Mathematical Methods for Physics III (Hilbert Spaces)

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    - www.ugr.es/~jsantiago/Docencia/MMIIIen/

Mathematical Methods for Physics III (Hilbert Spaces)

- Main Literature:
  - G. Helmberg, *Introduction to spectral theory in Hilbert space*, Dover, 1997.
  - P. Roman, *Some modern mathematics for physicists and other outsiders*, vol. 2, Pergamon, 1975.
  - P. Lax, *Functional Analysis*, Wiley 2002.
  - L. Abellanas y A. Galindo, *Espacios de Hilbert*, Eudema, 1987.
  - A. Vera López y P. Alegría Ezquerra, *Un curso de Análisis Funcional. Teoría y problemas,* AVL, 1997.
  - A. Galindo y P. Pascual, *Mecánica Cuántica*, Eudema, 1989.
  - E. Romera et al, *Métodos Matemáticos*, Paraninfo, 2013.
- Lecture notes are very succinct: examples, proofs and relevant comments on the blackboard (take your own notes)

Motivation

- Postulates of Quantum Mechanics

1st Postulate: EVERY PHYSICAL SYSTEM IS ASSOCIATED TO A COMPLEX SEPARABLE HILBERT SPACE AND EVERY PURE STATE IS DESCRIBED BY A RAY $|\Psi\rangle$ IN SUCH SPACE

2nd Postulate: EVERY OBSERVABLE IN A SYSTEM IS ASSOCIATED TO A SELF-ADJOINT LINEAR OPERATOR IN THE HILBERT SPACE WHOSE EIGENVALUES ARE THE POSSIBLE OUTCOMES OF A MEASURE OF THE OBSERVABLE

3rd Postulate: THE PROBABILITY OF GETTING A VALUE (a) WHEN MEASURING AN OBSERBABLE (A) IN A PURE STATE $|\Psi\rangle$ IS $\langle\Psi|P_{A,a}|\Psi\rangle$ WHERE $P_{A,a}$ IS THE PROJECTOR ON THE EIGENVALUE PROPER SUBSPACE

- But not only QM, also differential and integral equations, ...

Motivation

- But not only QM, also differential and integral equations, ...
- More generally, Hilbert Spaces are the mathematical structure needed to generalize $\mathbb{R}^n$ (or $\mathbb{C}^n$), including its geometrical features and operations with vectors to infinite dimensional vector spaces

Outline

- Linear and metric spaces
- Normed and Banach spaces
- Spaces with scalar product and Hilbert spaces
- Spaces of functions. Eigenvector expansions
- Functionals and dual space. Distribution theory
- Operators in Hilbert Spaces
- Spectral theory

Why Hilbert Spaces?

- They generalize the properties of $\mathbb{R}^n$ to spaces of infinite dimension

Linear Space

- (Finite) linear combinations of vectors. Linear independence. Linear basis.

Metric Space

- Infinite linear combinations require limits: notion of distance

Normed Space

- Translational invariant distance: it is enough with distance to the origin (norm)

(pre)Hilbert Space

- Generalization of $\mathbb{R}^n$ we need geometry (ortogonality, angles). Scalar product

Linear Space

- Definition: Liear (or vector) space over a field Λ is a triad (L, +, .) formed by a non-empty set L and two binary operations (addition and scalar multiplication) that satisfy:

$$+ : L \times L \longrightarrow L \qquad \cdot : \Lambda \times L \longrightarrow L$$

$$(i)\ (L,+) \text{ additive group} \quad \begin{cases} (ia)\ x+y=y+x \\ (ib)\ (x+y)+z=x+(y+z) \\ (ic)\ \exists 0 \in L / x+0=x \\ (id)\ \forall x \in L, \exists(-x) \in L / x+(-x)=0 \end{cases}$$

$$(ii)\ \lambda \cdot (x+y) = \lambda \cdot x + \lambda \cdot y$$

$$(\text{iii})\ \ \lambda \cdot (\mu \cdot x) = (\lambda\mu) \cdot x$$

$$(iv)\ (\lambda+\mu) \cdot x = \lambda \cdot x + \mu \cdot x$$

$$(v)\ 1 \cdot x = x$$

$$\forall\ x, y, z \in L$$

$$\forall\ \lambda, \mu \in \Lambda$$

Linear Space

- Trivial properties:

$$(i)\ \alpha \cdot 0 = 0$$

$$(ii)\ 0 \cdot x = 0$$

$$(iii)\ -x = (-1) \cdot x$$

$$(iv)\ x + y = x + z \Rightarrow y = z$$

$$(v)\ \alpha \cdot x = \alpha \cdot y, \alpha \neq 0 \Rightarrow x = y$$

$$(vi)\ \alpha \cdot x = \beta \cdot x, x \neq 0 \Rightarrow \alpha = \beta$$

$$(vii)\ \alpha \cdot x = 0 \Rightarrow \alpha = 0 \text{ o } x = 0$$

- Notation:

$$A + B = \{x + y, \forall x \in A, \forall y \in B\}$$

$$\lambda A = \{\lambda \cdot x, \forall x \in A\}$$

$$\Lambda x = \{\lambda \cdot x, \forall \lambda \in \Lambda\}$$

$$\Lambda A = \{\lambda \cdot x, \forall \lambda \in \Lambda, \forall x \in A\}$$

Linear Space

- Definition: Linear subspace. Non-trivial subset of a linear space with the structure of a linear space.

$$M \subset L \text{ (L linear space, } M \neq \emptyset) \text{ linear subspace if}$$

$$\alpha x + \beta y \in M, \qquad \forall \alpha, \beta \in \Lambda, \ \forall x, y \in M$$

- Properties:

$$\{M_\alpha\}_{\alpha \in A} \text{ subsp.} \Rightarrow \ \bigcap_\alpha M_\alpha, \ \sum_{i=1}^n M_i \text{ subsp.}$$

$$\sum_{i=1}^n \lambda_i x_i \in M, \ \forall n \text{ finite}, \forall x_1, \ldots x_n \in M$$

- Definition. Linear span: $\text{let } S \subset L$

$$[S] = \{\sum_{i=1}^n \alpha_i x_i, \forall n \text{ finite}, \forall x_i \in S, \forall \alpha_i \in \Lambda\} \text{ (it is linear subsp.)}$$

- Properties:

[S] is the smaller subsp. that contains S

$$[S] = \bigcap_i M_i, \ \{M_i\} \text{ set of subsp. that contain S}$$

Linear Space

- Definition: Linear independence.

  $X \subset L$ is linearly independent (l.i.) if

  $\sum_{i=1}^{n} \alpha_i x_i = 0,\ x_i \in X, \alpha_i \in \Lambda \Rightarrow \alpha_1 = \ldots = \alpha_n = 0$

- Definition: Hamel (or linear) basis. Maximal l.i. set (i.e. that it is not contained in any other l.i. set).
- Properties:

  Every l.i. set can be extended to a Hamel basis

  Every Hamel basis of L has the same number of elements (linear dimension)

  L=[B], $\forall$ B Hamel basis of L

  B Hamel basis of L $\Rightarrow x = \sum_{i=1}^{n} \alpha_i x_i, \alpha_i \in \Lambda, x_i \in B$ is unique

Linear Space

- Definition: Subspace direct sum. Let $\{M_i\}_{i=1}^{n}$ be subsps. of L

  $$L = M_1 \vec{\oplus} \ldots \vec{\oplus} M_n \text{ (L is direct sum of } M_i\text{) if}$$

  $$\forall x \in L \ \exists! \ x_1 \in M_1, \ldots x_n \in M_n \ / \ x = x_1 + \ldots + x_n$$

- Theorem: Let $L=M_1+M_2$

  $$L = M_1 \vec{\oplus} M_2 \Leftrightarrow M_1 \bigcap M_2 = \{0\}$$

  $$[M_2 \text{ is the linear complement of } M_1 \text{ in L}]$$

- More generally, if $L=M_1+...+M_n$

  $$L = M_1 \vec{\oplus} \ldots \vec{\oplus} M_n \Leftrightarrow M_i \bigcap \sum_{j \neq i} M_j = \{0\}$$

Linear Space

- Summary:
  - Linear (sub)space: (L,+,.)
  - Linear span: [S]={FINITE linear combinations of elements of S}
  - Linear independence: finite linear combination=0 $\Rightarrow$ al coeffs=0
  - Hamel basis: Maximal l.i. set. Unique cardinal (linear dimension). Unique linear expansion of elements of L in terms of elements of B.
  - Directa sum of subspaces: sum of subspaces with null intersection (to the sum of the remaining subspaces).
- Other results and definitions (mappings, inverse mapping, isomorphisms, projectors, …) can be defined here but we will postpone it to Hilbert spaces

Metric spaces

- Definition: Metric space is a pair (X,d) where X is an arbitrary but non-empty set and $d : X \times X \to \mathbb{R}$ is a function (distance or metric) that satisfies:

$$(i)\ d(x,y) \geq 0$$

$$(ii)\ d(x,y) = 0 \Leftrightarrow x = y$$

$$(iii)\ d(x,y) = d(y,x) \qquad \forall x,y,z \in X$$

$$(iv)\ d(x,z) \leq d(x,y) + d(y,z)$$

- Properties

$$(i)\ d(x_1,x_n) \leq d(x_1,x_2) + d(x_2,x_3) + \ldots + d(x_{n-1},x_n)$$

$$(ii)\ |d(x,z) - d(y,z)| \leq d(x,y)$$

$$(iii)\ Y \subset X,\ d'(y_1,y_2) = d(y_1,y_2) \Rightarrow (Y,d')\ \text{metric space with induced metric}\ d'$$

Metric spaces

- Definitions: Let (X,d) be a metric space.
  - Open ball of radius r centered at x: $B(x,r) = \{y \in X \ / \ d(x,y) < r\}$
  - Closed ball of radius r centered at x: $\bar{B}(x,r) = \{y \in X \ / \ d(x,y) \leq r\}$
  - Let $A \subset X$, $x \in A$ is an interior point if $\exists r > 0 / B(x,r) \subset A$
  - Interior of A: $\text{int A} = \{x \in X \ / \ x \text{ is an interior point of } A\}$
  - A is open $\text{int}\, A = A$
  - Given $A \subset X, \ x \in X$ is an adherence point if $\forall r > 0, \ B(x,r) \cap A \neq \emptyset$
  - Closure of A: $\bar{A} = \{x \in X \ / \ x \text{ is an adherence point in } A\}$
  - Closed subspace: $A \subset X$ is closed si $A = \bar{A}$
  - Dense subspace: $A \subset X$ is dense in X if $\bar{A} = X$

Metric spaces

- Properties of open and closed subspaces:
  - Let (X, d) be a metric space and $A \subset X$

$$\emptyset, X \text{ are closed (and open)}$$

$$A \text{ open} \;\Leftrightarrow\; A^c \text{ closed}$$

$$\cap_{i \in I} A_i \text{ closed if } A_i \text{ closed}$$

$$\cap_{i=1}^{n} A_i \text{ open if } A_i \text{ open}$$

$$\cup_{i=1}^{n} A_i \text{ closed if } A_i \text{ closed}$$

$$\cup_{i \in I} A_i \text{ open if } A_i \text{ open}$$

Metric spaces

- Definition: Convergent sequence

$$\{x_n\}_1^\infty \subset X \text{ converges to } x \text{ in } X,\ x_n \to x,\ \text{if } \forall r > 0, \exists N / x_n \in B(x,r), \forall n > N$$

$$\text{(equivalent: the sequence of real numbers } \{d(x_n, x)\} \text{ converges to 0)}$$

- Definition: Cauchy sequence

$$\{x_n\}_1^\infty \subset X \text{ is Cauchy if } \forall r > 0, \exists N / d(x_n, x_m) < r, \forall n, m > N$$

- Property: Every convergent sequence is a Cauchy sequence

$$d(x_n, x_m) \le d(x_n, x) + d(x_m, x) \to 0$$

- Definition: A metric space is complete if every Cauchy sequence is convergent. A subspace $S \subset X$ is complete if every Cauchy sequence in S converges in S

- Properties: Let $S \subset X,\ x \in X$

$$x \in \bar{S} \Leftrightarrow \exists \{x_n\}_1^\infty \subset S / x_n \to x$$

$$\text{Let } X \text{ be complete: } S \text{ is complete} \Leftrightarrow\ S \text{ closed}$$

Metric spaces

- Summary:
  - Metric (sub)space: (X,d)
  - Open and closed balls
  - Interior point: open ball centered in x inside A
  - int A= set of all interior points of A. Open subspace.
  - Adherence point of A, every open ball centered in x has non-zero intersection with A. Closure of A. Closed subspace. Dense subspace in X
  - Convergent sequence
  - Cauchy sequence
  - Complete metric space: Cauchy $\Rightarrow$ convergent
- Other properties (maps, continuity, boundedness, …) can be defined here but we will do it in Hilbert spaces.

Normed spaces

- Definition: Normed space is a pair (X,||.||) where X is a linear space and $||.|| : X \to R$ is a function (norm) with the following properties:

$$(i)\ ||x|| \geq 0$$

$$(ii)\ ||x|| = 0 \Leftrightarrow x = 0$$

$$(iii)\ ||\alpha x|| = |\alpha|\ ||x||$$

$$(iv)\ ||x+y|| \leq ||x|| + ||y||\ \text{(triangle inequality)}$$

  - Every linear subspace of a normed space X is a normed subspace with the norm of X.
- Relation between normed and metric spaces
  - Every nomed space is a metric space with the distance d(x,y)=||x-y||
  - The associated distance satisfies $d(x+z, y+z) = d(x,y),\ d(\alpha x, \alpha y) = |\alpha| d(x,y)$
  - Every metric linear space with these properties is a normed space with ||x||=d(x,0)
- Definition: Banach space. Complete normed space.

Normed spaces

- Properties: $(X, ||.||)$ normed space

  $(i)\ \Big| ||x|| - ||y|| \Big| \leq ||x-y||,\ \forall x, y \in X$

  $(ii)\ B(x_0, r) = x_0 + B(0, r),\ \forall x_0 \in X,\ r > 0$

  $(iii)\ X$ Banach $\iff \{a_n\}_1^\infty \in X, \sum_n ||a_n|| < \infty \Rightarrow \sum_n a_n$ converges in X

  (iv) Let X be Banach, a subspace Y is complete $\Leftrightarrow$ Y is closed in X

- Completion theorem:
  - Every normed linear space $L = (L, ||.||)$ admits a completion $\tilde{L}$, Banach space, unique up to norm isomorphisms, such that L is dense in $\tilde{L}$ and $||x||_{\tilde{L}} = ||x||_L$

- Inifinite sums in normed spaces

$$v_n \in X,\ v = \sum_{n=1}^{\infty} v_n \text{ si } \exists v \in X / \left|\left| \sum_{n=1}^{k} v_n - v \right|\right| \underset{k\to\infty}{\longrightarrow} 0$$

Normed spaces

- Hölder inequality (for sums):

$$\sum_{j=1}^{\infty} |a_j b_j| \leq \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |b_j|^q \right)^{1/q}$$

$$p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1 \quad \{a_j\}_1^{\infty} \in l_{\Lambda}^p \qquad \{b_j\}_1^{\infty} \in l_{\Lambda}^q$$

- Minkowski inequality (for sums):

$$\left(\sum_{j=1}^{\infty} |a_j + b_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{1/p} + \left(\sum_{j=1}^{\infty} |b_j|^p \right)^{1/p}$$

$$p \geq 1 \qquad \{a_j\}_1^{\infty}, \{b_j\}_1^{\infty} \in l_{\Lambda}^p$$

Normed spaces

- Summary:
  - Normed (sub)space: (X,||.||)
  - Relation norm $\underset{\not\longleftarrow}{\longrightarrow}$ distance
  - Banach space (complete normed space)
  - Absolute convengence $\Longrightarrow$ convergence in Banach spaces
  - A subspace of a Banach space is Banach $\Longleftrightarrow$ it is closed
  - Completion theorem: every normed space can be made complete in a unique way
  - An infinite sum converges in (X,||.||) to v if the sequence of partial sums converges to v
  - Hölder and Minkowski inequalities

Hilbert Space

- Definition: A pre-Hilbert space is a linear space with an associated scalar product.
  - Scalar product: $\langle .,. \rangle : X \times X \to \Lambda$ with the following properties

$$(i)\ \langle v, v \rangle \geq 0, \langle v, v \rangle = 0 \Leftrightarrow v = 0$$

$$(ii)\ \langle v, v_1 + v_2 \rangle = \langle v, v_1 \rangle + \langle v, v_2 \rangle$$

$$(iii)\ \langle v, \lambda w \rangle = \lambda \langle v, w \rangle$$

$$(iv)\ \langle v, w \rangle = \overline{\langle w, v \rangle}$$

$$\forall v, v_1, v_2, w \in L,\ \forall \lambda \in \Lambda$$

  - In particular we have

$$\langle \lambda_1 v_1 + \lambda_2 v_2, v \rangle = \bar{\lambda}_1 \langle v_1, v \rangle + \bar{\lambda}_2 \langle v_2, v \rangle$$

$$\langle v, w \rangle = 0\ \forall w \in L \Rightarrow v = 0$$

$$\langle v_1, w \rangle = \langle v_2, w \rangle\ \forall w \in L \Rightarrow v_1 = v_2$$

Hilbert Space

- Property: A pre-Hilbert space is a normed space with the norm associated to the scalar product $||v|| = +\sqrt{(v,v)}$
- Definition: A Hilbert space is a pre-Hilbert space that is complete with the norm associated to the scalar product (rather the distance associated to the norm).
- Properties: Let $(X, \langle .,. \rangle)$ be a pre-Hilbert space and $||\cdot||$ the associated norm:

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2) \text{ (Parallelogram identity)}$$

$$\mathrm{Re}\Big[\langle x, y\rangle\Big] = \frac{1}{4}[||x+y||^2 - ||x-y||^2]$$

$$\text{Polarization identity}$$

$$\mathrm{Im}\Big[\langle x, y\rangle\Big] = -\frac{1}{4}[||x+\mathrm{i}y||^2 - ||x-\mathrm{i}y||^2] \text{ (if X is complex)}$$

- Relation between scalar product and norm: a normed space $(X, ||.||)$ that satisfies the parallelogram identity is a pre-Hilbert space with a scalar product that satisfies $||x|| = +\sqrt{\langle x, x\rangle}$

Hilbert Space

- Properties: Let $(X, \langle ., . \rangle)$ be a pre-Hilbert space and $||\cdot||$ the associated norm:
  - Schwarz-Cauchy-Buniakowski inequality

$$|\langle v, w \rangle| \leq ||v||\,||w||,\ \forall v, w \in X,\ (\text{"="} \Leftrightarrow v, w \text{ lin. dep.})$$

  - Triangle inequality

$$||x + y|| \leq ||x|| + ||y||,\ \forall x, y \in X\ (\text{"="} \Leftrightarrow y = 0 \text{ o } x = cy, c \geq 0)$$

  - Continuity of the scalar product

$$x_n \to x,\ y_n \to y \Rightarrow \langle x_n, y_n \rangle \to \langle x, y \rangle$$

$$\{x_n\}_1^\infty,\ \{y_n\}_1^\infty \text{ are Cauchy in } X \Rightarrow \{\langle x_n, y_n \rangle\}_1^\infty \text{ is Cauchy in } \Lambda$$

Hilbert Space

- Properties: Let $(X, \langle .,. \rangle)$ be a pre-Hilbert space and $||\cdot||$ the associated norm:
  - $v, w \in X$ are orthogonal if $\langle v, w \rangle = 0$
  - $S = \{v_\alpha\}_{\alpha \in A} \subset X$ is an orthogonal set if $\langle v_\alpha, v_\beta \rangle = 0 \ \forall \alpha \neq \beta$
  - $S = \{v_\alpha\}_{\alpha \in A} \subset X$ is an orthonormal set if $\langle v_\alpha, v_\beta \rangle = \delta_{\alpha\beta}$
  - Every orthogonal set of non-vanishing vectors is l.i. (the inverse is not true)
- (Generalized) Pythagora's Theorem: Let $\{v_j\}_1^n$ be orthonormal in X

$$||v||^2 = \sum_{j=1}^{n} |\langle v_j, v \rangle|^2 + ||v - \sum_{j=1}^{n} \langle v_j, v \rangle v_j||^2, \quad \forall v \in X$$

- Pythagora's theorem

$$\left|\left| \sum_{j=1}^{n} v_j \right|\right|^2 = \sum_{j=1}^{n} ||v_j||^2, \text{ si } \langle v_i, v_j \rangle = 0 \quad (i \neq j)$$

Hilbert Space

- Properties:
  - Finite Bessel inequality: let $\{v_j\}_1^n$ be an orthonormal set

$$||v||^2 \geq \sum_{j=1}^{n} |\langle v_j, v\rangle|^2,\ \forall v \in X$$

  - Let $\{v_\alpha\}_{\alpha \in A}$ be an arbitrary orthonormal set

$$A^{(v)} \equiv \{\alpha \in A / \langle v_\alpha, v\rangle \neq 0\} \text{ is finite or numerable infinite}$$

  - Infinite Bessel inequality: let $\{v_\alpha\}_{\alpha \in A}$ be an arbitrary orthonormal set

$$||v||^2 \geq \sum_{\alpha \in A} |\langle v_\alpha, v\rangle|^2,\ \forall v \in X$$

- Completion Theorem:

> For any pre-Hilbert space $\big(X, \langle ., .\rangle\big)$, there is a Hilbert space H (unique up to isomorphisms) and an isomorphism $A : X \to W$ with W dense en H

Hilbert Space

- Definition: orthogonal complement $\text{Let } H \text{ Hilbert and } M \subset H,\ M \neq \emptyset$

$$M^{\perp} \equiv \{v \in H / v \perp M\} \ (\text{also } M^{\perp} = H \ominus M)$$

- Properties of the orthogonal complement

$$(i)\ M^{\perp} \text{ is a closed linear subspace } \forall M \subset H,\ H \text{ Hilbert}$$

$$(ii)\ M \cap M^{\perp} \subset \{0\}$$

$$(iii)\ M^{\perp\perp} \equiv (M^{\perp})^{\perp} \supset M$$

$$(iv)\ M^{\perp} = (\overline{M})^{\perp} = [M]^{\perp} = (\overline{[M]})^{\perp}$$

$$(v)\ \{0\}^{\perp} = H,\ \ H^{\perp} = \{0\}$$

Hilbert Space

- Theorem of orthogonal projection

Let M be a closed linear subspace of a Hilbert space H, then
$\forall v \in H : \exists! v_1 \in M, \exists! v_2 \in M^{\perp} / v = v_1 + v_2$ (v_1: orthogonal projection of v over M)

Equivalent:

Let M be a closed linear subspace of a Hilbert space H, then
$\forall v \in H : \exists! v_1 \in M \;/\; ||v - v_1|| = \inf\{||v - y||, y \in M\},\; v - v_1 \in M^{\perp}$

Hilbert Space

- Properties:
  - Definition: Orthogonal direct sum

    Let M, N be closed linear subspaces of H Hilbert

    $$H = M \oplus N \text{ si } H = M \vec{\oplus} N \text{ y } M \perp N$$

  - $H = M \oplus M^{\perp}, \forall$ closed linear subspace $M \subset H$

  - Orthogonal projector over M: $P_M : H \to M$

    $$P_M v = v_1,\ v = v_1 + v_2 \text{ con } v_1 \in M,\ v_2 \in M^{\perp}$$

    $$P_M + P_{M^{\perp}} = 1_H, \quad P_M P_{M^{\perp}} = P_{M^{\perp}} P_M = 0, \quad P_M^2 = P_M, \quad P_{M^{\perp}}^2 = P_{M^{\perp}}$$

  - $S^{\perp\perp} = \overline{[S]}\ \forall S \subset H,\ S \neq \emptyset$ (S closed subspace $\Rightarrow$ $S^{\perp\perp}$=S)

  - S linear subspace of H is dense in H $\Leftrightarrow S^{\perp} = \{0\}$

Hilbert Space

- Theorem: Let $\{x_n\}_1^\infty$ be an orthonormal set in H (Hilbert) y $\{\lambda_n\}_1^\infty \subset \Lambda$, then:

$$\sum_1^\infty \lambda_n x_n \text{ converges } \Leftrightarrow \sum_1^\infty |\lambda_n|^2 < \infty$$

- Theorem: Let $S = \{x_\alpha\}_{\alpha \in A}$ be an orthonormal set in H (Hilbert). Let $M \equiv \overline{[S]}$

$$(i)\ x_M \equiv \sum_{\alpha \in A} \langle x_\alpha, x \rangle x_\alpha \in M$$

$$(ii)\ x_M \text{ is the only vector that satisfies } x - x_M \perp M$$

$$(iii)\ x \in M \Rightarrow x = x_M$$

$$(iv)\ d(x, M) \equiv \inf_{y \in M} ||x - y|| = d(x, x_M)$$

> The best approximation of a vector x by elements of $M = \overline{[\{x_\alpha\}_{\alpha \in A}]}$ orthonormal is given by $P_M x$

Hilbert Space

- Orthonormalization theorem: Gram-Schmidt method

Let $\{v_j\}_{j\in J} \subset H$ a l.i. set, with J finite or numerable infinite $(\mathbb{N})$

$\exists \{u_j\}_{j\in J}$ orthonormal such that:

$(i)\ u_i \in \left[\{v_j\}_{j\in J}\right],\ v_i \in \left[\{u_j\}_{j\in J}\right]$ $\qquad (ii)\ \overline{\left[\{u_j\}_{j\in J}\right]} = \overline{\left[\{v_j\}_{j\in J}\right]}$

Solution:

$$u_m \equiv \frac{w_m}{||w_m||},\ \text{con } w_m \equiv v_m - \sum_{k=1}^{m-1} \langle u_k, v_m \rangle u_k$$

- Definition: Orthonormal basis

Maximal orthonormal set $S = \{v_\alpha\}_{\alpha \in A} \subset H$

- Theorem: Existence of orthonormal basis

Every Hilbert space $\neq \{0\}$ has an orthonormal basis

Hilbert Space

- Theorem: Characterization of orthonormal basis:

Let $S = \{v_\alpha\}_{\alpha \in A} \subset H \neq \{0\}$ an orthonormal set. The following statements are equivalent:

(i) S is an orthonormal basis of H

(ii) $\overline{[S]} = H$

(iii) $v \perp v_\alpha,\ \forall \alpha \in A \Rightarrow v = 0 \quad S^\perp = \{0\}$

(iv) $\forall v \in H \Rightarrow v = \sum_\alpha \langle v_\alpha, v \rangle v_\alpha$ (Fourier expansion)

(v) $\forall v, w \in H \Rightarrow \langle v, w \rangle = \sum_\alpha \langle v, v_\alpha \rangle \langle v_\alpha, w \rangle$ (Parseval identity)

(vi) $\forall v \in H \Rightarrow ||v||^2 = \sum_\alpha |\langle v_\alpha, v \rangle|^2$ (Parseval identity)

Hilbert Space

- Definition: Separable topologial (and metric) space:
  - A topological space X is separable if it contains a numerable subset dense in X.
  - A metric space M is separable if and only if it has a numerable basis of open subsets.
- Separability criterion in Hilbert spaces

A Hilbert space $H \neq \{0\}$ is separable	$\Leftrightarrow$	it admits a numerable orthonormal basis (finite or numerable infinite)

- Proposition:
  - All orthonormal basis of a Hilbert space H have the same cardinal (Hilbert dimension of H).

Hilbert Space

- Theorem of Hilbert Space classification

Definition: Two Hilbert spaces, H_1, H_2 over Λ are isomorphic if

$$\exists\, U : H_1 \to H_2,\ U \text{ linear isomorphism } / \langle Ux, Uy\rangle_{H_2} = \langle x, y\rangle_{H_1},\ \forall x, y \in H_1$$

Theorem:

Every Hilbert space $H \neq \{0\}$ is isomorphic to $l^2_\Lambda(A)$ where the cardinal of $A =$ the Hilbert dimension of H

Corolaries:

- A Hilbert space of finite Hilbert dimension, n, is isomorphic to $\mathbb{C}^n$
- A separable Hilbert space of infinite Hilbert dimension is isomorphic to $l^2_\Lambda(\mathbb{N})$
- Let H be a separable Hilbert space of Hilbert dimension h and linear dimension l
  - $h < \infty \Rightarrow l = h$ and any orthonormal basis is a linear basis
  - $h = \infty \Rightarrow l > h$ and no orthonormal basis is a linear basis

Hilbert Space

- Summary:
  - (Pre-)Hilbert space: Complete linear space with scalar product
  - Hilbert $\underset{\not\leftarrow}{\rightarrow}$ Normed
  - Parallelogram and polarization identities
  - Schwarz and triangle inequality, continuity of scalar product
  - Orthonormality. Pythagora's theorem and Bessel inequality
  - Completion theorem
  - Orthogonal complement and orthogonal projector. Best approximation to a vector.
  - Gram-Schmidt orthonormalization method
  - Orthonormal basis. Separable space
  - Theorem of Hilbert Space classification

Space of functions

- Some of the most important Hilbert spaces are spaces of functions.
  - Examples:

$\big(C_\Lambda[a,b], ||.||_\infty\big)$ complete, not pre-Hilbert

$\big(C_\Lambda[a,b], ||.||_p\big),\ p \geq 1$ not complete ($p=2$ pre-Hilbert)

$\big(B(\mathbb{R}), ||.||_\infty\big)$ complete, not pre-Hilbert

$\big(R^p(\mathbb{R}), ||.||_p\big),\ p \geq 1$ not complete ($p=2$ pre-Hilbert)

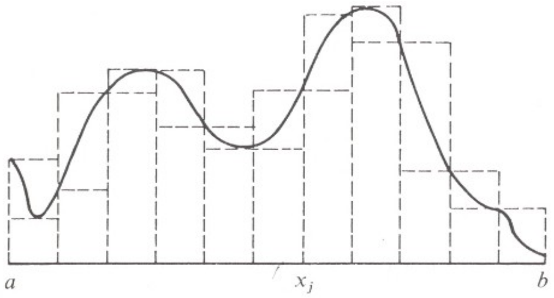
  - Example of not completeness of $\big(C_\Lambda[a,b], ||.||_2\big)$

$$f_n(x) = \begin{cases} 0, & x \leq \frac{1}{2} - \frac{1}{n}, \\ nx - \frac{n}{2} + 1, & \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2}, \\ 1, & \frac{1}{2} \leq x, \end{cases} \quad \text{is Cauchy but does not converge in } \big(C_\mathbb{R}[0,1], ||.||_2\big)$$

  - We can enlarge the space with the limits of all Cauchy sequences to complete it. We need a new concept of integral for that.

Space of functions

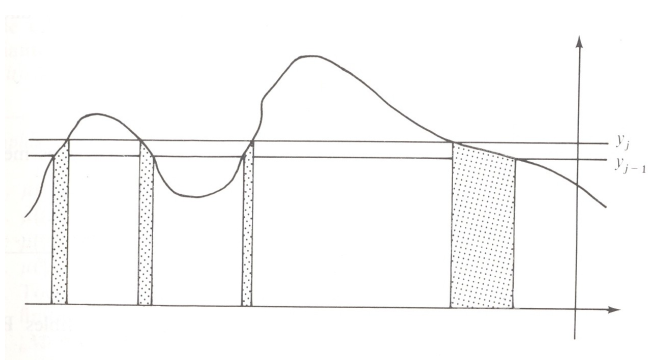
- Riemann integral:
  - Partition of the “x axis” and common convergence of upper and lower integrals



$$\int_a^b f(x)dx = I$$

$$\text{si } I = \lim_{|\pi|\to 0} \sum_1^n R_k^{\text{inf}} = \lim_{|\pi|\to 0} \sum_1^n R_k^{\text{sup}} < \infty$$

- Lebesgue integral:
  - Partition of the “y axis” and measure of subsets of the “x axis”



$$\int_{\mathbb{R}} f(x)dx \equiv \lim_{|\pi|\to 0} \Sigma_\pi(f)$$

$$\Sigma_\pi(f) \equiv \sum_{j=1}^n y_{j-1}\mu\{f^{-1}([y_{j-1}, y_j]\}$$

Space of functions

- We need a new concept of "measure"
  - Borel set: Element of $\mathcal{B}$, minimal family of subsets of $\mathbb{R}$ that contains all the open intervals (a,b) and satisfies:

$$(i)\ \{B_j\}_1^\infty \subset \mathcal{B} \Rightarrow \bigcup_{j=1}^{\infty} B_j \subset \mathcal{B} \qquad (ii)\ B \subset \mathcal{B} \Rightarrow \mathbb{R} - B \subset \mathcal{B}$$

  - Borel-Lebesgue measure (of a borel set B): $\mu(B) \equiv \inf_{I \supset B} l(I)$

$$I = \bigcup_{j=1}^{\infty} (a_j, b_j)\ \text{(union of disjoint open intervals)} \qquad l(I) \equiv \sum_{j=1}^{\infty} |b_j - a_j|$$

    - Properties:

$$B \in \mathcal{B} \Rightarrow \mu(B) = \inf\{\mu(A), A \text{ open} \supset B\} = \sup\{\mu(C), C \text{ compact} \subset B\}$$

$$B_n \in \mathcal{B},\ n \geq 1,\ \text{disjoint in pairs} \Rightarrow \mu\left(\cup_1^\infty B_n\right) = \sum_1^\infty \mu(B_n)$$

Space of functions

- We need a new concept of "measure"
  - Borel measurable function: $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable if $f^{-1}(B) \in \mathcal{B},\ \forall B \in \mathcal{B}$
  - f complex is Borel if both its real and imaginary parts are
  - Let $f, g,$ be real: $f + g, \lambda f (\lambda \in \mathbb{R}), fg, |f|$ are borel
  - Characterization of Borel measurable functions:

$$a)\ f : \mathbb{R} \to \mathbb{R} \text{ is Borel } \Leftrightarrow f^{-1}\{(a,b)\} \in \mathcal{B},\ \forall a, b$$

$$b)\ f_n(x) \to f(x),\ \forall x,\ f_n \text{ Borel} \Rightarrow f \text{ Borel}$$

$$c)\ f : \mathbb{R} \to \mathbb{R} \text{ is Borel } \Leftrightarrow \{x / f(x) < b\} \in \mathcal{B},\ \forall b$$

- Lebesgue integral let $f \geq 0$, bounded and Borel measurable. Its Lebesgue integral is

$$\int_{\mathbb{R}} f\, dx \equiv \lim_{|\pi| \to 0} \Sigma_\pi(f)$$

$$\pi : 0 = y_0 < y_1 < \ldots < y_n = \sup f \text{ partition of the range of } f$$

$$\Sigma_\pi(f) \equiv \sum_{j=1}^{n} y_{j-1} \mu\{f^{-1}([y_{j-1}, y_j])\}$$

Easy to extend to more general functions

Space of functions

- Lebesgue integrable functions

$$f \in \mathcal{L}^1_{\mathbb{R}}(\mathbb{R}) \text{ if} \int_{\mathbb{R}} |f|\, dx < +\infty, \quad \int_{\mathbb{R}} f\, dx \equiv \int_{\mathbb{R}} \frac{|f|+f}{2}\, dx - \int_{\mathbb{R}} \frac{|f|-f}{2}\, dx$$

$$f \in \mathcal{L}^1_{\mathbb{C}}(\mathbb{R}) \text{ if} \int_{\mathbb{R}} |f|\, dx < +\infty, \quad \int_{\mathbb{R}} f\, dx \equiv \int_{\mathbb{R}} \mathrm{Re}(f)\, dx + \mathrm{i} \int_{\mathbb{R}} \mathrm{Im}(f)\, dx$$

- Properties almost everywhere (a.e.).

A property $P(x),\ x \in \mathbb{R}$ is satisfied almost everywhere (a.e.) if the set $\{x/P(x) \text{ false}\}$ has vanishing measure

For instance $f_1 = f_2$ a.e. $\Leftrightarrow \int_{\mathbb{R}} |f_1 - f_2| dx = 0$

- L^1 Spaces.

$L^1(\mathbb{R})$ is the set of equivalence classes of functions in $\mathcal{L}^1(\mathbb{R})$ with the equivalence relation: $f_1 = f_2$ a.e.

Space of functions

- L^p spaces:

$$f \in \mathcal{L}^p(B) \text{ if } ||f||_p \equiv \left| \int_B |f|^p dx \right|^{1/p} < +\infty, \quad 1 \leq p < +\infty$$

- Definition: $L^p(B)$ set of equivalence classes of functions $f \in \mathcal{L}^p(B)$ with equivalence relation $f = g$ a.e.

- Properties:

(i) $(L^p(\mathbb{R}), ||.||_p)$, $(L^p(B), ||.||_p)$, are Banach

(ii) $C[a,b]$ is dense in $(L^p([a,b]), ||.||_p)$

(iii) $(L^p([a,b]), ||.||_p)$ is the completion of $C[a,b]$ $\big($same $[a,b] \to \mathbb{R}\big)$

(iv) $L^2(\mathbb{R})$ is Hilbert with the scalar product

$$\langle f, g \rangle \equiv \int_{\mathbb{R}} \bar{f}(x) g(x)\, dx, \text{ (same for } [a,b])$$

Space of functions

- (Integral) Hölder and Minkowski inequalities

Let $f, h \in L^p(X),\ g \in L^q(X),\ 1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$

Hölder inequality

$$\int_X |fg|\, dx \leq \left\{ \int_X |f|^p\, dx \right\}^{1/p} \cdot \left\{ \int_X |g|^q\, dx \right\}^{1/q}$$

Minkowski inequality

$$\left\{ \int_X |f + h|^p\, dx \right\}^{1/p} \leq \left\{ \int_X |f|^p\, dx \right\}^{1/p} + \left\{ \int_X |h|^p\, dx \right\}^{1/p}$$

Space of functions

- Some relevant orthonormal bases in L²:
  - Legendre's basis

$$P_n(x) \equiv \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \text{ (Legendre's Polynomials)}$$

$$\left\{\sqrt{n+1/2}P_n\right\}_0^\infty \text{ is an orthonormal basis of } L^2[-1,1]$$

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0,\ n = 0,1,\ldots \text{ (Legendre's eq.)}$$

  - Hermite's basis

$$H_n(x) \equiv (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \text{ (Hermite's polynomials)}$$

$$\left\{(\sqrt{\pi}2^n n!)^{-1/2}e^{-x^2/2}H_n\right\}_0^\infty \text{ is an orthonormal basis of } L^2(\mathbb{R})$$

$$H_n'' - 2xH_n' + 2nH_n = 0,\ n = 0,1,\ldots \text{ (Hermite's eq.)}$$

Space of functions

- Some relevant orthonormal bases in L²:
  - Laguerre's basis

$$L_n(x) \equiv \frac{1}{n!} e^x \frac{d^n}{dx^n}(e^{-x}x^n) \text{ (Laguerre's polynomial)}$$

$$\left\{e^{-x/2}L_n\right\}_0^\infty \text{ is an orthonormal basis of } L^2[0,\infty)$$

$$xL_n'' + (1-x)L_n' + nL_n = 0,\ n = 0, 1, \ldots \text{ (Laguerre's eq.)}$$

  - Orthonormal bases of polynomial associated to a weight function

Let $0 \neq \rho \in L^1(R)$, non-negative / $\exists \alpha > 0$, for which $\int_{\mathbb{R}} e^{|\alpha| t} \rho(t)\, dt < \infty$

If $\{p_n(t)\}_0^\infty$ are orthonormal polynomial with respect to the scalar product

$\langle f, g \rangle_\rho \equiv \int_{\mathbb{R}} \bar{f} g \rho$, obteined from $\{t^n\}_0^\infty$ through the Gram-Schmidt method, then

$\{p_n(t)\rho^{1/2}(t)\}_0^\infty$ is an orthonormal basis of $L^2(\text{sop } \rho)$

Space of functions

- Some relevant orthonormal bases in L^2:
  - Fourier's basis

$$\{e^{\mathrm{i}2\pi nx/L}/\sqrt{L}\}_{-\infty}^{+\infty} \text{ is an orthonormal basis in } L^2[a,a+L]$$

$$\left\{\frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi nx}{L}\right), \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi nx}{L}\right), \right\} \ (n=1,2,\ldots) \text{ is an orthonormal basis in } L^2[a,a+L]$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}\frac{2\pi nx}{L}} = a_0 + \sum_{n=1}^{\infty}\left[2a_n\cos\left(\frac{2\pi nx}{L}\right) + 2b_n\sin\left(\frac{2\pi nx}{L}\right)\right]$$

$$c_n = \frac{1}{L}\int_a^{a+L} e^{-\mathrm{i}\frac{2\pi nx}{L}} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_a^{a+L}\cos\left(\frac{2\pi nx}{L}\right) f(x)\,dx, \quad b_n = \frac{1}{L}\int_a^{a+L}\sin\left(\frac{2\pi nx}{L}\right) f(x)\,dx$$

Space of functions

- Some relevant orthonormal bases in L²:
  - Fourier's basis   Convergencia en L^2 (c.d.)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}\frac{2\pi n x}{L}} = a_0 + \sum_{n=1}^{\infty}\left[2a_n \cos\left(\frac{2\pi n x}{L}\right) + 2b_n \sin\left(\frac{2\pi n x}{L}\right)\right]$$

  - Jordan convergence criterion

Let $f \in L^2_{\mathbb{C}}[a,b]$ with bounded variation in (a,b), then the Fourier series converges at every point x$\in (a,b)$ to $\lim_{\epsilon\to 0}\frac{f(x+\epsilon)+f(x-\epsilon)}{2}$

  - Bases with only sines or cosines

$f \in L^2_{\mathbb{C}}[a,b]$ can be expanded in Fourier series using only sines or only cosines by expanding antisymmetric or symmetric extension of the function

Space of functions

- Expansion in eigenvectors
  - Consider the following differential operator

$$\mathcal{O} \equiv \frac{d^2}{dx^2}$$

every function $f \in L^2[a, a+L]$ can be expanded in eigen-functions of $\mathcal{O}$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n f_n(x)$$

with

$$f_n(x) = e^{\mathrm{i}\,\frac{2\pi n x}{L}}, \quad \mathcal{O} f_n = -\left(\frac{2\pi n}{L}\right)^2 f_n$$

↑ Eigenvalues

Space of functions

- Summary:
  - Borel sets. Borel-Lebesgue measure. Borel measurable functions.
  - Lebesgue integral.
  - Lebesgue integrable functions. $\mathcal{L}^1$ Spaces
  - Properties almost everywhere. L^p Spaces
  - $L^2(B)$ is a Hilbert space (completion of $C(B)$)
  - Hölder and Minkowski integral inequalities
  - Orthonormal polynomials in $L^2(B)$
  - Fourier basis. Fourier expansion.
  - Expansion in eigenvectors.

Linear forms

- Definitions: Let L be a linear space over the field Λ
  - A linear form (or functional) is a linear mapping $F : L \to \Lambda$

$$F(x+y) = F(x) + F(y), \quad F(\alpha x) = \alpha F(x), \ \forall x, y \in L, \ \forall \alpha \in \Lambda$$

  - A linear form in a normed space is continuous if

$$\forall \{x_n\} \to x \Rightarrow \{F(x_n)\} \to F(x), \ \forall x \in L$$

$$\forall \epsilon > 0 \ \exists \delta > 0 / ||x - y|| < \delta \Rightarrow |F(x) - F(y)| < \epsilon$$

  - A linear form in a normed space is bounded if

$$\exists M \geq 0 / |F(x)| \leq M||x||, \ \forall x \in L$$

$$||F|| = \sup_{x \neq 0} \frac{|F(x)|}{||x||} = \sup_{||x||=1} |F(x)| = \inf\{M \geq 0 / |F(x)| \leq M||x||\}$$

- Theorem: Let F be a linear form in a normed space

$$F \text{ is bounded} \Leftrightarrow F \text{ is continuous}$$

Linear forms

- Definition: Dual space of a Hilbert space (H,<,>) is the set of all continuous functional forms in H.

$$\tilde{H} = \{F : H \to \Lambda / F \text{ linear and continuous}\} \equiv \mathcal{A}(H, \Lambda)$$

  It is a Hilbert space (as we will see)

- Proposition: Let (H,<,>) be a Hilbert space of finite dimension:
  - All functionals in H are continuous
  - $\dim\ \tilde{H} = \dim\ H$

- Riesz-Fréchet representation theorem: Let (H,<,>) be a Hilbert space (separable or not)

$$\forall F : H \to \Lambda \text{ linear and continuous}$$
$$\exists! f \in H / F(g) = \langle f, g \rangle,\ \forall g \in H$$

Linear forms

- Properties:
  - Let $F \neq 0 \Rightarrow \dim(M_0^{\perp}) = 1 \qquad (M_0 \equiv \{h \in H / F(h) = 0\})$
  - Let $\{e_j\}_1^n$ be an orthonormal basis of Λ^n, $\forall \phi : H \to \Lambda^n$ linear and continuous

    $$\exists x_1, \ldots, x_n \in H / \phi(y) = \sum_1^n \langle x_j, y \rangle e_j$$
  - $||F_x||_{\mathcal{A}(H,\Lambda)} = ||x||_H$
  - F linear form in a Hilbert space is continuous $\Leftrightarrow$ its kernel M_0 is closed in H
  - $\tilde{H}$ is a Hilbert space with the scalar product associated to H

    $$\langle ., . \rangle : \tilde{H} \times \tilde{H} \to \Lambda$$

    $$F_f,\ F_g \to \langle F_f, F_g \rangle \equiv \langle g, f \rangle$$
  - The mapping $f \in H \to F_f \in \tilde{H}$ with $F_f(g) = \langle f, g \rangle$,

    is an anti-linear isometric bijection

Linear forms

- Bilinear forms: let $(H, \langle .,. \rangle)$ be a Hilbert space over Λ
  - Bilinear form (rather sesquilinear): mapping $\phi : H \times H \to \Lambda$ such that

$$(i)\ \phi(\alpha x, \beta y) = \bar{\alpha}\beta\phi(x,y),\ \forall \alpha, \beta \in \Lambda,\ \forall x, y \in H$$

$$(ii)\ \phi(x_1 + x_2, y) = \phi(x_1, y) + \phi(x_2, y)$$

$$(iii)\ \phi(x, y_1 + y_2) = \phi(x, y_1) + \phi(x, y_2)$$

  - A bilinear form is bounded if $\exists k \geq 0 / |\phi(x,y)| \leq k||x||\,||y||,\ \forall x, y \in H$

$$||\phi|| = \sup_{x \neq 0 \neq y} \frac{|\phi(x,y)|}{||x||\,||y||} \text{ (it is a norm)}$$

  - Theorem: let $\phi : H \times H \to \Lambda$, be a bilinear form bounded in H (Hilbert).

$\exists! A \in \mathcal{A}(H)$ (bounded linear mapping $A : H \to H$) such that

$$\phi(x,y) = \langle x, Ay \rangle,\ \forall x, y \in H$$

$$\text{and } ||\phi|| = ||A|| \equiv \sup_{0 \neq x \in H} \frac{||Ax||}{||x||} < +\infty$$

Linear forms

- Strong convergence (in norm) $x_n \stackrel{s}{\to} n \Leftrightarrow ||x_n - x|| \to 0$
- Weak convergence $x_n \stackrel{w}{\to} n \Leftrightarrow F(x_n) \to F(x), \ \forall F \in \tilde{H}$
- Theorems:

$$x_n \stackrel{s}{\to} x \Rightarrow x_n \stackrel{w}{\to} x$$

$$\left.\begin{array}{c} x_n \stackrel{w}{\to} x \\ ||x_n|| \to ||x|| \end{array}\right\} \Leftrightarrow x_n \stackrel{s}{\to} x$$

$$\left.\begin{array}{c} x_n \stackrel{w}{\to} x \\ x_n \stackrel{s}{\to} x' \end{array}\right\} \Rightarrow x_n \stackrel{w}{\to} x'$$

Distributions

- Test function spaces:
  - Test functions of compact support

$$\mathcal{D}(\mathbb{R}) = \{f \in \mathcal{C}^\infty(\mathbb{R}) / \mathrm{supp}(f) \text{ bounded of } \mathbb{R}\}, \quad (\mathrm{supp}(f) = \{x / f(x) \neq 0\})$$

it is a linear space and algebra of functions.

    - Convergence

$$f_n \stackrel{\mathcal{D}}{\to} f \text{ if } \begin{cases} i) \ \mathrm{supp}(f_n) \subset K \text{ bounded and independent of } n \\ ii) \ ||f_n^{(p)} - f^{(p)}||_\infty \underset{n\to\infty}{\longrightarrow} 0, \ \forall p \geq 0 \end{cases}$$

  - Test functions of rapid decrease

$$\mathcal{S}(\mathbb{R}) = \{f \in \mathcal{C}^\infty(\mathbb{R}) / \sup_{k,m\in\mathbb{N}} ||x^k f^{(m)}||_\infty < \infty\}$$

it is a semi-normed space ($||f||_{km} = ||x^k f^{(m)}||_\infty$ is semi-norm)

    - Convergence

$$f_n \stackrel{\mathcal{S}}{\to} f \text{ si } ||x^k f_n^{(m)}(x) - x^k f^{(m)}(x)||_\infty \underset{n\to\infty}{\longrightarrow} 0, \ \forall k, m \in \mathbb{N}$$

  - Properties

$$f_n \stackrel{\mathcal{D}}{\to} f \Rightarrow f_n \stackrel{\mathcal{S}}{\to} f, \quad \mathcal{D} \text{ is dense in } \mathcal{S}$$

Distributions

- Definitions and properties:
  - Distribution: $T : \mathcal{D}(\mathbb{R}) \to \Lambda$ linear and continuous (in the sense of $\mathcal{D}$)

    $T(\alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1 T(\phi_1) + \alpha_2 T(\phi_2), \ \forall \alpha_{1,2} \in \Lambda, \ \forall \phi_{1,2} \in \mathcal{D}$

    $\phi_n \stackrel{\mathcal{D}}{\to} \phi \Rightarrow T(\phi_n) \to T(\phi)$
  - Space of distributions: $\widetilde{\mathcal{D}(\mathbb{R})} = \{T/T \text{ distribution}\}$
  - Sufficient condition for T to be continuous

    $\exists M > 0 \text{ indep. of } \phi / |T(\phi)| \leq M ||\phi||_\infty, \ \forall \phi \in \mathcal{D}(\mathbb{R}) \Rightarrow T \text{ continuous in the sense of } \mathcal{D}$
  - Tempered distribution: $T : \mathcal{S}(\mathbb{R}) \to \Lambda$ linear and continuous (in the sense of $\mathcal{S}$)
  - Space of tempered distributions: $\widetilde{\mathcal{S}(\mathbb{R})}$
  - The sufficient condition for continuity applies the same.
  - Property:

$$\widetilde{\mathcal{S}(\mathbb{R})} \subset \widetilde{\mathcal{D}(\mathbb{R})}$$

Distributions

- Operations with distributions
  - Multiplication by a function:

$$\rho T : \phi \to T(\rho\phi) \text{ is an element of } \widetilde{\mathcal{D}(\mathbb{R})},\ \forall \rho \in C^\infty$$

$$\text{is an element of } \widetilde{\mathcal{S}(\mathbb{R})},\ \forall \rho \in C^\infty \text{ of slow growth}$$

$$\forall m, \exists N_m / ||\rho^{(m)}/(1+|x|^2)^{N_m}||_\infty < \infty$$

  - Derivative of a distribution:

$$T^{(m)} : \phi \to T\big((-1)^m \phi^{(m)}\big)$$

  - Shift:

$$T_a : \phi \to T(\phi_{-a}) \text{ with } \phi_a(x) \equiv \phi(x-a)$$

- These operations are continuous with respect to the following definition of convercend of distributions

$$T_n \to T \Leftrightarrow T_n(\phi) \to T(\phi),\ \forall \phi \in \mathcal{D}(\mathcal{S})$$

With this notion of convergence $\tilde{D}$ and $\tilde{S}$ are complete and $\tilde{S}$ is dense en $\tilde{D}$

Distributions

- Examples of distributions:
  - Dirac's delta $\delta_{x_0} : \phi \to \phi(x_0)$ (tempered distribution)

$$\text{Normally introduced as a "function": } \delta_{x_0}(\phi) = \int \delta(x - x_0)\phi(x)\, dx$$

$$\delta(x - x_0) = \begin{cases} \infty, & x = x_0 \\ 0, & x \neq x_0 \end{cases}$$

and as the limit of a sequence of functions

$$\delta_0 = \lim_{\lambda\to\infty} \sqrt{\frac{\lambda}{\pi}} e^{-\lambda x^2} = \lim_{\epsilon\downarrow 0} (\pi \mathrm{i}\epsilon)^{-1/2} e^{\mathrm{i}x^2/\epsilon} = \lim_{\lambda\to\infty} \frac{\sin \lambda x}{\pi x}$$

$$\delta(x - x_0) = \frac{d}{dx}\theta(x - x_0), \quad \theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \quad \text{(Heaviside step function)}$$

Let $f(x)$ be a function with a finite number of simple zeroes, then

$$\delta\big(f(x)\big) = \sum_1^n \frac{\delta(x - x_i)}{|f'(x_i)|}, \quad f(x_i) = 0$$

Distributions

- Examples of distributions:
  - Principal value of $\frac{1}{x}$ (tempered distribution) $\mathrm{PV}\frac{1}{x}(\phi) = \lim_{\epsilon\downarrow 0}\int_{|x|\geq\epsilon} dx\,\frac{\phi}{x}$

$$\text{We have } \mathrm{PV}\frac{1}{x} = \frac{d}{dx}\ln|x|$$

$$\text{Taking derivatives of } \lim_{\epsilon\downarrow 0}\ln(\epsilon + \mathrm{i}x) = \ln|x| - \mathrm{i}\frac{\pi}{2} + \mathrm{i}\pi\theta(x), \text{ we find}$$

$$\frac{1}{x\mp i0} \equiv \lim_{\epsilon\downarrow 0}\frac{1}{x\mp \mathrm{i}\epsilon} = \mathrm{VP}\frac{1}{x} \pm \mathrm{i}\pi\delta(x)$$

  - Characteristic distribution (distribution) $\text{Sea } X\subset\mathbb{R}$

$$\chi_X : \phi \to \chi_X(\phi) = \int_X \phi(x)\,dx$$

$$\text{Usually presented as a "function" } \chi_X(x) = \begin{cases} 0, & x\notin X, \\ 1, & x\in X \end{cases}$$

Distributions

- Regularity theorem

$$\forall T \in \widetilde{\mathcal{D}(\mathbb{R})},\ \exists f \text{ continuous in } \mathbb{R},\ \exists n \in \mathbb{N}/T = T_f^{(n)}$$

$$\text{where } T_f(\phi) \equiv \int_{\mathbb{R}} \bar{f}(x)\phi(x)\,dx$$

- Fourier transform

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\mathrm{i}kx} f(x)\,dx,\ \text{(direct transform)}$$

$$\check{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}kx} f(y)\,dy,\ \text{(inverse transform)}$$

$$\text{we have } \check{\hat{f}} = \hat{\check{f}} = f$$

- Fourier transform of distributions

$$\hat{T}(\phi) \equiv T(\check{\phi}),\ \forall T \in \widetilde{\mathcal{D}(\mathbb{R})}$$

Linear forms and distributions

- Summary:
  - Linear forms $T : H \to \Lambda$, bounded and continuous
  - Dual space: bounded linear forms
  - Riesz-Fréchet theorem: representation of linear forms in Hilbert spaces
  - Bilinear forms and their representation in Hilbert spaces
  - Spaces of test functions (bounded support and rapid decrease)
  - (Tempered) distribution: linear form in spaces of test functions
  - Operations with distributions: multiplication by a function, derivative, shift
  - Examples of distributions: delta, step, PV(1/x), characteristic distribution
  - Regularity theorem
  - Fourier transform (of distributions).

Operators in Hilbert spaces

- Definition:

(Anti)linear operator. (anti)linear univalued mapping between Hilbert spaces

$$T : D(T) \subset H_1 \to R(T) \subset H_2$$

$$T(\alpha x + \beta y) = \begin{cases} \alpha T(x) + \beta T(y), & \text{(linear)} \\ \bar{\alpha} T(x) + \bar{\beta} T(y), & \text{(anti-linear)} \end{cases} \quad \forall x, y \in D(T),\ \forall \alpha, \beta \in \Lambda$$

- Properties:
  - $D(T),\ R(T)$, $\text{Ker}(T)$ are linear subspaces
  - M linear subspace of $H_1 \Rightarrow TM \equiv \{Tx/x \in M\}$ is a linear subspace H_2
  - $\mathcal{L}(H_1, H_2) \equiv \{T : D(T) \subset H_1 \to H_2/T \text{ linear}\}$ is a linear space with

$$(T_1 + T_2)x = T_1 x + T_2 x, \quad (\alpha T)x = \alpha(Tx)$$

  - $\mathcal{L}(H) \equiv \mathcal{L}(H, H)$

Operators in Hilbert spaces

- Definition: $\text{Bounded operador. Let } T \in \mathcal{L}(H_1, H_2),\ D(T) = H_1$

$$T \text{ is bounded if } \exists M \geq 0 / ||Tx||_{H_2} \leq M||x||_{H_1},\ \forall x \in H_1$$

  - $\mathcal{A}(H_1, H_2) = \{T : H_1 \to H_2 / T \text{ bounded linear}\}$ is a normed space

$$\text{with the norm } ||T|| \equiv \sup_{x \neq 0} \frac{||Tx||}{||x||}$$

- Definición: Continuous operador.

  $T \in \mathcal{L}(H_1, H_2)$ is continuous in $x \in H_1$ if

$$\forall \{x_n\} \to x \Rightarrow \{Tx_n\} \to Tx,\ \left[||x_n - x|| \to 0 \Rightarrow ||Tx_n - Tx|| \to 0\right]$$

- $T \in \mathcal{L}(H_1, H_2)$ is continuous if it is $\forall x \in H_1 \quad T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n$
- Theorem: $T \in \mathcal{L}(H_1, H_2),\ H_{1,2}$ Hilbert spaces

$$T \in \mathcal{A}(H_1, H_2) \Leftrightarrow T \text{ continuous } \Leftrightarrow T \text{ continuous at any point of } H_1$$

- Dual space: $\tilde{H} = \mathcal{A}(H, \Lambda)$

Operators in Hilbert spaces

- Property: $T \in \mathcal{A}(H_1, H_2) \Rightarrow \mathrm{Ker}(T)$ is closed
- Definition: $T : D(T) \neq H_1 \to H_2$ is bounded in its domain if

$$\exists M \geq 0 / ||Tx|| \leq M||x||, \ \forall x \in D(T), \quad ||T|| = \sup_{0 \neq x \in D(T)} \frac{||Tx||}{||x||}$$

- Theorem (extension of operators bounded in a dense domain):

Let $T \in \mathcal{L}(H_1, H_2)$ bounded in its domain, dense in H_1 $(\overline{D(T)} = H_1)$

$\exists! \tilde{T} \in \mathcal{A}(H_1, H_2)$ that extends T to all H_1 and $||\tilde{T}|| = ||T||$

solution $\tilde{T}x = \begin{cases} Tx, & x \in D(T), \\ \lim\limits_{n\to\infty} Tx_n, & x_n \in D(T), \ \lim\limits_{n\to\infty} x_n = x \notin D(T) \end{cases}$

- Properties:
  - $\mathcal{A}(H)$ is a Banach space and algebra of functions with $ST(x) = S\big(T(x)\big)$
  - $||ST|| \leq ||S||\,||T||$
  - Commutator of operators: $[S, T] = ST - TS \neq 0$ in general

Operators in Hilbert spaces

- Definition: Let $T \in \mathcal{L}(H_1, H_2)$, we define the inverse operator (when it exists)

$T^{-1} : R(T) \subset H_2 \to D(T) \subset H_1$ such that $\begin{array}{l} T^{-1}Tx = x,\ \forall x \in D(T) = R(T^{-1}) \\ TT^{-1}y = y,\ \forall y \in R(T) = D(T^{-1}) \end{array}$

- Criterion of existence of the inverse operator Let $T \in \mathcal{L}(H_1, H_2)$

$$\exists\, T^{-1} \in \mathcal{L}(H_2, H_1) \Leftrightarrow T \text{ is injective } \Leftrightarrow Tx = 0 \Rightarrow x = 0$$

Note: Let $T \in \mathcal{A}(H_1, H_2),\ R(T) = H_2,\ T$ injective $\not\Rightarrow T^{-1} \in \mathcal{A}(H_2, H_1)$

- Theorem (criterion of inversion with boundedness):

Let $T \in \mathcal{A}(H_1, H_2),\ R(T) = H_2,\ H_{1,2} \neq \{0\}$ then

$$T^{-1} \in \mathcal{A}(H_2, H_1) \Leftrightarrow \exists k > 0 \ /\ ||Tv|| \geq k||v||,\ \forall v \in H_1$$

- Corolary: Let $T \in \mathcal{A}(H)$ bijective, with $H \neq \{0\}$. Then

$$T^{-1} \in \mathcal{A}(H) \Leftrightarrow \exists k > 0 \ /\ ||Tv|| \geq k||v||,\ \forall v \in H$$

Operators in Hilbert spaces

- Topologies en $\mathcal{A}(H)$: $\text{let } \{A_n \in \mathcal{A}(H)\}_1^\infty$
  - Uniform (or norm) topology

$$A_n \xrightarrow{u} A \Leftrightarrow ||A_n - A|| \underset{n\to\infty}{\longrightarrow} 0$$

  - Strong topology

$$A_n \xrightarrow{s} A \Leftrightarrow A_n v \underset{n\to\infty}{\longrightarrow} Av, \ \forall v \in H$$

  - Weak topology

$$A_n \xrightarrow{w} A \Leftrightarrow \langle w, A_n v\rangle \underset{n\to\infty}{\longrightarrow} \langle w, Av\rangle, \ \forall v, w \in H$$

- In finite dimension (dim of H is finite) they are all equivalent
- In infinite dimension

$$\text{Uniform top.} \underset{\neq}{>} \text{Strong top.} \underset{\neq}{>} \text{Weak top.}$$

Operators in Hilbert spaces

- Some interesting operators
  - Operators in finite dimension

$$T \in \mathcal{L}(H) \Rightarrow \text{ matrix in } \Lambda^n$$

$$\mathcal{A}(H_n) = \mathcal{L}(H_n) \text{ [all linear operators are bounded)]}$$

  - Destruction, creation and number operators (in l^2_Λ)

$$\begin{aligned} a \;\; &: (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots) \to (\alpha_1, \sqrt{2}\alpha_2, \ldots, \sqrt{n+1}\alpha_{n+1}, \ldots) \\ a^+ &: (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots) \to (0, \alpha_0, \sqrt{2}\alpha_1, \ldots, \sqrt{n}\alpha_{n-1}, \ldots) \\ N \;\; &: (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots) \to (0, \alpha_1, 2\alpha_2, \ldots, n\alpha_n, \ldots) \end{aligned}$$

  - Rotation operador (in $L^2(\mathbb{R}^3)$). Let R be a rotation in $\mathbb{R}^3$ around the origin

$$U(R) : f(x) \to f(R^{-1}x)$$

  - Shift operator (in $L^2(\mathbb{R}^n)$). Let a be a vector $\in \mathbb{R}^n$ fijo

$$U_a : f(x) \to f(x-a)$$

Operators in Hilbert spaces

- Some interesting operators
  - Position operador (en $L^2(B)$).

$$Q : f(x) \to xf(x)$$

  - Derivative operador. Let $\mathcal{S}(\mathbb{R}) = \{f \text{ of rapid decrease}\}$, dense in $L^2(\mathbb{R})$

$$P : f(x) \in \mathcal{S}(\mathbb{R}) \to -\mathrm{i}\frac{d}{dx}f(x)$$

  - Properties: let us define the position and momentum operators in $\mathcal{S}(\mathbb{R}^n)$

$$Q_j : f(x) \to x_j f(x)$$

$$(j, k = 1, 2, \ldots, n)$$

$$P_k : f(x) \in \mathcal{S}(\mathbb{R}) \to -\mathrm{i}\frac{\partial}{\partial x_k}f(x)$$

we have

$$Q_j,\ P_k \text{ are not bounded and satisfy } [Q_j, P_k] = i\delta_{jk}1_{\mathcal{S}}$$

Operators in Hilbert spaces

- Adjoint operador

Given $A \in \mathcal{A}(H)$ with H a Hilbert space, the adjoint operator is defined as the only operator $A^\dagger$ $(\in \mathcal{A}(H))$ that satisfies

$$\boxed{\langle w, Av\rangle = \langle A^\dagger w, v\rangle} \qquad \forall v, w \in H$$

- Properties:

$i)$ The mapping $A \to A^\dagger$ is an anti-linear isometric bijection of $\mathcal{A}(H)$

$$\left[||A^\dagger|| = ||A||, \; (\alpha A + \beta B)^\dagger = \bar{\alpha} A^\dagger + \bar{b} B^\dagger \right]$$

$ii)$ $(AB)^\dagger = B^\dagger A^\dagger$

$iii)$ $(A^\dagger)^\dagger = A$

$iv)$ $A, A^{-1} \in \mathcal{A}(H) \Rightarrow (A^\dagger)^{-1} = (A^{-1})^\dagger$

$v)$ $||A^\dagger A|| = ||A||^2$

$vi)$ $A^\dagger = (A^{\mathrm{T}})^*$ in finite dimension

Operators in Hilbert spaces

- Equality of operators

$$A = B \text{ if } D(A) = D(B) = D,\ Ax = Bx,\ \forall x \in D$$

$$(\text{equiv. if } D(A) = D(B) = D,\ \langle y, Ax\rangle = \langle y, Bx\rangle\ \forall x \in D,\ \forall y \in H)$$

- Some special types of operators: $\text{Let } T : D(T) \text{ dense in } H \to H$
  - Symmetric or hermitian operator

$$T \subset T^\dagger \quad \left[D(T) \underset{\neq}{\subset} D(T^\dagger),\ \langle x, Ty\rangle = \langle Tx, y\rangle,\ \forall x, y \in D(T)\right]$$

  - Self-adjoint operator

$$T = T^\dagger \quad \left[D(T) = D(T^\dagger),\ \langle x, Ty\rangle = \langle Tx, y\rangle,\ \forall x, y \in D(T)\right]$$

  - Bounded self-adjoint operator

$$A \in \mathcal{A}(H)\ /\ A = A^\dagger \qquad \left[A = A^\dagger \Leftrightarrow \langle x, Ax\rangle \in \mathbb{R},\ \forall x \in H\right]$$

Operators in Hilbert spaces

- Properties of bounded self-adjoint operators

$$\text{Let } A, B \in \mathcal{A}(H),\ A = A^\dagger,\ B = B^\dagger$$

$$i)\ ||A|| = \sup_{x \neq 0} \frac{|\langle x, Ax \rangle|}{||x||^2}$$

$$ii)\ \alpha A + \beta B \text{ is a bounded self-adjoint operator } \forall \alpha, \beta \in \mathbb{R}$$

$$iii)\ AB \text{ is a bounded self-adjoint operator } \Leftrightarrow [A, B] = 0$$

$$iv)\ ||A^n|| = ||A||^n$$

- Isometric operator

$$T : D(T) \subset H \to H / ||Tx|| = ||x||,\ \forall x \in D(T)$$

property

$$\boxed{T \text{ isometric}} \ \underset{\not\Leftarrow}{\Rightarrow} \ \boxed{T \text{ bounded in its domain with } ||T|| = 1}$$

Operators in Hilbert spaces

- Unitary operator

$$U \in \mathcal{A}(H) \;/\; U^\dagger = U^{-1}$$

Note:

$$U \in \mathcal{A}(H) \text{ isometric} \Leftrightarrow U^\dagger U = 1 \quad \left[UU^\dagger = 1 \Leftrightarrow R(U) = H \right]$$

$$U \in \mathcal{A}(H) \text{ unitary} \Leftrightarrow U^\dagger U = UU^\dagger = 1$$

- Characterization of a unitary operator. Let $U \in \mathcal{A}(H)$. They are equivalent

$$i)\ U \text{ unitary}$$

$$ii)\ R(U) = H, \langle Ux, Uy \rangle = \langle x, y \rangle,\ \forall x, y \in H$$

$$iii)\ R(U) = H, ||Ux|| = ||x||,\ \forall x \in H$$

$$iv)\ \{e_\alpha\}_{\alpha \in A} \text{ orthonormal basis of } H \Rightarrow \{Ue_\alpha\}_{\alpha \in A} \text{ orthonormal basis of } H$$

$$v)\ U^\dagger \text{ is unitary}$$

Operators in Hilbert spaces

- Orthogonal projector

$$P \in \mathcal{A} \text{ is an orthogonal projector if } P^2 = P = P^\dagger$$

- Theorem: Let P be an orthogonal projector, then

$$\exists\, M \text{ closed linear subspace in } H \text{ such that } P \text{ is the orthogonal projector over } M$$

- Normal operator

$$A : D(A) \text{ dense in } H \to H \;/\; D(AA^\dagger) = D(A^\dagger A),\; [A, A^\dagger] = 0$$

Note: $A \in \mathcal{A}(H) \Rightarrow A^\dagger \in \mathcal{A}(H) \Rightarrow D(AA^\dagger) = D(A^\dagger A) = H$

$$A \in \mathcal{A}(H) \text{ normal} \Leftrightarrow ||Av|| = ||A^\dagger v||,\; \forall v \in H$$

- Properties:

$$A \text{ self-adjoint } \Rightarrow A \text{ normal } (AA^\dagger = A^\dagger A = A^2)$$
$$A \text{ hermitian } \not\Rightarrow A \text{ normal } (D(AA^\dagger) \neq D(A^\dagger A))$$
$$A \text{ unitary } \Rightarrow A \text{ normal } (AA^\dagger = A^\dagger A = 1)$$
$$A \text{ isometric } \not\Rightarrow A \text{ normal } (D(AA^\dagger) \neq D(A^\dagger A))$$

Operators in Hilbert spaces

- Summary
  - Operator $\mathcal{L}(H_1, H_2)$, bounded $\mathcal{A}(H_1, H_2)$ and bounded in its domain
  - Continuos operador $\Leftrightarrow$ bounded
  - Theorem of extension of bounded operators with a dense domain
  - Inverse operator. Existence of inverse operator (with boundedness)
  - Uniform, strong and weak topologies in $\mathcal{A}(H)$
  - Examples of operators (finite dim., creation, destruction, number, position, derivative)
  - Adjoint operator
  - Hermitian, self-adjoin, isometric, unitary, normal operator
  - Orthogonal projector

Spectral theory

- Definition: Spectrum and resolvent of linear operators

  Let $A \in \mathcal{L}(H)$, with dense domain in H, separable Hilbert space over $\mathbb{C}$

  - $\mathbb{C}$ can be split in the following subsets, depending on the behavior of the operator $(A - \lambda I)^{-1}$

$$\mathbb{C} = \rho \cup \sigma \equiv \rho \cup \sigma_p \cup \sigma_r \cup \sigma_c, \text{ disjoint in pairs}$$

$\lambda \in \mathbb{C}$	$(A - \lambda I)^{-1}$	$R(A - \lambda I)$	$(A - \lambda I)^{-1}$
$\sigma_p(A)$	does not exist	-	-
$\sigma_r(A)$	exists	not dense in H	-
$\sigma_c(A)$	exists	dense en H	not bounded in its domain
$\rho(A)$	exists	dense en H	bounded in its domain

Spectral theory

- Properties:
  - Eigenvectors and eigenvalues $\lambda \in \sigma_p(A) \Leftrightarrow \exists v_\lambda \neq \text{ in } D(A)/Av_\lambda = \lambda v_\lambda$
  - Linear independence of eigenvectors with different eigenvalues

$$\{\lambda_i\}_1^n \subset \sigma_p(A),\ Av_i = \lambda_i v_i,\ \lambda_i \neq \lambda_j\ (i \neq j) \Rightarrow \{v_i\} \text{ l.i.}$$

  - Topological properties of the spectrum and resolvent

$$\forall A \in \mathcal{L}(H) \Rightarrow \rho(A) \text{ open},\ \sigma(A) \text{ closed in } \mathbb{R}^2$$

  - Spectrum of the adjoint operator: $\text{let } A \in \mathcal{A}(H)$

$$i)\ \lambda \in \rho(A) \Leftrightarrow \bar{\lambda} \in \rho(A^\dagger)$$

$$ii)\ \lambda \in \sigma_p(A) \Rightarrow \bar{\lambda} \in \sigma_p(A^\dagger) \cup \sigma_r(A^\dagger)$$

$$iii)\ \lambda \in \sigma_r(A) \Rightarrow \bar{\lambda} \in \sigma_p(A^\dagger)$$

$$iv)\ \lambda \in \sigma_c(A) \Leftrightarrow \bar{\lambda} \in \sigma_c(A^\dagger)$$

Spectral theory

- Properties:
  - Spectrum of normal operators: $\text{let } A \in \mathcal{A}(H) \text{ normal}$

$$a) Av = \lambda v \Leftrightarrow A^\dagger v = \bar{\lambda} v$$

$$b) Av_i = \lambda_i v_i,\ \lambda_i \neq \lambda_j \Rightarrow v_i \perp v_j$$

$$c) \sigma_r(A) = \emptyset$$

  - Spectrum of unitary operators (are normal):

$$U \text{ unitary} \Rightarrow \sigma(U) = \sigma_p(U) \cup \sigma_c(U) \subset \{\lambda / |\lambda| = 1\}$$

  - Spectrum of isometric operators (not normal in general):

$$A \text{ isometric} \Rightarrow \sigma_p(A) \subset \{\lambda / |\lambda| = 1\},\ \left[\text{in general } \sigma(A) \not\subset \{|\lambda| = 1\},\ \sigma_r \neq \emptyset\right]$$

Spectral theory

- Properties:
  - Spectrum of orthogonal projectors:

$$\sigma(0) = \{0\}, \quad \sigma(1) = \{1\}, \text{ all other orthogonal projectors satisfy}$$

$$P \in \mathcal{A}(H), P^2 = P = P^\dagger,\ 0 \neq P \neq 1, \Rightarrow \sigma(P) = \sigma_p(P) = \{0, 1\}$$

  - Spectrum of self-adjoint operators: sea $A \in \mathcal{A}(H)$ autoadjunto

$$1)\ \sigma(A) \subset \mathbb{R},\ \sigma_r(A) = \emptyset$$

$$2)\ \sigma(A) \subset \left[\inf_{||v||=1} \langle v, Av \rangle, \sup_{||v||=1} \langle v, Av \rangle \right]$$

$$3)\ M_\lambda(A) = \{v \in H / Av = \lambda v\} \text{ closed linear subspace}$$

$$4)\ \forall A \in \mathcal{A}(H),\ A = A^\dagger \Rightarrow \exists \{v_i\} \text{ orthonormal, maximal } / Av_i = \lambda_i v_i$$

$$\text{(not necessarily complete in } H\text{)}$$

Spectral theory

- Definition: $A \in \mathcal{L}(H_1, H_2)$ is compact $\big(A \in \mathcal{C}(H_1, H_2)\big)$

  if $\overline{A(X)}$ is compact in $H_1,\ \forall X \subset H_1,\ X$ bounded $(\sup\limits_{x \in X} ||x|| < \infty)$

  - If $\dim(H) < \infty \to \mathcal{L}(H) = \mathcal{A}(H) = \mathcal{C}(H)$

- Theorem: $\forall A \in \mathcal{C}(H)$

  1) $\sum\limits_{\lambda \in \sigma_p / |\lambda| > k} \dim M_\lambda(A) < +\infty,\ \forall k > 0$

  2) $\sigma_p(A)$ is at most numerable, with 0 as the only possible limit point

  3) $\mathbb{C} - \{0\} \subset \sigma_p(A) \cup \rho(A)$

  4) $0 \in \sigma(A)$

  5) $\sigma_r(A) \cup \sigma_c(A) \subset \{0\}$