Spaces with scalar product and Hilbert spaces

Problem 1: Spaces of finite dimension

Let X be a linear space of finite dimension and $\{e_i\}_1^n$ a linear basis. Show that any scalar product over X is fixed by the set of scalars $\gamma_{ij} = \langle e_i, e_j \rangle$, $i, j = 1, \ldots, n$. Is it possible to choose an arbitrary set of scalars γ_{ij}?

Problem 2: Pythagora's theorem

Let H be a Hilbert space over the body of scalars Λ. Show that two orthogonal vectors fulfil the property $x_1 \perp x_2 \Rightarrow ||x_1 + x_2|| = ||x_1 - x_2||$.
Is the inverse of Pythagora's theorem true? That is, any set of vectors that satisfy $||\sum_i x_i||^2 = \sum_i ||x_i||^2$ are necessarily orthogonal?

Problem 3: Pre-Hilbert and normed spaces

Let X be a pre-Hilbert space. Show that

$$||\lambda x + (1-\lambda)y|| = ||x||,\ \forall \lambda \in [0,1] \Rightarrow x = y.$$

Is that true for an arbitrary normed space?

Problem 4: Norm and scalar product

Let L be the set of 2×2 traceless complex hermitian matrices (equal to the complex conjugate of its transpose). Using that L is a normed space with the norm

$$||A||^2 = \frac{1}{2}\mathrm{Tr}(A \cdot A),$$

check if this norm comes from a scalar product and compute it in case it does. Find an orthonormal basis.

Problem 5: Norm and scalar product

Let $\mathcal{M}_{n\times n}$ the set of complex matrices with the usual sum and product by a scalar. Compute the scalar product associated to the following norm

$$||A|| = \frac{1}{n}\left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}.$$

Find an orthonormal basis for $n = 2$ that contains the matrix

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and compute the projection of the identity matrix on the orthogonal complement of $[S]$.

Problem 6: Norm in Hilbert spaces

Let H be a Hilbert space and v a vector in H. Show that

$$||v|| = \sup_{||w||=1} |\langle v, w \rangle|.$$

Problem 7: Pre-Hilbert spaces and orthogonal direct sum

Show that the space $C[-1, 1]$ with scalar product

$$\langle f, g \rangle = \int_{-1}^{1} dt \bar{f}(t) g(t),$$

is pre-Hilbert. Is it Hilbert? Show that it can be written as the orthogonal direct sum of the set of even and odd functions in $[-1, 1]$.

Problem 8: Pre-Hilbert spaces and orthogonal direct sum

Let X be the subspace of l^2 of sequences with a finite support (they only have a finite number of non-vanishing elements). Show that X is not complete. Let M be the following subspace of X

$$M = \{y \in X / y \perp x_0\}, \quad x_0 = \left\{ \frac{1}{n} \right\}_1^\infty .$$

Show that M is closed bu that $X \neq M \oplus M^\perp$. What would happen if X was complete?

Problem 9: Orthogonal direct sum

Show that $Y = \{\{\xi_n\}_1^\infty \in l^2 / \xi_{2n} = 0\}$ is a closed subspace of l^2 and find $Y^\perp$. Is $l^2 = Y \oplus Y^\perp$?

Problem 10: Best approximatino

Let $H = \mathbb{C}^4$ (Hilbert space). Find the distance of the vector $v = (1, 2, 1 + \mathrm{i}, -4)$ to the subspace generated by the two vectors $\{(0, 1, 0, 0), (0, 0, 0, 1)\}$.

Problem 11: $l^2_{\mathbb{C}}$ space

Discuss if the following sequences belong to $l^2_{\mathbb{C}}$:

a) $x = (1, 2 - 1/2, 2 - 1/3, \ldots, 2 - 1/n, \ldots)$,

b) $y = (1/\mathrm{i}, 0, 3/2\mathrm{i}, \ldots, (2n-1)/(n\mathrm{i}), \ldots)$

Problem 12: $l^2_{\mathbb{C}}$ space

Let M be the following subspace of l^2

$$M = \left\{ \{x_k\}_1^\infty / x_{2k-1} = a_k + b_k \cos\frac{1}{k}, x_{2k} = b_k \sin\frac{1}{k}, \{a_k\}, \{b_k\} \in l^2 \right\},$$

Show that it is not closed by proving that $\overline{M} = l^2$ and that the following vector of l^2

$$z = \left\{ \{z_k\}_1^\infty / z_{2k-1} = 0, z_{2k} = \sin\frac{1}{k} \right\}$$

does not belong to M.

Problem 13: Dense subspaces

Let $\{e_k\}_1^\infty$ be an orthonormal basis of a Hilbert space. Consider the following subspaces

$$M_1 = [\{x_k\}_1^\infty], \quad M_2 = [\{u_k\}_1^\infty], \quad M_3 = [\{w_k\}_2^\infty], \quad M_4 = [\{y_k\}_1^\infty],$$

where we have defined the vectors

$$x_k = e_k + e_{k+1}, \quad v_k = \sum_{j=1}^{k} e_j, \quad w_k = (-1)^k e_k - e_1, \quad y_k = e_k/(k+1) - e_{k+1}.$$

Are M_i dense in H? Are the closed? Can we define the best approximation of any vector of H within M_i?

Problem 14: Orthogonal projection

Show that the following statements are equivalent in a Hilbert space

i) P is an orthogonal projector.

ii) P is idempotent, that is $P^2 = P$, and it has $R(P) \perp R(I-P)$ (R codomain or image of the projector).

iii) P is idempotent and $\big(R(P)\big)^\perp = \{x \in H / P(x) = 0\}$.

Problem 15: Orthonormal bases

Let $\{e_n\}$ be an orthonormal basis of a complex separable Hilbert space H of dimension N (can be numerable infinite). Show that

$$N \text{ finite } \Leftrightarrow \exists v \in H,\ v \neq 0 / \{e_j - v\} \text{ is orthonormal.}$$

Problem 16: Best approximation (Problem to hand)

Let $H = l^2_{\mathbb{C}}$ and $B = \{e_1, e_2 + e_3, e_1 + e_2 - e_3\}$, with $\{e_n\}_{n=1}^{\infty}$ the canonical basis of H. Which one is the smallest subspace M that contains B? Is B an orthonormal basis of M? If it is not, find such a basis. Find the distance of the following vectors to M:

$$\begin{aligned} x_1 &= (1,1,1,1,0,\ldots), \\ x_2 &= (0,0,0,1,0,\ldots), \\ x_3 &= (1,0,0,0,0,\ldots). \end{aligned}$$

Problem 17: Best approximation (Problem to hand)

Consider the Hilbert space of real 2×2 matrices with the scalar product

$$\langle A, B \rangle = \frac{1}{2}\mathrm{Tr}(A^T B).$$

Find the distance of the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

to the subspace generated by the matrices

$$S = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$