The local picture theorem on the scale of topology (Preliminary version)

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Abstract

In this paper we prove a descriptive structure theorem of the extrinsic geometry of an embedded minimal surface in a Riemannian three-manifold in any small intrinsic neighborhood of a point of *concentrated topology.* This structure theorem includes a new limit object which we call a *minimal parking garage structure* on $\mathbb{R}^3$, whose beginning theory we also develop. We apply this topological structure theorem to prove that any complete, embedded minimal surface with finite genus and a countable number of ends in $\mathbb{S}^3$ is compact.

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Contents

1 Introduction.

Fundamental theorems on the nonexistence of singularities in mathematical and physical problems represent cornerstones for building powerful theories which can then become stepping stones to future theoretical advances and which can create the insights needed for deep applications to other related areas. Recent work by Colding and Minicozzi [3, 4, 9, 10] on removable singularities for certain limit minimal laminations of $\mathbb{R}^3$, and subsequent applications by Meeks and Rosenberg [39, 40] demonstrate the fundamental importance of these removable singularity results for obtaining a deep understanding of the geometry of complete, embedded minimal surfaces in three-manifolds. Removable singularity theorems

for limit minimal laminations also play a central role in our papers [26, 27, 29, 31, 32] where we obtain topological bounds and descriptive results for complete, embedded minimal surfaces of finite genus in $\mathbb{R}^3$.

An important building block of this emerging removable singularity theory is a basic compactness result which we present here. This result, Theorem 10.1 in section 10, associates to certain minimal surfaces in three-manifolds limit objects which are properly embedded minimal surfaces in $\mathbb{R}^3$, minimal parking garage structures on $\mathbb{R}^3$ and possibly, certain singular minimal laminations of $\mathbb{R}^3$ with restricted geometry; the concept of parking garage structure is developed at the beginning of section 10. We refer to this result as the Local Picture Theorem on the Scale of Topology. The usefulness of this theorem is that it gives for the first time a view of the extrinsic geometric structure of an arbitrary embedded minimal surface M in a small intrinsic neighborhood of a point of concentrated topology, where M lies in any homogeneously regular three-manifold. The results in the recent series of papers [3, 4, 9, 10] by Colding and Minicozzi and the recent minimal lamination closure theorem by Meeks and Rosenberg [40] play important roles in deriving this basic compactness result.

Two direct and deep consequences of our Local Picture Theorem on the Scale of Topology for embedded minimal surfaces in a complete, locally homogeneous[1] three-manifold N are:

1. The closure of any complete, embedded minimal surface with finite topology M in N has the structure of a minimal lamination. Thus, when N is compact, then M has bounded second fundamental form.

2. In the case the universal cover of N is topologically the three-space with a metric of non-negative scalar curvature, then any complete, embedded minimal surface in N of finite genus and a countable number of ends must be compact.

The second statement answers in the negative the long standing question as to whether the unit radius three-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ admits a complete, non-compact, embedded minimal surface of finite topology (since the early 1980's, it was known that there exist complete, embedded minimal annuli of unbounded curvature in the three-sphere with respect to certain metrics of positive scalar curvature, and see White [56] for such examples). This question is partly motivated by work of Lawson [21] who proved that for every non-negative integer k, there exists a compact, embedded minimal surface of genus k in $\mathbb{S}^3$. These applications of the Local Picture Theorem on the Scale of Topology appear in section 11 and we refer the reader to [27, 40] for further applications. Also, see section 7 for the closely related Local Picture Theorem on the Scale of Curvature which describes

[1] A Riemannian manifold N is *locally homogeneous* if given any two points $p, q \in N$, there exists an $\varepsilon > 0$ such that the balls $B_N(p, \varepsilon), B_N(q, \varepsilon)$ are isometric.

the geometry of an embedded minimal surface in a homogeneously regular three-manifold, locally around a point of large curvature.

The first main theorem of this paper is a crucial local removable singularity theorem for certain minimal laminations with isolated singularities in a Riemannian three-manifold, see Theorem 1.2 below. We obtain here a number of applications of this result to the classical theory of minimal surfaces, that we explain below. Also, this local removable singularity theorem has led us to make the following general conjecture and to obtain substantial positive partial results on it in [29]; these partial results lead naturally to the deep results in [26] on bounds for the index and for the number of ends for complete, embedded minimal surfaces in $\mathbb{R}^3$ with finite topology and fixed genus.

Conjecture 1.1 (Fundamental Singularity Conjecture) *Suppose $\mathcal{S} \subset \mathbb{R}^3$ is a closed set whose one-dimensional Hausdorff measure is zero. If $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3 - \mathcal{S}$, then $\mathcal{L}$ extends across $\mathcal{S}$ to a minimal lamination of $\mathbb{R}^3$.*

Since the union of a catenoid with a plane passing through its waist circle is a singular minimal lamination of $\mathbb{R}^3$ whose singular set is the intersecting circle, the above conjecture represents the strongest possible conjecture.

We point out to the reader that Conjecture 1.1 has a global nature, because there exist interesting minimal laminations of the open unit ball in $\mathbb{R}^3$ punctured at the origin which do not extend across the origin, see section 2. In hyperbolic three-space $\mathbb{H}^3$, there are rotationally invariant global minimal laminations which have a similar unique isolated singularity. The existence of these global singular minimal laminations of $\mathbb{H}^3$ demonstrate that the validity of Conjecture 1.1 must depend on the metric properties of $\mathbb{R}^3$. However, we do obtain the following local removable singularity result in any Riemannian three-manifold N for certain possibly singular minimal laminations.

Given a three-manifold N and a point $p \in N$, we will denote by d the distance function in N to p and $\mathbb{B}_N(p, r)$ the metric ball of center p and radius $r > 0$. For a lamination $\mathcal{L}$ of N, we will denote by $|K_{\mathcal{L}}|$ the absolute curvature function on the leaves of $\mathcal{L}$.

Theorem 1.2 (Local Removable Singularity Theorem) *A minimal lamination $\mathcal{L}$ of a punctured ball $\mathbb{B}_N(p, r) - \{p\}$ in a Riemannian three-manifold N extends to a minimal lamination of $\mathbb{B}_N(p, r)$ if and only if there exists a positive constant c such that $|K_{\mathcal{L}}|d^2 < c$ in some subball*[2].

Since stable minimal surfaces have local curvature estimates which satisfy the hypothesis of Theorem 1.2 and the universal covers limit leaves of a minimal lamination are stable, we obtain the next extension result for the sublamination of limit leaves of any minimal lamination in a countably punctured three-manifold.

[2]Equivalently by the Gauss theorem, for some positive constant c', $|A_{\mathcal{L}}|d < c'$, where $|A_{\mathcal{L}}|$ is the norm of the second fundamental form of $\mathcal{L}$.

Corollary 1.3 *Suppose that N is a Riemannian three-manifold, which is not necessarily complete. If $W \subset N$ is a closed countable subset and $\mathcal{L}$ is a minimal lamination of $N - W$, then:*

1. *The sublamination of $\mathcal{L}$ consisting of the closure of any collection of its stable leaves extends across W to a minimal lamination of N.*
2. *The sublamination of $\mathcal{L}$ consisting of its limit leaves extends across W to a minimal lamination of N.*
3. *If $\mathcal{L}$ is a minimal foliation of $N - W$, then $\mathcal{L}$ extends across W to a minimal foliation of N.*

We remark that the natural generalizations of the above local removable singularity theorem and of Conjecture 1.1 fail badly for codimension-one minimal laminations of $\mathbb{R}^n$, for $n = 2$ and for $n > 3$. In the case $n = 2$, consider the cone $\mathcal{C}$ over any two non-antipodal points on the unit circle; $\mathcal{C}$ consists of two infinite rays making an acute angle at the origin. The punctured cone $\mathcal{C} - \{\vec{0}\}$ is totally geodesic and so the norm of the second fundamental form of $\mathcal{C} - \{\vec{0}\}$ is zero but $\mathcal{C}$ is not a smooth lamination at the origin. In the case $n = 4$, let $\mathcal{C}$ denote the cone over the Clifford torus $\mathbb{S}^1(\sqrt{2}) \times \mathbb{S}^1(\sqrt{2}) \subset \mathbb{S}^3 \subset \mathbb{R}^4$. The punctured cone $\mathcal{C} - \{\vec{0}\}$ is a properly embedded minimal hypersurface of $\mathbb{R}^4 - \{\vec{0}\}$ which does not extend across $\{\vec{0}\}$ to a minimal hypersurface of $\mathbb{R}^4$. Since the norm of the second fundamental form of the Clifford torus is constant, then the norm of the second fundamental form of $\mathcal{C} - \{\vec{0}\}$ multiplied by the distance function to the origin is also a constant function on $\mathcal{C} - \{\vec{0}\}$. For $n > 4$, one can consider cones over any embedded compact minimal hypersurface in $\mathbb{S}^{n-1}$ which is not an equator. These examples demonstrate that Theorem 1.2 is precisely a two-dimensional result.

A fundamental application of our local removable singularity result is to characterize of all complete, embedded minimal surfaces of quadratic decay of curvature (see Theorem 1.4 below). Given a properly embedded minimal surface $M \subset \mathbb{R}^3$, this characterization result leads naturally to a dynamical theory for the space $D(M)$ which consists of properly embedded, non-flat minimal surfaces which are smooth dilation[3] limits of M. In section 9, we indicate how this dynamical theory can be used as a tool to obtain insight and simplification strategies for solving several fundamental outstanding problems in the classical theory of minimal surfaces. It is our hope that these dynamics on $D(M)$ will soon be better understood and that they can eventually be refined into a tool for proving Conjecture 1.1.

For the statement of the next theorem, we first recall that a complete Riemannian surface M has *intrinsic quadratic curvature decay constant $C > 0$ with respect to a point*

[3] A *dilation* of $\mathbb{R}^3$ is the composition of a translation and a homothety

$p \in M$, if the absolute curvature function $|K_M|$ of M satisfies

$$|K_M(q)| \leq \frac{C}{d_M(p,q)^2},$$

for all $q \in M$, where d_M denotes the Riemannian distance function. Note that if such a Riemannian surface M is a complete surface in $\mathbb{R}^3$ with $p = \vec{0} \in M$, then it also has extrinsic quadratic decay constant C with respect to the radial distance R to $\vec{0}$, i.e. $|K_M|R^2 \leq C$ on M. For this reason, when we say that a minimal surface in $\mathbb{R}^3$ has *quadratic decay of curvature*, we will always refer to curvature decay with respect to the extrinsic distance R to $\vec{0}$, independently of whether or not M passes through $\vec{0}$.

Theorem 1.4 (Quadratic Curvature Decay Theorem) *A complete, embedded minimal surface in $\mathbb{R}^3$ with compact boundary (possibly empty) has quadratic decay of curvature if and only if it has finite total curvature. In particular, a complete, connected embedded minimal surface $M \subset \mathbb{R}^3$ with compact boundary and quadratic decay of curvature is properly embedded in $\mathbb{R}^3$. Furthermore, if C is the maximum of the logarithmic growths of the ends of M, then*

$$\lim_{R\to\infty} \sup_{M-\mathbb{B}(R)} |K_M|R^4 = C^2,$$

where $\mathbb{B}(R)$ denotes the extrinsic ball of radius R centered at $\vec{0}$.

In the next theorem we examine the set of all non-flat, properly embedded minimal surfaces in $\mathbb{R}^3$ which arise as dilation limits of a fixed properly embedded minimal surface. In order to clarify its statement, we need some definitions.

Definition 1.5 Let $M \subset \mathbb{R}^3$ be a non-flat, properly embedded minimal surface. Then:

1. M is *periodic,* if it is invariant under a nontrivial translation or a screw motion symmetry.

2. M is *quasi-translation-periodic,* if there exists a divergent sequence $\{p_n\}_n \subset \mathbb{R}^3$ such that $\{M - p_n\}_n$ converges on compact subsets of $\mathbb{R}^3$ to M (note that every periodic surface is also quasi-translation-periodic, even in the case the surface is invariant under a screw motion symmetry).

3. M is *quasi-dilation-periodic,* if there exists a sequence of homotheties $\{h_n\}_n$ and a divergent sequence $\{p_n\}_n \subset \mathbb{R}^3$ such that $\{h_n(M - p_n)\}_n$ converges in a C^1-manner on compact subsets of $\mathbb{R}^3$ to M. Since M is not flat, it is not stable and, thus, the convergence of such a sequence $\{h_n(M - p_n)\}_n$ to M has multiplicity one by Lemma 3.4.

4. Let $D(M)$ be the set of non-flat, properly embedded minimal surfaces in $\mathbb{R}^3$ which are obtained as C^1-limits of a divergent sequence of dilations of M (i.e. the translational part of the dilations diverges). A non-empty subset $\Delta \subset D(M)$ is called *D-invariant,* if for any $\Sigma \in \Delta$, then $D(\Sigma) \subset \Delta$. A D-invariant subset $\Delta \subset D(M)$ is called a *minimal D-invariant set,* if it contains no proper, non-empty D-invariant subsets. We say that $\Sigma \in D(M)$ is a *minimal element,* if Σ is an element of a minimal D-invariant subset of $D(M)$.

The following result deals with the space $D(M)$ of dilation limits of a properly embedded minimal surface $M \subset \mathbb{R}^3$.

Theorem 1.6 (Dynamics Theorem) *Let $M \subset \mathbb{R}^3$ be a properly embedded, non-flat minimal surface. Then, $D(M) = \emptyset$ if and only if M has finite total curvature. Now assume that M has infinite total curvature, and consider $D(M)$ endowed with the topology of C^k-convergence on compact sets of $\mathbb{R}^3$ for all k. Then:*

1. *$D_1(M) = \{\Sigma \in D(M) \mid \vec{0} \in \Sigma,\ |K_\Sigma| \leq 1,\ |K_\Sigma|(\vec{0}) = 1\}$ is a non-empty compact subspace of $D(M)$.*

2. *For any $\Sigma \in D(M)$, $D(\Sigma)$ is a closed D-invariant set of $D(M)$. If $\Delta \subset D(M)$ is a D-invariant set, then its closure $\overline{\Delta}$ in $D(M)$ is also D-invariant.*

3. *Suppose that $\Delta \subset D(M)$ is a non-empty minimal D-invariant set which does not consist of exactly one surface of finite total curvature. If $\Sigma \in \Delta$, then $D(\Sigma) = \Delta$ and the closure in $D(M)$ of the path connected subspace of all dilations of Σ equals Δ. In particular, any minimal D-invariant set is connected and closed in $D(M)$.*

4. *Any non-empty D-invariant subset of $D(M)$ contains minimal elements. In particular, since $D(M)$ is D-invariant, $D(M)$ always contains some minimal element.*

5. *Let $\Delta \subset D(M)$ be a non-empty D-invariant subset. If no $\Sigma \in \Delta$ has finite total curvature, then $\Delta_1 = \{\Sigma \in \Delta \mid \vec{0} \in \Sigma,\ |K_\Sigma| \leq 1,\ |K_\Sigma|(\vec{0}) = 1\}$ contains a minimal element Σ' with $\Sigma' \in D(\Sigma')$ (which in particular, is a quasi-dilation-periodic surface of bounded curvature).*

6. *If a minimal element Σ of $D(M)$ has finite genus, then either Σ has finite total curvature, Σ is a helicoid, or Σ has genus zero, two limit ends, bounded curvature and is quasi-translation-periodic.*

7. *If $D(M)$ contains a surface which is not a helicoid, then $D_T(M) = \{\Sigma \in D(M) \mid I_\Sigma(\vec{0}) = 1,\ I_\Sigma \geq 1\}$ is a non-empty subset of $D(M)$, where I_Σ is the injectivity radius function of Σ. Furthermore, if $D(M)$ does not contain a helicoid, then:*

(a) *$D_T(M)$ is compact;*

(b) *there exists a positive constant T_M such that the absolute Gaussian curvature of every element $\Sigma \in D_T(M)$ is bounded from above by T_M and the absolute Gaussian curvature of $\Sigma \cap \mathbb{B}(1)$ is at least $\frac{1}{T_M}$ at some point;*

(c) *if Δ is any non-empty D-invariant subset of $D(M)$, then $\Delta \cap D_T(M) \neq \emptyset$ or else Δ contains a surface of finite total curvature, which after composing with a fixed dilation of $\mathbb{R}^3$, lies in $D_T(M)$.*

8. *Every middle end of a minimal element Σ of $D(M)$ is asymptotic to the end of a plane or catenoid.*

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2 Examples of nontrivial minimal laminations.

2.1 Minimal laminations with isolated singularities.

We first construct examples in the closed unit ball of $\mathbb{R}^3$ centered the origin with the origin as the unique non-removable singularity. We then show how these examples lead to related singular minimal laminations in the homogeneous spaces $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$.

Example I. *Catenoid type laminations.* Consider the sequence of horizontal circles $C_n = \mathbb{S}^2(1) \cap \{x_3 = \frac{1}{n}\}$, $n \geq 2$. Note that each pair C_{2k}, C_{2k+1} bounds a compact unstable catenoid $M(k)$. Clearly, $M(k) \cap M(k') = \emptyset$ if $k \neq k'$. The sequence $\{M(k)\}_k$ converges with multiplicity two outside of the origin $\vec{0}$ to the closed horizontal disk $\overline{\mathbb{D}}$ of radius 1 centered at $\vec{0}$. Thus, $\{M(k)\}_k \cup \{\overline{\mathbb{D}} - \{\vec{0}\}\}$ is a minimal lamination of the closed ball minus the origin, which does not extend through the origin, see Figure 1.

Example II. *Colding-Minicozzi examples.* In their paper [6], Colding and Minicozzi construct a sequence of compact embedded minimal disks $D_n \subset \mathbb{B}(\vec{0}, 1)$ with boundary in $\mathbb{S}^2(1)$, that converges to a singular minimal lamination $\overline{\mathcal{L}}$ of the closed ball $\overline{\mathbb{B}}(\vec{0}, 1)$ which has an isolated singularity at $\vec{0}$. The related lamination $\mathcal{L}$ of $\overline{\mathbb{B}}(\vec{0}, 1) - \{\vec{0}\}$ consists of a unique limit leaf which is the punctured closed disk $\overline{\mathbb{D}} - \{\vec{0}\}$, together with two non-proper leaves that spiral into $\overline{\mathbb{D}} - \{\vec{0}\}$ from opposite sides, see Figure 2.

Consider the exhaustion of $\mathbb{H}^3$ (naturally identified with $\mathbb{B}(\vec{0}, 1)$) by hyperbolic balls of hyperbolic radius n centered at the origin, together with compact minimal disks with boundaries on the boundaries of these balls, similar to the compact Colding-Minicozzi disks. We conjecture that these examples produce a similar limit lamination of $\mathbb{H}^3 - \{\vec{0}\}$ with three leaves, one which is totally geodesic and the other two

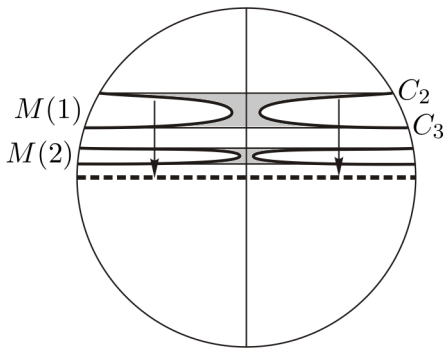

Figure 1: A catenoid type lamination.

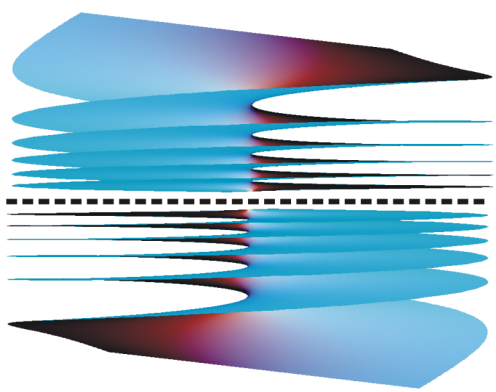

Figure 2: A Colding-Minicozzi type lamination in a cylinder.

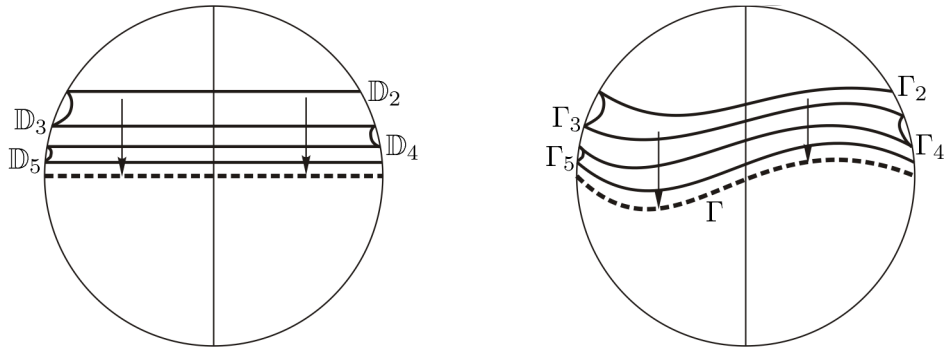

Figure 3: Left: Almost flat minimal disks joined by small bridges. Right: A similar example with a non-flat limit leaf.

which are not proper and that spiral into the first one. We remark that one of the main results of Colding-Minicozzi theory (Theorem 0.1 in [10]) insures that such an example cannot be constructed in $\mathbb{R}^3$.

Example III. *Catenoid type example in* $\mathbb{H}^3$ *and in* $\mathbb{H}^2 \times \mathbb{R}$. As in example I, consider the circles $C_n = \mathbb{S}^n(1) \cap \{x_3 = \frac{1}{n}\}$, where $\mathbb{S}^2(1)$ is now viewed as the boundary at infinity of $\mathbb{H}^3$. Then each pair of circles C_{2k}, C_{2k+1} is the asymptotic boundary of a properly embedded, annular, unstable minimal surface $M(k)$, which is a surface of revolution called a catenoid. The sequence $\{M(k)\}_k$ converges with multiplicity two outside of $\vec{0}$ to the horizontal totally geodesic subspace $\mathbb{D}$ at height zero. Thus, $\{M(k)\}_k \cup \{\mathbb{D} - \{\vec{0}\}\}$ is a minimal lamination of $\mathbb{H}^3 - \{\vec{0}\}$, which does not extend through the origin. A similar catenoidal construction can be done in $\mathbb{H}^2 \times \mathbb{R}$, where we consider $\mathbb{H}^2$ in the disk model of the hyperbolic plane. Note that the Half-space Theorem [19] excludes this type of singular minimal lamination in $\mathbb{R}^3$.

2.2 Minimal laminations with limit leaves.

Example IV. *Simply-connected bridged examples.* Consider the sequence of horizontal closed disks $\mathbb{D}_n = \overline{\mathbb{B}}(\vec{0}, 1) \cap \{x_3 = \frac{1}{n}\}$, $n \geq 2$. Connect each pair $\mathbb{D}_n, \mathbb{D}_{n+1}$ by a minimal small almost vertical bridge (in opposite sides for consecutive disks, as in Figure 3 left), and perturb slightly to obtain a complete, embedded, stable minimal surface with boundary in $\overline{\mathbb{B}}(\vec{0}, 1)$ (this is possible by the bridge principle [43]). We denote by M the intersection of this surface with $\mathbb{B}(\vec{0}, 1)$. Then the closure of M in $\mathbb{B}(\vec{0}, 1)$ is a minimal lamination of $\mathbb{B}(\vec{0}, 1)$ with two leaves, both being stable, one of which is $\mathbb{D}$ (this is a limit leaf) and the other one is not flat and not proper.

A similar example with a non-flat limit leaf can be constructed by exchanging the

horizontal circles by suitable curves in $\mathbb{S}^2(1)$. Consider a non-planar smooth Jordan curve $\Gamma \subset \mathbb{S}^2(1)$ which admits a one-to-one projection onto a convex planar curve in a plane Π. Let Γ_n be a sequence of smooth Jordan curves in $\mathbb{S}^2(1)$ converging toΓ, so that each Γ_n also projects injectively onto a convex planar curve in Π and $\{\Gamma_n\}_n \cup \{\Gamma\}$ is a lamination on $\mathbb{S}^2(1)$. Each of the Γ_n is the boundary of a unique minimal surface $\overline{M_n}$ which is a graph over its projection to Π. Now join slight perturbations of the $\overline{M_n}$ by thin bridges as in the preceding paragraph, to obtain a simply connected minimal surface in the closed unit ball. Let M be the intersection of this surface with $\mathbb{B}(\vec{0}, 1)$. Then, the closure of M in $\mathbb{B}(\vec{0}, 1)$ is a minimal lamination of $\mathbb{B}(\vec{0}, 1)$ with two leaves, both being non-flat and stable, and exactly one of them is properly embedded in $\mathbb{B}(\vec{0}, 1)$ and is a limit leaf (see Figure 3 right).

Example V. *Simply-connected bridged examples in* $\mathbb{H}^3$ *and* $\mathbb{H}^2 \times \mathbb{R}$. As in the previous subsection, the minimal laminations in example IV give rise to minimal laminations of $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ consisting of two stable, complete, simply connected minimal surfaces, one of which is proper and the other one which is not proper in the space, and either one is not totally geodesic or both of them are not totally geodesic, depending on the choice of the Euclidean model surface in Figure 3. In this case, the proper leaf is the unique limit leaf of the minimal lamination. More generally, Theorem 13 in [40] states that the closure of any complete, embedded minimal surface of finite topology in $\mathbb{H}^3$ or $\mathbb{H}^2 \times \mathbb{R}$ has the structure of a minimal lamination.

3 Stable minimal surfaces which are complete outside of a point.

Definition 3.1 A surface $M \subset \mathbb{R}^3 - \{\vec{0}\}$ is *complete outside the origin,* if every divergent path in M of finite length has as limit point the origin.

In sections 4 and 5 we study complete, embedded minimal surfaces $M \subset \mathbb{R}^3$ with quadratic decay of curvature. Our approach is to produce from M, via a sequence of homothetic shrinkings, a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{\vec{0}\}$ with a limit leaf L. Since L is a leaf of a minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$, then L is complete outside the origin. After passing to its universal cover $\widetilde{L}$, we can assume $\widetilde{L}$ is stable (see Lemma 3.4 below), orientable and complete outside the origin. The following lemma will be then used to show that the closure of $\widetilde{L}$ is a plane, which implies the same property for the closure of L. This planar leaf L will play a key role in proving that M must have finite total curvature.

Remark 3.2 The line of arguments in the last paragraph is inspired by ideas in our previous paper [32], where we proved that a properly embedded minimal surface of finite genus in $\mathbb{R}^3$ cannot have one limit end. A key lemma in the proof of this result states that

if such a surface M exists, then some sequence of homothetic shrinkings of M converges to a minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$. Furthermore, this lamination is contained in a closed half-space and contains a limit leaf L, which is different from the boundary of the half-space. Since L is a leaf of a minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$, then it is complete outside $\vec{0}$ and as it is a limit leaf, its universal cover must be stable by item *1* of Lemma 3.4 below. We then proved that the closure of L must be a plane. Using the plane $\overline{L}$ as a guide for understanding the lamination, we obtained a contradiction.

Before stating the stability lemma, we will set some specific notation to be used throughout the paper. Let $R\colon \mathbb{R}^3 \to \mathbb{R}$ be the distance function to the origin $\vec{0} \in \mathbb{R}^3$. Given $r > 0$, $\mathbb{B}(r)$ will stand for the open ball centered at $\vec{0}$ with radius r. The boundary and closure of $\mathbb{B}(r)$ will be respectively denoted by $\partial\mathbb{B}(r) = \mathbb{S}^2(r)$ and $\overline{\mathbb{B}}(r)$. $\mathbb{S}^1(r)$ will represent the circle $\{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$. For a surface $M \subset \mathbb{R}^3$, K_M will denote its curvature function.

If M is a complete, stable, orientable minimal surface in $\mathbb{R}^3$, then M must be a plane [13, 14, 46]. The following lemma extends this result to the case where M is complete outside the origin. This result was found independently by Colding and Minicozzi [4].

Lemma 3.3 (Stability Lemma) *Let $L \subset \mathbb{R}^3 - \{\vec{0}\}$ be a stable orientable minimal surface which is complete outside the origin. Then, $\overline{L}$ is a plane.*

Proof. If $\vec{0} \notin \overline{L}$, then L is complete and so, it is a plane. Assume now that $\vec{0} \in \overline{L}$. Consider the metric $\widetilde{g} = \frac{1}{R^2} g$ on L, where g is the metric induced by the usual inner product $\langle , \rangle$ of $\mathbb{R}^3$. Note that if L were a plane through $\vec{0}$, then $\widetilde{g}$ would be the metric on L of an infinite cylinder of radius 1 with ends at $\vec{0}$ and at infinity. We will show that in general, this metric is complete on L and that the assumption of stability can be used to show that (L, g) is flat. Since $(\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$ with $\widehat{g} = \frac{1}{R^2}\langle , \rangle$, is isometric to $\mathbb{S}^2(1) \times \mathbb{R}$, then $(L, \widetilde{g}) \subset (\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$ is complete.

We now prove that (L, g) is flat. The laplacians and Gauss curvatures of $g, \widetilde{g}$ are related by the equations $\widetilde{\Delta} = R^2\Delta$ and $\widetilde{K} = R^2(K_L + \Delta \log R)$. Since $\Delta \log R = \frac{2(1-\|\nabla R\|^2)}{R^2} \geq 0$,

$$-\widetilde{\Delta} + \widetilde{K} = R^2(-\Delta + K_L + \Delta \log R) \geq R^2(-\Delta + K_L).$$

Since $K_L \leq 0$ and (L, g) is stable, $-\Delta + K_L \geq -\Delta + 2K_L \geq 0$, and so, $-\widetilde{\Delta} + \widetilde{K} \geq 0$ on $(L, \widetilde{g})$. As $\widetilde{g}$ is complete, the universal covering of L is conformally $\mathbb{C}$ (Fischer-Colbrie and Schoen [14]). Since (L, g) is stable, there exists a positive Jacobi function u on L. Passing to the universal covering $\widehat{L}$, $\Delta\widehat{u} = 2K_{\widehat{L}}\widehat{u} \leq 0$, and so, $\widehat{u}$ is a positive superharmonic on $\mathbb{C}$, and hence constant. Thus, $0 = \Delta u - 2K_L u = -2K_L u$ on L, which means $K_L = 0$. $\square$

The following basic result is Lemma 18 in [40].

Lemma 3.4 (Stability of Leaves Lemma) *Suppose $\mathcal{L}$ is a minimal lamination of a Riemannian three-manifold N. Then the following statements hold:*

1. *If L is a limit leaf of $\mathcal{L}$, then the universal cover $\widetilde{L}$ of L is a stable minimal surface.*
2. *If M is a leaf of $\mathcal{L}$ and L is a leaf of the sublamination $L(M) \subset \mathcal{L}$ of limit points of M such that the holonomy representation of L on a side containing M has subexponential growth (amenable holonomy group) on compact subdomains of L, then L is stable. (For example, if the holonomy representation has image group isomorphic to a finitely generated abelian group.)*
3. *If M is a leaf of $\mathcal{L}$ and L is a leaf of the sublamination $L(M) \subset \mathcal{L}$ and there is an open set O_L containing L such that $O_L \cap L(M) = L$, then L is stable.*
4. *If N has positive Ricci curvature, then $\mathcal{L}$ has no limit leaves. If N has non-negative sectional curvature and L is a complete limit leaf of $\mathcal{L}$, then L is simply connected or 1-connected, totally geodesic and stable.*
5. *If $\{M_n\}_n$ is a sequence of embedded minimal surfaces in N that converges to $\mathcal{L}$ and the convergence to a non-limit leaf L of $\mathcal{L}$ is of multiplicity greater than one, then L is stable.*

We will need the following two corollaries, which follow immediately from Lemmas 3.3 and 3.4.

Corollary 3.5 *If L is a limit leaf of a minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$, then $\overline{L}$ is a plane.*

Corollary 3.6 *If $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3$ (resp. of $\mathbb{R}^3 - \{0\}$) which is a limit of embedded minimal surfaces M_n and L is a leaf of $\mathcal{L}$ whose multiplicity is greater than one as a limit of the sequence $\{M_n\}_n$, then L (resp. $\overline{L}$) is a plane.*

4 Minimal laminations with quadratic decay of curvature.

In this section we will obtain a preliminary description of any non-flat minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{\vec{0}\}$ with quadratic decay of curvature, see Definition 4.2 below. We first consider the simpler case where $\mathcal{L}$ consists of a properly embedded minimal surface in $\mathbb{R}^3$. When the decay constant is small, then the topology and geometry of the surface is simple, as shown in the next lemma.

Lemma 4.1 *There exists $C \in (0,1)$ such that if $M \subset \mathbb{R}^3 - \mathbb{B}(1)$ is a properly embedded, connected minimal surface with non-empty boundary $\partial M \subset \mathbb{S}^2(1)$ such that $|K_M|R^2 \leq C$ on M, then M is an annulus which has a planar or catenoidal end.*

Proof. First let C be any positive number less than 1 and we will check that M is an annulus.Let $f = R^2$ on M. Its critical points occur at those $p \in M$ where M is tangent to $\mathbb{S}^2(|p|)$. The hessian $\nabla^2 f$ at such a critical point p is $(\nabla^2 f)_p(v,v) = 2\left(|v|^2 - \sigma_p(v,v)\langle p, N\rangle\right)$, $v \in T_pM$, where σ is the second fundamental form of M and N its Gauss map. Taking $|v| = 1$, we have $\sigma_p(v,v) \leq |\sigma_p(e_i,e_i)| = \sqrt{|K_M|}(p)$, where e_1, e_2 is an orthonormal basis of principal directions at p. Since $\langle p, N\rangle \leq |p|$, we have

$$(\nabla^2 f)_p(v,v) \geq 2\left[1 - (|K_M|R^2)^{1/2}\right] \geq 2(1-\sqrt{C}) > 0. \tag{1}$$

Hence, all critical points of f in the interior of M are nondegenerate local minima on M. Since M is connected, f has no local minima except along ∂M where it obtains its global minimum value. By Morse theory, M intersects every sphere $\mathbb{S}^2(r)$, $r \geq 1$, transversely in a connected simple closed curve, which proves that M is an annulus.

If M has finite total curvature, then it must be asymptotic to an end of a plane or of a catenoid, thus either the lemma is proved or M has infinite total curvature. Note that since M is a properly embedded minimal annulus in $\mathbb{R}^3$ with compact boundary, then Collin's Theorem [11] implies that M has finite total curvature, thereby finishing the proof of a stronger result (we can exchange "there exists $C \in (0,1)$" in the statement of the lemma by "for all $C \in (0,1)$"). We will give an alternative proof, which does not use Collin's theorem, and that works for a constant $C \in (0,1)$ sufficiently small. This second proof will help the reader to understand the more difficult generalization to the case of a minimal lamination, which we will prove in section 5.

A general technique which we will use to obtain compactness of sequences of minimal surfaces is the following (see e.g. [39]): If $\{M_n\}_n$ is a sequence of minimal surfaces properly embedded in an open set $B \subset \mathbb{R}^3$, with their curvature functions K_{M_n} uniformly bounded, then a subsequence converges uniformly on compact subsets of B to a minimal lamination of B with leaves that have the same bound on the curvature as the surfaces M_n.

Suppose that the lemma fails. In this case, there exists a sequence of positive numbers $C_n \to 0$ and minimal annuli M_n satisfying the conditions of the lemma, such that M_n has infinite total curvature and $|K_{M_n}|R^2 \leq C_n$. Since the M_n are annuli with infinite total curvature, the Gauss-Bonnet formula implies that there exists a sequence of numbers $R_n \to \infty$ such that the total geodesic curvature of the outer boundary of $M_n \cap \mathbb{B}(R_n)$ is greater than n. After extracting a subsequence, the $\widetilde{M}_n = \frac{1}{R_n} M_n$ converge to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ by parallel planes (since the curvature of the leaves is zero) and the convergence is smooth outside $\vec{0}$. Furthermore, $\mathcal{L}$ contains a plane Π passing through $\vec{0}$. Consider the great circle $\Gamma = \Pi \cap \mathbb{S}^2(1)$ and let $\Gamma(\varepsilon)$ be the ε-neighborhood of Γ in $\mathbb{S}^2(1)$, for a small number $\varepsilon > 0$. Each $\widetilde{M}_n$ transversely intersects $\mathbb{S}^2(1)$ in a simple closed curve α_n and the Gauss map of $\widetilde{M}_n$ along α_n is almost constant and parallel to the unit normal vector to Π. Clearly, for n sufficiently large, either $\widetilde{M}_n \cap \Gamma(\varepsilon)$ contains long spiraling curves that join points in the two components of $\partial\Gamma(\varepsilon)$ or it consists of a single closed curve which

is C^2-close to Γ. This last case contradicts the assumption that the total geodesic curvature of $M_n \cap \mathbb{S}^2(R_n)$ is unbounded. Hence, we must have spiraling curves in $\widetilde{M_n} \cap \Gamma(\varepsilon)$. In this case, there are planes Π_+, Π_- in $\mathcal{L}$, parallel to Π, such that $\partial\Gamma(\varepsilon) = (\Pi_+ \cup \Pi_-) \cap \mathbb{S}^2(1)$. In a small neighborhood U of $(\Pi_+ \cup \Pi_-) \cap \mathbb{B}(2)$ which is disjoint from Π, the surfaces $\widetilde{M_n} \cap U$ converge smoothly to $\mathcal{L} \cap U$. Since $(\Pi_+ \cup \Pi_-) \cap \mathbb{B}(2)$ is simply connected, then a standard monodromy lifting argument implies $\widetilde{M_n} \cap \overline{\mathbb{B}}(1)$ contains two compact disks in U which are close to $(\Pi_+ \cup \Pi_-) \cap \overline{\mathbb{B}}(1)$. This contradicts the fact that each M_n intersects $\mathbb{S}^2(1)$ transversely in just one simple closed curve (see the first paragraph of this proof). This contradiction completes the proof of the lemma. □

Definition 4.2 The curvature function of a lamination $\mathcal{L}$ will be denoted by $K_{\mathcal{L}} \colon \mathcal{L} \to \mathbb{R}$. $\mathcal{L}$ is said to have *quadratic decay of curvature* if $|K_{\mathcal{L}}|R^2 \leq C$ on $\mathcal{L}$ for a number $C > 0$.

Our main result in this section will be the following proposition, which will be improved in Corollary 6.3.

Proposition 4.3 *Let $\mathcal{L}$ be a non-flat minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$ with quadratic decay of curvature. Then, any leaf of $\mathcal{L}$ is a properly embedded minimal surface in $\mathbb{R}^3 - \{\vec{0}\}$, and $\mathcal{L}$ does not contain flat leaves.*

Proof. The key step in the proof of this proposition is to show that any non-flat leaf of $\mathcal{L}$ is properly embedded in $\mathbb{R}^3 - \{\vec{0}\}$. The proof of this result occupies several pages. Arguing by contradiction, suppose $L \in \mathcal{L}$ is a non-flat leaf which is not proper in $\mathbb{R}^3 - \{\vec{0}\}$.

We claim that $\mathcal{L}$ contains a plane passing through $\vec{0}$. As L is not proper in $\mathbb{R}^3 - \{\vec{0}\}$, there exists $p \in \lim(L) = \{\text{limit points of } L\} \subset \mathbb{R}^3 - \{\vec{0}\}$. Let $L' \in \mathcal{L}$ be the leaf that contains p. Since $L' \cap \lim(L)$ is closed and open in L', then L' is a limit leaf of L contained in the closure of L. In particular, by Corollary 3.5, L' is either a plane or a plane punctured at the origin, and L is contained in one of the half-spaces determined by L'. If L' does not pass through $\vec{0}$, then L' has an ε-neighborhood $L'(\varepsilon)$ at positive distance from $\vec{0}$. Since $|K_L|R^2 \leq C$ for certain $C > 0$, then $L \cap L'(\varepsilon)$ has bounded curvature, which is impossible by the statement and proof of Lemma 1.3 in [39]; for the sake of completeness we now sketch the argument. Taking ε small, each component Ω of $L \cap L'(\varepsilon)$ is a multigraph. Actually Ω is a graph over its projection on L' by a separation argument. Thus, Ω is proper in $L'(\varepsilon)$, and the proof of the Half-space Theorem [19] gives a contradiction. Hence, the plane $\overline{L'}$ passes through $\vec{0}$.

Let $H^+ \subset \mathbb{R}^3$ be the open half-space of $\mathbb{R}^3 - \overline{L'}$ that contains L. After a rotation, we will assume $H^+ = \{x_3 > 0\}$. Since L is a leaf of the lamination $\mathcal{L}$, L is complete outside $\vec{0}$. As $\vec{0} \in \overline{L'}$ and $L' \subset \lim(L)$, then $\vec{0} \in \overline{L}$ as well.

We now check that L is proper in H^+ and $L' = \lim(L)$. Assume L is not proper in H^+. Then there exists a limit leaf $\widetilde{L}$of $\mathcal{L}$ contained in $H^+ \cap \overline{L}$. By the above arguments, $\widetilde{L}$ is a

plane. Since L is connected, L is proper in the open slab bounded by L' and $\widetilde{L}$ (otherwise there exists a plane in this slab which is disjoint from L and with points of L at both sides). Since $\widetilde{L}$ is a plane in H^+, it is at positive distance from $\vec{0}$ with L limiting to it, and so, we can apply the previous arguments to obtain a contradiction. Since $L' \subset \lim(L)$ and L is proper in H^+, it follows $L' = \lim(L)$.

Given $\delta > 0$, let $C_\delta = \{(x_1, x_2, x_3) \mid x_3^2 = \delta^2(x_1^2 + x_2^2)\} \cap H^+$ (positive half-cone) and C_δ^- the region of H^+ below C_δ. A consequence of $|K_L|R^2 \leq C$ is that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $p \in L \cap C_\delta^-$, then the angle that the tangent space to L at p makes with the horizontal is less than ε. Assume for the moment that L satisfies the following stronger property, whose proof we postpone to the end of the proof of this proposition.

> ($\star$) For any $\varepsilon > 0$ small, there exists a $\delta > 0$ such that in $L \cap C_\delta^-$, the inequality $|\nabla_L x_3| \leq \varepsilon \frac{x_3}{R}$ holds.

Thus for $\varepsilon > 0$ fixed and small, and $\delta > 0$ sufficiently small, L intersects C_δ transversely in a small angle that is uniformly bounded away from zero and each component of $L \cap C_\delta$ is locally a radial graph over the circle $C_\delta \cap \{x_3 = 1\}$. Furthermore, in the natural polar coordinates in C_δ, the radial lines intersect the collection of curves $L \cap C_\delta$ almost orthogonally. Let Λ be the set of components of $L \cap C_\delta$. Then any $\Gamma \in \Lambda$ is of one of the following types, see Figure 4:

Type I. Γ is a closed almost horizontal curve. In this case, any other $\Gamma' \in \Lambda$ is also of type I, and there are an infinite number of these curves, converging to $\{\vec{0}\}$.

Type II. Γ is a spiraling curve (with almost horizontal tangent vector) limiting down to $\{\vec{0}\}$. Γ rotates infinitely many times around C_δ, and any other $\Gamma' \in \Lambda$ is of type II. Note that in this case, $\Lambda = \{\Gamma_1, \ldots, \Gamma_n\}$ has a finite number of these spiraling components.

Note also that $L \cap C_{\delta'}$ has the same pattern as Λ, for each $\delta' \in (0, \delta)$.

Our next goal is to prove that L is recurrent for Brownian motion, which implies that L does not admit a non-constant positive harmonic function. However, x_3 gives a non-constant positive harmonic function on L. This contradiction will show that any non-flat leaf $L \in \mathcal{L}$ is properly embedded in $\mathbb{R}^3 - \{\vec{0}\}$. To see that L is recurrent for Brownian motion, we will conformally change the induced metric on L to obtain a complete Riemannian metric $\widetilde{g}$ such that $(L, \widetilde{g})$ has quadratic area growth. Since complete Riemannian manifolds of quadratic volume growth are recurrent (see e.g. [15]), we will obtain the recurrency of L.

Consider the metric $\widehat{g} = \frac{1}{R^2}\langle , \rangle$ on $\mathbb{R}^3 - \{\vec{0}\}$ and let $\widetilde{g} = \widehat{g}|_L$. As we noticed in the proof of Lemma 3.4, $(\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$ is isometric to $\mathbb{S}^2(1) \times \mathbb{R}$ and $(L, \widetilde{g}) \subset (\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$ is complete. Also note that a positive vertical half-cone C minus its vertex at $\vec{0}$ corresponds to the flat

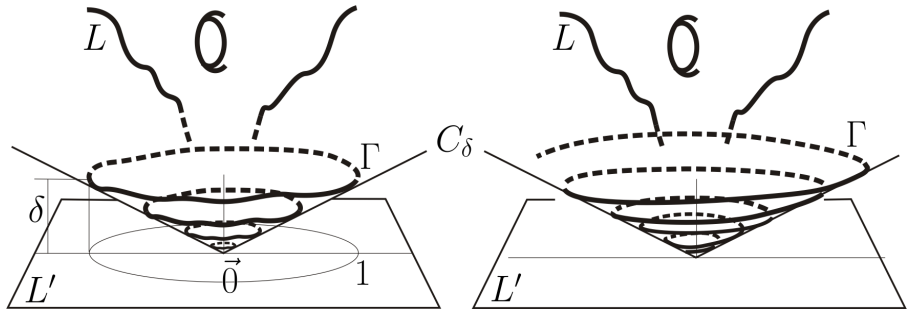

Figure 4: A curve $\Gamma \in \Lambda$ of type I (left) and of type II (right).

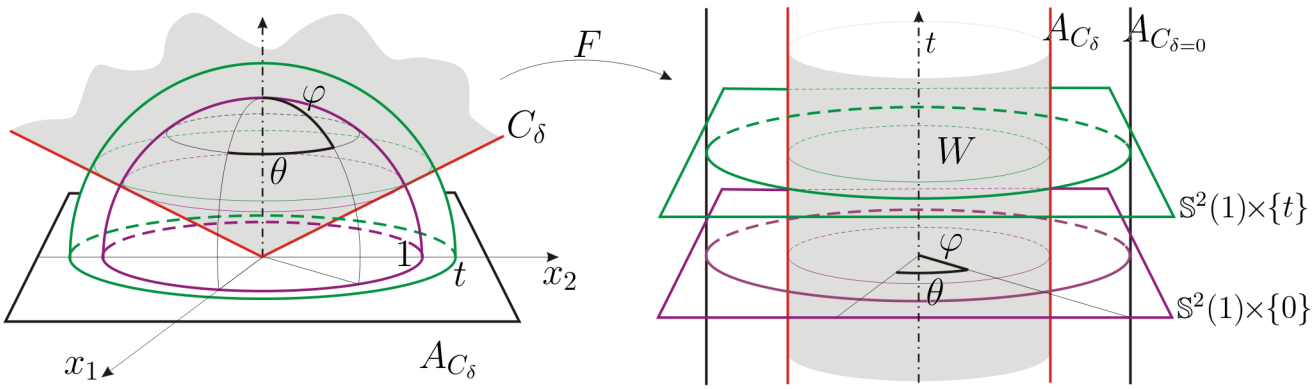

Figure 5: $F(p) = (\frac{p}{R(p)}, \log R(p))$ is an isometry between $(\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$ and $\mathbb{S}^2(1) \times \mathbb{R}$.

cylinder $A_C = (C \cap \mathbb{S}^2(1)) \times \mathbb{R}$. In the particular case where C is the (x_1, x_2)-plane, the corresponding cylinder A_C is totally geodesic and equal to $\mathbb{S}^1(1) \times \mathbb{R}$. We endow $\mathbb{S}^2(1) \times \mathbb{R}$ with global coordinates (φ, θ, t) so that (φ, θ) are the natural spherical coordinates on $\mathbb{S}^2(1)$ and t denotes the linear coordinate in $\mathbb{S}^2(1) \times \mathbb{R}$ (we will consider t to be the vertical height in $\mathbb{S}^2(1) \times \mathbb{R}$); recall that $\varphi \in [0, \pi]$ measures the angle with the positive vertical axis in $\mathbb{R}^3$ and $\theta \in [0, 2\pi)$. Let $W \subset \mathbb{S}^2(1) \times \mathbb{R}$ be the region corresponding to $\{x_3 > 0\} - C_\delta^-$, see Figure 5.

Suppose the curves in Λ are of type I.
Given $\Gamma \in \Lambda$, Γ corresponds in this model to an almost horizontal circle $\widetilde{\Gamma}$ in $A_{C_\delta} = \partial W$, and let $\widetilde{\Lambda}$ denote the collection of these curves. Let $E(\Gamma)$ be the component of $L \cap C_\delta^-$ with $\partial E(\Gamma) = \Gamma$. Since each of the components in $L \cap C_{\delta'}$ is of the same type as Γ for any $\delta' \in (0, \delta)$, then $E(\Gamma)$ is an annulus. Let $\widetilde{E}(\Gamma) \subset \mathbb{S}^2(1) \times \mathbb{R}$ be the related submanifold. Note that $\widetilde{E}(\Gamma)$ is asymptotic to the end of $\mathbb{S}^1(1) \times [0, \infty)$ and $\widetilde{E}(\Gamma)$ is a small φ-graph with small gradient over its projection to $\mathbb{S}^1(1) \times \mathbb{R}$, see Figure 6.

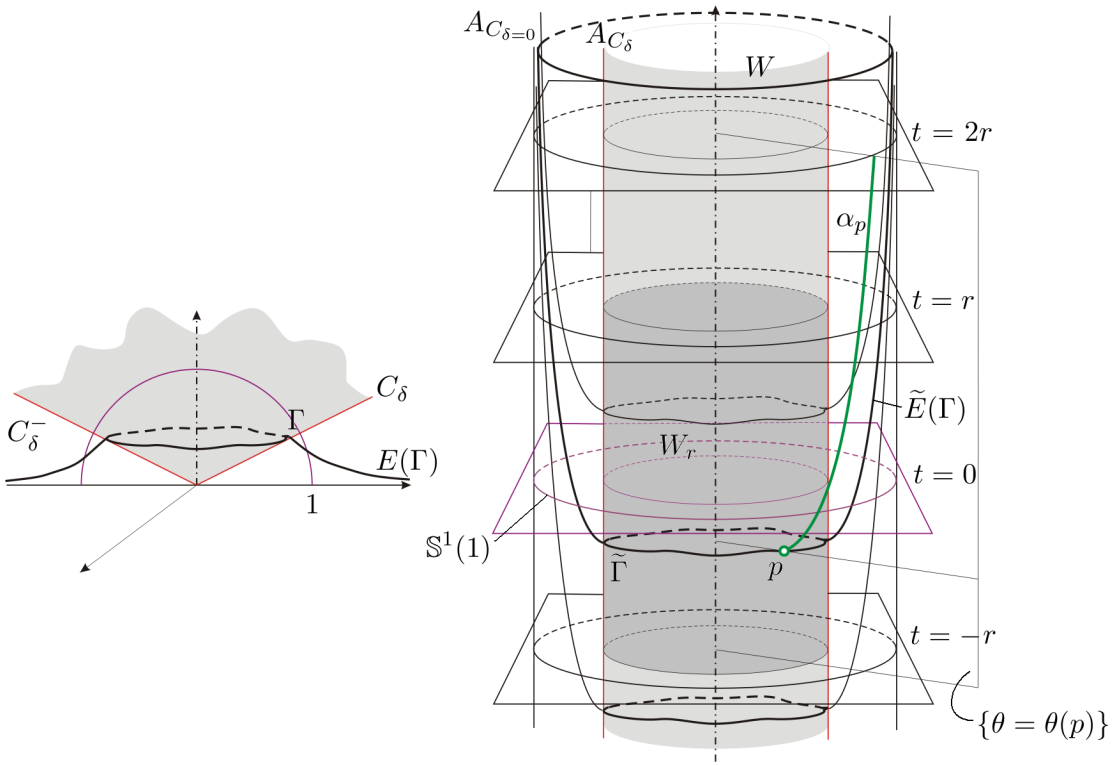

Figure 6: Quadratic area growth in the case of curves in Λ of type I.

For $t > 0$, let $[\widetilde{E}(\Gamma)](t)$ be the subset of $\widetilde{E}(\Gamma)$ consisting of those points at intrinsic distance at most t from $\widetilde{\Gamma} = \partial\widetilde{E}(\Gamma)$. Then for $\delta > 0$ sufficiently small and $t \geq 1$, the area of $[\widetilde{E}(\Gamma)](t)$ is less than $3\pi t$ and the limit as $t \to \infty$ of such an area divided by t is 2π.

For $r_0 > 0$ fixed and for $r \in \mathbb{R}$, we claim that the $\widetilde{g}$-area of $(L - C_\delta^-) \cap \{r < R < rr_0\}$ is bounded independently of r. This is equivalent to proving that the area of $(L - C_\delta^-) \cap \{r < R < rr_0\}$ divided by r^2 is bounded in r. Otherwise, there is a sequence $r_n > 0$ such that the area of $(L - C_\delta^-) \cap \{r_n < R < r_n r_0\}$ is bigger than any constant times r^2 for n large enough. Then the area of the surfaces $\frac{1}{r_n}\left[(L - C_\delta^-) \cap \{r_n < R < r_n r_0\}\right]$ becomes unbounded in a compact region of $\{x_3 > 0\}$. It follows that a subsequence of $\frac{1}{r_n}L$ converges to a new limit lamination with a limit leaf or leaf of infinite multiplicity $L_1 \subset \{x_3 > 0\}$ which must be a horizontal plane, see Corollaries 3.5 and 3.6. However, the existence of such a plane and the fact that below C_δ, L consists of annular graphs, implies that the lamination of which L is a leaf contains a graph, which must be a horizontal plane of positive height. But at the beginning of the proof of this proposition we ruled out the existence of such a plane, which proves our claim. It follows that for $r_1 > 0$ fixed and for $r \in \mathbb{R}$, the area of $\widetilde{L} \cap W \cap (\mathbb{S}^2(1) \times [r, r + r_1])$ is bounded independently of r. Furthermore, since the angle between ∂W and $\widetilde{L}$ is small but bounded away from zero, then the total length of components of Λ that intersect the region $\mathbb{S}^2(1) \times [r, r + 1]$ is bounded from above independently of r as is the number of these components.

For $r > 0$, let $W_r = W \cap (\mathbb{S}^2(1) \times [-r, r])$ and let $\widetilde{\Lambda}(r)$ denote the collection of those elements of $\widetilde{\Lambda}$ which intersect $\partial W(r)$. For r large, let $F(r) \subset \widetilde{L}$ be union of $\widetilde{L} \cap W_r$ with the region

$$V_r = \bigcup_{p \in \widetilde{\Gamma},\ \widetilde{\Gamma} \in \widetilde{\Lambda}(r)} \alpha_p$$

where for each $p \in \widetilde{\Gamma}$ with $\widetilde{\Gamma} \in \widetilde{\Lambda}(r)$, α_p is the component of $\widetilde{E}(\Gamma) \cap \{\theta = \theta(p)\}$ whose end points are p and a point at $\{t = 2r\}$, see Figure 6. Since there exists a universal constant $c > 0$ such that for $r > 1$, the number of $\widetilde{\Gamma}$-curves such that $\partial\widetilde{E}(\Gamma)$ intersects $\mathbb{S}^2(1) \times [-r, r]$, divided by r, is less than c, then the $\widetilde{g}$-area of V_r is certainly less than $3\pi \cdot 4r \cdot cr = 12\pi c r^2$ for r large. Since the $\widetilde{g}$-area of $\widetilde{L} \cap W_r$ grows linearly in r, then the $\widetilde{g}$-area of $F(r)$ is at most $13\pi c r^2$ for r large.

Fix $p_0 \in \widetilde{L} \cap W \cap (\mathbb{S}^2(1) \times \{0\})$ and let $B_{\widetilde{L}}(p_0, r)$ be the intrinsic open ball of radius $r > 0$ centered at p_0. We will show that $B_{\widetilde{L}}(p_0, r) \subset F(r)$. First note that $B_{\widetilde{L}}(p_0, r)$ is contained in the region $\mathbb{S}^2(1) \times (-r, r)$. Let p be a point in $B_{\widetilde{L}}(p_0, r)$. If $p \in W_r$, then $p \in F(r)$. Suppose $p \notin W_r$ and let $\gamma \subset \widetilde{L}$ be a curve of $\widetilde{g}$-length less than r joining p with p_0. Since $\partial F(r)$ does not intersect the region $\mathbb{S}^2(1) \times (-r, r)$, then γ does not intersect the boundary of $F(r)$. As $p_0 \in F(r)$, we conclude $\gamma \subset F(r)$. Hence, $p \in F(r)$. Finally, since $B_{\widetilde{L}}(p_0, r) \subset F(r)$, we deduce that the intrinsic area growth of $\widetilde{L}$ is at most quadratic.

Suppose the curves in Λ are of type II.
As in the previous case, define the similar objects $\widetilde{\Gamma}$, $E(\Gamma)$, $\widetilde{E}(\Gamma)$ and $\widetilde{\Lambda} = \{\widetilde{\Gamma}_1, \ldots, \widetilde{\Gamma}_n\}$. Note that each $\widetilde{\Gamma}$ is now is an infinite spiral on the cylinder $\partial W = A_{C_\delta}$ and each $\widetilde{E}(\Gamma)$ is a topological half-plane. Also, note that $\widetilde{E}(\Gamma)$ is a small φ-multigraph with small gradient over $\mathbb{S}^1(1) \times \mathbb{R}$ and it is asymptotic as a set to $\mathbb{S}^1(1) \times \mathbb{R}$ as well. For $r > 1$, define $\widetilde{\Lambda}(r)$ to be the set of components in $\widetilde{\Lambda} \cap W_r$ which contain an end point at height r and the other end point at height $-r$. Since the curves in $\widetilde{\Lambda}$ are very horizontal, for any $r > 1$ the set $\widetilde{\Lambda}(r)$ consists of n spiraling arcs $\widetilde{\Gamma}_1(r), \ldots, \widetilde{\Gamma}_n(r)$. For r large, let $F(r) \subset \widetilde{L}$ be union of $\widetilde{L} \cap W_{2r}$ with the region

$$V_r = \bigcup_{p \in \widetilde{\Gamma}(2r),\ \widetilde{\Gamma}(2r) \in \widetilde{\Lambda}(2r)} \alpha_p$$

where for each $p \in \widetilde{\Lambda}(2r)$, α_p is the component of $\widetilde{E}(\Gamma) \cap \{\theta = \theta(p)\}$ whose end points are p and a point at $\{t = 2r\}$.

The bounded area density argument in the previous case implies that there exists a universal constant $c > 0$ such that for $r > 0$ large, the sum of the lengths of the curves in $\widetilde{\Lambda}(2r)$, divided by r, is less than c. Since the $\widetilde{g}$-area of $\widetilde{L} \cap W_{2r}$ grows linearly in r, the $\widetilde{g}$-area of $F(r)$ is at most $2cr^2$ for r large.

Let $p_0 \in \widetilde{L} \cap W_r$ be a point at height 0. It remains to check that $B_{\widetilde{L}}(p_0, r) \subset F(r)$ in order to conclude that the intrinsic area growth of $\widetilde{L}$ is at most quadratic. In this case, however the boundary of $F(r)$ intersects the region W_r along a finite collection $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$ of almost vertical arcs $\alpha_i \subset \alpha_{p_i}$, where p_i is the extremum of $\widetilde{\Gamma}_i(2r)$ at height $t = -2r$. By the previous argument for type I curves, in order to prove $B_{\widetilde{L}}(p_0, r) \subset F(r)$ it suffices to show that the length of any curve $\sigma \subset \widetilde{L} \cap [\mathbb{S}^2(1) \times (-r, r)]$ joining a point of $\Sigma = \partial F(r) \cap (\mathbb{S}^2(1) \times (-r, r))$ to any point in an element of $\widetilde{\Lambda}(r)$ has length greater than r. Since the curves in $\widetilde{\Lambda}$ are non-compact, we can assign to each one of them a well-defined real angle valued function $\widehat{\theta}$ which coincides with θ **mod** 2π and this function extends to a continuous function on each $\widetilde{E}(\Gamma)$. Since the curves in $\widetilde{\Lambda}$ are almost horizontal and $\widehat{\theta} \circ \sigma$ is continuous, the absolute difference between the $\widehat{\theta}$-values of the end points of σ is much larger than r. This forces the length of σ to be much larger than r as well, which proves our claim. This concludes the proof that the leaf L is recurrent, which is a contradiction. Henceforth, any non-flat leaf of $\mathcal{L}$ is properly embedded in $\mathbb{R}^3$.

Next we show that none of the leaves of $\mathcal{L}$ are flat. Suppose $\widetilde{L} \in \mathcal{L}$ is a flat leaf, and let $\widehat{L} \in \mathcal{L}$ be a non-flat leaf. If $\widehat{L}$ does not limit to $\vec{0}$, then $\widehat{L}$ has bounded curvature, and so, it is properly embedded in $\mathbb{R}^3$. By the Half-space Theorem, we then obtain a contradiction. Hence, $\vec{0}$ is a limit point of $\widehat{L}$.

Now consider the sublamination $\widetilde{\mathcal{L}} = \{\widehat{L}, \widetilde{L}\}$. Suppose that $\widetilde{L}$ has $\vec{0}$ in its closure and we will obtain a contradiction. After a rotation, assume that $\widetilde{L}$ is the (x_1, x_2)-plane and the third coordinate of $\widehat{L}$ is positive. Also, note that $\widetilde{L}$ is properly embedded in $H = \{(x_1, x_2, x_3) \mid x_3 \geq 0\} - \{\vec{0}\}$. Since $\widehat{L}$ is proper in H and disjoint from the compact

set $\mathbb{S}^1(1)\times\{0\}\subset\partial H$, then the distance from $\mathbb{S}^1(1)\times\{0\}$ to $\widehat{L}$ is greater than some $\varepsilon>0$. In particular, the cylinder $\mathbb{S}^1(1)\times[0,\varepsilon]$ is disjoint from $\widehat{L}$. Since the curves $\mathbb{S}^1(1)\times\{\varepsilon\}$ and $\mathbb{S}^1(\frac{1}{n})\times\{0\}$ are homotopic in a component of $H-\widehat{L}$, then using $\widehat{L}$ as a barrier we obtain a least area annulus $A(n)$ disjoint from $\widehat{L}$ with these curves as boundary. The annulus $A(n)$ is a catenoid but no such catenoid exists for n large. This gives the desired contradiction.

So we may assume that $\widetilde{L}$ is a plane which does not pass through $\vec{0}$. By the proof of the Half-space Theorem [19], the distance between $\widehat{L}$ and $\widetilde{L}$ is positive. Consider the plane Π parallel to $\widetilde{L}$ at distance 0 from $\widehat{L}$. Since $\widehat{L}$ is not a plane, Π must go through the origin, and we finish as before.

Finally, we prove property $(\star)$**.** Suppose the inequality $(\star)$ fails. Then there exists a sequence of points $p_n\in L$ such that $\frac{x_3(p_n)}{|p_n|}\to 0$ as $n\to\infty$ and $|\nabla_L x_3|(p_n)|p_n|\geq\varepsilon x_3(p_n)$ for all n. Fix $k\in\mathbb{N}$, $k\geq 2$, and let $A(k)=\{(x_1,x_2)\mid \frac{1}{k}\leq(x_1^2+x_2^2)^{1/2}\leq k\}$. Fix $\delta>0$ small such that $L\cap C_\delta^-$ is locally an almost horizontal graph. This graphical property implies that either $\Delta(k):=L\cap C_\delta^-\cap x_3^{-1}(A(k))$ only contains compact components which are graphs over their projections to $A(k)$ (infinitely many of them), or $\Delta(k)$ contains a finite number of components, some of which are non-compact multigraphs over $A(k)$ with one end. We will only consider the second case, as the first case can be carried out in a similar manner. We may assume that all points p_n lie on a fixed noncompact component $C(k)$ of $\Delta(k)$. A standard renormalization argument shows that we may also assume that $|p_n|=1$ for all n (one works instead with the sequence $\frac{1}{|p_n|}C(k)$). The component $C(k)$ can be expressed as a graph over a representative of one of the two ends of the universal cover $\widehat{A}(k)$ of $A(k)$ (endowed with the covering metric g). Note that if $\widetilde{A}(k)$ denotes $\widehat{A}(k)$ endowed with the (lifted) metric $\frac{1}{R^2}g$, then $\widetilde{A}(k)$ becomes isometric to the strip $S(k)=\{(x_1,x_2)\mid -\log k\leq x_2\leq\log k\}$ with its usual flat metric. Similarly, let $\widetilde{C}(k)$ denote $C(k)$ with the conformally related metric $\frac{1}{R^2}g_{C(k)}$. Also note that the vertical projection $\pi_k\colon\widetilde{C}(k)\to\widetilde{A}(k)=S(k)$ converges to an isometry as one approaches the end of $\widetilde{C}(k)$. Consider the harmonic function $H_{k,n}\colon\widetilde{C}(k)\to(0,\infty)$ given by $H_{k,n}=\frac{1}{x_3(p_n)}x_3$, which satisfies $H_{k,n}(p_n)=1$ and $|\widetilde{\nabla}H_{k,n}|(p_n)\geq\varepsilon$. Let $\widehat{H}_{k,n}=H_{k,n}\circ(\pi_k)^{-1}$ defined on a representative of one of the ends of $S(k)$. Let $\widetilde{p}_{k,n}=\pi_k(p_n)\in S(k)$ and consider the function $F_{k,n}(x)=\widehat{H}_{k,n}(x-\widetilde{p}_{k,n})$. Then a subsequence of $\{F_{k,n}\}_n$ converges as $n\to\infty$ to a harmonic function F_k on $S(k)$, and $F_k(0,0)=1$, $|\nabla F_k|(0,0)\geq\varepsilon$. After extracting a subsequence, these harmonic functions F_k converge as $k\to\infty$ to a positive non-constant harmonic function on the complex plane, which is impossible. This proves the property $(\star)$ and so, this concludes the proof of Proposition 4.3. $\square$

5 The proof of the Local Removable Singularity Theorem.

In this section we prove Theorem 1.2.

Theorem 5.1 *Let $\overline{\mathbb{B}}(p, R_1)$ be a compact Riemannian ball of radius R_1 centered at a point p. Suppose $\mathcal{L} \subset \overline{\mathbb{B}}(p, R_1) - \{\vec{p}\}$ is a minimal lamination such that there exists $C > 0$ with $|K_{\mathcal{L}}|R^2 \leq C$. Then, $\mathcal{L}$ extends to a minimal lamination of $\overline{\mathbb{B}}(p, R_1)$. In particular,*

1. *The curvature of $\mathcal{L}$ is bounded in a neighborhood of $\vec{p}$.*
2. *If $\mathcal{L}$ consists of a single leaf $M \subset \overline{\mathbb{B}}(p, R_1) - \{\vec{p}\}$ which is a properly embedded minimal surface with $\emptyset \neq \partial M \subset \partial\mathbb{B}(R_1)$, then M extends smoothly through $\vec{p}$.*

Proof. We will first prove the theorem in the $\mathbb{R}^3$ setting where $p = \vec{0}$ and $\overline{\mathbb{B}}(p, R_1) = \overline{\mathbb{B}}(R_1) = \{x \in \mathbb{R}^3 \mid \|x\| \leq R_1\}$. We first consider the special case where $\mathcal{L}$ consists of a single leaf M which is properly embedded in $\overline{\mathbb{B}}(R_1) - \{\vec{0}\}$. In this case it is known that the area of M is finite and M satisfies the monotonicity formula, see for instance [17]. For the sake of completeness, we give a self-contained proof in our setting.

For $0 < r \leq R \leq R_1$, let $A_M(R) = \text{Area}(M \cap \mathbb{B}(R))$, $l_M(R) = \text{Length}(M \cap \mathbb{S}^2(R)) \in (0, \infty]$ and $A_M(r, R) = \text{Area}(M \cap [\mathbb{B}(R) - \mathbb{B}(r)]) \in (0, \infty)$. The Divergence Theorem applied to the vector field $p^T = p - \langle p, N\rangle N$ gives

$$2A_M(r, R) = \int_{M\cap[\mathbb{B}(R)-\mathbb{B}(r)]} \text{Div}(p^t) = \int_{\partial_r} \langle p, \nu\rangle + \int_{\partial_R} \langle p, \nu\rangle,$$

where $\partial_r = M \cap \mathbb{S}^2(r)$, $\partial_R = M \cap \mathbb{S}^2(R)$ and ν is the unit exterior conormal vector to $M \cap [\mathbb{B}(R) - \mathbb{B}(r)]$ along its boundary. The first integral is not positive, and Schwarz inequality applied to the second one gives $2A_M(r, R) \leq R\, l_M(R)$. Taking $r \to 0$, we have

$$2A_M(R) \leq R\, l_M(R). \tag{2}$$

In particular, the area of M is finite. Next we observe that the monotonicity formula holds in our setting (i.e. $R^{-2}A_M(R)$ is not decreasing). To see this, note that

$$R^3 \frac{d}{dR}\left(\frac{A_M(R)}{R^2}\right) = R\ A'_M(R) - 2A_M(R). \tag{3}$$

The coarea formula applied to the function R gives

$$A'_M(R) = \int_{\partial_R} \frac{ds}{|\nabla R|} \geq l_M(R) \tag{4}$$

where ∇R is the intrinsic gradient of R and ds the length element along ∂_R. Now (3), (4), (2) imply the monotonicity formula.

As an important consequence of the finiteness of its area, M has limit tangent cones at the origin under expansions. To prove that M extends to a smooth minimal surface in $\overline{\mathbb{B}}(R_1)$, we discuss two situations separately. In the first one we will deduce that M

has finite topology, in which case the removability theorem is known (see [2], although we will also provide a proof of the removability of the singularity in this situation), and to conclude the proof in this first case where $\mathcal{L}$ consists of a single leaf M which is properly embedded in $\overline{\mathbb{B}}(R_1) - \{\vec{0}\}$, we will show that the second case cannot hold.

1. Suppose there exist $C_1 < 1$ and $R_2 \leq R_1$ such that $|K_M|R^2 \leq C_1$ in $M \cap \mathbb{B}(R_2)$. Using the arguments in the proof of Lemma 4.1, we deduce that $M \cap \mathbb{B}(R_2)$ consists of a finite number of annuli with compact boundary, transverse to spheres centered at the origin and having $\vec{0}$ in their closure, together with a finite number of compact disks. Since the collection of disks is compact, for R_2 sufficiently small we may assume that there are no such disk components. Let A be one of the annuli in $M \cap \mathbb{B}(R_2)$. If A is conformally $\{\varepsilon < |z| \leq 1\}$ for some $\varepsilon > 0$, then each coordinate function of A can be reflected in $\{|z| = \varepsilon\}$ (Schwarz's reflection principle), defining a conformal branched harmonic map that maps the entire curve $\{|z| = \varepsilon\}$ to a single point, which is impossible. Thus, A is conformally $\mathbb{D}^* = \mathbb{D} - \{\vec{0}\}$, and so, its coordinate functions extend smoothly across $\vec{0}$, defining a possibly branched minimal surface A_0 that passes through $\vec{0}$. If $\vec{0}$ is a branch point of A_0, then A cannot be embedded in a punctured neighborhood of $\vec{0}$, which is a contradiction. Hence, A_0 is a smooth embedded minimal surface passing through $\vec{0}$. Since M is embedded, the usual maximum principle for minimal surfaces implies that there exists only one such an annulus A_0, and the theorem holds in this case.

2. Now assume that there exists a sequence $\{p_n\}_n \subset M$ converging to $\vec{0}$ such that $1 \leq |K_M|R^2(p_n)$ for all n, and we will obtain a contradiction. The expanded surfaces $\widetilde{M}_n = \frac{1}{|p_n|}M \subset \mathbb{R}^3 - \{\vec{0}\}$ also have $|K_{\widetilde{M}_n}|R^2 \leq C$. After choosing a subsequence, the $\widetilde{M}_n$ converge to a minimal lamination $\mathcal{L}_1$ of $\mathbb{R}^3 - \{\vec{0}\}$ with $|K_{\mathcal{L}_1}|R^2 \leq C$. Furthermore, $\mathcal{L}_1$ contains a non-flat leaf L passing through a point in $\mathbb{S}^2(1)$, where it has absolute Gaussian curvature at least 1. By Proposition 4.3, L is a properly embedded minimal surface in $\mathbb{R}^3 - \{\vec{0}\}$. Finally, we obtain the desired contradiction. By the monotonicity formula, $R^{-2}A_M(R)$ is bounded as $R \to 0$. Geometric measure theory implies that any sequence of expansions of M converges (up to a subsequence) to a minimal cone over a configuration of geodesic arcs in $\mathbb{S}^2(1)$. Since any smooth point of such a minimal cone is flat, we contradict the existence of the non-flat minimal leaf L.

Suppose that $\mathcal{L}$ is a possibly non-connected, properly embedded minimal surface in $\overline{\mathbb{B}}(R_1) - \{\vec{0}\}$, where properness refers to the intrinsic Riemannian topology. Consider the intersection of $\mathcal{L}$ with the closed ball of radius $R_2 \in (0, R_1)$. By the maximum principle, every component of $\mathcal{L}$ intersects $\partial\overline{\mathbb{B}}(R_2)$. Since $\partial\overline{\mathbb{B}}(R_2)$ is compact, there are at most a finite number of such components. If there are two components of $\mathcal{L} \cap \overline{\mathbb{B}}(R_2)$ which have $\vec{0}$ in their closure, then each of these components extends smoothly across the origin and we contradict the maximum principle for minimal surfaces. Therefore, at most one component has the origin in its closure, and the other components, which are compact, do

not intersect a certain ball $\overline{\mathbb{B}}(R_3)$ for some $R_3 \in (0, R_2)$. Hence, $\mathcal{L}$ extends in this case.

In the $\mathbb{R}^3$ setting, it remains to prove the theorem in the case $\mathcal{L}$ is a minimal lamination such that $\mathcal{L}$ does not intersect any small punctured neighborhood of $\vec{0}$ in a properly embedded surface. Thus, under our hypotheses, every punctured neighborhood of $\vec{0}$ intersects a limit leaf of $\mathcal{L}$. Since the set of limit leaves of $\mathcal{L}$ is closed, it follows that $\mathcal{L}$ contains a limit leaf F with $\vec{0}$ in the closure of F.

We claim that any blow-up of $\mathcal{L}$ from $\vec{0}$ converges outside $\vec{0}$ to a flat lamination of $\mathbb{R}^3$ by planes. Since $|K_{\mathcal{L}}|R^2$ is scale invariant, our claim follows by proving that for any $\varepsilon > 0$, there is $r(\varepsilon) \in (0, 1)$ such that $|K_{\mathcal{L}}|R^2 < \varepsilon$ on $\mathcal{L} \cap \mathbb{B}(r(\varepsilon))$. Arguing by contradiction, suppose there exists a sequence of points $q_n \in \mathcal{L}$ converging to $\vec{0}$ with $(|K_{\mathcal{L}}|R^2)(q_n)$ bounded away from zero. Then, after expansion by $\frac{1}{|q_n|}$ and taking a subsequence, we obtain a non-flat minimal lamination $\mathcal{L}_1$ of $\mathbb{R}^3 - \{\vec{0}\}$ which satisfies the hypotheses in Proposition 4.3. In particular, $\mathcal{L}_1$ does not contain flat leaves. The limit leaf F in $\mathcal{L}$ produces under expansion a leaf F_1 (whose universal cover is stable) in $\mathcal{L}_1$, which is complete outside the origin, and by Corollary 3.5, F_1 is a plane, which contradicts Proposition 4.3. Now our claim is proved.

By the above claim, we know that any blow-up limit of $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$ by parallel planes. It follows that for $\varepsilon > 0$ sufficiently small, in the annular domain $A = \{x \in \mathbb{R}^3 \mid \frac{1}{2} \leq |x| \leq 2\}$ the normal vectors to the leaves of $\mathcal{L}_\varepsilon = \frac{1}{\varepsilon}\mathcal{L} \cap A$ are almost parallel. Hence, for such a sufficiently small ε, each component C of $\mathcal{L}_\varepsilon$ that intersects $\mathbb{S}^2(1)$ is of one of the following types:

1. A compact disk with boundary in $\mathbb{S}^2(2)$;

2. A compact planar domain with one boundary curve in $\mathbb{S}^2(2)$ and at least two other boundary curves in $\mathbb{S}^2(\frac{1}{2})$ and where the outer boundary curve bounds a compact disk in $\frac{1}{\varepsilon}\mathcal{L}$;

3. A compact annulus with one boundary curve in $\mathbb{S}^2(\frac{1}{2})$ and the other boundary curve in $\mathbb{S}^2(2)$;

4. An infinite multigraph whose limit set consists of two compact annular components described in 3.

It follows that if for some sufficiently small ε_0, $\mathcal{L}_{\varepsilon_0}$ has a component of type 4, then this multigraph component persists for positive $\varepsilon < \varepsilon_0$, varying in a continuous manner in terms of ε. Thus, the existence of a multigraph component in $\mathcal{L}_{\varepsilon_0}$ implies that $\mathcal{L} \cap \mathbb{B}(\varepsilon_0)$ has two properly embedded annular leaves in $\mathbb{B}(\varepsilon_0) - \{\vec{0}\}$. By our previously considered case, these two annular leaves extend smoothly to two minimal disks that intersect at the origin, thereby, contradicting the maximum principle for minimal surfaces. This contradiction shows that only components of types 1, 2, 3 can occur in $\mathcal{L}_{\varepsilon_0}$. But in each of these cases,

the outer boundary curve of a component C of $\mathcal{L}_{\varepsilon_0}$ bounds a disk or a properly embedded annulus in $\mathbb{B}(\varepsilon_0)-\{\vec{0}\}$ that extends to a minimal disk (there can only be one such annulus by the maximum principle). Rado's theorem implies that these surfaces are graphs and so, by curvature estimates, have bounded curvature in a neighborhood of the origin. Hence, the closure of $\mathcal{L}$ in $\overline{\mathbb{B}}(R_1)-\{\vec{0}\}$ is a minimal lamination of $\overline{\mathbb{B}}(R_1)$, which proves the theorem in the case $\mathbb{R}^3$ setting.

Assume now that $\overline{\mathbb{B}}(p,R_1)$ is not necessarily equipped with a flat metric. We now explain how to modify the arguments applied in the $\mathbb{R}^3$ setting to the three-manifold setting. First consider the case, where for some small R_2 the exponential map on the R_1-ball in $T_p(\overline{\mathbb{B}}(p,R_1))$ is a diffeomorphism yielding $\mathbb{R}^3$ coordinates on $\mathbb{B}(p,R_2)$ centered at $p\equiv\vec{0}$. Suppose that for some R_2, $0<R_2\leq R_1$, $\mathcal{L}\cap\mathbb{B}(p,R_2)$ is non-compact properly embedded minimal surface M in $\overline{\mathbb{B}}(p,R_2)-\{p\}$. It follows from the monotonicity formula of area for a minimal surface that M has finite area, and hence, under homothetic rescaling of coordinates has minimal limit tangent cones at $\vec{0}$.

If there exists an $\varepsilon>0$ and a sequence $\{p_n\}_n\subset M$ converging to p such that $\varepsilon\leq|K_M|R^2(p_n)$ for all n, then a subsequence of the expanded surfaces $\widetilde{M_n}=\frac{1}{|p_n|}M$ in $\frac{1}{|p_n|}\overline{\mathbb{B}}(p,R_1)$ converges to a non-flat minimal lamination $\mathcal{L}_\infty$ of $\mathbb{R}^3$. Since $\mathcal{L}_\infty$ is not flat at some point of $\mathbb{S}^2(1)$, it has a leaf which is not a cone, which is a contradiction to the conclusion of the previous paragraph. Hence, any sequence of homothetic blow-ups of M has a subsequence which converges smoothly to a plane passing through the origin in $\mathbb{R}^3$. In particular, M has a finite number of annular ends, each of which has linear area growth with respect to the complete metric $\frac{1}{R^2}\langle,\rangle$. Hence, the ends of M are conformally punctured disks. Standard regularity theory implies the harmonic map of M into $\mathbb{B}(p,R_2)$ extends smoothly across the punctured disks. In particular, as in the $\mathbb{R}^3$ setting, we see that for R_2 sufficiently small $\mathcal{L}\cap(\mathbb{B}(p,R_2)-\{p\})$ is a punctured disk that extends smoothly to a minimal lamination of $\mathbb{B}(p,R_2)$.

Assume now for all R_2 with $0<R_2\leq R_1$, that $\mathcal{L}\cap(\mathbb{B}(p,R_2)-\{p\})$ is not a properly embedded minimal surface in $\mathbb{B}(p,R_2)-\{p\}$. In particular, $\mathcal{L}$ contains a limit leaf L with $p\in\overline{L}$. The proof of this case is essentially identical to the proof given in the $\mathbb{R}^3$ setting. This completes the proof of Theorem 5.1. □

Remark 5.2 Applying the same techniques as those used in the proof of Theorem 5.1,it is not difficult to prove a removable singularity result for a minimal lamination of quadratic curvature decay, which is a minimal lamination of a neighborhood of infinity in $\mathbb{R}^3$. This result states that, after a rotation, outside of some large ball, the leaves of $\mathcal{L}$ are graphs asymptotic to ends of horizontal planes or to ends of vertical catenoids or almost horizontal multigraphs over annular domains in the (x_1,x_2)-plane.

Corollary 5.3 *Let M be a stable embedded minimal surface in a Riemannian three-manifold N, which is complete outside a countable closed set of N. Then, the closure*

of M has the structure of a minimal lamination of N, and the intrinsic metric completion of M is a leaf of this lamination. In particular, if N is $\mathbb{R}^3$, then the closure of M is a plane.

Proof. Let $S \subset N$ be the closed countable set such that M is complete outside S. Note that the closure of M in $N-S$ is a minimal lamination of $N-S$. By Theorem 1.2 and the curvature estimates for stable minimal surfaces (see Ros [48] and Schoen [52]), M extends smoothly through the subset of isolated points in S. Thus, we can assume S has no isolated points. Since S is a closed and countable subset of the complete metric space N, then S is a complete, countable metric space in the induced metric. But a complete, countable metric space always has an isolated point (simple application of Baire's Theorem, since otherwise the substraction of the first n points of a listing of the space would be a countable dense subset S_n but the intersection $\cap_n S_n$ is empty), and so, S has isolated points. This contradiction proves the corollary. □

We now prove Theorem 1.2 stated in the introduction. By Theorem 5.1, we need only to prove items *1, 2, 3* of Theorem 1.2. Consider a collection $\mathcal{L}_1$ of stable leaves of a minimal lamination $\mathcal{L}$ of $\mathbb{B}_N(p,r)-\{p\}$. The closure of $\mathcal{L}_1$ forms a minimal sublamination of $\mathcal{L}$, with the same curvature estimates as those of $\mathcal{L}_1$. By curvature estimates for stable minimal surfaces [52, 47], $|K_{\mathcal{L}_1}|d^2 \leq c$ for some $c > 0$. Hence, the first part of this Theorem 1.2 insures that the closure of $\mathcal{L}_1$ is a minimal lamination of $\mathbb{B}_N(p,r)$. This proves item *1* in Theorem 1.2. Suppose now that $\mathcal{L}_2$ is the sublamination of $\mathcal{L}$ consisting of its limit leaves. By item *1* of Lemma 3.4, the universal covering of any leaf of $\mathcal{L}_2$ is stable, and so, the leaves of $\mathcal{L}_2$ satisfy the same curvature estimates as stable minimal surfaces. Now item *2* follows similarly as item *1* above. Finally, suppose $\mathcal{F}$ is a possibly singular minimal foliation of a Riemannian manifold N with at most a countable number of singularities. In this case, every leaf of $\mathcal{F}$ is a limit leaf and so, the set of isolated singularities of $\mathcal{F}$ is empty. Arguing as in the proof of the above corollary, we deduce that $\mathcal{F}$ has no singularities, which proves item *3* of Theorem 1.2. □

In any flat three-torus $\mathbb{T}^3$, there exists a sequence $\{M_n\}_n$ of embedded minimal surfaces of genus three, with area diverging to infinity [22] (a similar result holds for any genus $g \geq 3$, see Traizet [54]). After choosing a subsequence these surfaces converge to a minimal foliation of $\mathbb{T}^3$ and the convergence is smooth away from two points. Since by the Gauss-Bonnet formula, these surfaces have absolute total curvature 8π, this example demonstrates a special case of the following result.

Corollary 5.4 *Suppose $\{M_n\}_n$ is a sequence of complete, embedded minimal surfaces in a Riemannian three-manifold N, such that there exists a open covering of N and $\int_{B\cap M_n}|A_n|^2$ is uniformly bounded for any open set B in this covering (here A_n denotes the second*

fundamental form of M_n). Then, there exists a subsequence of $\{M_n\}_n$ that converges to a $C^{1,\alpha}$-minimal lamination $\mathcal{L}$ of N, and the singular set of convergence $S(\mathcal{L})$ is closed and discrete. If L is a limit leaf of $\mathcal{L}$ or a leaf with infinite multiplicity as a limit, then this leaf it totally geodesic. If each M_n is connected and N is compact, then $\mathcal{L}$ is compact and connected in the subspace topology (not necessarily path-connected).

Proof. Let q be a point in N. We will say that q is a *bad* point for the sequence $\{M_n\}_n$ if there exists a subsequence $\{M_{n_k}\}_k \subset \{M_n\}_n$ such that the total curvature of every M_{n_k} in $\mathbb{B}_N(q, \frac{1}{k})$ is at least 2π. First note that we can replace the covering in the statement by a countable open covering of N by balls $\mathbb{B}_i$, $i \in \mathbb{N}$. Assume for the moment that $\mathbb{B}_1$ contains a bad point q_1 for $\{M_n\}_n$. We claim that $\mathbb{B}_1$ has a finite number of bad points after replacing $\{M_n\}_n$ by a subsequence. To see this, since q_1 is a bad point for $\{M_n\}_n$, there exists a subsequence $\{M'_k = M_{n_k}\}_k \subset \{M_n\}_n$ such that the total curvature of every M'_k in $\mathbb{B}_N(q_1, \frac{1}{k})$ is at least 2π. Suppose that $q_2 \in \mathbb{B}_1$ is a bad point for $\{M'_k\}_k$. Then we find a subsequence $\{M''_j = M_{k_j}\}_j \subset \{M'_k\}_k$ such that the total curvature of every M''_j in $\mathbb{B}_N(q_2, \frac{1}{j})$ is at least 2π. In particular, for j large, there are disjoint neighborhoods of q_1 and q_2 in $\mathbb{B}_1$, each with total curvature of M''_j at least 2π. By our hypothesis, this process of finding bad points and subsequences in $\mathbb{B}_1$ stops after a finite number of steps, which proves our claim. A standard diagonal argument then shows that after replacing the M_n by a subsequence, the set of bad points $A \subset N$ for $\{M_n\}_n$ is a discrete closed set in N.

Suppose that $q \in N - A$. We claim that $\{M_n\}_n$ has bounded curvature in a neighborhood of q.Arguing by contradiction, suppose there exist points $p_n \in M_n$ converging to q and such that $|K_{M_n}|(p_n) \to \infty$ as $n \to \infty$. Let $\varepsilon_q = \frac{1}{2} d_N(q, A) > 0$. By the Local Picture Theorem on the Scale of Curvature (Theorem 7.1 in the introduction, which will be proved in section 7), we may assume for n large that

$$\int_{\mathbb{B}_N(q,r_n)\cap M_n} |A_n|^2 > 2\pi,$$

where $r_n \searrow 0$ satisfies $d_N(q, p_n) < r_n < \frac{\varepsilon_q}{2}$. This clearly contradicts that $q \in N - A$, and so, our claim holds. Therefore, there exist a neighborhood U_q and a minimal lamination $\mathcal{L}_q$ of U_q such that a subsequence of the M_n converges to $\mathcal{L}_q$ in U_q.

Another standard diagonal argument proves that after extracting a subsequence, the M_n converge to a minimal lamination $\mathcal{L}$ of $N - A$. Note that the curvature function $K_{\mathcal{L}}$ of L does not grow faster than quadratically at any point of A (in terms of the inverse of the distance function to that point): otherwise, there exists a sequence of blow-up points $p_n \in \mathcal{L}$ converging to a point $q \in A$ with $|K_{L_n}|(p_n) d_N(p_n, q)$ unbounded, where L_n is the leaf of $\mathcal{L}$ passing through p_n. Using again the Local Picture Theorem on the Scale of Curvature, we deduce that there exist disjoint small neighborhoods $V(p_n)$ of p_n in L_n, such that the total curvature of L_n in $V(p_n)$ is at least 2π. Since M_n converges to $\mathcal{L}$, this

contradicts our hypothesis. Once we know that $K_{\mathcal{L}}$ does not grow faster than quadratically at any point of A, our local removable singularity theorem (Theorem 1.2) implies $\mathcal{L}$ extends to a $C^{1,\alpha}$-minimal lamination of N. The proofs of the remaining statements in the corollary are straightforward. □

Remark 5.5 Theorem 1.2 supports the conjecture that a properly embedded minimal surface in a punctured ball extends smoothly through the puncture. This is one of the fundamental open problems in minimal surface theory, and a special case of our fundamental removable singularity conjecture stated in the introduction. A partial result for this conjecture was obtained by Gulliver and Lawson [16], who proved it in the special case the surface is stable. Note that by curvature estimates for stable minimal surfaces, an embedded stable minimal surface M in a punctured ball in a Riemannian manifold which is complete away from the puncture (in the sense that every divergent path with finite length limits to the puncture or to the boundary of the ball), satisfies the curvature estimate in Theorem 5.1, and so, its closure in the open ball is a minimal lamination of the open ball. In general, the closure such a stable minimal surface can contain other leaves, as seen in example IV of section 2, where we constructed an embedded stable disk in $\mathbb{B}(1) - \{\vec{0}\}$ which is complete, has its boundary in $\partial\mathbb{B}(1)$, and whose closure is a minimal lamination of $\mathbb{B}(1)$ with two non-flat leaves.

6 The characterization of minimal surfaces with quadratic decay of curvature.

In this section we will prove Theorem 1.4 stated in the introduction.

Proposition 6.1 *Let $\mathcal{L}$ be a non-flat minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$ with quadratic decay of curvature. Then, $\mathcal{L}$ consists of a single leaf, which extends to a connected, properly embedded minimal surface in $\mathbb{R}^3$.*

Proof. By Proposition 4.3, each leaf L of $\mathcal{L}$ is a minimal surface which is properly embedded in $\mathbb{R}^3 - \{\vec{0}\}$. Applying Theorem 5.1 to each $L \in \mathcal{L}$, we deduce that L extends to a properly embedded minimal surface in $\mathbb{R}^3$. Finally, $\mathcal{L}$ consists of a single leaf by the maximum principle applied at the origin and the Strong Half-space Theorem [19]. □

Theorem 6.2 *Let $M \subset \mathbb{R}^3$ be a complete, embedded, non-flat minimal surface with compact boundary (possibly empty). If M has quadratic decay of curvature, then M is properly embedded in $\mathbb{R}^3$ with finite total curvature.*

Proof. We first check that M is proper when ∂M is empty. In this case, the closure $\mathcal{L}$ of M in $\mathbb{R}^3-\{\vec{0}\}$ is a minimal lamination of $\mathbb{R}^3-\{\vec{0}\}$ satisfying the conditions in Proposition 6.1. It follows that M is a properly embedded minimal surface in $\mathbb{R}^3$ with bounded curvature.

We now prove that M is also proper when $\partial M \neq \emptyset$. Since ∂M is compact, we may assume $\vec{0} \notin \partial M$ by removing a compact subset from M. Therefore, there exists an $\varepsilon > 0$ such that $\partial M \subset \mathbb{R}^3 - \mathbb{B}(\varepsilon)$. Thus, Theorem 5.1 gives that $\overline{M} \cap (\mathbb{B}(\varepsilon) - \{\vec{0}\})$ has bounded curvature, and so, M does as well (in order to apply Theorem 5.1 we need $M \cap (\mathbb{B}(\varepsilon) - \{\vec{0}\})$ to be non-empty; but otherwise M would have bounded curvature so we would arrive to the same conclusion). If M were not proper in $\mathbb{R}^3$, then $\overline{M} - \partial M$ has the structure of a minimal lamination of $\mathbb{R}^3 - \partial M$ with a limit leaf L which is disjoint from M. Since we may assume, after possibly removing an intrinsic neighborhood of ∂M, that $\overline{L} \cap \partial M = \emptyset$, then L is complete and stable, and hence, L is a plane. Since M limits to L and has bounded curvature, we easily obtain a contradiction to the proof of the Half-space Theorem. Hence, M is proper independently of whether or not ∂M is empty.

From now on, we will assume that M is non-compact and properly embedded in $\mathbb{R}^3$. Since ∂M is compact (possibly empty), there exists an $R_1 > 0$ such that $\partial M \subset \mathbb{B}(R_1)$. It remains to show that M has finite total curvature.

Let $C_1 \in (0,1)$ be the constant given by the statement of Lemma 4.1. Suppose first that there exists $R_2 > R_1$ such that $|K_M|R^2 \leq C_1$ in $M - \mathbb{B}(R_2)$. Applying Lemma 4.1 to each component of $M - \mathbb{B}(R_2)$, such components are annular ends with finite total curvature. Since M is proper, there are a finite number of such components and $M \cap \mathbb{B}(R_2)$ is compact. Thus, M has finite total curvature, which proves the theorem in this case.

Now assume that there exists a sequence $\{p_n\}_n \subset M$ diverging to ∞ such that $C_1 \leq |K_M|R^2(p_n)$ for all n, and we will find a contradiction. The homothetically shrunk surfaces $\widetilde{M}_n = \frac{1}{|p_n|}M$ also have curvature decaying quadratically and their boundaries collapse to $\vec{0}$. Thus, a subsequence of $\widetilde{M}_n$ converges to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{\vec{0}\}$, whose curvature function decays quadratically. Since $|K_{\widetilde{M}_n}|(\frac{1}{|p_n|}p_n) \geq C_1$ and we can assume $\frac{1}{|p_n|}p_n \to \widetilde{p}_\infty \in \mathbb{S}^2(1)$, there exists a leaf $L \in \mathcal{L}$ which is non-flat with $\widetilde{p}_\infty \in L$. By Proposition 6.1, $\mathcal{L} = \{L\}$ and $\overline{L}$ is properly embedded in $\mathbb{R}^3$. If the convergence of the $\widetilde{M}_n$ to $\mathcal{L}$ has multiplicity greater than one, then L would be flat (see Lemma 3.2 and Lemma 3.4), but it is not. Also note that $\overline{L}$ is connected, and so, it must pass through the origin. Since $\overline{L}$ is properly embedded of multiplicity one and $\vec{0} \in \overline{L}$, we have $\lim_{r\to 0} r^{-2}\text{Area}(\overline{L} \cap \mathbb{B}(r)) = \pi$ and for any $\delta > 0$, there exists $r(\delta) > 0$ such that $\pi < r(\delta)^{-2}\text{Area}(\overline{L} \cap \mathbb{B}(r(\delta))) < \pi + \delta$. This implies $(r(\delta)|p_n|)^{-2}\,\text{Area}(M \cap \mathbb{B}(r(\delta)|p_n|)) < \pi + \delta$ for all n large. Since δ can be taken arbitrarily small, we deduce that $R^{-2}\text{Area}(M \cap \mathbb{B}(R))$ is at most π for some sequence $\{R_n\}_n \to \infty$. Since $R^{-2}\text{Area}(M \cap \mathbb{B}(R))$ is not decreasing (monotonicity formula), $R^{-2}\text{Area}(M \cap \mathbb{B}(R))$ must be at most π for all R. This inequality implies $R^{-2}\text{Area}(\overline{L} \cap \mathbb{B}(R)) \leq \pi$ for all R, which by the monotonicity formula implies L is a plane. This contradiction proves the theorem. $\square$

Corollary 6.3 *Let $\mathcal{L}$ be a non-flat minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$. If $\mathcal{L}$ has quadratic decay of curvature, then $\mathcal{L}$ consists of a single leaf, which extends to a properly embedded minimal surface with finite total curvature in $\mathbb{R}^3$.*

Proof. This follows easily from Proposition 6.1 and Theorem 6.2. □

Theorem 1.4 follows immediately from Theorem 6.2. We just remark that the last statement in Theorem 1.4 follows from the finite total curvature assumption, since a non-flat, complete, embedded, non-compact minimal surface of finite total curvature has a positive number of catenoidal ends and possibly finitely many planar ends. A simple calculation shows that the growth constant C^2 in Theorem 1.4 depends on the maximum logarithmic growth C of the catenoidal ends of M.

Remark 6.4 Given $r > 0$, $\mathbb{S}^2(r)$ denotes the sphere of radius r centered at the origin. For $C > 0$, let $\mathcal{F}_C$ denote the family of all complete, embedded, connected minimal surfaces $M \subset \mathbb{R}^3$ with quadratic curvature decay constant C, normalized so that the maximum of the function $|K_M|R^2$ occurs at a point of $M \cap \mathbb{S}^2(1)$. In [28] we will apply Theorem 1.4 to prove that $\mathcal{F}_C$ is naturally a compact metric space and that for C fixed, there is a bound on the topology of surfaces in $\mathcal{F}_C$ and that the subsets of $\mathcal{F}_C$ with fixed topology are compact.

7 The Local Picture Theorem on the Scale of Curvature.

Every complete, embedded, constant mean curvature surface in $\mathbb{R}^3$ with bounded curvature is properly embedded (Meeks and Rosenberg [39]), here we include minimal surfaces as being those with constant mean curvature zero. The next theorem shows that any complete, minimal surface in $\mathbb{R}^3$ that is not properly embedded, has natural limits under dilations, which are properly embedded, constant mean curvature surfaces. By dilation, we mean the composition of a homothety and a translation.

Theorem 7.1 (Local Picture on the Scale of Curvature) *Suppose M is a complete, embedded, constant mean curvature surface with unbounded curvature in a homogeneously regular three-manifold N. Then, there exists a sequence of points $p_n \in M$ and positive numbers $\varepsilon_n \to 0$, such that the following statements hold.*

1. *For all n, the component M_n of $\mathbb{B}_N(p_n, \varepsilon_n) \cap M$ that contains p_n is compact with boundary $\partial M_n \subset \partial \mathbb{B}_N(p_n, \varepsilon_n)$.*

2. *Let $\lambda_n = \sqrt{|K_{M_n}|(p_n)}$. The absolute curvature function $|K_{M_n}|$ satisfies $\frac{\sqrt{|K_{M_n}|}}{\lambda_n} \leq 1 + \frac{1}{n}$ on M_n, with $\lim_{n\to\infty} \varepsilon_n \lambda_n = \infty$.*

3. *The metric balls $\lambda_n \mathbb{B}_N(p_n, \varepsilon_n)$ of radius $\lambda_n \varepsilon_n$ converge uniformly to $\mathbb{R}^3$ with its usual metric (so that we identify p_n with $\vec{0}$ for all n), and, for any $k \in \mathbb{N}$, the surfaces $\lambda_n M_n$ converge C^k on compact subsets of $\mathbb{R}^3$ and with multiplicity one to a connected, properly embedded minimal surface M_∞ in $\mathbb{R}^3$ with $\vec{0} \in M_\infty$, $|K_{M_\infty}| \leq 1$ on M_∞ and $|K_{M_\infty}|(\vec{0}) = 1$.*

In the above theorem, we obtain a local picture or description of the local geometry of a constant mean curvature surface in an extrinsic neighborhood of a point p_n of concentrated curvature. Certainly, if for any positive ε the intrinsic ε-balls of a minimal surface are not always disks, then the curvature blows up as $\varepsilon \to 0$ at some points in these non-simply connected intrinsic ε-balls. It follows in this case that the injectivity radius of the surface is zero, i.e. there exists a divergent sequence of points where the injectivity radius function of the surface tends to zero; such points are called *points of concentrated topology.* In section 10 we prove a local picture theorem on the scale of topology for complete, embedded, constant mean curvature surfaces with zero injectivity radius, which has some similarities with Theorem 7.1.

In this section, we will study the local geometry of embedded, constant mean curvature surfaces in a homogenously regular Riemannian three-manifold in neighborhoods of certain points of large curvature. We will prove a local structure result that will be crucial in obtaining interesting global results and applications in the following sections. More specifically, we will consider a complete, embedded minimal surface M of unbounded curvature in a homogeneously regular three-manifold N, and obtain from M certain limits which we can consider to be properly embedded minimal surfaces in $\mathbb{R}^3$. Our goal here is to prove Theorem 7.1 in the introduction, which describes in detail how we will obtain these limits.

The proof of Theorem 7.1 uses a standard blow-up technique, where the scaling factors are the inverse of the square root of the absolute curvature at points of almost maximal curvature, a concept which we develop below. After the blowing-up process, we will find a limit which is a complete minimal surface with bounded Gaussian curvature, conditions which are known to imply properness for the limit. This properness will lead to the conclusions of Theorem 7.1. In particular, the proof of Theorem 7.1 is completely independent of Corollary 5.4, in whose proof we used this local picture theorem on the scale of curvature.

Recall that $M \subset N$ is a complete, embedded constant mean curvature surface with unbounded curvature in a homogeneously regular three-manifold. After a fixed constant scaling of the metric of N, we may assume that the injectivity radius of N is greater than 1. The first step in the proof of Theorem 7.1 is to obtain special points $p'_n \in M$, called *blow-up points* or *points of almost maximal curvature.* First consider an arbitrary sequence of points $q_n \in M$ such that $|K_M|(q_n) \geq n^2$, which exists since K_M is unbounded. Let $p'_n \in B_M(q_n, 1)$ be a maximum of $h_n = |K_M| d_M(\cdot, \partial B_M(q_n, 1))^2$, where $B_M(q_n, 1)$ denotes the intrinsic metric ball in M centered at q_n with radius 1 and d_M stands for the intrinsic

distance on M.

We define $\lambda'_n = \sqrt{|K_M|(p'_n)}$. Note that

$$\lambda'_n \geq \lambda'_n d_M(p'_n, \partial B_M(q_n,1)) = \sqrt{h_n(p'_n)} \geq \sqrt{h_n(q_n)} = \sqrt{|K_M|(q_n)} \geq n.$$

Fix $t>0$. Since $\lambda'_n \to \infty$ as $n\to\infty$, the sequence $\{\lambda'_n \mathbb{B}_N(p'_n, \frac{t}{\lambda'_n})\}_n$ converges to the ball $\mathbb{B}(t)$ of $\mathbb{R}^3$ with its usual metric, where we have used geodesic coordinates centered at p'_n and identified p'_n with $\vec{0}$. Similarly, we can consider $\{\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})\}_n$ to be a sequence of embedded, constant mean curvature surfaces with boundary, all passing through $\vec{0}$ with curvature -1 at this point. We claim that the curvature of $\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})$ is uniformly bounded. To see this, pick a point $z_n \in B_M(p'_n, \frac{t}{\lambda'_n})$. Note that for n large enough, z_n lies in $B_M(q_n,1)$. Then,

$$\frac{\sqrt{|K_M|(z_n)}}{\lambda'_n} = \frac{\sqrt{h_n(z_n)}}{\lambda'_n d_M(z_n, \partial B_M(q_n,1))} \leq \frac{d_M(p'_n, \partial B_M(q_n,1))}{d_M(z_n, \partial B_M(q_n,1))}. \tag{5}$$

By the triangle inequality, $d_M(p'_n, \partial B_M(q_n,1)) \leq \frac{t}{\lambda'_n} + d_M(z_n, \partial B_M(q_n,1))$, and so,

$$\begin{aligned} \frac{d_M(p'_n, \partial B_M(q_n,1))}{d_M(z_n, \partial B_M(q_n,1))} &\leq 1 + \frac{t}{\lambda'_n d_M(z_n, \partial B_M(q_n,1))} \\ &\leq 1 + \frac{t}{\lambda'_n \left(d_M(p'_n, \partial B_M(q_n,1)) - \frac{t}{\lambda'_n} \right)} \leq 1 + \frac{t}{n-t}, \end{aligned} \tag{6}$$

which tends to 1 as $n\to\infty$.

It follows that after extracting a subsequence, that the surfaces $\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})$ converge smoothly to a compact embedded minimal surface $M_\infty(t)$ contained in $\mathbb{B}(t)$ with bounded curvature, that passes through $\vec{0}$ and with curvature -1 at the origin (perhaps the boundary of $M_\infty(t)$ is not smooth). Consider the compact surface $M_\infty(1)$ together with the surfaces $\lambda'_n B_M(p'_n, \frac{1}{\lambda'_n})$ that converge to it (after passing to a subsequence). Note that $M_\infty(1)$ is contained in $M_\infty = \bigcup_{t\geq 1} M_\infty(t)$, which is a complete, injectively immersed minimal surface in $\mathbb{R}^3$.

By construction, M_∞ has bounded curvature, so it is properly embedded in $\mathbb{R}^3$ [39]. It follows that for all $R>0$, there exist $t>0$ and $k\in\mathbb{N}$ such that if $m\geq k$, then the component of $\left[\lambda'_m B_M(p'_m, \frac{t}{\lambda'_m})\right] \cap \mathbb{B}(R)$ that passes through $\vec{0}$ is compact and has its boundary on $\mathbb{S}^2(R)$. Applying this property to $R_n = \sqrt{\lambda'_n}$, we obtain $t(n)>0$ and $k(n)\in\mathbb{N}$ satisfying that if we let M_n denote the component of $B_M(p'_{k(n)}, \frac{t(n)}{\lambda'_{k(n)}}) \cap \mathbb{B}_N(p'_{k(n)}, \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}})$ that contains $p'_{k(n)}$, then M_n is compact and has its boundary on $\partial\mathbb{B}_N(p'_{k(n)}, \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}})$. Clearly

this compactness property remains valid if we increase the value of $k(n)$. Hence, we may assume without loss of generality that

$$t(n)(n+1) < k(n) \text{ for all } n, \qquad \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}} \to 0 \text{ as } n \to \infty.$$

We now define $p_n = p'_{k(n)}$, $\varepsilon_n = \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}}$ and $\lambda_n = \lambda'_{k(n)}$. Then it is easy to check that the $p_n, \varepsilon_n, \lambda_n$ and M_n satisfy the conclusions stated in Theorem 7.1 (in order to prove item 2 in the statement of Theorem 7.1, simply note that equations (7) and (8) imply that $\frac{\sqrt{|K_{M_n}|}}{\lambda_n} = \frac{\sqrt{|K_{M_n}|}}{\lambda'_{k(n)}} \leq 1 + \frac{t(n)}{k(n)-t(n)} < 1 + \frac{1}{n}$, where the last inequality follows from $t(n)(n+1) < k(n)$). This finishes the proof of Theorem 7.1.

Remark 7.2 If the surface $M \subset N$ in Theorem 7.1 were properly embedded, then the argument needed to carry out its proof could be formulated in a more standard manner by using the techniques developed in the papers [31, 39]; it is the non-properness of M that necessitates our being more careful here in defining the limit surface M_∞. Equally important as its elegant statement is the flexibility of applying this statement and the proof of Theorem 7.1, as well as the related Local Theorem on the Scale of Topology (Theorem 10.1), in novel ways. We now point out several of these important applications.

First consider a properly embedded minimal surface Σ in $\mathbb{R}^3$. The Quadratic Curvature Decay Theorem implies that if Σ is not minimal with finite total curvature, there exists a sequence of positive numbers $\{\lambda_n\}_n$ converging to zero such that the sequence of homothetically shrunk surfaces $\{\lambda_n\Sigma\}_n$ do not have bounded curvature in arbitrarily small neighborhoods of $\mathbb{S}^2$. To apply Theorem 7.1 directly to this situation, we consider the manifold N to be a countable number of copies of $\mathbb{R}^3$ and M to be the embedded disconnected surface whose components are the surfaces in $\{\lambda_n\Sigma\}_n$. The statement and proof of Theorem 7.1 implies that M has a local picture arising from points $p_n \in \Sigma_n$ near $\mathbb{S}^2$, in other words, under a divergent sequence of translations and homotheties, the surface Σ has a limiting surface M_∞ in $\mathbb{R}^3$. The usefulness of this remark is exploited in the next section. Similar applications of the related Local Picture Theorem on the Scale of Topology are useful for understanding the asymptotic behavior of limit ends of genus zero for any properly embedded minimal surface in $\mathbb{R}^3$, see [27].

The point of view in the last paragraph is also useful for obtaining curvature estimates for constant mean curvature surfaces; in the minimal case see [31] for such applications. When the constant mean curvature is non-zero, we call the constant mean curvature surface a CMC surface. Consider the collection of all complete, embedded, simply connected CMC surfaces in complete, locally homogeneous three-manifold N. One of the main results of Meeks and Tinaglia in [41] is that a surface M in $\mathcal{M}(N)$ with positive constant mean curvature H_M has absolute Gaussian curvature less than $\mathbb{K}_N/H_M$, for some constant

$\mathbb{K}_N$. Otherwise, there exists a sequence $\{\lambda_n\Sigma\}_n$ of complete, embedded, simply connected CMC surfaces in N with unbounded curvature. Since in this case there is no assumed bound on the constant mean curvature surfaces in the sequence, the related local picture theorem on the scale of curvature produces a limit surface M_∞. Still, the arguments in the proof of Theorem 7.1 imply that M_∞ satisfies all of the properties of the theorem except that it may have non-zero, constant mean curvature and may self-intersect. After a study of the geometry of M_∞, then Meeks and Tinaglia obtain a contradiction, which proves the desired curvature estimate. For the sake of accurate reference, we explain the complete statement of the needed generalization of Theorem 7.1 in this application; we call this generalization the *CMC Local Picture Theorem on the Scale of Curvature.* One technical problem that arises in this CMC local picture theorem is that the limit surface M_∞ in $\mathbb{R}^3$, while being properly immersed is not necessarily embedded. However, the limit surface is always *strongly Alexandrov embedded*, as given in the next definition. The hypothesis of the new theorem allows each component of M to have constant mean curvature and the limit surface is allowed to be strongly Alexandrov embedded. With minor changes, the proof of Theorem 7.1 works in this more general setting and we leave the details to the reader.

Definition 7.3 Suppose W is a complete flat three-manifold with boundary $\partial W = \S$ together with an isometric immersion $f\colon W \to \mathbb{R}^3$ such that f restricted to the interior of W is injective. We will call the image surface $f(\S)$ *strongly Alexandrov embedded* if W lies on the mean convex side of $\S$.

8 The space $D(M)$ of dilation limits and the Dynamics Theorem.

We now prove Theorem 1.6 stated in the introduction. Suppose M is a non-flat, properly embedded minimal surface in $\mathbb{R}^3$. Recall that we defined in the introduction the set $D(M)$ of all properly embedded minimal surfaces in $\mathbb{R}^3$ which are C^1-limits (with multiplicity one) of divergent sequences of dilations of M. A natural topology on $D(M)$ is the usual topology of uniform C^k-convergence on compact subsets of $\mathbb{R}^3$ for all k.

We will now define a metric space structure on the subspace $D_1(M) = \{\Sigma \in D(M) \mid \vec{0} \in \Sigma,\ |K_\Sigma| \leq 1,\ |K_\Sigma|(\vec{0}) = 1\}$, which generates a topology that coincides with the topology of uniform C^k-convergence on compact subsets of $\mathbb{R}^3$. We first prove that there exists some $\varepsilon > 0$ such that $\overline{\mathbb{B}(\varepsilon)}$ intersects every surface $\Sigma \in D_1(M)$ in a unique component which is a graphical disk over its projection to the tangent space to Σ at $\vec{0}$ and with gradient less than 1. Otherwise, there exists a sequence $\{\Sigma_n\}_n \subset D_1(M)$ such that this property fails in the ball $\overline{\mathbb{B}}(\frac{1}{n})$. By the uniform graph lemma [45], for n large there exists $\delta > 0$ such that Σ_n intersects $\overline{\mathbb{B}}(\delta)$ in at least two components, one of which passes through $\vec{0}$ and

the other one intersects $\mathbb{B}(\frac{1}{n})$, and such that both components are graphical over domains in the tangent space of Σ_n at $\vec{0}$ with small gradient. Hence, a subsequence of these Σ_n (denoted in the same way) converges to a minimal lamination of $\mathbb{R}^3$ with a stable leaf passing through $\vec{0}$ (see items 1 and 5 of Lemma 3.4). Since such a stable leaf is a plane, we contradict that the convergence is smooth and the curvature of the Σ_n is -1 at $\vec{0}$ for every n. This proves our claim on the existence of ε. Given $\Sigma_1, \Sigma_2 \in D_1(M)$, we define the distance between Σ_1 and Σ_2 as

$$d(\Sigma_1, \Sigma_2) = d_{\mathcal{H}}\left(\Sigma_1 \cap \overline{\mathbb{B}}(\frac{\varepsilon}{2}), \Sigma_1 \cap \overline{\mathbb{B}}(\frac{\varepsilon}{2})\right),$$

where $d_{\mathcal{H}}$ denotes the Hausdorff distance. Standard elliptic theory implies that the metric topology on $D_1(M)$ associated to the distance d agrees with the topology of the uniform C^k-convergence on compact sets of $\mathbb{R}^3$ for any k.

In the introduction, we defined a D-invariant subset $\Delta \subset D(M)$ as a non-empty subset such that $D(\Sigma) \subset \Delta$ for all $\Sigma \in \Delta$. Furthermore, Δ is a minimal D-invariant set of $D(M)$ if contains no proper non-empty D-invariant subsets. Any element Σ in a minimal D-invariant subset of $D(M)$ is called a minimal element of $D(M)$.

The Dynamics Theorem implies that every minimal element Σ of $D(M)$ which does not have finite total curvature, is an element of $D(\Sigma)$, and so it satisfies the following periodicity property. Fix any $R > 0$, and let Σ_R denote the portion of Σ inside the open ball of radius R centered at the origin. Then, for all small $\varepsilon > 0$, there exists a collection $\{\mathbb{B}_n = \mathbb{B}(p_n, R_n)\}_n$ of disjoint open balls such that: the surfaces $\Sigma_n = \frac{R}{R_n}((\Sigma \cap \mathbb{B}_n) - p_n)$ can be parametrized by Σ_R so that as mappings they are ε-close to Σ_R in the C^2-norm. By Zorn's lemma, any such collection of balls is contained in a maximal collection with the same property. Minimality of Σ also implies that there exists a $d_\Sigma(R, \varepsilon) > 0$ such that for any such maximal collection, the set of numbers

$$D_n = \left\{ \frac{\text{distance}(\mathbb{B}_n, \mathbb{B}_m)}{R_n} \right\}_{m \neq n}$$

is bounded from above by $d_\Sigma(R, \varepsilon)$. In particular, for a minimal element $\Sigma \in D(M)$ of infinite total curvature, each compact subdomain of the surface can be approximated with arbitrarily high precision (under dilation) by an infinite collection of pairwise disjoint compact subdomains of the surface.

As direct consequences of Definition 1.5 in the introduction, we have:

(i) If $\Sigma \in D(M)$ and $D(\Sigma) = \emptyset$, then $\{\Sigma\}$ is always a minimal D-invariant set.

(ii) $\Sigma \in D(M)$ is quasi-dilation-periodic if and only if $\Sigma \in D(\Sigma)$.

(iii) Any minimal element $\Sigma \in D(M)$ is contained in a unique minimal D-invariant set.

(iv) If $\Delta \subset D(M)$ is a minimal D-invariant set and $\Sigma \in \Delta$ satisfies $D(\Sigma) \neq \emptyset$, then $D(\Sigma) = \Delta$ (otherwise $D(\Sigma)$ would be a proper non-empty D-invariant subset of Δ). In particular, Σ is quasi-dilation-periodic.

(v) If $\Delta \subset D(M)$ is a D-invariant set and $\Sigma \in \Delta$ is a minimal element, then the (unique) minimal D-invariant subset Δ' of $D(M)$ which contains Σ satisfies $\Delta' \subset \Delta$ (otherwise $\Delta' \cap \Delta$ would be a proper non-empty D-invariant subset of Δ').

Next we start the proof of Theorem 1.6. Suppose M is a properly embedded, non-flat minimal surface in $\mathbb{R}^3$, and firstly assume that M has finite total curvature. Then, its total curvature outside of some ball in space is less than 2π, and so, any $\Sigma \in D(M)$ must have total curvature less than 2π, which implies Σ is flat. This implies $D(M) = \emptyset$.

Reciprocally, assume that $D(M) = \emptyset$ and M does not have finite total curvature. By Theorem 1.4, M does not have quadratic decay of curvature, and so, there exists a divergent sequence of points $z_n \in M$ with $(|K_M|R^2)(z_n) \to \infty$. Hence, there exists another divergent sequence of points $q_n \in M$ with $(|K_M|R^2)(q_n) \geq n^2$. Let p_n be a maximum of the function $h_n = |K_M| d_{\mathbb{R}^3}(\cdot, \partial\mathbb{B}(q_n, \frac{|q_n|}{2}))^2$. Note that $\{p_n\}_n$ diverges in $\mathbb{R}^3$ (because $|p_n| \geq \frac{|q_n|}{2}$). We define $\lambda_n = \sqrt{|K_M|(p_n)}$. By similar arguments as those in the previous section, λ_n diverges and the sequence $\{\lambda_n(M - p_n)\}_n$ converges (after passing to a subsequence) to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with a non-flat leaf L which passes through $\vec{0}$ with $|K_L|(\vec{0}) = 1$. Furthermore, the curvature function $K_{\mathcal{L}}$ of $\mathcal{L}$ satisfies $|K_{\mathcal{L}}| \leq 1$ and so, the leaf L of $\mathcal{L}$ passing through $\vec{0}$ is properly embedded in $\mathbb{R}^3$. By the Strong Half-space Theorem, $\mathcal{L}$ consists just of L, and the convergence of the surfaces $\lambda_n(M - p_n)$ to L has multiplicity one (since L is not flat). Therefore, $L \in D_1(M)$, which contradicts that $D(M) = \emptyset$. This proves the equivalence stated in Theorem 1.6.

Assume now that $D(M) \neq \emptyset$. The arguments in the last paragraph and the discussion in the second paragraph of this section show that $D_1(M) \neq \emptyset$ and that the topology on $D_1(M)$ of uniform C^1-convergence on compact sets agrees with the metric space topology on $D_1(M)$ induced from $D(M)$. Hence, compactness of $D_1(M)$ will follow from sequential compactness. Given a sequence $\{\Sigma_n\}_n \subset D_1(M)$, a subsequence converges to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$, which has bounded curvature and curvature -1 at $\vec{0}$. The same arguments given in the last paragraph imply that $\mathcal{L}$ consists just of the leaf L passing through $\vec{0}$, which is a properly embedded minimal surface in $\mathbb{R}^3$. Clearly, $L \in D_1(M)$, which proves item *1* of the theorem.

Using the definition of D-invariance,it is elementary to prove that for any $\Sigma \in D(M)$, $D(\Sigma)$ is a closed D-invariant set in $D(M)$; essentially, this is because the set of limit points of a set in a topological space forms a closed set. The same techniques prove that if $\Delta \subset D(M)$ is a D-invariant subset, then its closure in $D(M)$ is also D-invariant. This proves item *2* in the theorem.

Now assume that Δ is a minimal D-invariant set in $D(M)$. If Δ contains a surface

of finite total curvature, then the minimality of Δ implies Δ consists only of this surface, and so, it is connected and closed in $D(M)$. Suppose now that Δ contains no surfaces of finite total curvature. Then, for any $\Sigma \in \Delta$, $D(\Sigma)$ is a non-empty D-invariant subset of Δ which by minimality implies $D(\Sigma) = \Delta$. Since $D(\Sigma)$ is closed, Δ is closed as well. Since $D(\Sigma) = \Delta$, then Δ also contains the path connected subset S of all dilations of Σ. Since $D(\Sigma)$ is a closed set in $D(M)$, the definition of $D(\Sigma)$ implies that the closure of S in $D(M)$ equals $D(\Sigma)$, and so, $\Delta = D(\Sigma)$ is connected. This proves item *3* in the theorem.

Next we prove item *4*. Suppose $\Delta \subset D(M)$ is a D-invariant set. One possibility is that Δ contains a surface Σ of finite total curvature. By the main statement of this theorem, $D(\Sigma) = \emptyset$ and by item **(i)** above, Σ is a minimal element in Δ. Now assume Δ contains no surfaces of finite total curvature. Consider the set Λ of all closed D-invariant subsets of Δ. Note that this collection is non-empty, since for any $\Sigma \in \Delta$ (recall that Δ cannot be empty since it is D-invariant), the set $D(\Sigma) \subset \Delta$ is such a closed non-empty D-invariant set by the first statement in item *2*. Λ has a partial ordering induced by inclusion. We just need to check that any linearly ordered set in Λ has a lower bound, and then apply Zorn's Lemma to obtain item *3* of the theorem. Suppose $\Lambda' \subset \Lambda$ is a non-empty linearly ordered subset. We must check that the intersection $\bigcap_{\Delta' \in \Lambda'} \Delta'$ is an element of Λ. In our case, this means we need to prove that such an intersection is non-empty, because the intersection of closed (resp. D-invariant) sets is closed (resp. D-invariant).

Given $\Delta' \in \Lambda'$, recall that $\Delta'_1 = \{\Sigma \in \Delta' \mid \vec{0} \in \Sigma,\ |K_\Sigma| \leq 1,\ |K_\Sigma|(\vec{0}) = 1\}$. Note that Δ'_1 is a closed subset of $D(M)$, since Δ' and $D_1(M)$ are closed in $D(M)$. The set Δ'_1 is non-empty by the following argument. Let $\Sigma \in \Delta'$. Since Σ does not have finite total curvature and Δ' is D-invariant, $D(\Sigma)$ is a non-empty subset of Δ'. By item *1*, $D_1(\Sigma)$ is a non-empty subset of Δ'_1, and so, Δ'_1 is non-empty and the argument is finished. Now define $\Lambda'_1 = \{\Delta'_1 \mid \Delta' \in \Lambda'\}$. As $\bigcap_{\Delta'_1 \in \Lambda'_1} \Delta'_1 = \bigcap_{\Delta' \in \Lambda'} \Delta'_1 = \bigcap_{\Delta' \in \Lambda'} (\Delta' \cap D_1(M)) = \left(\bigcap_{\Delta' \in \Lambda'} \Delta'\right) \cap D_1(M)$, in order to check that $\bigcap_{\Delta' \in \Lambda'} \Delta'$ is nonempty, it suffices to show that $\bigcap_{\Delta'_1 \in \Lambda'_1} \Delta'_1$ is non-empty. But this is clear since each element of Λ'_1 is a closed subset of the compact metric space $D_1(M)$, and the finite intersection property holds for the collection Λ'_1.

Next we prove item *5*. Let $\Delta \subset D(M)$ be a D-invariant subset which contains no surfaces of finite total curvature. By item *3*, there exists a minimal element $\Sigma \in \Delta$. Since none of the surfaces of Δ have finite total curvature, it follows that $D(\Sigma) \neq \emptyset$. As Σ is a minimal element, there exists a minimal D-invariant subset $\Delta' \subset D(M)$ such that $\Sigma \in \Delta'$. By item **(iv)** above, $D(\Sigma) = \Delta'$. Note that Δ'_1 contains $D_1(\Sigma)$, which is non-empty since $D(\Sigma) \neq \emptyset$ (by item *1* of this theorem). Then there exists a surface $\Sigma_1 \in \Delta'_1$, which in particular is a minimal element (any element of Δ' is), and lies in Δ_1 (because $\Delta' \subset \Delta$ by item **(v)**). Finally, Σ_1 is dilation-periodic by item **(iv)**, thereby proving item *5* of the theorem.

Suppose now $\Sigma \in D(M)$ has does not have finite total curvature. By Theorem 1 in [32],

either Σ is a helicoid with handles or it has exactly two limit ends. On the other hand, if Σ is also a minimal element of $D(M)$, then item **(iv)** above implies that Σ is quasi-dilation-periodic, which means that there exists a sequence of dilations $d_n \colon \mathbb{R}^3 \to \mathbb{R}^3$ whose translation part diverges, such that $\{d_n(\Sigma)\}_n$ converges smoothly to Σ on compact sets. Since Σ has finite genus, we deduce that its genus is zero. Hence, if Σ has finite topology, then it is simply connected and, therefore, a helicoid (see also [39]). Now assume Σ has two limit ends and genus zero. Under these hypotheses, we proved in [31] that Σ has bounded curvature. It remains to prove that Σ is quasi-translation-periodic (we cannot deduce this from Theorem 1 in [31] since it only insures that there exists a divergent sequence $p_n \in \mathbb{R}^3$ such that after extracting a subsequence, $\Sigma - p_n$ converges on compact subsets of $\mathbb{R}^3$ to a surface with the same appearance as Σ, but that might be different from Σ). Theorem 1 in [31] also implies Σ has a well-defined non-zero flux vector $F \in \mathbb{R}^3$. If h_n is the homothety part of d_n, then clearly $h_n(F)$ is a flux vector of $d_n(\Sigma)$, which implies that the length of $h_n(F)$ converges to the length of F. Therefore, as $n \to \infty$, the homotheties h_n converge to the identity map, and so, Σ is quasi-translation-periodic.

This finishes the proof of Theorem 1.6.

Remark 8.1 Numerically based computer graphics experiments indicate that there exists an almost-finite genus (in the sense of Riemann surfaces, see e.g. Sario [50]) properly embedded minimal surface M with infinite genus. This surface appears as a limit of finite genus helicoids and contains the x_1 and x_3-axes. The handles of M are attached to a fixed vertical helicoid symmetrically along the x_2-axis, and outside a large ball, the handles are closely approximated by catenoids. $D(M)$ apparently consists of all vertical helicoids and all vertical catenoids. See Bobenko [1], Schmies [51] pages 82-83, Hoffman, Weber and Wolf [20].

A recently discovered singly-periodic, properly embedded minimal surface M (Rodriguez and Hauswirth [18]) has a connected dilation limit space $D(M)$ consisting of all dilations of M and one of the doubly periodic minimal surfaces $\widehat{M}$ in [44]. In this case, $\widehat{M}$ is the unique minimal element in $D(M)$ up to dilation. The method of construction of this example probably can be modified to construct a properly embedded minimal surface Σ such that $D(\Sigma)$ contains an infinite number of pairwise disjoint minimal D-invariant sets.

Many of the techniques that we have used in the proof of the Dynamics Theorem and in the proof of the Local Picture Theorem on the Scale of Curvature can be applied to obtain results for minimal hypersurfaces in $\mathbb{R}^{n+1}$, when $n > 2$. If M^n is a complete, (not necessarily proper) embedded minimal submanifold of $\mathbb{R}^{n+1}$ and has bounded second fundamental form in any ball in $\mathbb{R}^{n+1}$, then the closure $\overline{M^n}$ has the structure of a $C^{1,\alpha}$-minimal lamination of $\mathbb{R}^{n+1}$ by minimal hypersurfaces. If M^n does not have bounded second fundamental form, then the proof of the Local Picture Theorem on the Scale of

Curvature shows that there exists a divergent sequence of compact subdomains $M_k \subset M^n$, which, after translation and homothety, converge to a complete, embedded minimal submanifold $M^n_\infty \subset \mathbb{R}^{n+1}$ passing through the origin $\vec{0}$, with the norm of the second fundamental form of M^n_∞ at $\vec{0}$ being 1, and the norm of the second fundamental being bounded from above by 1. It follows that $\overline{M^n_\infty}$ is a non-flat minimal lamination of $\mathbb{R}^{n+1}$; however, we do not know in this case if $\overline{M^n_\infty}$ is a properly embedded surface in $\mathbb{R}^{n+1}$. If M^n is stable, then the same property holds for the universal cover of M^n_∞ and that of any leaf of $\overline{M^n_\infty}$.

Suppose now that M^n is a properly embedded minimal hypersurface in $\mathbb{R}^{n+1}$. If we denote by $D_1(M^n)$ the set of minimal, dilation limit laminations of M^n passing through the origin, with the lengths of their second fundamental forms equal to 1 at the origin and bounded by 1, then $D_1(M^n)$ becomes a compact metric space with the topology of uniform C^1-convergence on compact sets of $\mathbb{R}^{n+1}$. Note that $D_1(M^n)$ also makes sense if M^n is a minimal lamination of $\mathbb{R}^{n+1}$, instead of a properly embedded minimal submanifold. In this set up, one also obtains an interesting dynamics type result for $D_1(M^n)$ with the "minimal" elements being certain minimal laminations. It is interesting to contemplate how these results might play a role in understanding complete, stable, embedded minimal hypersurfaces in $\mathbb{R}^{n+1}$.

9 Applications of the Dynamics Theorem.

In this section, we present several different applications of the Dynamics Theorem 1.6, which are summarized in the statement of Theorem 9.1 below. We focus our attention on obtaining interesting properly embedded minimal surfaces in $\mathbb{R}^3$ which are dilation limits of a sequence of compact subdomains on a complete, (possibly non-proper) embedded minimal surface in $\mathbb{R}^3$, where this surface satisfies some interesting geometric constraint.

In certain cases, we can prove that for a given properly embedded, non-flat minimal surface $M \subset \mathbb{R}^3$, there are no minimal surfaces of finite total curvature in $D(M)$. In such a case, Theorem 1.6 implies that there exists a dilation limit of M which has bounded curvature and is quasi-dilation-periodic. We consider three such cases in the following theorem.

Theorem 9.1 *Suppose $M \subset \mathbb{R}^3$ is a complete, orientable, non-flat, embedded minimal surface which satisfies one of the following three properties.*

1. *The Gauss map of M misses a subset $\Delta \subset \mathbb{S}^2(1)$ which contains two non-antipodal points.*

2. *M has a nontrivial well-defined injective associate surface $f_\theta \colon M \to \mathbb{R}^3$ (this holds, for example, when M admits an intrinsic isometry which does not extend to an ambient isometry).*

3. M is a properly embedded minimal surface of quadratic area growth and neither M nor any element of $D(M)$ has finite total curvature.

Then:

(i) There exists a properly embedded, quasi-dilation-periodic minimal surface $\Sigma \subset \mathbb{R}^3$ with infinite genus and bounded curvature, with Σ being a minimal element in $D(\Sigma)$, which also satisfies the same property 1, 2 or 3 as M.

(ii) If M is properly embedded in $\mathbb{R}^3$, then Σ can be chosen to be any minimal element of $D_1(M)$. Otherwise, Σ can be obtained as one of the local pictures of M on the scale of curvature, via Theorem 7.1.

(iii) If M satisfies 3, then every minimal element $\Sigma \in D(M)$ is quasi-dilation-periodic, and every limit tangent cone at infinity of such a Σ is a cone over a finite collection of geodesic arcs which join two antipodal points of $\mathbb{S}^2(1)$.

Proof. Let $M \subset \mathbb{R}^3$ be a complete, embedded minimal surface that satisfies one of the properties *1, 2,* or *3*.

Note that each of the properties *1, 2* above implies that M does not have finite total curvature, and both properties are preserved by limits under translations and rescalings (thus such limits also have infinite total curvature). First assume that M is proper in $\mathbb{R}^3$. If M satisfies *1* or *2*, then we easily deduce that neither M nor any surface in $D(M)$ (which makes sense because M is proper) has finite total curvature. The same property is true if M satisfies *3*, by assumption. In any case, the Dynamics Theorem, Theorem 1.6, implies that $D(M) \neq \emptyset$ and item *4* of the same theorem applied to $\Delta = D(M)$ gives that there exists a minimal element $\Sigma \in D_1(M)$, which is a properly embedded, quasi-dilation-periodic minimal surface with bounded curvature. We claim that Σ has infinite genus. Otherwise, item *4* of Theorem 1.6 implies that Σ is a helicoid or a genus zero surface with two limit ends which is quasi-translation-periodic. The helicoid limit is clearly impossible if M satisfies properties *1, 2* or *3*. If Σ has genus zero with two limit ends, then Theorem 1 in [31] implies that its Gauss map omits exactly two antipodal directions (in contradiction with property *1*), Σ has a well-defined non-zero flux vector (which contradicts *2*) and Σ has cubical area growth (which contradicts *3*). Therefore, Σ has infinite genus. It is clear that Σ satisfies the same property *1, 2* or *3* as M. This proves part *(i)* of Theorem 9.1 under the additional hypothesis that M is proper in $\mathbb{R}^3$. Note also that the same argument demonstrates the first statement in part *(ii)* of the same theorem.

If M has bounded curvature, then it is proper in $\mathbb{R}^3$ (Theorem 1.6 in [39]) and we can apply the arguments in the previous paragraph. Now assume M has unbounded curvature. Applying Theorem 7.1, we conclude that there exists a properly embedded minimal surface Σ_1 which is a limit of compact regions of M under a sequence of dilations. As before,

$D(\Sigma_1)$ contains no surfaces of finite total curvature, so we can apply the preceding case (when M was assumed to be proper) to Σ_1, thereby proving part *(i)* of Theorem 9.1 and the second statement in part *(ii)*.

It remains to prove item *(iii)* of the theorem. As before, the fact that no surfaces in $D(M)$ have finite total curvature implies that any minimal element in $D(M)$ is quasi-dilation-periodic. A straightforward application of the monotonicity formula gives that if M has quadratic area growth constant $C > 0$, then a translated of M has quadratic area growth constant at most C. From here we deduce directly that if M has quadratic area growth constant $C > 0$, then any surface in $D(M)$ also has quadratic area growth constant at most C. If a minimal element $\Sigma \in D(M)$ does not satisfy the last statement of item *(iii)*, then Σ has a limit tangent cone at infinity $\mathcal{C}$ with a point $p \in \mathcal{C} \cap \mathbb{S}^2(1)$ where $\mathcal{C}$ is not smooth and such that the area density of $\mathcal{C}$ at p (counted with multiplicity as a limit of Σ) is strictly less than the area density of $\mathcal{C}$ at $\vec{0}$. Then there exists a sequence of homotheties $\{h_n(x) = \tau_n x\}_n$ with the positive numbers τ converging to zero, such that $\Sigma_n = h_n(\Sigma)$ converges to $\mathcal{C}$ as $n \to \infty$. Since $\mathcal{C}$ is not smooth at p, there exists a sequence $p_n \in \Sigma_n$ converging to p with $|K_{\Sigma_n}|(p_n) \to \infty$ as $n \in \infty$. After possibly exchanging p_n by points of almost maximal curvature on Σ_n converging to p (in the sense of section 7) and extracting a subsequence, the surfaces $\widetilde{\Sigma}_n = \sqrt{|K_{\Sigma_n}|(p_n)}(\Sigma_n - p_n)$ converge to a properly embedded minimal surface $\Sigma' \in \mathbb{R}^3$ that satisfies $\vec{0} \in \Sigma'$, $|K_{\Sigma'}| \leq 1$ and $|K_{\Sigma'}|(\vec{0}) = 1$. Since the numbers τ_n of h_n converge to zero and $p_n \to p \in \mathbb{S}^2(1)$, we deduce that $\widetilde{\Sigma}_n$ can be written in the form $\widetilde{\Sigma}_n = \lambda_n(\Sigma - q_n)$ for a divergent sequence $\{q_n\}_n \subset \mathbb{R}^3$ and some $\lambda_n > 0$. In particular, $\Sigma' \in D(\Sigma)$, and thus, $\Sigma' \in D_1(\Sigma)$. But Σ' has area growth constant strictly less than the area growth constant of Σ, which is a contradiction because $\Sigma \in D(\Sigma') = D(\Sigma)$. Now the proof of Theorem 9.1 is complete. $\square$

9.1 Classical conjectures related to the Dynamics Theorem.

Consider a complete, non-flat minimal surface $M \subset \mathbb{R}^3$ which satisfies one of the following three properties:

1. The Gauss map of M misses a subset $\Delta \subset \mathbb{S}^2(1)$ which contains two non-antipodal points.

2. M has a nontrivial well-defined injective associate surface $f_\theta \colon M \to \mathbb{R}^3$.

3. M is a properly embedded minimal surface of quadratic area growth and neither M nor any element in $D(M)$has finite total curvature.

By Theorem 9.1, M gives rise to special limit minimal surfaces, namely properly embedded, quasi-dilation-periodic minimal surfaces with infinite genus and bounded curvature, which satisfy the same hypothesis in 1, 2 or 3 as M. It is our hope that the condi-

tional existence of these quasi-dilation-periodic examples will lead to positive solutions of the following conjectures.

Conjecture 9.2 (Meeks, Pérez, Ros) *If M is a complete, non-flat, embedded minimal surface in $\mathbb{R}^3$, whose Gauss map misses a non-empty subset $\Delta \subset \mathbb{S}^2(1)$ which does not consist just one point or of exactly two antipodal points, then Δ is a pair of antipodal points and M is a singly or doubly-periodic Scherk minimal surface. On the other hand, if the Gauss map of M misses exactly 2 antipodal points, then M is a catenoid,, a helicoid, a Riemann minimal example or a doubly-periodic minimal surface with a natural quotient having genus one, total curvature -8π and parallel ends (these last surfaces have been recently classified by Pérez, Rodríguez and Traizet [44]).*

Conjecture 9.3 (Meeks) *Every intrinsic isometry of a complete, connected, embedded, constant mean curvature surface in $\mathbb{R}^3$ extends to an ambient isometry. More generally, the helicoid is the unique, complete, embedded constant mean curvature surface in $\mathbb{R}^3$ that has more than one isometric immersion with the same constant mean curvature (the similar conjecture to the first statement is false without assuming embeddedness, since it is false for Enneper's surface which is not embedded). Also see Corollary 10.3.*

Conjecture 9.4 (Meeks) *A complete, embedded, connected minimal surface $M \subset \mathbb{R}^3$ with quadratic area growth has a unique limit tangent cone at infinity. Furthermore, if M has quadratic area growth constant 2π, then M is a catenoid or a singly-periodic Scherk minimal surface (see the recent paper [42] by Meeks and Wolf for the solution of this second statement in the infinite symmetry case).*

10 The Local Picture Theorem on the Scale of Topology.

Recall from Theorem 7.1 in the introduction and its proof in section 7, that the Local Picture Theorem on the Scale of Curvature is a tool that allows to produce, after blowing-up a complete, embedded minimal surface M in a homogeneously regular three manifold, a non-flat, properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ with normalized curvature (in the sense that $|K_{M_\infty}| \leq 1$ on M_∞ and $\vec{0} \in M_\infty$, $|K_{M_\infty}|(\vec{0}) = 1$). The key ingredient to do this was to find points $p_n \in M$ of almost maximal curvature and then rescale the translated surfaces $M - p_n$ by $\sqrt{|K_M|(p_n)} \to \infty$ as $n \to \infty$. We will devote this section to obtain a somehow similar result for a surface whose injectivity radius is zero, by exchanging the role of the square root of $|K_M|$ by $1/I_M$, where I_M denotes the injectivity radius function on M. We will consider this rescaling ratio after evaluation at points $p_n \in M$ of *almost concentrated topology*, in a sense to be made precise later on. One of the difficulties of this generalization is that the limit objects that we can find after blowing-up might be not only properly embedded minimal surfaces in $\mathbb{R}^3$, but also new objects, namely limit

minimal parking garage structures (that we will study in Subsection 10.2) and certain kinds of singular minimal laminations of $\mathbb{R}^3$.

10.1 The statement of the main theorem.

The statement of the next theorem includes the term *minimal parking garage structure on* $\mathbb{R}^3$ which is defined in Subsection 10.2. Roughly stated, a parking garage structure is a limit object for a sequence of embedded minimal surfaces which converges to a minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by parallel planes, with singular set of convergence being a locally finite set of lines $S(\mathcal{L})$ orthogonal to $\mathcal{L}$, along which the limiting surfaces have the local appearance of a highly-sheeted double multigraph; the set of lines $S(\mathcal{L})$ are called the *columns* of the parking garage structure. For example, the sequence of homothetic shrinkings $\frac{1}{n}H$ of a vertical helicoid H converges to a minimal parking garage structure that consists of the minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes with singular set of convergence $S(\mathcal{L})$ being the x_3-axis.

We remark that some of the language associated to minimal parking garage structures, such as columns, appeared first in a paper of Traizet and Weber [55]. In this paper, they use this structure to produce certain one-parameter families of complete, embedded minimal surfaces, which are obtained by analytically untwisting the limit minimal parking garage structure through an application of the implicit function theorem. Their work also indicates that up to homothety and rigid motion that there are only a countable number of possible limiting minimal parking garage structures with a finite number of columns.

Theorem 10.1 (Local Picture on the Scale of Topology) *Suppose M is a complete, embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold N. Then, there exists a sequence of points $p_n \in M$ and positive numbers $\varepsilon_n \to 0$ such that the following statements hold.*

1. *For all n, the component M_n of $\mathbb{B}_N(p_n, \varepsilon_n) \cap M$ that contains p_n is compact, with boundary $\partial M_n \subset \partial \mathbb{B}_N(p_n, \varepsilon_n)$.*

2. *Let $\lambda_n = 1/I_{M_n}(p_n)$, where I_{M_n} denotes the injectivity radius function of M restricted to M_n. Then, $\lambda_n I_{M_n} \geq 1 - \frac{1}{n+1}$ on M_n, and $\lim_{n\to\infty} \varepsilon_n \lambda_n = \infty$.*

3. *The metric balls $\lambda_n \mathbb{B}_N(p_n, \varepsilon_n)$ of radius $\lambda_n \varepsilon_n$ converge uniformly to $\mathbb{R}^3$ with its usual metric (so that we identify p_n with $\vec{0}$ for all n).*

Furthermore, one of the following three possibilities occurs.

4. *The surfaces $\lambda_n M_n$ have uniformly bounded curvature on compact subsets of $\mathbb{R}^3$ and there exists a connected, properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ with $\vec{0} \in M_\infty$, $I_{M_\infty} \geq 1$ and $I_{M_\infty}(\vec{0}) = 1$ here I_{M_∞} denotes the injectivity radius function of M_∞),*

such that for any $k \in \mathbb{N}$, the surfaces $\lambda_n M_n$ converge C^k on compact subsets of $\mathbb{R}^3$ to M_∞ with multiplicity one as $n \to \infty$.

5. *The surfaces $\lambda_n M_n$ converge*[4] *to a limiting minimal parking garage structure on $\mathbb{R}^3$, consisting of a foliation $\mathcal{L}$ by planes with columns based on a locally finite set $S(\mathcal{L})$ of lines orthogonal to the planes in $\mathcal{L}$ (which is the singular set of convergence of $\lambda_n M_n$ to $\mathcal{L}$), and:*

   5.1 $S(\mathcal{L})$ contains a line L_1 which passes through the origin and another line L_2 at distance 1 from L_1.

   5.2 All of the lines in $S(\mathcal{L})$ have distance at least 1 from each other.

   5.3 If there exists a bound on the genus of the surfaces $\lambda_n M_n$, then $S(\mathcal{L})$ consists of just two components L_1, L_2 with associated limiting double multigraphs being oppositely handed.

6. *There exists a non-empty, closed set $\mathcal{S} \subset \mathbb{R}^3$ and a lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$ such that the surfaces $(\lambda_n M_n) - \mathcal{S}$ converge to $\mathcal{L}$ outside some singular set of convergence $S(\mathcal{L}) \subset \mathbb{R}^3 - \mathcal{S}$. Let $\Delta(\mathcal{L}) = \mathcal{S} \cup S(\mathcal{L})$. Then:*

   6.1 There exists $R_0 > 0$ such that sequence of surfaces $\{(\lambda_n M_n) \cap \mathbb{B}(\vec{0}, R_0)\}_n$ does not have bounded genus.

   6.2 The sublamination $\mathcal{P}$ of flat leaves in $\mathcal{L}$ is non-empty.

   6.3 The set $\Delta(\mathcal{L})$ is a closed set of $\mathbb{R}^3$ which is contained in the union of planes $\bigcup_{P \in \mathcal{P}} \overline{P}$. Furthermore, there are no planes in $\mathbb{R}^3 - \mathcal{L}$.

   6.4 If $P \in \mathcal{P}$, then the plane $\overline{P}$ intersects $\Delta(\mathcal{L})$ in an infinite set of points, which are at least distance 1 from each other in $\overline{P}$, and either $\overline{P} \cap \Delta(\mathcal{L}) \subset \mathcal{S}$ or $\overline{P} \cap \Delta(\mathcal{L}) \subset S(\mathcal{L})$.

The following corollary follows immediately from the above theorem, the Local Picture Theorem on the Scale of Curvature, and the fact that the set of properly embedded minimal surfaces in $\mathbb{R}^3$ with uniformly bounded Gaussian curvature and absolute Gaussian curvature 1 at the origin, forms a compact metric space; see the discussion at the beginning of section 8. By the regular neighborhood theorem in [34] or [53], the surfaces in this compact metric space all have cubical area growth CR^3 where $C > 0$ depends on the uniform bound of the curvature.

Corollary 10.2 *Suppose M is a complete, embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold N, and suppose M does not have a*

[4] This convergence must be understood similarly as those in Theorems **??** and **??** outside the singular set of convergence of $\lambda_n M_n$ to $\mathcal{L}$.

local picture on the scale of curvature which is a helicoid. Then every local picture on the scale of topology has curvature bounded by an uniform constant, and satisfies statement 4 in Theorem 10.1. Hence, the set of local pictures for M in statement 4 is compact, and there is a constant C such that every local picture on the scale of topology has area growth at most CR^3.

Recall that a non-flat minimal surface $f\colon M \to \mathbb{R}^3$ has zero flux if the integral of the conormal around any closed curve on M is zero. By the Weierstrass representation, such an M has non-zero flux if and only if $f\colon M \to \mathbb{R}^3$ is the unique isometric minimal immersion of M into $\mathbb{R}^3$ up to rigid motion. Since complete, embedded minimal surfaces in $\mathbb{R}^3$ with positive injectivity radius are properly embedded in $\mathbb{R}^3$ (see Theorem 10.5 below) and minimal surfaces which have local pictures corresponding to items *5* or *6* in the Local Picture Theorem on the Scale of Topology have non-zero flux, the next result is an immediate consequence of that theorem.

Corollary 10.3 *Suppose that M is a non-simply connected, complete, embedded minimal surface in $\mathbb{R}^3$ that has zero flux. Then there exists a properly embedded minimal surface of infinite genus, one end and zero flux.*

Remark 10.4 In [29] we will study further properties of singular laminations and are confident that we will prove that item *6* in Theorem 10.1 cannot occur; in other words, when the curvature functions of the surfaces $\lambda_n M_n$ become unbounded in a fixed compact set in $\mathbb{R}^3$, then item *5* must occur. Note that if M has finite genus or the sequence $\{\lambda_n M_n\}_n$ has uniformly bounded genus in fixed size small intrinsic metric balls, then item *6* does not occur, since *6.1* does not occur. This fact plays a crucial role in the proof of a bound on the number of ends for a complete, embedded minimal surface of finite topology in $\mathbb{R}^3$, that only depends on its genus, see [26]. Also in [29], we will apply Theorem 10.1 to give a general structure theorem for singular minimal laminations of $\mathbb{R}^3$ with a countable number of singularities.

The proof of Theorem 10.1 depends on the recent Minimal Lamination Closure Theorem by Meeks and Rosenberg [40], which we state below for the readers convenience.

Theorem 10.5 (Minimal Lamination Closure Theorem) *If M is a complete, embedded minimal surface with positive injectivity radius in a Riemannian three-manifold N, then the closure $\overline{M}$ of M has the structure of a $C^{1,\alpha}$-minimal lamination of N. Furthermore if N is $\mathbb{R}^3$ and M is not an infinite family of parallel planes, then M is properly embedded.*

In [40], Meeks and Rosenberg apply the above theorem, together with our Theorem 10.1, to prove that the closure of a complete, embedded minimal surface of finite

topology in a three-manifold N that is a product of a Riemannian surface Σ with $\mathbb{R}$ has the structure of a $C^{1,\alpha}$-minimal lamination. They then apply this result to prove that if Σ is a homogeneously regular surface of non-negative curvature which is not a flat torus, then every complete, embedded, connected minimal surface of finite topology in N must be properly embedded in N; if Σ is a flat torus, then they show that the only non-proper, complete minimal surfaces of finite topology in $\Sigma \times \mathbb{R}$ are totally geodesic. This last result generalizes a recent theorem of Colding and Minicozzi [3] who proved this result in the case $N = \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$. We will also need some of the recent results of Colding and Minicozzi contained in [10, 9, 4].

10.2 Parking garages and limiting parking garage structures on $\mathbb{R}^3$.

In order to understand the Local Picture Theorem on the Scale of Topology, we first need to develop the topological structure of a parking garage structure on $\mathbb{R}^3$ and relate this structure to how minimal surfaces converge to it.

In [55], Weber and Traizet produced an analytic method for constructing a one-parameter family of properly embedded, periodic minimal surfaces in $\mathbb{R}^3$, which in the limit are approximated by a finite number of regions on vertical helicoids in $\mathbb{R}^3$ that have been glued together in a consistent way. They referred to the limiting configuration as a parking garage structure on $\mathbb{R}^3$ with columns corresponding to the axes of the helicoids that they glue together. Most of the area of these surfaces, just before the limit, consists of very flat horizontal levels (almost horizontal densely packed horizontal planes) joined by the vertical helicoidal columns.

One can travel quickly up and down the horizontal levels of the limiting surfaces only along the helicoidal columns in much the same way that some parking garages are configured for traffic flow; hence, the name parking garage structure.

We now briefly describe the topological picture of a parking garage. Consider a possibly infinite, non-empty, locally finite set of points $P \subset \mathbb{R}^2$ and a related collection $\mathcal{D}$ of open round disks centered at the points of P such that the closures of these disks form a pairwise disjoint collection. Consider an onto representation $\sigma\colon H_1(\mathbb{R}^2 - \mathcal{D}) \to \mathbb{Z}$ such that σ takes the value of $+1$ or -1 on the homology classes represented by the boundary circles of the disks in $\mathcal{D}$. Let $\pi\colon M \to \mathbb{R}^2 - \mathcal{D}$ be the associated infinite cyclic covering space corresponding to the kernel of the composition of the natural map from $\pi_1(\mathbb{R}^2 - \mathcal{D})$ to $H_1(\mathbb{R}^2 - \mathcal{D})$ with σ. It is straightforward to embed M into $\mathbb{R}^3$ so that under the natural identification of $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\}$, the map π is the restriction to M of the orthogonal projection of $\mathbb{R}^3$ to $\mathbb{R}^2 \times \{0\}$. Furthermore, in this embedding, we may assume that the covering transformation of M corresponding to an $n \in \mathbb{Z}$ is given geometrically by translating M vertically by $(0, 0, n)$. In particular, M is a periodic surface with boundary in $\partial\mathcal{D} \times \mathbb{R}$. The surface M has exactly one boundary curve on each cylinder over the boundary circle of each disk in $\mathcal{D}$. We may assume that each of these boundary curves is

a helix.

Let $M(\frac{1}{2})$ be the vertical translation of M by $(0,0,\frac{1}{2})$ and note that $M \cup M(\frac{1}{2})$ is an embedded, disconnected periodic surface in $(\mathbb{R}^2 - \mathcal{D}) \times \mathbb{R}$ with a double helix on each boundary cylinder in $\partial\mathcal{D} \times \mathbb{R}$. The topological *parking garage* corresponding to the representation σ is now obtained by attaching to $M \cup M(\frac{1}{2})$ an infinite helicoidal strip in each of the solid cylinders in $\mathcal{D} \times \mathbb{R}$; by choosing M appropriately, the resulting surface G is smooth. The surface G gives the desired topological picture of a parking garage surface.

Since in minimal surface theory, we only see the parking garage structure in the limit, when the helicoidal strips in the cylinders of $\mathcal{D} \times \mathbb{R}$ become arbitrarily densely packed, it is useful in our construction of G to consider parking garages $G(t)$ invariant under translation by $(0,0,t)$ with $t \in (0,1]$ tending to zero. For $t \in (0,1]$, consider the affine transformation $F_t(x_1,x_2,x_3) = (x_1,x_2,tx_3)$. Then $G(t) = F_t(G)$. Note that our previously defined surface G is $G(1)$ in this new setup. As $t \to 0$, the $G(t)$ converge to the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes with singular set of convergence $S(\mathcal{L})$ consisting of the vertical lines in $P \times \mathbb{R}$. Also, note that M depends on the representation σ, so to be more specific, we could also denote $G(t)$ by $G(t,\sigma)$.

Finally, we remark on the topology of the ends of the periodic parking garage surface G in the case that P is a finite set, where $G = G(t,\sigma)$ for some t and σ. Suppose $\mathcal{D} = \{D_1,\ldots,D_n\}$. Then we associate to G an integer index:

$$I(G) = \sum_{i=1}^{n} \sigma([\partial D_i]) = k \in \mathbb{Z}.$$

Let $\overline{G}$ be the quotient orientable surface $\overline{G} = G/\mathbb{Z}$ in $\mathbb{R}^3/\mathbb{Z}$, where $\mathbb{Z}$ is generated by translation by $(0,0,t)$. The ends of $\overline{G}$ are annuli and there are exactly two of them. If $k = 0$, then these annular ends of $\overline{G}$ lift to graphical annular ends of G. If $k \neq 0$, then the universal cover of an end of $\overline{G}$ has $|k|$ orientation preserving lifts to G, each of which gives rise to an infinite multigraph over its projection to the end of $\mathbb{R}^2 \times \{0\}$. Finally, note that G has genus zero if and only if $n = 1$ and $k = \pm 1$ in which case G is simply connected, or $n = 2$ and $k = 0$ in which case G has an infinite number of annular ends with two limit ends (see the proof of Theorem 10.1 for a proof of this statement). Since G is periodic, then it has infinite genus precisely when $|k| > 1$ or $n > 2$ and in these cases it has exactly one end. To see why this last sentence holds, one argues as follows. Note that since $|k| > 1$ or $n > 2$, there exist at least two points $x_1, x_2 \in P$ with associated values $\sigma([\partial D_1]) = \sigma([\partial D_2])$ for the corresponding disks D_1, D_2 in $\mathcal{D}$ around x_1, x_2 (up to reindexing). Consider an embedded arc γ in $\mathbb{R}^2 - P$ joining x_1 to x_2. Then one can lift γ to consecutive levels of the parking garage G joined by short vertical segments on the columns over x_1 and x_2. Let $\widetilde{\gamma}$ denote this associated simple closed curve on G. Observe that if $\widetilde{\gamma}'$ is the related simple closed curve obtained by translating $\widetilde{\gamma}$ up exactly one level in G, then $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ have intersection number one. Thus, a small regular neighborhood of

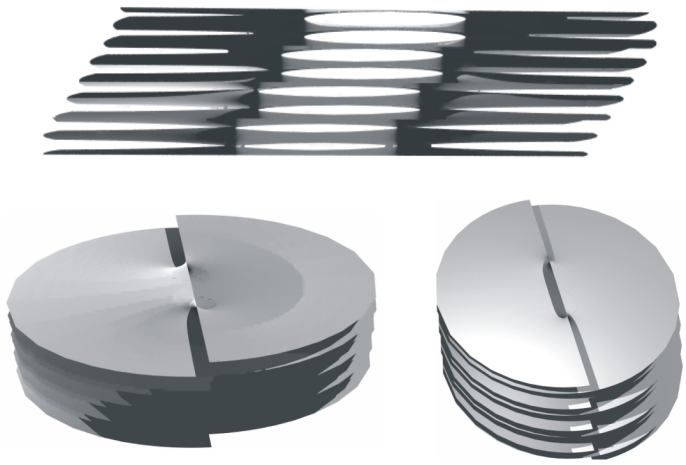

Figure 7: Three views of a minimal parking garage structure, constructed on a Riemann minimal example.

$\widetilde{\gamma} \cup \widetilde{\gamma}'$ on G has genus one. Since G is periodic, it has infinite genus.

The most famous example of a parking garage structure is obtained by taking the limit of homothetic shrinkings of a vertical helicoid and one obtains in this way the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes with a single column, or singular curve of convergence $S(\mathcal{L})$, being the x_3-axis.

There exists another well-known limiting *minimal* parking garage structure of $\mathbb{R}^3$ with two columns and with the columns oppositely oriented (corresponding to one right handed and one left handed helicoid), which is obtained as a limit of the classical periodic genus zero Riemann minimal examples $R_t, t \in (0, \infty)$, as the length of the horizontal flux component of R_t goes to infinity (see [30] for a proof of these properties). Each of these surfaces R_t has an infinite number of planar ends with two limit ends, and the limit parking garage structure has invariant $k = 0$, see Figure 7.

Finally, there is a well-known minimal parking garage structure of $\mathbb{R}^3$ with an infinite number of columns all of which are oriented the same way. This object can be obtained as a limit of the Scherk doubly-periodic minimal surfaces S_θ, $\theta \in (0, \frac{\pi}{2}]$ with lattice $\{((m+n)\cos\theta, (m-n)\sin\theta, 0) \mid m, n \in \mathbb{Z}\}$, as $\theta \to 0$. In this case, the surfaces converge to a foliation of $\mathbb{R}^3$ by planes parallel to the (x_1, x_3)-plane with columns of the same orientation being the horizontal lines parallel to the x_2-axis and passing through $\mathbb{Z} \times \{0\} \times \{0\}$. The parking garage S_θ for θ small is not recurrent for Brownian motion, but it is close to that condition, in the sense that it does not admit positive non-constant harmonic functions, see [33]. On the other hand, parking garage surfaces with a finite number of columns are recurrent for Brownian motion [33].

We refer the interested reader to [55] for further details and more examples of parking

garage structures that occur in minimal surface theory. We just note now that it also makes sense for a sequence of compact embedded minimal surfaces $M(n)$, with boundaries on the boundary of balls of radius n centered at the origin, to converge on compact subsets of $\mathbb{R}^3$ to a parking garage structure on $\mathbb{R}^3$ consisting of a foliation $\mathcal{L}$ of $\mathbb{R}^3$ by planes with a locally finite set of lines $S(\mathcal{L})$ orthogonal to the planes in $\mathcal{L}$, where $S(\mathcal{L})$ corresponds to the singular set of convergence of the $M(n)$ to $\mathcal{L}$. We note that each of the lines in $S(\mathcal{L})$ has an associated $+$ or $-$ sign corresponding to whether or not the associated forming helicoid along the line is right or left handed.

There are two other papers [25, 23] that clarify the notion of parking garage structures for the limit of a sequence of minimal surfaces in $\mathbb{R}^3$. Consider a sequence of compact embedded minimal surfaces $M(n)$ in $\mathbb{R}^3$ whose boundaries diverge in $\mathbb{R}^3$ and which are uniformly locally simply connected in the sense that for every point $p \in \mathbb{R}^3$, there exists an $\varepsilon > 0$ such that $B(p, \varepsilon) \cap M(n)$ consists of compact disks for n large. In such a case, the results of Colding and Minicozzi [10] show that a subsequence $M(n_i)$ of the $M(n)$ converges to a possibly singular minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with singular set of convergence $S(\mathcal{L})$. If $\mathcal{L}$ is non-singular and $S(\mathcal{L})$ is non-empty, then the curvature estimates of Colding and Minicozzi in [10] together with the regularity results of Meeks in [23, 25] show that $\mathcal{L}$ is a foliation of $\mathbb{R}^3$ by planes with $S(\mathcal{L})$ consisting of a locally finite collection of lines orthogonal to the planes of $\mathcal{L}$. In this case, one can then check (see Meeks [25]) that the subsequence $M(n_i)$ converging to $\mathcal{L}$ has, for n_i large, the appearance in compact subsets of $\mathbb{R}^3$ of highly sheeted helicoids along curves "parallel" and close to the lines in $S(\mathcal{L})$. Thus, one obtains a limiting minimal parking garage structure on $\mathbb{R}^3$ in this case.

10.3 The proof of the Local Picture Theorem on the Scale of Topology.

In this section we will describe the extrinsic geometry and topology of a complete, embedded minimal surface M in a homogenously regular three-manifold N, in a small intrinsic neighborhood of a point $p \in M$ where the injectivity radius of M is extremely small. We will prove Theorem 10.1 which shows that either M has the appearance of a properly embedded minimal surface (homothetically shrunk) in $\mathbb{R}^3$ near p, a limiting minimal parking garage structure of $\mathbb{R}^3$, or a special kind of singular minimal lamination of $\mathbb{R}^3$ (see the statement *6* of Theorem 10.1).

Let $M \subset N$ be a complete, embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold. After a fixed constant scaling of the metric of N, we may assume that the injectivity radius of N is greater than 1. The first step in the proof of Theorem 10.1 is to obtain special points $p'_n \in M$, called *points of concentrated topology.* First consider an arbitrary sequence of points $q_n \in M$ such that $I_M(q_n) \leq \frac{1}{n}$ (here I_M denotes the injectivity radius function of M), which exists since the injectivity radius of M is zero. Let $p'_n \in B_M(q_n, 1)$ be a maximum of $h_n = I_M^{-1} d_M(\cdot, \partial B_M(q_n, 1))$.

We define $\lambda'_n = I_M(p'_n)^{-1}$. Note that

$$\lambda'_n \geq \lambda'_n d_M(p'_n, \partial B_M(q_n, 1)) = h_n(p'_n) \geq h_n(q_n) = I_M(q_n)^{-1} \geq n.$$

Fix $t > 0$. Since $\lambda'_n \to \infty$ as $n \to \infty$, the sequence $\{\lambda'_n \mathbb{B}_N(p'_n, \frac{t}{\lambda'_n})\}_n$ converges to the ball $\mathbb{B}(t)$ of $\mathbb{R}^3$ with its usual metric, where we have used geodesic coordinates centered at p'_n and identified p'_n with $\vec{0}$. Similarly, we can consider $\{\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})\}_n$ to be a sequence of embedded minimal surfaces with boundary, all passing through $\vec{0}$ with injectivity radius 1 at this point. We claim that the injectivity radius function of $\lambda'_n M$ restricted to $\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})$ is greater that some positive constant. To see this, pick a point $z_n \in B_M(p'_n, \frac{t}{\lambda'_n})$. Since for n large enough, z_n belongs to $B_M(q_n, 1)$, we have

$$\frac{1}{\lambda'_n I_M(z_n)} = \frac{h_n(z_n)}{\lambda'_n d_M(z_n, \partial B_M(q_n, 1))} \leq \frac{d_M(p'_n, \partial B_M(q_n, 1))}{d_M(z_n, \partial B_M(q_n, 1))}. \tag{7}$$

By the triangle inequality, $d_M(p'_n, \partial B_M(q_n, 1)) \leq \frac{t}{\lambda'_n} + d_M(z_n, \partial B_M(q_n, 1))$, and so,

$$\begin{aligned} \frac{d_M(p'_n, \partial B_M(q_n, 1))}{d_M(z_n, \partial B_M(q_n, 1))} &\leq 1 + \frac{t}{\lambda'_n d_M(z_n, \partial B_M(q_n, 1))} \\ &\leq 1 + \frac{t}{\lambda'_n \left(d_M(p'_n, \partial B_M(q_n, 1)) - \frac{t}{\lambda'_n} \right)} \leq 1 + \frac{t}{n - t}, \end{aligned} \tag{8}$$

which tends to 1 as $n \to \infty$.

We now consider the following special case.

> $(*)$ *For every $t > 0$, the surfaces $\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})$ have uniformly bounded curvature.*

Under the above hypothesis, it follows that after extracting a subsequence, the $\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})$ converge smoothly to an embedded minimal surface $M_\infty(t)$ contained in $\mathbb{B}(t)$ with bounded curvature, that passes through $\vec{0}$. Consider the compact surface $M_\infty(1)$ together with the surfaces $\lambda'_n B_M(p'_n, \frac{1}{\lambda'_n})$ that converge to it (after passing to a subsequence). Note that $M_\infty(1)$ is contained in $M_\infty = \bigcup_{t \geq 1} M_\infty(t)$, which is a complete, injectively immersed minimal surface in $\mathbb{R}^3$.

We now remark on some properties of the minimal surface M_∞. By construction, the injectivity radius function of M_∞ has the value 1 at the origin. Since M_∞ has non-positive curvature, the exponential map $\exp_{\vec{0}} \colon T_{\vec{0}} M_\infty \to M_\infty$ is a submersion. Since the injectivity radius of M_∞ at $\vec{0}$ is 1, then $\exp_{\vec{0}}$ is injective on the open unit disk in $T_{\vec{0}} M_\infty$, but it is not injective on its boundary. Hence, there is an embedded geodesic loop γ in M_∞ of length 2 based at the origin (which is a limit of such loops of $\lambda'_n B_M(p'_n, \frac{2}{\lambda'_n})$). By the Gauss-Bonnet

formula, γ is homotopically nontrivial in M_∞. In particular, M_∞ is non-simply connected and so M_∞ is not a plane. Also note that M_∞ is the limit of surfaces with injectivity radius approaching 1 and so M_∞ has injectivity radius exactly 1.

Since $M_\infty \subset \mathbb{R}^3$ has positive injectivity radius, Theorem 10.5 implies that M_∞ is properly embedded in $\mathbb{R}^3$. It follows that for all $R > 0$, there exist $t > 0$ and $k \in \mathbb{N}$ such that if $m \geq k$, then the component of $\left[\lambda'_m B_M(p'_m, \frac{t}{\lambda'_m})\right] \cap \mathbb{B}(R)$ that passes through $\vec{0}$ is compact and has its boundary on $\mathbb{S}^2(R)$. Applying this property to $R_n = \sqrt{\lambda'_n}$, we obtain $t(n) > 0$ and $k(n) \in \mathbb{N}$ satisfying that if we let M_n denote the component of $B_M(p'_{k(n)}, \frac{t(n)}{\lambda'_{k(n)}}) \cap \mathbb{B}_N(p'_{k(n)}, \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}})$ that contains $p'_{k(n)}$, then M_n is compact and has its boundary on $\partial\mathbb{B}_N(p'_{k(n)}, \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}})$. Clearly, this compactness property remains valid if we increase the value of $k(n)$. Hence, we may assume without loss of generality that

$$t(n)(n+1) < k(n) \ \text{ for all } n, \qquad \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}} \to 0 \ \text{ as } n \to \infty.$$

We now define $p_n = p'_{k(n)}$, $\varepsilon_n = \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}}$ and $\lambda_n = \lambda'_{k(n)}$. Then in the case where our hypothesis $(*)$ holds, it is easy to check that the $p_n, \varepsilon_n, \lambda_n$ and M_n satisfy the first four conclusions stated in Theorem 10.1 (item 2 in the statement of Theorem 10.1 follows from equations (7) and (8) since

$$\frac{1}{\lambda_n(I_M)|_{M_n}} = \frac{1}{\lambda'_{k(n)}(I_M)|_{M_n}} \leq 1 + \frac{t(n)}{k(n) - t(n)} < 1 + \frac{1}{n},$$

where the last inequality follows from $t(n)(n+1) < k(n)$).

Suppose now that the hypothesis $(*)$ fails to hold. It follows, after extracting a subsequence, that for some fixed positive number $t_1 > 0$, the maximum absolute curvature of the surfaces $\lambda'_n B_M(p'_n, \frac{t_1}{\lambda'_n})$ diverges to infinity as $n \to \infty$.

Consider $\widetilde{M}(n) = \lambda'_n B_M(p'_n, \frac{t_n}{\lambda'_n})$ to be a subset of $\mathbb{R}^3$ with $p'_n = \vec{0}$, for an increasing sequence $t_n \to \infty$. After replacing this sequence $\{t_n\}_n$ by a sequence that goes to infinity more slowly, we may assume that $\widetilde{M}(n)$ has injectivity radius greater than $\frac{1}{2}$ at points of distance greater than $\frac{1}{2}$ from its boundary. Let $p \in \widetilde{M}(n)$ be a point such that $\overline{B}_{\widetilde{M}(n)}(p, \frac{1}{2}) \subset \widetilde{M}(n) - \partial\widetilde{M}(n)$. Note that $B_{\widetilde{M}(n)}(p, \frac{1}{2})$ is a disk. Letting $\Sigma = \overline{B}_{\widetilde{M}(n)}(p, \frac{1}{2})$ in the statement of Theorem 6 in [40], one obtains that there exist $\delta \in (0, \frac{1}{2})$ and $R_0 > 0$ (which we can assume to be less that $\frac{1}{2}$), both independent of n and p, such that if $B_\Sigma(x, R) \subset \Sigma - \partial\Sigma$ and $R \leq R_0$, then the component $\Sigma(x, \delta R)$ of $\Sigma \cap \mathbb{B}_{\lambda'_n N}(x, \delta R)$ passing through x satisfies $\Sigma(x, \delta R) \subset B_\Sigma(x, \frac{R}{2})$. Furthermore, $\Sigma(x, \delta R)$ is a compact embedded

minimal disk in $\mathbb{B}_{\lambda'_n N}(x, \delta R)$ with $\partial\Sigma(x, \delta R) \subset \partial\mathbb{B}_{\lambda'_n N}(x, \delta R)$. In particular, letting $x = p$ and $R = \frac{R_0}{2}$, one has that $\Sigma(p, \frac{\delta R_0}{2})$ is a compact embedded minimal disk in $\mathbb{B}_{\lambda'_n N}(p, \frac{\delta R_0}{2})$ with $\partial\Sigma(p, \frac{\delta R_0}{2}) \subset \partial\mathbb{B}_{\lambda'_n N}(p, \frac{\delta R_0}{2})$.

Choose a increasing divergent sequence of positive numbers $\{R(k)\}_k$ and let $\widetilde{M}(n, R(k))$ be the component of $\widetilde{M}(n) \cap \mathbb{B}(\vec{0}, R(k))$ that contains the origin. The proof of Proposition 10 in [40] (which is based on the proof of the similar Proposition 3.4 in [3]) shows that for every $k \in \mathbb{N}$, there exists an n_k such that for $n \geq n_k$, then $\widetilde{M}(n, R(k)) \subset \widetilde{M}(n) - \partial\widetilde{M}(n)$, and so, $\widetilde{M}(n, R(k))$ has its boundary in $\partial\mathbb{B}(\vec{0}, R(k))$. It follows that we can redefine $\{R(k)\}_k$ to be a increasing sequence with $\lim_{k\to\infty} R(k) = \infty$ and, for every $k \in \mathbb{N}$, $\partial\widetilde{M}(k, R(k)) \subset \mathbb{B}(\vec{0}, R(k))$.

Recall that a sequence of embedded compact minimal surfaces Σ_n in $\mathbb{R}^3$ with boundaries diverging in space, is uniformly locally simply connected if there is an $\varepsilon > 0$ such that for any ball of radius $\varepsilon > 0$ and for n sufficiently large, that ball intersects Σ_n in simply connected components. By the discussion above (see also Theorem 6 in [36] and Proposition 1.1 in [3]), the sequence of minimal surfaces $\widetilde{M}(n, R(n))$ can be considered to be uniformly locally simply connected (the metric balls containing the surfaces are converging to $\mathbb{R}^3$ with the usual metric). By Colding and Minicozzi [7, 8, 10], after replacing by a subsequence, $\{\widetilde{M}(n, R(n))\}_n$ converges on compact subsets of $\mathbb{R}^3$ to a possibly singular minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$. We now briefly discuss the local geometry of $\mathcal{L}$.

Let us denote by $\mathcal{S}$ the singular set of the lamination (i.e. $\mathcal{S}$ is the smallest closed set such that $\mathcal{L}$ is a regular lamination of $\mathbb{R}^3 - \mathcal{S}$), and let $S(\mathcal{L}) \subset \mathbb{R}^3 - \mathcal{S}$ be the singular set of convergence of $\widetilde{M}(n, R(n)) - \mathcal{S}$ to $\mathcal{L}$ (note that the leaves of $\mathcal{L}$ extend through $S(\mathcal{L})$, but not across $\mathcal{S}$). Let $\Delta(\mathcal{L}) = \mathcal{S} \cup S(\mathcal{L})$. Around each point $p \in \Delta(\mathcal{L})$, there exists a small ball $\mathbb{B}(p, \varepsilon)$ and a unique leaf $D(p, *)$ of $\mathcal{L} \cap \mathbb{B}(p, \varepsilon)$, whose closure $D(p)$ is a smooth compact minimal disk passing through p with $\partial D(p) \subset \partial\mathbb{B}(p, \varepsilon)$ and which is a small graph over its projection to $T_pD(p)$. If $p \in S(\mathcal{L})$, then there is a neighborhood of p in $\mathbb{B}(p, \varepsilon)$ foliated by similar disks and $S(\mathcal{L})$ intersects this family of disks transversely in a connected Lipschitz curve, which is in fact a $C^{1,1}$-curve orthogonal to the local foliation ([23, 25]). If $p \in \mathcal{S}$, then there exists a double cone $C_p \subset \mathbb{B}(p, \varepsilon)$ with vertex at p and axis orthogonal to $D(p)$ such that in the non-simply connected component $C_p(*)$ of $B(p, \varepsilon) - C_p$, then $D(p, *) \subset C_p(*)$, $C_p(*) \cap \Delta(\mathcal{L}) = \emptyset$ and $C_p(*) \cap \mathcal{L}$ consists of graphs and a positive number of pairs of multigraphs over $D(p, *)$ of small gradient which converge to $D(p, \varepsilon)$ from both sides; if $p \in \mathcal{S}$ is an isolated point, then for ε small, one has one pair of multigraphs and no annular graphs. It follows by a standard blowing up argument and the $C^{1,1}$-regularity theorem for the singular set of a Colding-Minicozzi minimal lamination [23] that the cone C_p can be chosen arbitrarily narrow by choosing ε arbitrarily small. We refer the reader to the papers in [8, 10] for further details on this local structure of $\mathcal{L}$ near a point $p \in \Delta(\mathcal{L})$ and for the related double multigraph picture for $\widetilde{M}(n, R(n)) \cap \mathbb{B}(p, \varepsilon)$ for n large.

Since the collection of surfaces $\widetilde{M}(n,t_1)$ have unbounded curvature and are contained in $\widetilde{M}(n,R(n))$ for $R(n)>t_1$, $\Delta(\mathcal{L})$ is non-empty and contains a point an extrinsic distance at most t_1 from the origin in $\mathbb{R}^3$.

From Colding and Minicozzi [7, 8, 10] or directly from the above local description, it follows that through each point $x\in\Delta(\mathcal{L})$ there passes a smooth limit leaf of $\mathcal{L}$ whose closure P_x is a complete minimal surface which contains the disk $D(x)$ and whose universal cover is stable. So P_x is a plane which we will assume is horizontal. Let

$$\mathcal{P}=\{P_x \mid x\in\Delta(\mathcal{L})\}.$$

Note that $\Delta(\mathcal{L})$ is a closed set in $\mathbb{R}^3$ and curvature estimates in [10] for minimal disks imply that a limit plane P of planes in $\mathcal{P}$ passes through $\Delta(\mathcal{L})$, and so, P is in $\mathcal{P}$. Therefore, the union of the planes in $\mathcal{P}$ is a closed subset of $\mathbb{R}^3$.

We claim that every plane in $\mathcal{L}$ lies in $\mathcal{P}$ and no plane in $\mathbb{R}^3-\mathcal{L}$ is disjoint from $\Delta(\mathcal{L})$. Suppose not and let P be such a plane. Since P does intersect $\Delta(\mathcal{L})$, the curvature estimates of Colding-Minicozzi (the leaves are uniformly locally simply connected) imply that a fixed size (size is independent of P) regular neighborhood $N(P)$ of the plane is disjoint from $\Delta(\mathcal{L})$ and $\mathcal{L}\cap N(P)$ has bounded curvature. From this bounded curvature hypothesis, a straightforward application of the proof of the Half-space Theorem or the proof of Lemma 1.3 in [39] implies that $N(P)\cap\mathcal{L}$ consists only of planes of $\mathcal{L}$. Let $\mathcal{L}'$ be the related singular minimal lamination obtained by enlarging $\mathcal{L}$ by adding to it all planes which are disjoint from it. Note that by curvature estimates in [10], each of these added on planes is a fixed minimal distance from $\Delta(\mathcal{L})$ and from any non-flat leaf of $\mathcal{L}$. Hence, the planes of $\mathcal{L}'$ which are not in $\mathcal{P}$ form a both open and closed subset of $\mathbb{R}^3$, but $\mathbb{R}^3$ is connected. Hence, this set is empty, which proves our claim.

Since the sequence $\{\widetilde{M}(n,R(n))\}_n$ is uniformly locally simply connected, the results in [10] imply that there exists an $\eta>0$ so that the distance between any two points of $P_x\cap\Delta(\mathcal{L})$ is a least η for all $x\in\Delta(\mathcal{L})$. Using the plane P_x as a guide, in the case $x_1,x_2\in P_x\cap\Delta(\mathcal{L})$ are distinct, one can produce homotopically nontrivial simple closed curves on the approximating surfaces of lengths converging to twice the distance between x_1 and x_2 (see for example [31]). Since this curve can be chosen to be a closed geodesic, our injectivity radius assumption implies that the spacing between x_1 and x_2 is at least 1.

Assume now that for some $x\in\Delta(\mathcal{L})$, the set $P_x\cap\Delta(\mathcal{L})$ is finite. Let D_x be a large round disk containing the set $P_x\cap\Delta(\mathcal{L})$ in its interior. In a neighborhood of each point $y\in P_x\cap\Delta(\mathcal{L})$ and outside a double vertical cone based at y, there are two multigraphs contained in the surfaces $\widetilde{M}(n,R(n))$, which, after choosing a subsequence, are always right or left handed (depending on y). Assign a number $n(y)=\pm1$ depending on whether the multigraphs are right or left handed. Let

$$I(x)=\sum_{y\in P_x\cap\Delta(\mathcal{L})} n(y),\ \text{and for } w\in\Delta(\mathcal{L}),\ \text{let } |I|(w)=\sum_{y\in P_w\cap\Delta(\mathcal{L})}|n(y)|.$$

Note that $|I|(w)$ takes the value of ∞ if and only if the number of point in $P_w \cap \Delta(\mathcal{L})$ is infinite. If $|I(w)| < \infty$ for a given $w \in \Delta(\mathcal{L})$, then $I(w) = \sum_{y \in P_w \cap \Delta(\mathcal{L})} n(y)$ makes sense.

We claim that $\mathcal{L}$ is a foliation by horizontal planes. If not, then there is a point $z \in \Delta(\mathcal{L})$ such that P_z is not a limit of planes in $\mathcal{P}$ at one side of P_z; suppose this side is the upside. Note that $\Delta(\mathcal{L}) \cap P_x = \mathcal{S} \cap P_x$. By curvature estimates for minimal disks given in [10], the number of points in $P_w \cap \Delta(\mathcal{L})$ is a locally constant function of $w \in \Delta(\mathcal{L})$. Since $|I|(w)$ and $I(w)$ are finite locally constant functions near any $v \in S(\mathcal{L})$ with $|I|(v) < \infty$, we can choose z so that $|I|(z) = |I|(x)$ and $I(z) = I(x)$. After replacing x by z, we may assume that $P_x \cap \Delta(\mathcal{L})$ is a finite set and P_x is not a limit of planes in $\mathcal{P}$ above P_x.

First suppose that $I(x) = 0$ and L is the leaf of $\mathcal{L}$ above P_x which has P_x in its limit set. In this case a flux argument gives a contradiction to the invariance of flux for ∇x_3. We now give this flux argument. Without loss of generality, assume $x = \vec{0}$. In this case $A = P_x - D_x$ is contained in the (x_1, x_2)-plane and let $R = A \times [0, \varepsilon]$ for some small $\varepsilon > 0$. By the curvature estimates in [10], we may assume that the curvature of the leaves in $R \cap L$ is almost zero. It then follows that each component G of $R \cap L$ is a graph over its projection to A with boundary contained in $(\partial A \times [0, \varepsilon]) \cup (A \times \{\varepsilon\})$. Let $\{G_1, G_2, \ldots, G_n, \ldots\}$ denote the set of these graphical components whose boundary contains a simple closed curve on the cylinder $\partial A \times [0, \varepsilon]$. Let $\partial_n \subset \partial G_n \cap (\partial A \times [0, \varepsilon])$ denote the corresponding closed curve. We may assume that each ∂_n is a graph over the circle $\partial A \times \{0\}$ and that the sequence $\{\partial_n\}_n$ is ordered decreasingly by their relative heights, converging to $\partial A \times \{0\}$ as $n \to \infty$.

Since graphs are proper, each G_n is properly embedded in $A \times [0, \varepsilon]$, and so, each G_n is a parabolic surface [12]. Since the inner product of ∇x_3 with the outward conormal vector to G_n along $G_n \cap (A \times \{\varepsilon\})$ is non-negative, then the Algebraic Flux Lemma in [24] implies that the flux of ∇x_3 across $\partial_n \subset \partial G_n$ is non-positive and negative if $\partial G_n \cap (A \times \{\varepsilon\}) \neq \emptyset$. Since $I(x) = 0$, then for some $k \in \mathbb{N}$, there exist a finite number k of pairs of points p_i, q_i such that $P_x \cap \Delta(\mathcal{L}) = P_x \cap \mathcal{S} = \{p_1, q_1, p_2, q_2, \ldots, p_k, q_k\}$ and where $n(p_i) = -n(q_i)$ for all $i, 1 \leq i \leq k$. Construct a collection $\{\delta_1, \ldots, \delta_k\}$ of pairwise disjoint simple closed embedded arcs in P_x with the end points of δ_i equal to p_i, q_i and the interior of each arc δ_i is disjoint from $\Delta(\mathcal{L})$. It is straightforward to construct a related sequence $\{\gamma_i(m)\}_{m \in \mathbb{N}}$ of simple closed curves in L consisting of lifts of δ_i to adjacent sheets of L over δ_i joined by short arcs of length ε_i near p_i and q_i, which converge with multiplicity 2 to δ_i and $\varepsilon_i \to 0$ as $i \to \infty$ (the existence of the short connecting arcs follows easily from the techniques and results in [31]). Being careful in choosing the indexing of these curves $\gamma_i(m)$, we obtain a collection $\Gamma_m = \{\gamma_1(m), \gamma_2(m), \ldots, \gamma_k(m)\}$, all at about the same level in the parking garage structure of L near P_x, which separates $L_\varepsilon = L \cap \{(x_1, x_2, x_3) \mid 0 < x_3 \leq \varepsilon\}$ and forms the boundary of the subregion $L(m)$ of L_ε that contains only a finite number of curves in $\{\partial_n\}_n$. By the Algebraic Flux Lemma, the flux of ∇x_3 across $\Gamma_m \subset \partial L(m)$ is negative and equal in absolute value to the flux of ∇x_3 across $\partial L(m) \cap \{x_3 = \varepsilon\}$. Since for $k \in \mathbb{N}$, $L(m) \subset L(m+k)$, the absolute value of the flux of ∇x_3 across $\Gamma_m \subset \partial L(m)$ is

positive and non-decreasing as $m \to \infty$. But, by construction, the integral of $|\nabla x_3|$ along Γ_m converges to zero as $m \to \infty$, which gives the desired contradiction.

Assume now that $I(x) \neq 0$. Then, there exist a finite positive number of multigraphs in $\mathcal{L}$ in the region just above $R = P_x - D_x$ (the number of these multigraphs is equal to $|2I(x)|$ by our discussion in Subsection 10.2). In this case, the argument used in [10] to produce a limit foliation of $\mathbb{R}^3$ by planes can be applied to obtain a contradiction; one shows that the sum of the angular fluxes of ∇x_3 of the multigraphs in $\mathcal{L}$ spiraling into R is uniformly bounded for any compact range of angles. We now give this argument.

Suppose $n = I(x) \in \mathbb{N}$. After rearrangement, we may assume that $P_x \cap \Delta(\mathcal{L}) = P_x \cap \mathcal{S} = \{p_1, q_1, p_2, q_2, \ldots, p_k, q_k, s_1, s_2, \ldots, s_n\}$, where $k \geq 0$, $n(p_i) = -n(q_i)$ and $n(s_i) = 1$ for $i \in \{1, 2, \ldots, n\}$. As in the previously considered case, for the pairs of points in $\{p_1, q_1, \ldots, p_k, q_k\}$, we can find connection loops $\Gamma(m) = \{\alpha_1(m), \alpha_2(m), \ldots, \alpha_k(m)\}$ at an approximate level $m \in \mathbb{N}$ in the surface $\Sigma = (D_x \times (0, \varepsilon]) \cap \mathcal{L}$ for some small $\varepsilon > 0$; this surface has the appearance of a parking garage surface with $2k + n$ columns for ε small and positive. Next choose pairwise disjoint arcs $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in D_x with $\partial\alpha_i \subset \partial D_x$ and $s_i \in \alpha_i$ for $i \in \{1, 2, \ldots, n\}$. From the parking garage structure on Σ, we can lift these arcs to obtain a collection $\Lambda(m) = \{\alpha_1(m), \alpha_2(m), \ldots, \alpha_n(m)\}$ of uniformly bounded length arcs in Σ at an approximate level $m \in \mathbb{N}$ and such that for each $m > 1$, $\Gamma(1) \cup \Lambda(1)\Gamma(m) \cup \Lambda(m)$ bounds a compact subdomain $\Sigma(k)$ of Σ with boundary $\Gamma(1) \cup \Lambda(1) \cup \Gamma(m) \cup \Lambda(m)$ together with $2n$ spiraling arcs, each making about m turns on the cylinder $\partial D_x \times (0, \varepsilon]$.

Suppose $\{G(1), \ldots, G(n)\}$ in $\mathcal{L} \cap (R \times (0, \varepsilon])$ are the corresponding multigraphs . Without loss of generality, we may assume that $\partial R = \mathbb{S}^1(1)$ and the boundary of each multigraph $G(i)$ contains a connected arc $\Gamma(i)$ in $\partial\Sigma$. Using natural polar coordinates (r, θ) with $r \in [1, \infty)$ and $\theta \in [0, \infty)$, we can parametrize these multigraphs as graphs $G_i = G_i(r, \theta)$ over their projection onto the domain $[1, \infty) \times [\beta_i, \infty)$, for a certain $\beta_i \in [0, 2\pi)$ where $[\Gamma(i)](\beta_i)$ is the end point of $\Gamma(i)$ at height ε.

Assume that each $\Gamma(i)$ is right-handed so it is parameterized by $G_i(1, \theta)$ for $\theta \in [\beta_i, \infty)$. Given $t \in [0, \infty)$, let $\Gamma(i, t)$ to be the arc $G_i(\{1\} \times [2\pi, 2\pi + t])$ on $\Gamma(i)$. Let $F_i(t) = \int_{\Gamma(i,t)} \frac{\partial x_3}{\partial \eta} ds$ be the flux of x_3 along $\Gamma(i, t)$ with the unit conormal η that points into the solid cylinder $\{x_1^2 + x_2^2 \leq 1\}$. Consider the set of numbers $\mathcal{F} = \{F(t) = \sum_{i=1}^k F_i(t) \mid t \in [0, \infty)\}$. Since the multigraph $G_{\Gamma(i)}$ coming out from any $\Gamma(i)$ is an ∞-valued positive multigraph over the annulus $\{1 \leq x_1^2 + x_2^2\}$, the proof of Theorem 0.6 in [5] implies that the set $\mathcal{F}$ cannot be bounded (also see the discussion after the statement of Corollary 0.7 in [5]). We now check that $\mathcal{F}$ is in fact a bounded set.

We will prove that $\mathcal{F}$ is bounded as a consequence of the Divergence Theorem, applied to the field ∇x_3 on the compact minimal surfaces $\Sigma(k)$ contained in L, similar to the application of Lemma 4.1 in [5]. Since the lengths of the part of $\partial\Sigma(k)$ which does not consist of spirals is bounded is uniformly bounded, the Divergence Theorem applied to

∇x_3 gives the boundedness of $\mathcal{F}$. This contradiction completes our proof that in the case we are considering, where some plane P_x intersects $\Delta(\mathcal{L})$ in a finite set of points, then $\mathcal{L}$ is the foliation of $\mathbb{R}^3$ by horizontal planes.

Now suppose $\mathcal{L}$ is a foliation of $\mathbb{R}^3$ by planes. By the curvature estimates in [10], $S(\mathcal{L})$ consists of Lipschitz curves transverse to $\mathcal{L}$ with one passing through each point in $P_x \cap S(\mathcal{L})$. By the $C^{1,1}$-regularity theorem for $S(\mathcal{L})$ in [23], $S(\mathcal{L})$ consists of vertical lines over $P_x \cap S(\mathcal{L})$. The surfaces $M(n, R(n))$ are now seen to converge on compact subsets of $\mathbb{R}^3$ to the limiting minimal parking garage structure on $\mathbb{R}^3$ consisting of horizontal planes and with vertical columns over the points $y \in P_x \cap S(\mathcal{L})$ with orientation numbers $n(y) = \pm 1$. .

Note that a subsequence of the geodesics loops of length 2 in $\widetilde{M}(n, R(n))$ based at the origin must converge to a straight line segment of length 1 on the horizontal plane passing through the origin with end points in $S(\mathcal{L})$. Hence, $S(\mathcal{L})$ has at least two components with distance between them equal to 1.

Now assume that for every fixed $t > 1$, the surfaces $\widetilde{M}(n, t)$ have uniformly bounded genus. We claim that $P_x \cap S(\mathcal{L})$ consists of exactly two points x_1, x_2 and these points satisfy $n(x_1) = 1$ and $n(x_2) = -1$. Note that we have already shown that $P_x \cap S(\mathcal{L})$ contains at least two points.

Were $P_x \cap S(\mathcal{L})$ to contain more than two points, then two such points x_1, x_2 would have the same sign: $n(x_1) = n(x_2)$. Thus, it suffices to prove x_1 and x_2 cannot have the same sign (when the genus of the surfaces $\widetilde{M}(n, t)$ are uniformly bounded for each t).

If the sign of x_1 and x_2 were the same, then consider an embedded arc γ in $P_x - S(\mathcal{L})$ joining x_1 to x_2. It is not difficult to prove that in a small neighborhood $N(\gamma)$ of γ in $\mathbb{R}^3$, the limiting surfaces have unbounded genus. The reason for this is that one can produce simple closed curves on the approximating surfaces which consist of two lifts of γ to consecutive levels of the forming parking garage structure joined by short vertical arcs on the columns over x_1 and x_2. Let $\widetilde{\gamma}$ denote this associated simple closed curve. Observe that if $\widetilde{\gamma}'$ is the related simple closed curve obtained by translating $\widetilde{\gamma}$ up one level in the parking garage structure, then $\widetilde{\gamma}'$ and $\widetilde{\gamma}$ have intersection number one. Thus, a small regular neighborhood of $\widetilde{\gamma} \cup \widetilde{\gamma}'$ on the approximating surface has genus one. Since the limiting minimal parking garage structure is essentially periodic under smaller and smaller vertical translations, we obtain a contradiction to our bounded genus hypothesis. In fact, for n large, using the forming parking garage structure and the minimal lamination metric theorem in [25], the curves $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ (and an arbitrary large number of "translated pairs") can be shown to lie in $\widetilde{M}(n, t)$ for $t = 5(|x_1| + |x_2|)$. This contradiction proves our claim that $P_x \cap S(\mathcal{L})$ consists of exactly two points x_1, x_2 and these points satisfy $n(x_1) = 1$ and $n(x_2) = -1$.

So far, we have shown that most of the consequences stated in the theorem hold. In choosing the required points p'_n, we make choices so that hypothesis $(*)$ always holds for

the sequence or always fails to hold. We choose the sequence of points so that the related surfaces converge to one of the types of limit objects that we are interested in: a properly embedded surface when $(*)$ holds, or otherwise, a minimal parking garage structure or a special singular minimal lamination. In the case, that $(*)$ holds we already defined the required data for p_n, ε_n and M_n. If the sequence of surfaces converges to a singular minimal lamination, then we can define the points p_n to be the new points p'_n, $\lambda_n = \lambda'_n$ and $\varepsilon_n = \frac{R(n)}{\lambda_n}$. In the case the surfaces $M(n, R(n))$ converge to a minimal parking garage structure, then we can also choose p_n to be certain points on the short geodesic loops based at p'_n, which lie a fixed distance at most $\frac{1}{\lambda'_n}$ from the points p'_n, and so that the other special properties in statement *5* of the theorem hold. This completes the proof of Theorem 10.1.

Remark 10.6 Our techniques used to prove Theorem 10.1 have other consequences. For example, suppose $\{M_n\}_n$ is a sequence of compact embedded minimal surfaces with $\vec{0} \in M_n$ whose boundaries lie in the boundaries of balls $\mathbb{B}(R_n)$, where $R_n \to \infty$. Suppose that there exists some $\varepsilon > 0$ such that for any ball B in $\mathbb{R}^3$ of radius ε, for n sufficiently large, $M_n \cap B$ consists of disks (this condition on the sequence of surfaces is called *uniformly locally simply connected* following Colding-Minicozzi [10]), and such that for some fixed compact set C, there exists a $d > 0$ such that for n large injectivity radius function of M_n is at most d at some point of $M_n \cap C$. Then the proof of Theorem 10.1 shows that, after replacing by a subsequence, there exists a sequence $\{R'_n\}_n$ with $R'_n \to \infty$ and $R'_n \leq R_n$ such that if we let M'_n denote the component of $M_n \cap \mathbb{B}(R'_n)$ containing $\vec{0}$, then the M'_n converge on compact subsets of $\mathbb{R}^3$ to one of the following:

1. A properly embedded, non-simply connected minimal surface M in $\mathbb{R}^3$. In this case, the convergence of the surfaces to M is smooth of multiplicity one on compact sets.

2. A minimal parking garage structure of $\mathbb{R}^3$ with at least two columns.

3. A singular minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with properties similar to the minimal lamination described in item *6* of Theorem 10.1 (Also see the related statement 6 of Theorem 1.5 in [29].).

11 Embedded minimal surfaces of finite genus in homogeneous three-manifolds.

We now apply the Local Picture Theorem on the Scale of Topology from the previous section to prove that the closure of a complete, embedded minimal surface of finite topology in a locally homogeneous three-manifold has the structure of a minimal lamination. As one application of this theorem, we prove that with respect to the standard constant curvature

metric on the three-sphere $\mathbb{S}^3$, any complete, embedded minimal surface of finite genus and a countable number of ends must be compact. Note that each of the eight Thurston geometries for compact three-three manifolds are represented by the geometries described in this theorem but also many other homogeneous geometries can arise for the same manifold. For example, any three-dimensional Lie group equipped with a left invariant metric is a complete homogenous three-manifolds.

Theorem 11.1 *Let N be a complete, connected, locally homogeneous three-manifold and let $M \subset N$ be a complete, embedded minimal surface of finite topology. Then:*

1. *The closure $\overline{M}$ has the structure of a minimal lamination of N with M as a leaf. In particular, if N is compact, then M has bounded curvature.*

2. *Suppose N has its geometry modeled on $\mathbb{S}^3$ with a locally homogenous metric of non-negative scalar curvature. Then M is compact. In fact, if $\Sigma \subset N$ is a complete, embedded minimal surface, then Σ cannot have any annular ends; in particular, if Σ has finite genus and a countable number of ends, then it is compact.*

3. *Suppose that N has its geometry modeled on $\mathbb{S}^2 \times \mathbb{R}$ with a metric of constant positive curvature on $\mathbb{S}^2$. Then M is a totally geodesic sphere or a projective plane, or M lifts to a properly embedded in $\widetilde{N} = \mathbb{S} \times \mathbb{R}$ and this lift has two annular ends, linear area growth and bounded curvature. More generally, if $\Sigma \subset \mathbb{S}^2 \times \mathbb{R}$ is a complete, embedded minimal surface, then each annular end of Σ is properly embedded and Σ has at most two ends which admit representatives with compact boundary and which are properly embedded.*

4. *Suppose that N is modeled on the flat Euclidean geometry $\mathbb{E}^3$. If either N is non-simply connected or if M has more than one end, then M has bounded curvature and finite total curvature. If $N = \mathbb{E}^3$ and M has one end, then M is asymptotic to a helicoid.*

In [40], Meeks and Rosenberg prove that the closure of any complete, embedded minimal surface in a complete three-manifold with non-positive sectional curvature has the structure of a minimal lamination (their proof of this theorem also uses our Local Picture Theorem on the Scale of Topology from the previous section). Also, the additional properties described in items *3* and *4* of the above theorem depend on previous results of Meeks and Rosenberg [35, 37, 39, 40] and of Collin [11].

We now prove Theorem 11.1. Let $\widetilde{N}$ be a complete, three-dimensional, simply connected, homogenous Riemannian manifold which is the universal cover of N. Note that in this case, the identity component $I(\widetilde{N})$ of the isometry group of $\widetilde{N}$ acts transitively on $\widetilde{N}$. It can be shown that the one-parameter subgroups generate space $\mathcal{K}$ of killing fields of dimension at least three. In [31] we prove that $\mathcal{K}$ on $\widetilde{N}$ satisfies the following key property,

which is easy to verify in any of the more familiar eight Thurston geometries which occur for compact three-manifolds:

Killing field property: There exists a positive constant $C(\widetilde{N})$ such that for any 2 points $p, q \in \widetilde{N}$ and unit tangent vector $v_p \in T_p(\widetilde{N})$, there exists a killing field $\mathbf{Y} \in \mathcal{K}$ such that $\mathbf{Y}(p) = v_p$ and $|\mathbf{Y}(q)| \leq C(\widetilde{N})$.

Suppose $M \subset N$ satisfies the hypotheses of the theorem. If M is compact, then $\overline{M} = M$ clearly has the structure of a minimal lamination. Assume now that M has a finite positive number of pairwise disjoint annular ends representatives $\{E_2, E_2, \ldots, E_n\}$. The closure $\overline{M}$ has the structure of a minimal lamination if and only if M has locally bounded curvature in the sense that M has bounded on compact subsets of N. On the other hand, by the results in [40], the failure of M to have locally bounded curvature in N is equivalent to the property that there exists a point $p \in N$ and a sequence of points $\{p_n\}_n \subset M$ converging to p and such that $I_M(p_n) \to 0$ as $n \to \infty$, where I_M is the injectivity radius function on M. Let M_∞ be a local picture for M arising from this sequence of points and given by the Local Picture Theorem on the Scale of Topology, which is Theorem 10.1. We will assume that the points p_n are the blow-up points p_n in the statement of this theorem.

Since in a sufficiently small ambient ball B centered at p, components of $M \cap B$ have genus zero, Theorem 10.1 implies that M_∞ is a properly embedded, minimal planar domain in $\mathbb{R}^3$ or M_∞ is a minimal parking garage structure on $\mathbb{R}^3$ with two columns. Recall that M_∞ arises from approximating "local pictures" $M_n \subset M$ and, without loss of generality, we may assume $M_n \subset E_1$. Note that if M_∞ is a properly embedded minimal surface in $\mathbb{R}^3$, then M_∞ has non-zero flux. In all cases, including the case where M_∞ is a minimal parking garage structure on $\mathbb{R}^3$, there exist oriented, simple closed curves $\gamma_n \subset M_n$ with $p_n \in \gamma_n$ with lengths L_n converging to zero, which have non-trivial flux vectors F_n (which are the integrals of the unit "conormals" in a fixed local coordinate system around p), and the sequence of normalized fluxes $\frac{F_n}{I_M(p_n)}$ converges to the finite non-zero flux F_∞ of the limit curve $\gamma \subset M_\infty$.

After a fixed rotation of coordinates, we first consider the case where M_∞ is a properly embedded minimal surface in $\mathbb{R}^3$ with horizontal limit tangent plane at infinity or M_∞ corresponds to a minimal parking garage structure with a foliation by horizontal planes. By Theorem*** in [32], the flux vector F_∞ is only vertical if M_∞ is a catenoid and if M_∞ is a surface, then we can assume that γ_n is the intersection of a "horizontal plane" with M_n and bounds a "horizontal" disk G_n in this plane. If M_∞ is a minimal parking garage structure, γ_n is a connection loop which bounds an embedded disk G_n whose area is much smaller that the length of its flux vector F_n.

After choosing a subsequence, we may assume one of the following two distinct cases occurs:

Case A: The curves $\gamma_n \subset E_1$ bound disks $D_n \subset E_1$.

Case B: The curves γ_n are homotopically non-trivial on E_1 and so, after orienting these curves, for each $n, k \in \mathbb{N}$, $\gamma_n - \gamma_{n+k}$ is the homological boundary of a compact annulus $A(n,k) \subset E_1$.

First note that in either of the above two cases, we may assume that the lengths of the curves γ_n go to zero as $n \to \infty$, and so the domains D_n and E_1 lift to domains $\widetilde{D}_n$ or $\widetilde{E_1}$ in the universal cover $\widetilde{N}$. In **Case B** let $\widetilde{A}(n,k)$ denote the lifts of the domains $A(n,k)$. After isometries which are given by a fixed left translations in the group, we may assume that these lifts lift the point p to some fixed lift $\widetilde{p}$ in $\widetilde{N}$. For these lifts, we may assume that the distance from the lifted points$\{\widetilde{p}_n\}_n$ of the surfaces to $\widetilde{p}$ is the same as the distances of p_n to p. Also let $\widetilde{G}_n$ denote the related lifts of the horizontal disks G_n. Finally, let U_n be the domain in $\widetilde{N}$ which bounds the cycle $\widetilde{D}_n \cup \widetilde{G}_n$ or $\widetilde{A}(n,k) \cup \widetilde{G}_n \cup \widetilde{G}_{n,k}$.

When $\mathbf{Y}$ is a smooth vector field on $\widetilde{N}$, write $\mathrm{DIV}\,\mathbf{Y}$ for its divergence on $\widetilde{N}$, and recall the relation to volume change as U_n deforms along the flow of $\mathbf{Y}$. In **Case A**, we compute:

$$\delta_{\mathbf{Y}}|U_n| = \int_{U_n} \mathrm{DIV}\ \mathbf{Y} = \int_{\partial U_n} \langle \nu, \mathbf{Y} \rangle = \int_{\widetilde{D}_n} \langle \nu, \mathbf{Y} \rangle + \int_{\widetilde{G}_n} \langle \nu, \mathbf{Y} \rangle,$$

where ν is the outward pointing unit normal to ∂U_n. In **Case B**, we compute:

$$\delta_{\mathbf{Y}}|U_n| = \int_{U_n} \mathrm{DIV}\ \mathbf{Y} = \int_{\partial U_n} \langle \nu, \mathbf{Y} \rangle = \int_{\widetilde{A}(n,k)} \langle \nu, \mathbf{Y} \rangle + \int_{\widetilde{G}_n} \langle \nu, \mathbf{Y} \rangle + \int_{\widetilde{G}_{k,n}} \langle \nu, \mathbf{Y} \rangle.$$

By our hypotheses $\widetilde{N}$ is unimodular, and so if $\mathbf{Y}$ is a left invariant vector field, then $\delta_{\mathbf{Y}}|U_n| = 0$.

In **Case A** we compute the variation of area of the lifted domains $\widetilde{D}_n$ with respect to the variation $\mathbf{Y}$, using the fact that $\mathrm{div}\ \mathbf{Y}^{\mathrm{N}} = 0$ by minimality:

$$\delta_{\mathbf{Y}}|\widetilde{D}_n| = \int_{\widetilde{D}_n} \mathrm{div}\ \mathbf{Y} = \int_{\widetilde{D}_n} \mathrm{div}\ \mathbf{Y}^{\mathrm{N}} + \int_{\widetilde{D}_n} \mathrm{div}\ \mathbf{Y}^{\mathrm{T}} = \int_{\partial \widetilde{D}_n} \langle \eta, \mathbf{Y} \rangle,$$

where η is the unit conormal to $\widetilde{D}_n$ and we have decomposed the divergence on $\widetilde{D}_n$ into tangential and normal components:

$$\mathrm{div}\ \mathbf{Y} = \mathrm{div}\ \mathbf{Y}^{\mathrm{T}} + \mathrm{div}\ \mathbf{Y}^{\mathrm{N}}.$$

Suppose now that we are in **Case A** and M_∞ is not a catenoid. In this case let $\mathbf{Y}$ denote the left invariant vector field corresponding to the horizontal component h of the

flux of M_∞; let h_n denote the corresponding horizontal flux of γ_n. Then the above two formulas show that

$$0 = \frac{1}{I_M(p_n)}\delta_{\mathbf{Y}}|U_n| = \frac{1}{I_M(p_n)}\int_{\widetilde{D}_n}\langle \nu, \mathbf{Y}\rangle + \frac{1}{I_M(p_n)}\int_{\widetilde{G}_n}\langle \nu, \mathbf{Y}\rangle = \frac{|h_n|}{I_M(p_n)} + \varepsilon_n,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. We obtain a contradiction since $\frac{1}{I_M(p_n)}|h_n|$ converges to $|h| \neq 0$. If M_∞ is a catenoid, then choose $\mathbf{Y}$ to be the left invariant vector field corresponding to the vertical tangent vector $e_3 \in T_e(\widetilde{N})$. Then

$$0 = \frac{1}{I_M(p_n)}\delta_{\mathbf{Y}}|U_n| = \frac{1}{I_M(p_n)}\int_{\widetilde{D}_n}\langle \nu, \mathbf{Y}\rangle + \frac{1}{I_M(p_n)}\int_{\widetilde{G}_n}\langle \nu, \mathbf{Y}\rangle = -2\pi \pm \pi \neq 0.$$

Since the length of the waist circle of a catenoid is 2π and the area of the disk which it bounds is π, one again arrives at a contradiction. Hence, **Case A** does not occur.

Since norm of any left invariant vector field on $\widetilde{N}$ is constant, **Case B** also cannot occur as follows from a straightforward application of the divergence theorem applied to the related region $U_{n,k}$ for n large and for k much greater than n, in much the same way as we applied the divergence theorem to resolve **Case A** (one uses the fact that the associated fluxes on $G_{n,k}$ and $\partial G_{n,k}$ are essentially negligible in comparison to the fluxes on G_n and ∂G_n). This completes our proof of item *1* in Theorem 11.1.

Now suppose that N has its geometry modeled on $\mathbb{S}^3$ with a homogenous metric of positive Ricci curvature. Suppose that $\Sigma \subset N$ is a complete, embedded minimal surface. The proof of item *1* actually generalizes to show that the curvature of any annular end representative $E \subset \Sigma$ has locally bounded curvature in N. Since N is compact and E is not, there exists a complete minimal surface L in the closure $\overline{E}$ which is a limit leaf of a related minimal lamination of N. By Lemma ***, the universal cover of L is a complete, stable minimal surface in N, which is impossible since N has positive Ricci curvature. This contradiction implies Σ does not have any annular ends. If Σ has finite genus and a countable number of ends, then Baire's theorem implies that the set of simple or non-limit ends of Σ are dense in the countable metric space of ends of Σ (see Lemma 2.1 in [27] for the proof). Hence, if Σ is non-compact, it has at least one annular end,and so it must be compact.

The arguments in the proof of the properties described in item*3* are similar to those used to prove item *2*, together with the results in [38] concerning the geometry of minimal annuli in an N satisfying the hypotheses of this item. Also, one needs to apply a theorem of Rosenberg [49] that implies that a possibly disconnected, properly embedded minimal surface $\Sigma \subset N$ with compact boundary in $N = \mathbb{S}^2 \times \mathbb{R}$ cannot have more that two ends.

The results described in the classical setting of item *4* follow from the result in [40] that a finite topology, complete, embedded minimal surface M in a complete, flat three-manifold is proper, and on previous results of Meeks and Rosenberg [35, 37, 39, 40] and

of Collin [11] concerning the geometry of properly embedded minimal surfaces in these manifolds. This completes our proof of Theorem 11.1.

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