg sum rules, together with vector-meson dominance and the large N limit, give, for the large Q^2 limit of the LR correlator

$$\Pi_{LR}(Q^2) \approx f_\pi^2 \left[1 + \frac{m_\rho^2}{m_{a_1}^2 - m_\rho^2} \frac{Q^2}{Q^2 + m_{a_1}^2} - \frac{m_{a_1}^2}{m_{a_1}^2 - m_\rho^2} \frac{Q^2}{Q^2 + m_\rho^2} + \ldots \right]. \quad (6.77)$$

Its integral can then be approximated by

$$\boxed{V(\pi) \approx \frac{3\alpha}{8\pi^2} \sin^2\left(\frac{\pi}{f_\pi}\right) \int_0^\infty dQ^2\, \Pi_{LR}(Q^2) \approx \frac{3\alpha}{8\pi^2} \sin^2\left(\frac{\pi}{f_\pi}\right) f_\pi^2 \frac{m_{a_1}^2 m_\rho^2}{m_{a_1}^2 - m_\rho^2} \log\left(\frac{m_{a_1}^2}{m_\rho^2}\right).} \quad (6.78)$$

Since we observe experimentally $m_{a_1} > m_\rho$, the integral is positive and the potential generated for the pion is minimized for

$$\sin\left(\frac{\langle\pi\rangle}{f_\pi}\right) = 0. \quad (6.79)$$

Thus, the radiative corrections align the vacuum along the $U(1)_Q$ preserving direction. This is in fact a more general result than the approximation we have used here to derive it. It is a consequence of the general property

$$\Pi_{LR}(Q^2) \geq 0, \text{ for } 0 \leq Q^2 \leq \infty, \quad (6.80)$$

as proved by Witten in the case of QCD [50].

Let us summarize the result we have obtained. Gauging the electromagnetic current explicitly breaks the symmetry, inducing a potential for the pseudo-Goldstone bosons (the pions) at the quantum level. The neutral pion is not affected by this and remains an exact Goldstone in this approximation. The potential generated for the charged pions induces a mass squared for them, which is positive, and therefore the electromagnetic gauge symmetry is not spontaneously broken in the process. All this was in the chiral limit. If we turn on the u, d quark masses, the neutral pion (and also the charged ones) will get a mass from the quark masses, but the mass difference between charged and neutral pions is still dominated by the one loop electromagnetic contribution we just computed. Our calculation gives,

$$m_{\pi^\pm}^2 - m_{\pi^0}^2 \approx \frac{3\alpha}{4\pi}\frac{m_{a_1}^2 m_\rho^2}{m_{a_1}^2 - m_\rho^2}\log\left(\frac{m_{a_1}^2}{m_\rho^2}\right). \tag{6.81}$$

This result was first obtained in [51] who also used $m_{a_1} \approx \sqrt{2}m_\rho$ to get

$$\Delta m_\gamma^2 \equiv m_{\pi^\pm}^2 - m_{\pi^0}^2 \approx \frac{3\alpha}{2\pi}m_\rho^2 \log 2 \approx 1430 \text{ MeV}^2. \tag{6.82}$$

This is the electromagnetic contribution to the charged pion mass. The explicit chiral symmetry breaking from the finite quark masses gives a common mass to all three pions on top of which the electromagnetic contribution sits. Thus we have

$$m_{\pi^\pm} = \sqrt{m_{\pi^0}^2 + \Delta m_\gamma^2} \approx m_{\pi^0} + \frac{\Delta m^2 \gamma}{2m_{\pi^0}} \approx m_{\pi^0} + 5.3 \text{ MeV}, \tag{6.83}$$

which is not far from the experimental result

$$m_{\pi^\pm} - m_{\pi^0} = 4.6 \text{ MeV}. \tag{6.84}$$

Using the large N limit of the correlators we can compute other observables. An example in the QCD chiral Lagrangian that be relevant in composite Higgs models is the kinetic mixing between left and right fields,

$$\begin{aligned}\mathcal{L}_\chi &= \frac{f_\pi^2}{4}\text{Tr}\left[(D_\mu\Sigma)^\dagger D^\mu\Sigma\right] + \ldots + L_{10}\text{Tr}\left[\Sigma^\dagger L_{\mu\nu}\Sigma R^{\mu\nu}\right] + \ldots \\ &= -\frac{1}{2}(P_T)_{\mu\nu}\left[f_\pi^2 - 4p^2 L_{10} + \ldots\right]\text{Tr}\left[\Sigma^\dagger L^\mu \Sigma R^\nu\right] + \ldots,\end{aligned} \tag{6.85}$$

so that the LR correlator, in terms of chiral coefficients reads

$$\Pi_{LR}(p^2) = f_\pi^2 - p^2 4L_{10} + \ldots, \tag{6.86}$$

or alternatively

$$L_{10} = -4\Big[\Pi'_{AA}(0) - \Pi'_{VV}(0)\Big], \tag{6.87}$$

where the prime denotes derivation with respect to p^2. The chiral coefficient L_{10} is the QCD analogue of the S parameter for electroweak symmetry breaking. Inserting in this expression Weinberg sum rules, we get

$$L_{10} = -4\left(\frac{f_\rho^2}{m_\rho^2} - \frac{f_{a_1}^2}{m_{a_1}^2}\right) = -4f_\pi^2\left(\frac{1}{m_\rho^2} + \frac{1}{m_{a_1}^2}\right) \leq 0. \tag{6.88}$$

Note that it has a well defined sign. In fact, we will obtain a similar result in composite Higgs models for the S parameter.

One loop potential in the minimal composite Higgs model

The previous discussion of the electromagnetic contribution to the pion mass can be translated almost unchanged to the Higgs potential in composite Higgs models. For simplicity, we will consider the contribution of the $SU(2)_L$ gauge fields, neglecting the smaller ($g' \ll g$) contribution from the $U(1)_Y$ one. The effective Lagrangian reads in that case

$$\mathcal{L}_{SU(2)_L} = \frac{1}{2}(P_T)^{\mu\nu}\left(\Pi_0 + \frac{s_h^2}{4}\Pi_1\right) L^a_\mu L^a_\nu. \tag{6.89}$$

Introducing again a generic ξ gauge,

$$\mathcal{L}_{GF} = -\frac{1}{2g^2\xi}(\partial_\mu L^{a\,\mu})^2,$$

we have for the propagator

$$i\Big[(P_T)_{\mu\nu}\frac{1}{\Pi_0(p^2)} - \xi\frac{g^2}{p^2}(P_L)_{\mu\nu}\Big], \tag{6.90}$$

and for the "mass terms"

$$i\frac{\sin^2(h/F_\pi)}{4}\Pi_1(p^2)(P_T)_{\mu\nu}. \tag{6.91}$$

The contribution of one loop diagrams with increasing number of mass insertions then gives

$$V(\pi) = \frac{9}{2}\int_0^\infty \frac{d^4Q}{(2\pi)^4}\log\Big[1 + \frac{\Pi_1(Q^2)}{\Pi_0(Q^2)}\frac{\sin^2(h/F_\pi)}{4}\Big]. \tag{6.92}$$

The factor of 9 stands for 3 polarizations times three $SU(2)_L$ gauge bosons. Once again, we have to argue on the convergence of this integral. Recall that Π_0 is related to the correlator of two unbroken currents

$$\langle J^\mu_a J^\nu_a\rangle = (P_T)^{\mu\nu}\Pi_0(p^2), \tag{6.93}$$

whereas Π_1 is to the difference between broken and unbroken

$$\langle J^\mu_a J^\nu_a\rangle - \langle J^\mu_{\hat a} J^\nu_{\hat a}\rangle = -\frac{1}{2}(P_T)^{\mu\nu}\Pi_1(p^2). \tag{6.94}$$

Note that much like Π_{LR} for QCD, Π_1 here is an order parameter of the symmetry breaking, which is sensitive to EWSB at energies of order F_π but goes to zero at momenta $Q^2 \gg F_\pi$. It is in this sense that composite models solve the hierarchy problem, since the Higgs potential is insensitive to scales much higher than the compositeness scale. The correct way of making these statements more precise is by using the OPE, as we did in the QCD case. The operator with the smallest dimension contributing to both broken and unbroken correlators is the identity operator,

$$\langle J^\mu_{a,\hat a} J^\nu_{a,\hat a}\rangle = (q^2\eta^{\mu\nu} - q^\mu q^\nu)\Big[C^{(1)}_{a,\hat a}(q^2) + \sum_{n>1} C^{(n)}_{a,\hat a}(Q^2)\langle\mathcal{O}_n\rangle\Big]. \tag{6.95}$$

For instance, if the strong sector is asymptotically free, we have, for the large Q^2 limit

$$C^{(1)}(Q^2) = -\frac{N}{24\pi^2}\log\frac{Q^2}{\mu^2} + \mathcal{O}(\alpha_s). \tag{6.96}$$

Thus, if we want the integral that determines the Higgs potential to be convergent, we need the first operator contributing to the difference of broken and unbroken generators to be dimension 5 or greater,

$$\langle J_a^\mu J_a^\nu\rangle - \langle J_{\hat{a}}^\mu J_{\hat{a}}^\nu\rangle = (q^2\eta^{\mu\nu} - q^\mu q^\nu)\Big[C^{(5)}(q^2)\langle\mathcal{O}_5\rangle + \dots\Big], \tag{6.97}$$

so that

$$\Pi_1(Q^2) \sim \frac{1}{Q^{n-2}} = \frac{1}{(Q^2)^{n/2-1}}, \quad n \geq 5 \text{ (for } Q^2 \to \infty). \tag{6.98}$$

Let us therefore assume that $\Pi_1(Q^2 \to \infty)$ goes to zero fast enough for the integral to converge, expanding the Logarithm we get,

$$V(h) = \frac{9}{8}\frac{g^2}{16\pi^2}\sin^2\left(\frac{h}{F_\pi}\right)\int_0^\infty dQ^2\,\Pi_1(Q^2). \tag{6.99}$$

The analogy with QCD suggests that $\Pi_1(Q^2)$ is positive, and therefore EWSB is not triggered by this correction (gauge interactions tend to align the vacuum with the symmetry preserving one). This was overcome in the original models by Georgi and Kaplan by introducing an extra gauged $U(1)_A$ group in the elementary sector, in addition to the SM gauge group $SU(2)_L \times U(1)_Y$. The crucial property to trigger EWSB was that the new gauge symmetry was not a subset of the unbroken H symmetry of the strong interaction (whereas the SM group is). This way the radiative corrections from $U(1)_A$ destabilized the EW-preserving vacuum and EWSB could be triggered. In that case, we would have

$$\epsilon = \frac{v}{F_\pi} = \sin\frac{\langle h\rangle}{F_\pi} \neq 0 \tag{6.100}$$

at the minimum of the potential.

The reason behind this extra complication of the model was that, at the time of the original models, the top quark was thought to be much lighter than the electroweak gauge bosons and therefore its possible contribution (which as we will see comes with the opposite sign) to the Higgs potential would be negligible. The fact that the top is so heavy changes the picture completely and it is in fact the top contribution the most important one. We will discuss such contribution in future lectures. Also, we will use five-dimensional models to compute the corresponding form factors and that way being able to fully compute the Higgs potential in a realistic composite Higgs model. For the moment, let us assume there is some extra contribution to the Higgs potential that induces EWSB, $\epsilon \neq 0$. As we mentioned, the S parameter is, in the electroweak theory, the analogue of the L_{10} chiral coefficient. From Eq.(6.26) we have

$$\mathcal{L} = -(P_T)_{\mu\nu}p^2\frac{s_h^2}{4}\Pi_1'(0)L_\mu^3 B_\nu + \dots\ , \tag{6.101}$$

which implies an S parameter (note that we have to canonically normalize the corresponding gauge fields)

$$S = 16\pi\Pi'_{3B}(0) = 16\pi\frac{s_h^2}{4}\Pi'_1(0)4\pi\epsilon^2\Pi'_1(0) = 8\pi\epsilon^2\left(\frac{f_\rho^2}{m_\rho^2} - \frac{f_{a_1}^2}{m_{a_1}^2}\right) = 4\pi\epsilon^2 F_\pi^2\left(\frac{1}{m_\rho^2} + \frac{1}{m_{a_1}^2}\right). \tag{6.102}$$

If we now use $\epsilon^2 F_\pi^2 = v^2$ and a relation similar to the QCD one between the hadronic masses $m_{a_1}^2 \sim \sqrt{2}m_\rho^2$, we get

$$S \approx 6\pi\left(\frac{v^2}{m_\rho^2}\right). \tag{6.103}$$

This expression for S in terms of v and m_ρ is in fact very similar for composite Higgs models and for technicolor models. The difference is in the relation between these two parameters. In technicolor we have

$$v = F_\pi \sim \frac{\sqrt{N}}{4\pi}m_\rho, \Rightarrow S_{TC} \approx \frac{3N}{8\pi}, \tag{6.104}$$

whereas in composite Higgs models we have

$$v = \epsilon F_\pi = \epsilon\frac{\sqrt{N}}{4\pi}m_\rho. \tag{6.105}$$

Thus, ϵ controls the ratio v/m_ρ and therefore the contribution to the S parameter

$$S_{CH} \sim \epsilon^2 S_{TC}. \tag{6.106}$$

A bound on S then turns into an upper bound on ϵ

$$S \leq 0.3 \Rightarrow \boxed{\epsilon^2 \leq \frac{48\pi}{60N} \approx \frac{1}{4}\left(\frac{10}{N}\right)}, \tag{6.107}$$

where we have defined $\frac{F_\pi^2}{m_\rho^2} \equiv \frac{N}{16\pi^2}$.

This bound on ϵ is mild enough to allow for phenomenologically viable composite Higgs models without too much fine-tuning. We have however not discussed yet one of the other big problems in technicolor theories, the generation of fermion masses and the associated presence of dangerous flavor violating processes. In fact, composite Higgs models suffer some of the same problems (mildly improved by the presence of the ϵ parameter) as technicolor models in regards of fermion masses, *if fermion masses are generated the same way as in technicolor.*

6.3 Fermions in Composite Higgs Models

Fermions can get a mass in composite Higgs models in a similar way to technicolor theories, by coupling to some composite operator which can be the Higgs itself or can couple to the Higgs so that electroweak symmetry breaking is propagated to the fermions and a mass for them is generated.

6.3.1 Fermion masses the wrong way

Let's assume, just as in technicolor, that the coupling to the composite operator is **bilinear**,

$$\mathcal{L} = \lambda \bar{q} q \mathcal{O}(x). \tag{6.108}$$

The operator $\mathcal{O}$ has the quantum numbers of the Higgs and therefore its square can generate a correction to the Higgs mass. In order for it not to be sensitive to UV physics (so that the hierarchy problem is not reintroduced), the anomalous dimensions has to be $[\mathcal{O}^2] \geq 4$ (the operator has to be irrelevant). At weak coupling or in the large N limit, this bound implies

$$[\mathcal{O}] \geq 2. \tag{6.109}$$

Then the fermion masses scale as

$$m_q \sim \Lambda_{TC} \left(\frac{\Lambda_{TC}}{\Lambda}\right)^{[\mathcal{O}]-1}, \quad \Lambda_{TC} \sim m_\rho. \tag{6.110}$$

The bound on the anomalous dimension of the corresponding operator means that Λ cannot be too large but that means that FCNC are not enough suppressed and therefore in conflict with observation,

$$\frac{(\bar{q}q)^2}{\Lambda^2}.$$

So far, we essentially have the same result as in technicolor theories. One difference with composite Higgs models is that F_π and therefore m_ρ can be made arbitrarily large by making ϵ small,

$$m_\rho \approx \frac{v}{\epsilon} \frac{4\pi}{\sqrt{N}}. \tag{6.111}$$

Thus, FCNC can be evaded by taking $\epsilon \ll 1$. Unfortunately, this can be done only at the expense of fine-tuning. This is the problem we have found already, we can technically take the SM limit in composite Higgs models and all the problems associated to technicolor like theories will disappear, but of course the Hierarchy problem will then reappear. We saw that with unitarity restoration in longitudinal gauge boson scattering, we saw it in the contribution to the S parameter and we see it now in the contribution to FCNC from the generation of fermion masses. The former two are mild enough to make the model compatible with just a mild fine-tuning. The FCNC however, is so restrictive that fully reintroduces the hierarchy problem. In fact, for $\epsilon \ll 1$, the following estimate holds,

$$\mu^2 \sim \frac{g^2}{16\pi^2} m_\rho^2, \quad \lambda \sim \frac{g^2}{N}, \tag{6.112}$$

so that the natural scale for the Higgs vev is

$$v^2 \sim \mu^2/\lambda \sim \frac{N}{16\pi^2} m_\rho^2 \sim f_\pi^2. \tag{6.113}$$

Thus in general we have several contributions that are naturally of order F_π^2 but their sum has to be fine-tuned to give a result $\epsilon^2 F_\pi^2$, the naive fine-tuning is therefore ϵ^2.

6.3.2 Fermion masses the right way

There is a way of solving the FCNC problem that we observed in the previous section. The solution is to couple the elementary fermions to the composite states linearly instead of quadratically [52]

$$\mathcal{L} = \lambda[\bar{q}\mathcal{O} + \text{h.c.}]. \tag{6.114}$$

In that case, the quark masses scale like

$$m_q \sim \epsilon m_\rho \left(\frac{m_\rho}{\Lambda}\right)^{2\gamma}, \tag{6.115}$$

where $\gamma = [\mathcal{O}\psi] - 4 = [\mathcal{O}] + \frac{3}{2} - 4 = [\mathcal{O}] - \frac{5}{2}$. The bound on $[\mathcal{O}]$ from the requirement of no UV instabilities is

$$[\bar{\mathcal{O}}\not\partial\mathcal{O}] \geq 4 \Rightarrow [\mathcal{O}] \geq \frac{3}{2} \text{ (large N)}. \tag{6.116}$$

Thus we get

$$2\gamma = 2[\mathcal{O}] - 5 \geq -2. \tag{6.117}$$

Recall that the unitarity bound ($[\mathcal{O}] \geq 3/2$) is compatible with no reintroduction of UV instabilities. In particular, we can now generate heavy enough masses with a large value of Λ provided γ is close enough to zero. The FCNC four-fermion operators involving elementary fields are then suppressed by the safely large scale Λ and therefore compatible with observations.

The linear coupling of elementary fermions to composite operators not only improves the constraints from FCNC, they also allow us to generate hierarchical fermion masses in a natural way. The reason is the following. Suppose that at some high scale $\Lambda \gg \Lambda_{EW}$, the strong sector is in the vicinity of an IR fixed points, so that the theory behaves like a conformal field theory at energies below Λ. In that case, the fermion masses can be estimated to be

$$m_q \sim \lambda_L(\mu)\lambda_R(\mu)\frac{N}{16\pi^2}\epsilon m_\rho, \tag{6.118}$$

where $\mu \sim \Lambda_{EW}$ and $\lambda_{L,R}$ are the corresponding Wilson coefficients for the couplings of $q_{L,R}$ to the corresponding composite Operators. If the corresponding anomalous dimensions $\gamma_{L,R}$ are small and positive, we have

$$\mu\frac{d}{d\mu}\lambda = \gamma\lambda \Rightarrow \lambda(\mu) \approx \lambda_0\left(\frac{\mu}{\Lambda}\right)^\gamma. \tag{6.119}$$

Given that $\mu/\Lambda \ll 1$, we have that small changes in the anomalous dimensions can induce very large changes in the low energy fermion masses

$$\boxed{m_q \sim \left(\frac{m_\rho}{\Lambda}\right)^{\gamma_L+\gamma_R}\frac{\sqrt{N}}{4\pi}v}. \tag{6.120}$$

Is this enough to generate the large top mass? The answer is yes, provided the anomalous dimensions are negative. In that case the corresponding operator is relevant and the coupling λ grows in the IR. The growth of λ at lower energies forces us to include higher orders in the RGE for λ (due to wave-function renormalization),

$$\mu\frac{d}{d\mu}\lambda = \gamma\lambda + c\frac{N}{16\pi^2}\lambda^3. \tag{6.121}$$

If c is positive, λ continues to grow in the IR until it hits a new IR fixed point

$$\lambda_* \approx \sqrt{\frac{-\gamma}{c}}\frac{4\pi}{\sqrt{N}}. \tag{6.122}$$

Note that the IR fixed point is still perturbative (in λ) in the large N limit. We can this way get a large enough mass to reproduce the top mass. For instance, if both $\mathcal{O}_{L,R}$ are relevant, we get

$$m_t \sim \frac{4\pi}{\sqrt{N}}v\sqrt{\gamma_L\gamma_R}. \tag{6.123}$$

Thus, the linear coupling of elementary fermions to composite operators allowed us to have a large UV scale Λ to suppress FCNC while still generating large enough fermion masses and even the top mass. This construction has even more interesting features that we will discuss in the next few sections.

Note that there are new possible contributions to FCNC induced by the exchange of composite states at the scale m_ρ. However, the coupling of any SM fermion to the composite sector occurs through the Wilson coefficients λ_i and therefore FCNC four-fermion interactions are proportional to the SM fermion masses,

$$\frac{q_i q_j q_k q_l}{\Lambda^2} \sim \frac{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}}{m_\rho^2}, \tag{6.124}$$

this naturally predict small flavor violation for light SM fermions and large violations in top couplings. The former, which are very stronly constrained experimentally, are suppressed enough to be compatible with experiment whereas the latter are essentially untested experimentally but will be subject to study at the LHC.

6.3.3 Partial compositeness

The linear coupling of elementary fermions to the composite operators induces another interesting feature: partial compositeness. Partial compositeness stands for the following property. Imagine the operator $\mathcal{O}$ to which the elementary fermion couples linearly has the right quantum numbers to excite a tower of massive fermionic composite states ξ_n,

$$\langle 0|\mathcal{O}|\xi_n\rangle = \Delta_n. \tag{6.125}$$

The linear coupling

$$\mathcal{L} = \bar{\psi}\mathcal{O} + \text{h.c.}, \tag{6.126}$$

induces then at low energies a mass mixing between the composite fermionic states and the elementary fermions

$$\mathcal{L}_{\text{mix}} = \sum_n \Delta_n(\bar{\psi}\xi_n + \text{h.c.}). \tag{6.127}$$

Note: The same coupling is induced between elementary gauge bosons and vector resonances that are excited by the corresponding conserved current,

$$\langle 0|J^\mu|\rho_n\rangle = \epsilon^\mu m_{\rho,n} f_{\rho,n}. \tag{6.128}$$

We then have

$$\mathcal{L}_{\text{mix}} = \sum_n m_{\rho,n} f_{\rho,n} A_\mu \rho_n^\mu, \tag{6.129}$$

which is known in the literature as ρ photon mixing.

This mass mixing means that the physical states, before EWSB, will be an admixture of elementary and composite states, thus the name partial compositeness. Take for simplicity only one resonance, the Lagrangian in the elementary/composite basis reads,

$$\mathcal{L} = \bar{\psi}_L i\not{\partial}\psi_L + \bar{\chi}(i\not{\partial} - m)\xi + \Delta(\bar{\psi}_L\xi_R + \text{h.c.}). \tag{6.130}$$

Due to the mixing term, ψ_L and χ are not mass eigenstates. They can be easily diagonalized by taking

$$\begin{pmatrix} \psi_L^l \\ \psi_L^h \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \psi_L \\ \xi_L \end{pmatrix}, \tag{6.131}$$

with $\tan\phi = \Delta/m$. The resulting Lagrangian reads

$$\mathcal{L} \to \bar{\psi}_L^l i\not{\partial}\psi_L^l + \bar{\psi}^h(i\not{\partial} - \frac{m}{\cos\phi})\psi^h. \tag{6.132}$$

$\psi^{l,h}$ represent the light (SM) and heavy physical fields, respectively. Note that the SM fermions are still massless (before EWSB), this is due to conservation of the fermionic index. Charged fermions can only get Dirac masses that mix a LH field with a RH field. The fermion content we started with was two LH fields and one RH field, thus, after rotation we will be left with a single LH field that has to be massless. Once EWS is broken, it will marry another massless fermion with the opposite chirality to get a mass. The angle ϕ parametrizes the degree of compositeness of the SM fermions:

- The larger ϕ the more composite a SM particle is.
- In order to solve the hierarchy problem, the Higgs should be totally composite.
- Heavier particles (after EWSB) couple more to the Higgs and therefore are more composite, lighter particles are more elementary

$$\lambda = g_* \sin\phi_L \sin\phi_R. \tag{6.133}$$

The same considerations apply to the gauge sector

$$\mathcal{L} = -\frac{1}{4g_e^2}F_{e,\mu\nu}^2 - \frac{1}{4g_*^2}F_{c,\mu\nu}^2 + \frac{M_*^2}{2}\left(\frac{g_e}{g_*}A_\mu^e - A_\mu^c\right)^2, \tag{6.134}$$

which can be diagonalized by

$$\begin{pmatrix} A_\mu \\ \rho_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_\mu^e \\ A_\mu^c \end{pmatrix}, \tag{6.135}$$

to give

$$\mathcal{L} = -\frac{1}{2}F_{\mu\nu}^2 + \frac{1}{2}\left(D_\mu\rho_\nu D_\nu\rho_\mu - D_\mu\rho_\nu D_\mu\rho_\nu\right) + \frac{M_*^2}{2\cos^2\theta}\rho_\mu^2 + \ldots \ . \tag{6.136}$$

This simple model is very useful to parametrized general composite Higgs models and their extra dimensional counterparts, see [53] for more details.

6.3.4 Fermion contribution to the Higgs potential

We saw in the previous section that the top gets its large mass as a consequence of it being partially composite, its couplings to the composite sector are large, $\lambda(\mu) \sim 4\pi\sqrt{-\gamma}/\sqrt{N}$, and that induces a large coupling to the composite sector. But this large coupling suggests that the top sector could significantly contribute to the Higgs potential at one loop. Recall that the top is an elementary field, external to the strong sector, that couples linearly (and strongly) to some of the composite states. This coupling breaks the global symmetry and induces a potential for the pseudo-Goldstone bosons (our composite Higgs) at one loop. Let us investigate such contribution.

First we have to choose a representation for the fermions. Each SM fermion will couple linearly to an operator of the strongly coupled theory which has to transform as a representation of $SO(5) \times U(1)_X$

$$\Delta\mathcal{L} = \lambda(\bar{\psi}\mathcal{O} + \text{h.c.}). \tag{6.137}$$

As we did with the gauge bosons, the simplest thing to do is to promote the SM to full representations of $SO(5) \times U(1)_X$ by means of spurions that we will set to zero at the right time. The smallest representations we can embed the SM fermions are the spinorial (4), fundamental (5) or adjoint (10) representations of $SO(5)$. The advantage of the latter two is that they allow us to implement a $L \leftrightarrow R$ parity within the custodial symmetry that can be used to protect the $Zb\bar{b}$ coupling. For simplicity we will consider just the top sector, as the bottom and other light fermions are not expected to give a sizable contribution to the Higgs potential. We choose to incorporate the SM fermions into fundamentals of $SO(5)$,

$$\Psi_{qL} = \begin{bmatrix} \begin{pmatrix} q'_L \\ q_L \end{pmatrix} \\ u'_L \end{bmatrix}, \quad \Psi_{tR} = \begin{bmatrix} \begin{pmatrix} q_R^t \\ q_R^{\prime t} \end{pmatrix} \\ t_R \end{bmatrix}, \tag{6.138}$$

where both multiplets have $Q_X = 2/3$ and only q_L and t_R are elementary fields (the SM third generation quark doublet and the RH top), all other fields are the spurions needed to complete full $SO(5)$ representations. The most general $SO(5)\times U(1)_X$ invariant Lagrangian, at the quadratic order and in momentum space, is

$$\begin{aligned} \Delta\mathcal{L} &= \sum_{r=q_L,t_R} \bar{\Psi}^i_r \not{p}\Big(\delta^{ij}\hat{\Pi}^r_0(p) + \Sigma^i\Sigma^j\hat{\Pi}^r_1(p)\Big)\Psi^j_r \\ &+ \bar{\Psi}^i_{qL}\Big(\delta^{ij}\hat{M}_0(p) + \Sigma^i\Sigma^j\hat{M}_1(p)\Big)\Psi^j_{uR} + \text{h.c.}, \end{aligned} \tag{6.139}$$

where $i = 1, \ldots, 5$ run over the $SO(5)$ indices in the fundamental representation, $\hat{\Pi}^{qL,uR}_{0,1}(p)$ and $\hat{M}_{0,1}(p)$ are form factors that parametrize the effect of the composite sector and we have

$$\Sigma = s_h(\hat{h}^1, \hat{h}^2, \hat{h}^3, \hat{h}^4, c_h/s_h). \tag{6.140}$$

Setting to zero the spution fields, we get the effective Lagrangian for the SM fields and the Higgs,

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \bar{q}_L\not{p}\left[\Pi^q_0(p^2) + \frac{s_h^2}{2}\Pi^q_1(p^2)\hat{H}^c\hat{H}^{c\dagger}\right]q_L + \bar{u}_R\not{p}\left[\Pi^u_0(p^2) + \frac{s_h^2}{2}\Pi^u_1(p^2)\right]u_R \\ &+ \frac{s_hc_h}{\sqrt{2}}M^u_1(p^2)\bar{q}_L\hat{H}^cu_R + \text{h.c.}, \end{aligned} \tag{6.141}$$

wher we have defined

$$\hat{H}^c = \begin{pmatrix} -\hat{h}^3 - i\hat{h}^4 \\ \hat{h}^1 + i\hat{h}^2 \end{pmatrix}, \tag{6.142}$$

and we have

$$\Pi^q_{0,1} = \hat{\Pi}^{qL}_{0,1}, \quad \Pi^u_0 = \hat{\Pi}^{uR}_0 + \hat{\Pi}^{uR}_1, \quad \Pi^u_1 = -\frac{1}{2}\hat{\Pi}^{uR}_1, \quad M^u_1 = \hat{M}_1. \tag{6.143}$$

From this effective Lagrangian we can give a good estimation of the top mass by setting $p^2 = 0$,

$$m_t \approx \frac{s_hc_h}{\sqrt{2}}\frac{M^u_1(0)}{\sqrt{Z_{uL}Z_{uR}}}, \tag{6.144}$$

where the wave function renormalization constants are

$$Z_{uL,R} = \Pi^{q,u}_0(0) + \frac{s_h^2}{2}\Pi^{q,u}_1(0). \tag{6.145}$$

The effective Lagrangian also allows us to compute the contribution to the Higgs effective potential, through the Coleman-Weinberg formula

$$\begin{aligned} V(h) &= -2N_c\int\frac{d^4Q}{(2\pi)^4}\left\{\log\left(1 + \frac{s_h^2}{2}\frac{\Pi^q_1}{\Pi^q_0}\right) + \log\left(1 + \frac{s_h^2}{2}\frac{\Pi^u_1}{\Pi^u_0}\right)\right. \\ &\left. + \log\left(1 - \frac{s_h^2c_h^2}{2}\frac{(M^u_1)^2}{(-Q^2)[\Pi^q_0 + \Pi^q_1s_h^2/2][\Pi^u_0 + \Pi^u_1s_h^2/2]}\right)\right\}. \end{aligned} \tag{6.146}$$

The first two contributions come from diagrams that only involve t_L or only involve t_R and they are similar to the ones we found for gauge bosons. The last contribution comes from diagrams that involve both t_L *and* t_R. In that case, the full term proportional to $\not{p}$ gives us the propagator

$$\frac{i}{\not{p}(\Pi_0^{q,u} + \Pi_1^{q,u} s_h^2/2)}, \tag{6.147}$$

and the term proportional to M_1^u gives us the vertex (note that in order to close the loop we need two of those, thus the square in the contribution to the Higgs potential). Now if we assume as usual that the integrals are convergent and that we can expand the logarithms, we get a potential of the form

$$V(h) \approx \alpha s_h^2 - \beta s_h^2 c_h^2 = (\alpha - \beta) s_h^2 + \beta s_h^4, \tag{6.148}$$

where

$$\alpha = -2N_c \int \frac{d^4Q}{(2\pi)^4} \left\{ \frac{1}{2}\frac{\Pi_1^q}{\Pi_0^q} + \frac{1}{2}\frac{\Pi_1^u}{\Pi_0^u} \right\}, \tag{6.149}$$

$$\beta = 2N_c \int \frac{d^4Q}{(2\pi)^4} \left\{ \frac{1}{2}\frac{(M_1^u)^2}{(Q^2)\Pi_0^q \Pi_0^u} \right\}. \tag{6.150}$$

In this approximation, we have

$$F_\pi V' = c_h[2(\alpha-\beta)s_h + 4\beta s_h^3] = s_{2h}[\alpha - \beta + 2\beta s_h^2], \tag{6.151}$$

$$F_\pi^2 V'' = 2[\alpha - \beta + 2\beta s_h^2] + s_{2h}[4\beta s_h c_h] = 2[\alpha - \beta + 2\beta s_h^2] + 8\beta s_h^2 c_h^2, \tag{6.152}$$

where a prime here denotes derivative with respect to h. In particular we have EWSB ($V''(0) < 0$) if $\alpha - \beta < 0$. In that case, there is a minimum of the potential with $0 < s_h < 1$ for $\beta > 0$ and $\beta > |\alpha|$

$$\langle s_h^2 \rangle = \epsilon^2 = \frac{\beta - \alpha}{2\beta}. \tag{6.153}$$

Recall that if the minimum is at $s_h = 1$ we go back to the technicolor limit. We have to add this top contribution to the gauge boson one, which is in general smaller (although for realistic values of the higgs vev becomes relevant). Whether one can actually get a reasonable vev for the Higgs or not depends on the actual values of the integrals. One thing we can do (we'll see it in future lectures) is to resort to five-dimensional models in warped extra dimensions, which are weakly coupled duals of these composite Higgs models. In the 5D picture, one can compute the form factors and the integrals and the following results can be found. In general $|\alpha_{L,R}|$ are parametrically larger than β, however, we naturally have that α_L and α_R have opposite signs and therefore a partial cancellation is possible, leading to $\beta > |\alpha_L + \alpha_R|$ and therefore to a natural realistic pattern EWSB.

Our potential also gives us an estimate of the Higgs mass,

$$m_h^2 = V''(h_{\text{min}}) = \frac{8\beta}{F_\pi^2} s_h^2 c_h^2. \tag{6.154}$$

Now we have

$$\beta = \int \frac{d^4Q}{(2\pi)^4} \frac{2N_c}{Q^2} F(Q^2), \tag{6.155}$$

where $F(Q^2)$ is defined in Eq. (6.150). If we define some scale Λ at which the integral is effectively cut-off

$$\Lambda^2 \equiv 2 \int_0^\infty dQ\, Q \frac{F(Q^2)}{F(0)}, \tag{6.156}$$

that we would expect to be of the order of the mass of the first fermionic resonances, and we use the approximate expression for the top mass

$$m_t^2 \approx \frac{s_h^2 c_h^2}{2} F(0), \tag{6.157}$$

we get for the Higgs mass

$$m_h^2 \approx \frac{N_c}{\pi^2} \frac{m_t^2}{v^2} \epsilon^2 \Lambda^2. \tag{6.158}$$

Numerically we get

$$m_h \approx 190 \text{ GeV } \left(\frac{\epsilon}{0.5}\right) \left(\frac{\Lambda}{1 \text{ TeV}}\right), \tag{6.159}$$

thus one expects a light Higgs in this scenario (in practice, 5D models typically give an even lighter Higgs, mostly because of the presence of light fermionic composites that imply a lower value of Λ).

Chapter 7

Models with Extra Dimensions

Literature: This section has been taken almost entirely from Roberto Contino's notes.

Models with more than four space-time dimensions offer new possibilities for a natural realization of EWSB. They can provide new mechanism to explain the stability of the EW scale against UV physics and also other observed hierarchies in nature, like fermion masses. We will be interested in them also as new realizations of strongly coupled theories and in particular of composite Higgs models.

The simplest way to make models with extra dimensions compatible with observation is to compactify the extra dimensions with a size below current experimental reach. Let us discuss how one can analyze these models. We will take as an example models with one extra dimension. Although more dimensions can give rise to new phenomenology, the simpler 5D models provide enough intuition and tools to study other models.

7.1 5D models compactified on a circle

7.1.1 Gauge theories in a circle

Let us consider a gauge theory in 5D with the fifth dimension compactified on a circle of radius R. The action reads,

$$\begin{aligned} S &= \int d^4x dy \{-\frac{1}{4}F_{MN}F^{MN}\} = \int d^4x dy \{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}F_{\mu 5}F_5^{\mu}\} \\ &= \int d^4x dy \{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}A_\mu \partial_5^2 A^\mu + \frac{1}{2}\partial_\mu A_5 \partial^\mu A_5 - \partial_5 A^\mu \partial_\mu A_5\}, \end{aligned} \tag{7.1}$$

where we have denoted with M, N all five dimensions, μ, ν the four extended dimensions (denoted with an x, whereas the extra dimension is denoted with a y) and used a mostly minus metric $\eta_{MN} = \text{diag}(1, -1, -1, -1, -1)$. We have also integrated by parts when necessary (in the circle there are no remaining boundary terms). Let us now insert the Fourier expansion of A_M,

$$A_M(x,y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \left[\frac{1}{1+\delta_{n,0}} A_M^{e(n)}(x) \cos(ny/R) + A_M^{o(n)}(x) \sin(ny/R) \right], \tag{7.2}$$

in the action and integrate over the extra dimension y to get

$$S = \int d^4x \sum_n \{ -\frac{1}{4} F^{e(n)}_{\mu\nu} F^{e(n)\mu\nu} - \frac{1}{4} F^{o(n)}_{\mu\nu} F^{o(n)\mu\nu} + \frac{1}{2}(m_n A_\mu^{e(n)} + \partial_\mu A_5^{o(n)})^2 + \frac{1}{2}(m_n A_\mu^{o(n)} - \partial_\mu A_5^{e(n)})^2 \}, \tag{7.3}$$

where $m_n \equiv n/R$ and we have used the orthonormality of the Fourier basis. Note that, except for $n=0$, $A_5^{o,e(n)}$ is the eaten Goldstone boson of $A_\mu^{e,o(n)}$ to give it a mass. Thus, we have described the 5D gauge boson as a massless 4D gauge boson, $A_\mu^{e(0)}$, a 4D massless scalar $A_5^{e(0)}$, and two towers of massive gauge bosons with masses $m_n = n/R$. We have considered here a $U(1)$ gauge group as an example. Since we are solving only for the quadratic terms so far, the result is identical for non-abelian gauge groups, although in that case, the massless scalar would transform in the adjoint representation of the corresponding gauge group. Also note that in the $R \to 0$ limit, all massive modes decouple and we are only left with 4D massless fields (the so called zero modes) which is the mathematical way of saying that we cannot see the extra dimension if it is small enough. This also means that at energies below the inverse of the size of the extra dimension, we just see the world as four-dimensional. Only when we probe energies high enough, we will start perceiving the 5D nature (or equivalently, the tower of KK modes).

Gauges

Consider the case of a non compact extra dimension ($R \to \infty$). We can then get to the $A_5 = 0$ gauge by means of the gauge transformation

$$\Omega = \mathcal{P} \exp \left[ig \int_{-\infty}^{y} dy' A_5(x,y') \right], \quad \partial_y \Omega = igA_5\Omega. \tag{7.4}$$

Under this transformation, A_5 transforms as,

$$A_5' = \frac{i}{g}\Omega^{-1}[\partial_5 - igA_5]\Omega = \frac{i}{g}\Omega^{-1}[igA_5 - igA_5]\Omega = 0. \tag{7.5}$$

However, if R is finite, Ω is not periodic, due to $A_5^{e(0)}(x)$ (the lower limit of the integral would be in that case 0). We can still use $\tilde{\Omega} \equiv \Omega \exp[-igyA_5^{e(0)}]$, which is periodic. In that case we have

$$A_5' = A_5^{e(0)}(x). \tag{7.6}$$

Thus, we can eliminate the y dependent part of A_5 but not the zero mode (which agrees with the fact that it is not an eaten Goldstone boson). This corresponds then to the unitary gauge, in which all would-be Goldstone bosons are eaten and only physical modes appear.

Anoter alternative is to go to an R_ξ type gauge by adding the following gauge-fixing term

$$\mathcal{L}_{GF} = \int d^4xdy \left[-\frac{1}{2\xi}(\partial_\mu A^\mu - \xi \partial_5 A_5)^2 \right], \tag{7.7}$$

which eliminates the $A_\mu - A_5$ mixing

$$\mathcal{L}+\mathcal{L}_{GF} = \int d^4xdy \left\{ -\frac{1}{2}A_\nu \left[\left(1-\frac{1}{2\xi}\right)\partial^\mu\partial^\nu - g^{\mu\nu}\partial^\rho\partial_\rho \right] A_\mu - \frac{1}{2}A_\mu\partial_5^2 A^\mu - \frac{1}{2}A_5\partial^\rho\partial_\rho A_5 + \frac{\xi}{2}A_5\partial_5^2 A_5 \right\}. \tag{7.8}$$

7.1.2 Fermions in a circle

Let us now consider fermions in our 5D space. One important feature of fermions in 5D is that the smallest spinorial representation of the Lorentz group in 5D is a four-component Dirac fermion. The Gamma matrices in 5D are

$$\Gamma^M = (\gamma^\mu, -i\gamma^5), \tag{7.9}$$

so that γ^5 is part of Lorentz transformation and Ψ_L and Ψ_R are related by them. This also means that we can actually write a Dirac mass term. The action reads,

$$S = \int d^4xdy\{\bar{\Psi}(i\Gamma^M\partial_M - M)\Psi\} = \int d^4xdy\{\bar{\Psi}_L i\partial\!\!\!/ \Psi_L + \bar{\Psi}_R i\partial\!\!\!/ \Psi_R - \bar{\Psi}_L[\partial_5+M]\Psi_R + \bar{\Psi}_R[\partial_5 - M]\Psi_L\}. \tag{7.10}$$

Inserting again the corresponding Fourier expansion, we get

$$\begin{aligned} S \;=\; & \int d^4xdy \sum_n \{\bar{\psi}_L^{e(n)} i\partial\!\!\!/ \psi_L^{e(n)} + \bar{\psi}_L^{o(n)} i\partial\!\!\!/ \psi_L^{o(n)} + \bar{\psi}_R^{e(n)} i\partial\!\!\!/ \psi_R^{e(n)} + \bar{\psi}_R^{o(n)} i\partial\!\!\!/ \psi_R^{o(n)} \qquad (7.11) \\ & - m_n[\bar{\psi}_L^{e(n)}\psi_R^{o(n)} - \bar{\psi}_L^{o(n)}\psi_R^{e(n)} + \text{h.c.}] - M[\bar{\psi}_L^{e(n)}\psi_R^{e(n)} + \bar{\psi}_L^{o(n)}\psi_R^{o(n)} + \text{h.c.}]\}. \end{aligned}$$

The quadratic terms are not diagonal in the Fourier basis, however, they only mix level by level with a mass matrix

$$\begin{pmatrix} \bar{\psi}_L^{e(n)} & \bar{\psi}_L^{o(n)} \end{pmatrix} \begin{pmatrix} M & m_n \\ -m_n & M \end{pmatrix} \begin{pmatrix} \psi_R^{e(n)} \\ \psi_R^{o(n)} \end{pmatrix} + \text{h.c.}, \tag{7.12}$$

which can be diagonalized with the rotation

$$\begin{pmatrix} \psi_{L,R}^{e(n)} \\ \psi_{L,R}^{o(n)} \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i} & 1 \\ -\mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} \psi_{L,R}^{e(n)} \\ \psi_{L,R}^{o(n)} \end{pmatrix}, \quad n \neq 0. \tag{7.13}$$

The action reads, in the new basis,

$$
\begin{aligned}
S \quad \to \quad & \int d^4x dy \sum_n \{ \bar{\psi}_L^{e(n)} i\partial\!\!\!/ \psi_L^{e(n)} + \bar{\psi}_L^{o(n)} i\partial\!\!\!/ \psi_L^{o(n)} + \bar{\psi}_R^{e(n)} i\partial\!\!\!/ \psi_R^{e(n)} + \bar{\psi}_R^{o(n)} i\partial\!\!\!/ \psi_R^{o(n)} \\
& -[(M + im_n)\bar{\psi}_L^{e(n)} \psi_R^{e(n)} + (M - im_n)\bar{\psi}_L^{o(n)} \psi_R^{o(n)} + \text{h.c.}]\}, \qquad (7.14)
\end{aligned}
$$

so that we end up with a vector-like "zero mode", $\psi_{L,R}^{e(0)}$, with mass M and two towers of "massive KK modes" with masses, $\sqrt{M^2 + \frac{n^2}{R^2}}$. As with the gauge bosons, even if $M = 0$, we have just zero modes below energies $\sim R^{-1}$ so that for small enough extra dimensions they are undetectable.

7.1.3 Couplings in extra dimensions

So far we have described the quadratic terms in the action of 5D fields in terms of infinite towers of 4D KK modes. Let us consider now the interactions among different fields. As an example, we take the coupling of fermions and gauge bosons. The relevant part of the action reads

$$
\Delta S = \int d^4x dy g_5 \bar{\Psi} \gamma^\mu \Psi A_\mu = \sum_{mnr} \int d^4x \frac{g_5}{\sqrt{\pi R}} \Big[\int dy \frac{f_m^{\Psi_L} f_n^{\Psi_L} f_r^A}{\pi R} \Big] \bar{\psi}_L^{(m)} \gamma^\mu \psi_L^{(n)} A_\mu^{(r)} + L \to R, \qquad (7.15)
$$

where we have used the general KK expansions

$$
A_\mu = \sum_n \frac{f_n^A}{\sqrt{\pi R}} A_\mu^{(n)}, \quad \Psi_{L,R} = \sum_n \frac{f_n^{\Psi_{L,R}}}{\sqrt{\pi R}} \psi_{L,R}^{(n)}. \qquad (7.16)
$$

The mass dimensions of different fields and couplings are not the same if 5D as in 4D. Taking into account that the action has to be dimensionless, we get the following mass dimensions

$$
[A_\mu] = \frac{3}{2}, \quad [\Psi] = 2, \quad [g_5] = -\frac{1}{2}. \qquad (7.17)
$$

In particular, $g_5/\sqrt{\pi R}$ is dimensionless. The structure of the coupling above is generic, the effective 4D coupling is given by the 5D coupling, properly normalized, times the overlap of the corresponding KK mode wave functions.

7.2 5D models compactified on an interval

The model we just outlined above is not realistic. For once, we have obtained vector-like fermions contrary to what we observe in the SM fermions, which are chiral. Second, we have obtained a massless scalar, which is the remnant of a higher-dimensional gauge field (and a closer look at its properties shows that the field is massless because a protection from the 5D gauge invariance), and therefore an excellent candidate for a naturally light

Higgs boson. However, it transforms in the adjoint representation of the corresponding gauge group, again different from what we observe in nature (the Higgs is a doublet, as opposed to a triplet of $SU(2)_L$). These problems can be both solved if we compactify on a singular space, instead of a smooth one like the circle. The simplest possibility is to assume the extra dimension is compactified not in a circle but in an S^1/Z_2 orbifold, which is just a circle with the points $y \to -y$ identified. This is a smooth manifold except at the two fixed points $y = 0, \pi R$. Fields can be classified into even or odd under the $y \to -y$ Z_2 symmetry so that, out of the two towers (sines and cosines) of the Fourier expansion, only half of it, the one with the right parity properties, will survive. In fact, it is common to consider more general orbifolds, like $S^1/(Z_2 \times Z_2')$, in which independent parities at $y = 0$ and $y = \pi R$ are assigned to the different fields. We will in fact consider a compactification that is equivalent to the most general 5D orbifold, an interval $0 \leq y \leq L$. We now have two boundaries in our extra dimension, which can have their corresponding boundary operators and that will receive contributions from boundary terms whenever we integrate by parts. Our fields will have boundary conditions that have to be compatible with the variation of the boundary action.

7.2.1 Models in an interval

Let us start with the example of a fermion in an interval. Assuming for the moment no boundary actions, we have

$$S = \int d^4x dy \Big\{ \frac{1}{2}(\bar{\Psi} i \Gamma^M \partial_M \Psi - i(\partial_M \Psi)^\dagger \Gamma^0 \Gamma^M \Psi) - M\bar{\Psi}\Psi \Big\}, \tag{7.18}$$

where we have written the action in an explicitly hermitian way. The variation of the action then reads

$$\begin{aligned} \delta S &= \int d^4x dy \Big\{ \frac{1}{2}(\delta\bar{\Psi} i\Gamma^M \partial_M \Psi + \bar{\Psi} i \Gamma^M \partial_M \delta\Psi - i(\partial_M \delta\Psi)^\dagger \Gamma^0 \Gamma^M \Psi - i(\partial_M \Psi)^\dagger \Gamma^0 \Gamma^M \delta\Psi) \\ &\qquad - M\delta\bar{\Psi}\Psi - M\bar{\Psi}\delta\Psi \Big\} \\ &= \int d^4x dy \Big\{ \delta\bar{\Psi}(i\Gamma^M \partial_M - M)\Psi + [\overline{(i\partial_M \Psi)}\Gamma^M - M\bar{\Psi}]\delta\Psi \Big\} \\ &+ \frac{1}{2}\int d^4x \Big[\bar{\Psi}\gamma^5 \delta\Psi - \delta\bar{\Psi}\gamma^5 \Psi \Big]_0^L, \end{aligned} \tag{7.19}$$

where the last term corresponds to the boundary contribution from integration by parts. The variation of the bulk part gives the bulk equations of motion

$$i\partial\!\!\!/ \Psi_{L,R} + (\pm \partial_5 - M)\Psi_{R,L} = 0, \tag{7.20}$$

whereas the boundary part of the variation provides the consistent boundary conditions. As an example, we can choose Dirichlet boundary conditions for either of the two chiralities, but once the b.c. for one chirality is chosen, the b.c. for the other chirality is fixed by the equations of motion,

$$\Psi_{R,L}(y_i) = 0 \Rightarrow (\partial_5 \pm M)\Psi_{L,R}\Big|_{y_i} = 0. \tag{7.21}$$

We denote these simple fermionic boundary conditions by $[\pm, \pm]$, where the first sign refers to the b.c. at $y = 0$ and the second at $y = L$ with a $[+]$ denoting $\Psi_R = 0$ whereas a $[-]$ denotes $\Psi_L = 0$. More general boundary conditions can be imposed if boundary terms are present in the action. Let us for the moment see where this simple choice takes us. Let us consider a general KK expansion

$$\Psi_{L,R}(x, y) = \sum_n f_n^{L,R}(y)\psi_{L,R}^{(n)}(x), \tag{7.22}$$

where we assume that the 4D fields satisfy the 4D Dirac equation

$$i\partial\!\!\!/\psi_{L,R}^{(n)} = m_n\psi_{R,L}^{(n)}. \tag{7.23}$$

Inserting this expansion in the equations of motion we obtain the equation for the fermionic profiles

$$(\pm\partial_5 - M)f_n^{R,L} = -m_n f_n^{L,R}, \tag{7.24}$$

which is a system of two coupled first order differential equations. We can decouple them by iteration to obtain

$$[\partial_5^2 \pm (\partial_5 M) - M^2 + m_n^2]f_n^{L,R} = 0. \tag{7.25}$$

The solutions depend on the value of M and the choice of boundary conditions. Let us discuss the presence of possible massless zero modes. If $m_0 = 0$ then the tow first order equations are already decoupled and we can solve them immediately,

$$f_0^{L,R} = A_{L,R}e^{\mp}My, \tag{7.26}$$

with $A_{L,R}$ the normalization constant to be determined by the normalization condition

$$\int_0^L dy f_m^L f_n^L = \int_0^L dy f_m^R f_n^R = \delta_{mn}. \tag{7.27}$$

Now, a Dirichlet b.c. at any boundary forces the corresponding normalization constant to vanish so that we will only have a zero mode for the following b.c.

$$[++] \qquad \Rightarrow f_0^L(y) = Ae^{-My}, \tag{7.28}$$

$$[--] \qquad \Rightarrow f_0^R(y) = Ae^{My}, \tag{7.29}$$

$$[\pm, \mp] \qquad \Rightarrow \text{ no zero mode.} \tag{7.30}$$

Thus, contrary to the circle, the interval allows for chiral zero modes.

A similar game can be played with gauge bosons. In that case, simple boundary conditions compatible with the variation of the action are $[+] \equiv A_5 = 0$ and $[-] \equiv A_\mu = 0$. Again, independent boundary conditions can be imposed at each brane. The result is,

$$[++] \quad \Rightarrow A_\mu^{(0)}, \tag{7.31}$$
$$[--] \quad \Rightarrow A_5^{(0)}, \tag{7.32}$$
$$[\pm, \mp] \quad \Rightarrow \text{ no zero mode.} \tag{7.33}$$

In particular we now have zero modes for A_μ and A_5 along different gauge directions, which means that now the scalar zero mode does not need to transform under the adjoint representation of the gauge group with $[++]$ b.c.. In fact, these choices of b.c. allow us to break the gauge group in a spontaneous way. The existence of A_μ zero modes corresponds to unbroken gauge symmetries at low energies. Twisted b.c. correspond to the symmetries being broken locally at the corresponding boundary. Finally, symmetries broken at both branes result in zero modes for A_5. Recall that this latter zero mode is physical and cannot be gauged away.

Let us consider a general gauge group G broken to a subgroup H_0 at the boundary $y = 0$ and to a subgroup H_1 at $y = L$. This is accomplished by imposing the following boundary conditions

$$A^a\ [++] \quad \Rightarrow \quad T^a \in \text{Alg}\{H_0 \cap H_1\}, \tag{7.34}$$
$$A^{\bar{a}}\ [+-] \quad \Rightarrow \quad T^{\bar{a}} \in \text{Alg}\{H_0/(H_0 \cap H_1)\}, \tag{7.35}$$
$$A^{\dot{a}}\ [-+] \quad \Rightarrow \quad T^{\dot{a}} \in \text{Alg}\{H_1/(H_0 \cap H_1)\}, \tag{7.36}$$
$$A^{\hat{a}}\ [--] \quad \Rightarrow \quad T^{\hat{a}} \in \text{Alg}\{G/(H_0 \cup H_1)\}. \tag{7.37}$$

In particular we have that the low energy gauge symmetry group is $H_0 \cap H_1$ and $A_5^{(0)}$ lives on the coset space $G/(H_0 \cup H_1)$. We could now use this structure to realize a pattern of symmetry breaking that we are familiar with. Let us consider a bulk $SO(5) \times U(1)_X$ symmetry, broken to $SO(4) \times U(1)_X$ at $y = L$ and to $SU(2)_L \times U(1)_Y$ (with $Y = T_R^3 + X$) at $y = 0$. In that case we have

$$A^a\ [++] \ \in \ \text{Alg}\{SU(2)_L \times U(1)_Y\}, \tag{7.38}$$
$$A^{\dot{a}}\ [-+] \ \in \ \text{Alg}\{(SO(4) \times U(1)_X)/(SU(2)_L \times U(1)_Y)\}, \tag{7.39}$$
$$A^{\hat{a}}\ [--] \ \in \ \text{Alg}\{SO(5)/SO(4)\}. \tag{7.40}$$

This theory has an $SU(2)_L \times U(1)Y$ unbroken gauge invariance, a massless scalar $A_5^{\hat{a}}$ transforming as a 4 of $SO(4)$ (an $SU(2)_L \times SU(2)_R$ bidoublet), plus a tower of massive vectors forming split $SO(5) \times U(1)_X$ adjoints. The fact that the boundary conditions explicitly break the $SO(4)$ symmetry imply that a potential for A_5 will be generated at the loop level. Locality and gauge invariance in the extra dimension however guarantees that this potential is finite, as the only gauge invariant contribution has to come from a non-local Wilson line. In order to make the connection with composite Higgs models, we will compute the one loop contribution to the A_5 potential from bulk fermions. We could do it by inserting all KK modes in the corresponding Coleman-Weinberg expression. However, there is another method that will give the result in a way that is completely paralel to the one we used for composite Higgs models.

7.2.2 Holographic Method

Our goal is to compute the effective Lagrangian for the low energy fields, including the A_5, in a way that can be directly parametrized by the effective Lagrangian we used in composite Higgs models. That way we can borrow our calculation of the Higgs potential from last chapter and we will have directly our answer. The idea is to compute the effective Lagrangian in two steps.

1. First, we integrate out the bulk degrees of freedom by using the 5D equations of motion (at the classical level) for fixed values of the fields on one boundary, say $y = 0$.

$$Z = \int d\Phi e^{iS[\Phi]} = \int d\phi_0 e^{iS_0[\phi_0]} \int_{\phi_0} e^{i(S[\Phi]-S_0[\phi_0])} = \int d\phi_0 e^{iS_0[\phi_0]+iS_{\text{eff}}[\phi_0]}. \qquad (7.41)$$

   The resulting effective action $S_{\text{eff}}[\phi_0]$ is in general a 4D non-local action for the boundary fields ϕ_0.

2. Using the 4D boundary action $(S_0 + S_{\text{eff}})$ we compute the A_5 potential by performing loops of the boundary matter fields ϕ_0. Since we aim at its potential, we can treat $A_5^{(0)}$ as a classical external background (its legs have no momentum).

This procedure is equivalent to the KK calculation since we know that hte potential is a non-local effect, that requires any 5D loop to propagate from one brane to the other and therefore all 5D loops receive a contribution from loops of ϕ_0.

Another important observation is the following. First, we can always go to an almost axial gauge in which $A_5 = A_5^{(0)}$. Then we can always make a field redefinition of the form,

$$\Phi \to \Phi' \equiv \Omega\Phi, \quad A_M \to A'_M \equiv \frac{i}{g}\Omega^{-1}[\partial_M - igA_M]\Omega, \qquad (7.42)$$

where

$$\Omega = e^{i\theta(x,y)} = e^{-ig\int_y^L dy'\, A_5^{(0)}(x)}. \qquad (7.43)$$

This field redefinition is a gauge transformation except that it does not satisfy the corresponding boundary condition at $y = 0$ (it does not vanish). At every other point of the extra dimension it coincides with the b.c. that makes $A_5 = 0$. Thus, with this field redefinition, we eliminat A_5 from everywhere in the bulk and at the $y = L$ brane, leaving all the dependence of $A_5^{(0)}$ at the $y = 0$ brane. A_5 disappears from everywhere except at the corresponding b.c., which is now

$$\phi'_0 \equiv \Phi'(y = 0) = e^{i\theta(y=0)}\phi_0 \equiv e^{i\theta_0}\phi_0, \qquad (7.44)$$

where $\theta_0 \equiv -La_0 A_5^{(0)}$.

To summarize, the theory with A_5 and boundary conditions ϕ_0 is equivalent to a theory with vanishing A_5 and boundary conditions $\phi_0' = e^{i\theta_0}\phi_0$. Thanks to this observation, we can perform the calculation of the potential by deriving the boundary action for the fields ϕ_0' (with $A_5 = 0$), and at the end of the calculation reinterpret $\phi_0' = e^{i\theta_0}\phi_0$.

Let us apply this holographic method to bulk fermions and then we will use the results to compute the effective potential in a toy model of electroweak symmetry breaking. The bulk fermionic action reads

$$S = \frac{1}{g_5^2}\int d^4x dy \Big\{ \frac{1}{2}(\bar{\Psi} i\Gamma^M \partial_M \Psi - i(\partial_M \Psi)^\dagger \Gamma^0 \Gamma^M \Psi) - M\bar{\Psi}\Psi \Big\}, \tag{7.45}$$

where we have written the action in an explicitly hermitian way and we have factored out a global factor g_5^{-2} so that the 5D fermions have mass dimension $3/2$ as in 4D. Recall that the variation of the action then reads

$$\begin{aligned} g_5^2 \delta S &= \int d^4x dy \Big\{ \frac{1}{2}(\delta\bar{\Psi} i\Gamma^M \partial_M \Psi + \bar{\Psi} i\Gamma^M \partial_M \delta\Psi - i(\partial_M \delta\Psi)^\dagger \Gamma^0 \Gamma^M \Psi - i(\partial_M \Psi)^\dagger \Gamma^0 \Gamma^M \delta\Psi) \\ &\qquad - M\delta\bar{\Psi}\Psi - M\bar{\Psi}\delta\Psi \Big\} \\ &= \int d^4x dy \Big\{ \delta\bar{\Psi}(i\Gamma^M \partial_M - M)\Psi + [\overline{(i\partial_M \Psi)}\Gamma^M - M\bar{\Psi}]\delta\Psi \Big\} \\ &+ \frac{1}{2}\int d^4x \Big[\bar{\Psi}\gamma^5 \delta\Psi - \delta\bar{\Psi}\gamma^5 \Psi \Big]_0^L. \end{aligned} \tag{7.46}$$

Now, instead of using Dirichlet b.c. for one of the two chiralities as we did when expanding in KK modes (this choice guaranteed the variation of the boundary action vanished), we want to choose a holographic boundary condition. Again, we cannot choose independent b.c. for each chirality and we will choose,

$$\Psi_L(y=0) = \psi_L^0, \tag{7.47}$$

whereas Ψ_R is free to vary at $y = 0$. At $y = L$ on the other hand we will choose standard Dirichlet b.c. for one of the two chiralities. Note that with this choice of b.c., the boundary action no longer vanishes, since we now have, at $y = 0$, $\delta\Psi_L = 0$ but $\Psi_L \delta\Psi_R \neq 0$. Thus, we have to add an extra boundary action so that our boundary conditions are compatible with the variational principle of the action. The required boundary action is [54]

$$S_0 = \frac{1}{2}\frac{1}{g_5^2}\int d^4x dy \, [\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L]\Big|_{y=0}. \tag{7.48}$$

Following the holographic approach, we have to solve the bulk equations of motion with b.c. as in Eq. (7.47) and insert the solution back in the action. The bulk equation of motion is the Dirac equation that when inserted back in the 5D bulk action makes it vanish, thus the only remaining part of the action after inserting the EoM is the boundary term we had to add

$$S_{\text{eff}} = \frac{1}{2g_5^2} \int d^4x [\bar{\psi}_L^0 \psi_R^0 + \bar{\psi}_R^0 \psi_L^0], \tag{7.49}$$

where we have defined $\psi_R^0 \equiv \Psi_R(y = 0)$, to be determined in terms of ψ_L^0 by the corresponding solution of the equations of motion.

The bulk EoM read,

$$-\not{p}\Psi_{L,R} + (\pm\partial_5 - M)\Psi_{R,L} = 0, \tag{7.50}$$

where we have Fourier transformed the four extended dimensions into momentum space, denoted by p^μ. We will solve these equations by using the ansatz

$$\Psi_{L,R}(p, y) = f_{L,R}(p, y)\psi_{L,R}(p), \tag{7.51}$$

with $p \equiv \sqrt{p_\mu p^\mu}$. We can always require that the four dimensional fields satisfy the off-shell 4D Dirac equation,

$$\not{p}\psi_{L,R}(p) = p\psi_{R,L}(p), \tag{7.52}$$

so that we obtain EoM for the 5D profiles

$$-\not{p} f_{L,R} + (\pm\partial_5 - M)f_{R,L} = 0. \tag{7.53}$$

These two coupled first order equations can be decoupled by iteration

$$(-\partial_5^2 + M^2)f_{L,R} = p^2 f_{L,R} \Rightarrow (\partial_5^2 + \omega^2)f_{L,R} = 0, \tag{7.54}$$

where we have defined $\omega^2 \equiv p^2 - M^2$. The precise solution depends on the b.c. at $y = L$, we have two possibilities:

- Case L_+, $\Psi_R(L) = 0$. The solution reads

$$\begin{aligned} f_L(p, y) &= \frac{1}{p}\big\{\omega \cos[\omega(y - L)] - M \sin[\omega(y - L)]\big\}, & (7.55)\\ f_R(p, y) &= \sin[\omega(y - L)]. & (7.56) \end{aligned}$$

- Case L_-, $\Psi_L(L) = 0$. The solution reads

$$\begin{aligned} f_L(p, y) &= \sin[\omega(y - L)], & (7.57)\\ f_R(p, y) &= -\frac{1}{p}\big\{\omega \cos[\omega(y - L)] + M \sin[\omega(y - L)]\big\}. & (7.58) \end{aligned}$$

We can now write the contribution to the boundary action. Recall

$$\psi_R^0 = \Psi_R(p, 0) = f_R(p, 0)\psi_R(p) = f_R(p, 0)\frac{\not{p}}{p}\psi_L(p) = \frac{f_R(p, 0)}{f_L(p, 0)}\frac{\not{p}}{p}\psi_L^0, \tag{7.59}$$

so that the boundary action reads

$$S_{\text{eff}} = \frac{1}{g_5^2}\int d^4x\, \bar{\psi}_L^0 \frac{f_R(p,0)}{f_L(p,0)}\frac{\not{p}}{p}\psi_L^0 \equiv \frac{L}{g_5^2}\int d^4x\, \bar{\psi}_L^0 \Sigma(p)\not{p}\psi_L^0, \tag{7.60}$$

where we have defined

$$\Sigma(p) \equiv \frac{1}{pL}\frac{f_R(p,0)}{f_L(p,0)}. \tag{7.61}$$

Let us now apply this formalism to a toy model of EWSB. Instead of a realistic $SO(5)/SO(4)$ symmetry breaking pattern, we will consider a bulk $SO(2)$ gauge theory broken on both branes and an $SO(2)$ bulk fermion doublet. The boundary conditions are

$$A_\mu \sim [--], \qquad \begin{pmatrix} \Psi^1 \sim [++] \\ \Psi^2 \sim [--] \end{pmatrix}. \tag{7.62}$$

The effective action for the boundary fields reads

$$S_{\text{eff}} = \frac{L}{g_5^2}\int d^4x\, \begin{pmatrix} \bar{\psi}_L^{0(1)} & \bar{\psi}_L^{0(2)} \end{pmatrix} \not{p} \begin{pmatrix} \Sigma^{(+)}(p) & 0 \\ 0 & \Sigma^{(-)}(p) \end{pmatrix} \begin{pmatrix} \psi_L^{0(1)} \\ \psi_L^{0(2)} \end{pmatrix}, \tag{7.63}$$

where we have defined the Σ functions for the $L_\pm$ branches,

$$\Sigma^{(+)} = -\frac{1}{ML + \omega L\cot(\omega L)}, \tag{7.64}$$

$$\Sigma^{(-)} = \frac{\omega\cot(\omega L) - M}{p^2 L}. \tag{7.65}$$

Now we take into account that the fields we are using are really the rotated ones, $\psi_L^{0(i)} \to (e^{i\theta_0})_{ij}\psi_L^{0(j)}$, with

$$e^{i\theta_0} = e^{-Ah(x)/f} = \begin{pmatrix} \cos(h/f) & -\sin(h/f) \\ \sin(h/f) & \cos(h/f) \end{pmatrix}, \tag{7.66}$$

where $f \equiv 1/\sqrt{Lg_5^2}$, h is the properly normalized $A_5^{(0)}$ and

$$A \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7.67}$$

The now Higgs dependent effective action reads

$$S_{\text{eff}} = \frac{L}{g_5^2}\int d^4x\, \begin{pmatrix} \bar{\psi}_L^{0(1)} & \bar{\psi}_L^{0(2)} \end{pmatrix} \not{p} \begin{pmatrix} \Sigma^{(+)} - s_h^2(\Sigma^{(+)} - \Sigma^{(-)}) & -s_h c_h(\Sigma^{(+)} - \Sigma^{(-)}) \\ -s_h c_h(\Sigma^{(+)} - \Sigma^{(-)}) & \Sigma^{(-)} + s_h^2(\Sigma^{(+)} - \Sigma^{(-)}) \end{pmatrix} \begin{pmatrix} \psi_L^{0(1)} \\ \psi_L^{0(2)} \end{pmatrix}, \tag{7.68}$$

where we have defined $s_h \equiv \sin h/f$ and $c_h \equiv \cos h/f$. In order to impose the $[-,.]$ b.c. for Ψ^2 we have to set $\psi_L^{0(2)}$ to zero and keep only $\psi_L^{0(1)}$ as a dynamical variable. The resulting action is

$$S_{\text{eff}} = \frac{L}{g_5^2}\int d^4x\, \bar{\psi}_L^{0(1)} \not{p}\big[\Sigma^{(+)} - \sin^2\frac{h(x)}{f}(\Sigma^{(+)} - \Sigma^{(-)})\big]\psi_L^{0(1)}. \tag{7.69}$$

This expression closely resembles the effective Lagrangian for fermions and the Higgs in composite Higgs models. In both cases the Higgs enters as an "angular variable", as corresponds to a Higgs that is originally a Goldstone boson. Also the dynamics of the new physics (composite states in one case, KK modes in the other) is encoded in the form factors $\Sigma^{(\pm)}(p)$. The difference now is that we have explicit expressions for the form factors and can therefore compute exactly the Higgs potential. In fact, we can directly use the results we obtained for general form factors in the case of composite Higgs models to write down the contribution to the Higgs potential in our extra dimensional toy model,

$$V(h) = -2\int \frac{d^4Q}{(2\pi)^4} \log\left[1 - \sin^2\left(\frac{h}{f}\right)\frac{\Sigma^{(+)}(Q) - \Sigma^{(-)}(Q)}{\Sigma^{(+)}(Q)}\right], \tag{7.70}$$

where $Q^2 = -p^2$ is the Euclidean momentum. The convergence of the integral depens on the convergence properties of $\Sigma^{(+)} - \Sigma^{(-)}$, which as usual is an order parameter. In our case we have, in Euclidean momentum

$$\frac{\Sigma^{(+)}(Q) - \Sigma^{(-)}(Q)}{\Sigma^{(+)}(Q)} = 1 - \frac{1}{Q^2}\Big[\omega_E^2 \coth^2(\omega_E L) - M^2\Big], \tag{7.71}$$

so that at large momentum, this order parameter vanishes exponentially,

$$\frac{\Sigma^{(+)}(Q) - \Sigma^{(-)}(Q)}{\Sigma^{(+)}(Q)} \sim -4\left(1 + \frac{M^2}{Q^2}\right)e^{-2QL}, \quad \text{(for } QL, Q/M \gg 1). \tag{7.72}$$

This exponential convergence corresponds, from a 4D conformal field theory point of view, to the conformal symmetry being spontaneously broken by an operator of infinite scaling dimension. Since the convergence is so strong, we get a good approximation by expanding the logarithm, getting

$$V(h) \approx \frac{-2}{8\pi^2}\int_0^\infty dQQ(Q^2 + M^2)(\coth^2\omega_E L - 1)\sin^2 h/f = -\frac{1}{4\pi^2}\frac{1}{L^4} f(m)\sin^2\frac{h}{f}, \tag{7.73}$$

where we have defined

$$f(m) \equiv \int_m^\infty d\omega_E \omega_E^3(\coth^2\omega_E - 1), \tag{7.74}$$

with, for instance $f(0) = \frac{3}{2}\zeta(3) \approx 1.8$.

This calculation shows how the corresponding Higgs potential is finite and how fermions can give a contribution to the Higgs potential with non-vanishing minimum (note the sign of the contribution). More realistic models include realistic symmetry structure (for instance $SO(5)/SO(4)$) and warped extra dimensions instead of flat space. A minimal realistic model of natural electroweak symmetry breaking in models with warped extra dimensions was proposed in [55].

Chapter 8

Conclusions

The goal of these lectures was to introduce the main implications of symmetries and symmetry breaking and their application to the exciting case of electroweak symmetry breaking. After a discussion of general topics related to symmetries, we have focused on EWSB in the standard model and beyond. The hierarchy problem and natural realizations of EWSB have been the main guiding principle. Of course, even a full semester course cannot be enough to cover the vast amount of literature and model building that has taken place in the last few decades on the subject of EWSB. Thus, the choice of topics has been quite biased. The main deciding factor has been the extremely good set of lecture notes by R. Contino, that we kindly allowed me to use as a draft script. Although there have been quite a number of additions and some subtractions, the main line of argumentation and choice of topics are mostly due to him. Of course I have tried to give it my personal point as often as possible (and in doing so I might have introduced some mistakes in the notes that are only my responsibility). In particular I hope that the emphasis in ideas rather than particular models will be enough to overcome the lack of discussion in some interesting developments in theoretical particle physics and EWSB that we did not have time to cover in the lectures.

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