Structure theorems for singular minimal laminations

William H. Meeks III∗ Joaquín Pérez Antonio Ros†

January 25, 2008

Abstract

In this paper, we apply our local removable singularity theorem and local structure theorems for embedded minimal surfaces and minimal laminations in $\mathbb{R}^3$ proven in [16, 15], to obtain global structure theorems for certain possibly singular minimal laminations of $\mathbb{R}^3$. We will use Theorems 1.3 and Theorem 1.6 below in [14] to prove that a complete, embedded minimal surface in $\mathbb{R}^3$ with finite genus and a countable number of ends is proper. Theorem 1.6 will also be applied in [13] to obtain bounds on the index and the topology of complete, embedded minimal surfaces of fixed genus and finite topology in $\mathbb{R}^3$.

Mathematics Subject Classification: Primary 53A10, Secondary 49Q05, 53C42

Key words and phrases: Minimal surface, stability, curvature estimates, local picture, minimal lamination, removable singularity, limit tangent cone, minimal parking garage structure, injectivity radius, locally simply connected.

1 Introduction.

Recent work by Colding and Minicozzi [1, 2, 5, 6] on removable singularities for certain limit minimal laminations of $\mathbb{R}^3$, and subsequent applications by Meeks and Rosenberg [19, 20] demonstrate the fundamental importance of removable singularities results for obtaining a deep understanding of the geometry of complete, embedded minimal surfaces in three-manifolds. Removable singularities theorems for limit minimal laminations also play a central role in our papers [13, 14, 17, 18] where we obtain topological bounds and descriptive results for complete, embedded minimal surfaces of finite genus in $\mathbb{R}^3$.

In this article, we will extend some of these results. We will prove global theorems on the structure of certain possibly singular minimal laminations of $\mathbb{R}^3$. These theorems

∗This material is based upon work for the NSF under Award No. DMS - 0405836. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

†Research partially supported by a MEC/FEDER grant no. MTM2004-02746.
depend on the local theory of embedded minimal surfaces and minimal laminations developed in our papers [16, 15]. Besides having important applications (see [13, 14]), these structure theorems help to provide important geometrical insight for possibly resolving the following fundamental conjecture in [16], at least in the case the set $S$ described in it is countable.

**Conjecture 1.1 (Fundamental Singularity Conjecture)**

Suppose $S \subset \mathbb{R}^3$ is a closed set whose one-dimensional Hausdorff measure is zero. If $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3 - S$, then $\overline{\mathcal{L}}$ has the structure of a $C^{1,\alpha}$-minimal lamination of $\mathbb{R}^3$.

Since the union of a catenoid with a plane passing through its waist circle is a singular minimal lamination of $\mathbb{R}^3$ whose singular set is the intersecting circle, the above conjecture represents the strongest possible removable singularities conjecture. We now give a formal definition of a singular lamination and the set of singularities associated to a leaf of a singular lamination. Given an open set $A \subset \mathbb{R}^3$ and $B \subset A$, we will denote by $\overline{B}^A$ the closure of $B$ in $A$. In the case $A = \mathbb{R}^3$, we simply denote $\overline{B}^{\mathbb{R}^3}$ by $\overline{B}$.

**Definition 1.2** A singular lamination of an open set $A \subset \mathbb{R}^3$ with singular set $S \subset A$ is the closure $\overline{\mathcal{L}}^A$ of a lamination $\mathcal{L}$ of $A - S$, such that for each point $p \in S$, then $p \in \overline{\mathcal{L}}^A$, and in any open neighborhood $U_p \subset A$ of $p$, $\overline{\mathcal{L}}^A \cap U_p$ fails to have an induced lamination structure in $U_p$. It then follows that $S$ is closed in $A$. The singular lamination $\overline{\mathcal{L}}^A$ is said to be minimal if the leaves of the related lamination $\mathcal{L}$ of $A - S$ are minimal surfaces.

For a leaf $L$ of $\mathcal{L}$, we call a point $p \in \overline{\mathcal{L}}^A \cap S$ a singular leaf point of $L$ if for some open set $V \subset A$ containing $p$, then $L \cap V$ is closed in $V - S$. We let $\mathcal{S}_L$ denote the set of singular leaf points of $L$. Finally, we define $\overline{\mathcal{L}}^A(L) = L \cup \mathcal{S}_L$ to be the leaf of $\overline{\mathcal{L}}^A$ associated to the leaf $L$ of $\mathcal{L}$.

In particular, the leaves of the singular lamination $\overline{\mathcal{L}}^A$ are of the following two types.

- If for a given $L \in \mathcal{L}$ we have $\mathcal{L}^A \cap S = \emptyset$, then $L$ a leaf of $\overline{\mathcal{L}}^A$.
- If for a given $L \in \mathcal{L}$ we have $\mathcal{L}^A \cap S \neq \emptyset$, then $\overline{\mathcal{L}}^A(L)$ is a leaf of $\overline{\mathcal{L}}^A$.

Note that since $\mathcal{L}$ is a lamination of $A - S$, then $\overline{\mathcal{L}}^A = \mathcal{L} \cup S$ (disjoint union). Hence, the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ considered to be a subset of $\mathbb{R}^3$ is $\overline{\mathcal{L}} = \mathcal{L} \cup S \cup (\partial A \cap \overline{\mathcal{L}})$. In contrast to the behavior of (regular) laminations, it is possible for distinct leaves of a singular lamination $\overline{\mathcal{L}}^A$ of $A$ to intersect. For example, the union of two orthogonal planes in $\mathbb{R}^3$ is a singular lamination $\overline{\mathcal{L}}$ of $A = \mathbb{R}^3$ with singular set $S$ being the line of intersection of the planes. In this example, the above definition yields a related lamination $\mathcal{L}$ of
$\mathbb{R}^3 - S$ with four leaves which are open half-planes and $\mathcal{L}$ has four leaves which are the associated closed half-planes that intersect along $S$; thus, $\mathcal{L}$ is not the disjoint union of its leaves. However, the Colding-Minicozzi Example II in Section 2 of [16] describes a singular minimal lamination $\mathcal{L}_1$ of the open unit ball $A = \mathbb{B}(1) \subset \mathbb{R}^3$ with singular set $S_1$ being the origin $\{0\}$; the related (regular) lamination $L_1$ of $\mathbb{B}(1) - \{0\}$ consists of three leaves, which are the punctured unit disk $\mathbb{D} - \{0\} = \{(x_1, x_2, 0) \mid 0 < x_1^2 + x_2^2 < 1\}$ and two spiraling, non-proper disks $L^+ \subset \{x_3 > 0\}$ and $L^- \subset \{x_3 < 0\}$. In this case, $0$ is a singular leaf point of $\mathbb{D} - \{0\}$ (hence $\mathcal{L}_1(\mathbb{D} - \{0\})$ equals the unit disk $\mathbb{D}$), but $0$ is not a singular leaf point of either $L^+$ or $L^-$ (because $L^+ \cap V$ fails to be closed in $V - S_1$ for any open set $V \subset \mathbb{B}(1)$ containing $0$), and so $\mathcal{L}_1(L^+) = L^+$ and analogously $\mathcal{L}_1(L^-) = L^-$. Hence, $\mathcal{L}_1$ is the disjoint union of its leaves in this case.

Conjecture 1.1 is motivated by a number of results that we obtained in [16, 15] and by the structure theorems presented here. The first of these results, which we prove in Section 2, is Theorem 1.3 below; this is a general structure theorem for possibly singular minimal laminations of $\mathbb{R}^3$ whose singular set is countable. Theorem 1.3 is useful in applications because of the following situation. Suppose that $L$ is a non-planar leaf of a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - S$, with $S \subset \mathbb{R}^3$ being closed. In this case, its closure $\mathcal{L}$ has the structure of a possibly singular minimal lamination of $\mathbb{R}^3$, which under rather weak hypotheses, can be shown to have a countable singular set. Then, if $L$ can also be shown to have finite genus, then item 7 of the next theorem demonstrates that $\mathcal{L} = \mathcal{L} = \{\mathcal{L}\}$ is a smooth, properly embedded minimal surface in $\mathbb{R}^3$.

Throughout the paper, $\mathbb{B}(p, R)$ will denote the open Euclidean ball of radius $R > 0$ centered at a point $p \in \mathbb{R}^3$, and $\mathbb{B}(R) = \mathbb{B}(0, R)$. For a surface $\Sigma \subset \mathbb{R}^3$, $K_\Sigma$ will denote its Gaussian curvature function.

**Theorem 1.3 (Structure Theorem for Singular Minimal Laminations of $\mathbb{R}^3$)**

Suppose that $\mathcal{L} = \mathcal{L} \cup S$ is a possibly singular minimal lamination of $\mathbb{R}^3$ with a countable set $S$ of singularities. Then:

1. The set $\mathcal{P}$ of leaves in $\mathcal{L}$ which are planes forms a closed subset of $\mathbb{R}^3$.
2. The set $\text{Lim}(\mathcal{L})$ of limit leaves of $\mathcal{L}$ is closed in $\mathbb{R}^3$ and satisfies $\text{Lim}(\mathcal{L}) \subset \mathcal{P}$.
3. If $P$ is a plane in $\mathcal{P} - \text{Lim}(\mathcal{L})$, then there exists a $\delta > 0$ such that $P(\delta) \cap \mathcal{L} = P$, where $P(\delta)$ is the $\delta$-neighborhood of $P$. In particular, $S \cap (\mathcal{P} - \text{Lim}(\mathcal{L})) = \emptyset$.
4. Suppose $p \in S - \bigcup_{p \in \mathcal{P}} P$. Then for $\varepsilon > 0$ sufficiently small, $\mathcal{L}(p, \varepsilon) = \mathcal{L} \cap \mathbb{B}(p, \varepsilon)$ has the following description.

4.1. $\mathcal{L}(p, \varepsilon)$ has finite area and consists of a finite number of leaves, each of which is properly embedded in $\mathbb{B}(p, \varepsilon) - S$. 

3
4.2. Each point \( q \in B(p, \varepsilon) \cap S \) represents the end of a unique leaf \( L_q \) of \( \mathcal{L}(p, \varepsilon) \) and this end has infinite genus (\( L_q = L_{q'} \) may occur if \( q, q' \) are distinct points in \( B(p, \varepsilon) \cap S \), for example this occurs if \( B(p, \varepsilon) \cap S \) is infinite). In particular, if \( p \) is an isolated point of \( S \), then \( \varepsilon \) can be chosen small enough so that \( \mathcal{L}(p, \varepsilon) \) consists of a single, smooth non-compact leaf with infinite genus and exactly one end.

4.3. All of the components of \( \mathcal{L}(p, \varepsilon) \) lie on the same leaf of \( \overline{\mathcal{L}} \).

From now on, suppose that there exists a non-flat leaf \( L \) in the lamination \( \mathcal{L} \) of \( \mathbb{R}^3 - S \).

5. One of the following possibilities holds.

5.1. \( \mathcal{L} = \{ L \} \), in which case \( \overline{\mathcal{L}}(L) \) is properly embedded in \( \mathbb{R}^3 \) and \( \overline{\mathcal{L}} = \{ \overline{L} \} \).

5.2. \( \mathcal{L} \neq \{ L \} \). In this case, \( \overline{L} \) has the structure of a possibly singular minimal lamination of \( \mathbb{R}^3 \) (with singular set contained in \( \overline{L} \cap S \)), which consists of the leaf \( \overline{L}(L) \) together with a set \( P(L) \) of one or two planar leaves of \( \overline{L} \). Furthermore, \( \overline{L}(L) \) is properly embedded (not necessarily complete) in a component \( C(L) \) of \( \mathbb{R}^3 - P(L) \), \( C(L) \cap L = L \), and every open slab \( \varepsilon \)-neighborhood \( P(\varepsilon) \) of a plane \( P \in P(L) \) intersects \( \overline{L}(L) \) in a connected surface with infinite genus and unbounded Gaussian curvature.

In particular, \( \overline{L} \) is the disjoint union of its leaves, regardless of whether case 5.1 or 5.2 occurs.

6. If \( L \) has finite genus, then \( L \) is a smooth, properly embedded minimal surface in \( \mathbb{R}^3 \) (thus \( \mathcal{L} = \overline{L} = \{ \overline{L} \} \) and \( S = \emptyset \)).

In the next theorem, we will consider the case where the possibly singular minimal lamination arises as a limit of a sequence of embedded, possibly non-proper, minimal surfaces in \( \mathbb{R}^3 \), which satisfies the locally positive injectivity radius property described in the next definition.

**Definition 1.4** Consider a closed set \( W \subset \mathbb{R}^3 \) and a sequence of embedded minimal surfaces \( \{ M_n \} \) (possibly with boundary) in \( A = \mathbb{R}^3 - W \). We will say that this sequence has **locally positive injectivity radius** in \( A \), if for every \( q \in A \), there exists \( \varepsilon_q > 0 \) and \( n_q \in \mathbb{N} \) such that for \( n > n_q \), the restricted functions \( I_{M_n} \big|_{B(q, \varepsilon_q) \cap M_n} \) are uniformly bounded away from zero, where \( I_{M_n} \) is the injectivity radius function of \( M_n \).

By Proposition 1.1 in Colding and Minicozzi [1], the property that a sequence \( \{ M_n \} \) has locally positive injectivity radius in an open set \( A \) is equivalent to the property that \( \{ M_n \} \) is locally simply connected in \( A \), in the sense that around any point \( q \in A \), we...
can find a ball \( \mathbb{B}(q, \delta_q) \subset A \) such that for \( n \) sufficiently large, \( \mathbb{B}(q, \delta_q) \) intersects \( M_n \) in components which are disks with boundaries on \( \partial \mathbb{B}(q, \delta_q) \).

Note that if the \( M_n \) have boundary and \( \{M_n\}_n \) has locally positive injectivity radius in \( A = \mathbb{R}^3 - W \), then for any \( p \in A \) there exists \( \varepsilon_p > 0 \) and \( n_p \in \mathbb{N} \) such that \( \partial M_n \cap \mathbb{B}(p, \varepsilon_p) = \emptyset \) for \( n > n_p \), i.e., the boundary of \( M_n \) must eventually diverge in space or converge to a subset of \( W \).

In [13], we will apply the following Theorem 1.5 in an essential way to prove that for each non-negative integer \( g \), there exists a bound on the number of ends of a complete, embedded minimal surface in \( \mathbb{R}^3 \) with finite topology and genus at most \( g \). This topological boundedness result implies that the stability index of a complete, embedded minimal surface of finite index has an upper bound that depends only on its finite genus. In this application of Theorem 1.5, the set \( W \) will be finite. Also Theorem 1.5 (as well as Theorem 1.6 below) will be applied in [14] to prove that a complete, embedded minimal surface in \( \mathbb{R}^3 \) with an infinite number of ends, finite genus and compact boundary (possibly empty), is properly embedded in \( \mathbb{R}^3 \) if and only if it has a countable number of limit ends if and only if it has two limit ends (limit ends are the limit points in the space of ends endowed with its natural topology).

**Theorem 1.5** Suppose \( W \) is a countable closed subset of \( \mathbb{R}^3 \) and \( \{M_n\}_n \) is a sequence of embedded minimal surfaces (possibly with boundary) in \( A = \mathbb{R}^3 - W \), which has locally positive injectivity radius in \( A \). Then, after replacing by a subsequence, \( \{M_n\}_n \) converges on compact subsets of \( A \) to a possibly singular minimal lamination \( \mathcal{L}^A = \mathcal{L} \cup S^A \) of \( A \) (here \( \mathcal{L}^A \) denotes the closure in \( A \) of a minimal lamination \( \mathcal{L} \) of \( A - S^A \), and \( S^A \) is the singular set of \( \mathcal{L}^A \)). Furthermore, the closure \( \overline{\mathcal{L}} \) in \( \mathbb{R}^3 \) of \( \bigcup_{\mathcal{L} \in \mathcal{L}} \mathcal{L} \) has the structure of a possibly singular minimal lamination of \( \mathbb{R}^3 \), with the singular set \( S \) of \( \overline{\mathcal{L}} \) satisfying

\[
S \subset S^A \cup (W \cap \overline{\mathcal{L}}).
\]

Let \( S(\mathcal{L}) \subset \mathcal{L} \) denote the singular set of convergence of the \( M_n \) to \( \mathcal{L} \). Then:

1. The set \( \mathcal{P} \) of planar leaves in \( \overline{\mathcal{L}} \) forms a closed subset of \( \mathbb{R}^3 \).

2. The set \( \text{Lim}(\overline{\mathcal{L}}) \) of limit leaves of \( \overline{\mathcal{L}} \) is closed in \( \mathbb{R}^3 \) and satisfies \( \text{Lim}(\overline{\mathcal{L}}) \subset \mathcal{P} \).

3. If \( P \) is a plane in \( \mathcal{P} - \text{Lim}(\overline{\mathcal{L}}) \), then there exists \( \delta > 0 \) such that \( P(\delta) \cap \overline{\mathcal{L}} = P \), where \( P(\delta) \) is the \( \delta \)-neighborhood of \( P \).

4. For each point \( q \in S^A \cup S(\mathcal{L}) \), there passes a plane \( P_q \in \text{Lim}(\overline{\mathcal{L}}) \). Furthermore, \( P_q \) intersects \( S^A \cup S(\mathcal{L}) \cup W \) in a countable closed set.

5. Through each point of \( p \in W \) satisfying one of the conditions (5.A),(5.B) below, there passes a plane \( P_p \in \mathcal{P} \).
(5.A) The area of \( \{M_n \cap R_k\}_n \) diverges to infinity for all \( k \) large, where \( R_k \) is the ring \( \{x \in \mathbb{R}^3 \mid \frac{1}{k+1} < |x - p| < \frac{1}{k}\} \) (in this case, \( P_p \in \text{Lim}(\mathcal{L}) \) or the convergence of the \( M_n \) to \( P_p \) has infinite multiplicity).

(5.B) The convergence of the \( M_n \) to some leaf of \( \mathcal{L} \) having \( p \) in its closure is of multiplicity greater than one.

6. Suppose that there exists a leaf \( L \) of \( \mathcal{L} \) which is not contained in \( \mathcal{P} \). Then, \( L \cap (\mathcal{S}^A \cup S(\mathcal{L})) = \emptyset \) (note that \( L \) might contain singular points which belong to \( W \)), the convergence of portions of the \( M_n \) to \( L \) is of multiplicity one, and one of the following two possibilities holds:

6.1. \( L \) is proper in \( \mathbb{R}^3 \), \( \mathcal{P} = \emptyset \) and \( \mathcal{L} = \{L\} \).

6.2. \( L \) is not proper in \( \mathbb{R}^3 \) and \( \mathcal{P} \neq \emptyset \). In this case, there exists a subcollection \( \mathcal{P}(L) \subset \mathcal{P} \) consisting of one or two planes such that \( \mathcal{L} = L \cup \mathcal{P}(L) \), and \( L \) is proper in a component \( C(L) \) of \( \mathbb{R}^3 - \mathcal{P}(L) \) and \( L = C(L) \cap \mathcal{L} \).

In particular, \( \mathcal{L} \) is the disjoint union of its leaves, regardless of whether case 6.1 or 6.2 occurs (if case 6.2 occurs, then each leaf of \( \mathcal{L} \) is either a plane or a minimal surface possibly with singularities in \( W \), which is properly embedded, not necessarily complete, in an open half-space or open slab of \( \mathbb{R}^3 \)).

7. Suppose that the surfaces \( M_n \) have uniformly bounded genus and \( \mathcal{S} \cup S(\mathcal{L}) \neq \emptyset \). Then \( \mathcal{L} \) contains a non-empty foliation \( \mathcal{F} \) of an open slab of \( \mathbb{R}^3 \) by planes and \( \overline{S(\mathcal{L})} \cap \mathcal{F} \) consists of 1 or 2 straight line segments orthogonal to these planes, intersecting every plane in \( \mathcal{F} \). Furthermore, if there are 2 different line segments in \( \overline{S(\mathcal{L})} \cap \mathcal{F} \), then in the related limiting minimal parking garage structure of the slab, the limiting multigraphs along the 2 columns are oppositely oriented. If the surfaces \( M_n \) are compact with boundary, then \( \mathcal{L} = \mathcal{F} \) is a foliation of all of \( \mathbb{R}^3 \) by planes and \( \overline{S(\mathcal{L})} \) consists of complete lines orthogonal to \( \mathcal{F} \).

We refer the reader to [15] for a discussion of the notion of a limiting minimal parking garage structure of \( \mathbb{R}^3 \) or see [2] for a related discussion. In item 7 of the above theorem, we refer to the “related limiting minimal parking garage structure of the slab” which has not been defined precisely because the sequence of the surfaces \( \{M_n\}_n \) only converges to a minimal lamination \( \mathcal{L} \) in \( \mathcal{F} - W \), rather than to a minimal lamination of \( \mathcal{F} \). If \( \mathcal{F} \) is a union of planar leaves of \( \mathcal{L} \) which forms an open slab, then \( \mathcal{F} \cap \mathcal{S} = \emptyset \) and for \( n \) large, \( M_n \cap \mathcal{F} \) has the appearance of a parking garage structure away from the small set \( W \). In spite of this problem that arises from \( W \), we feel that our language here appropriately describes the behavior of the limiting configuration. Next we explain an example that can clarify this situation. In Example II of Section 2 of [16] we mentioned a construction by Colding and Minicozzi [3] of a sequence \( \{D_n\}_n \) of compact minimal disks contained in the unit
ball $\mathbb{B}(1)$ of $\mathbb{R}^3$, with $\partial D_n \subset \partial \mathbb{B}(1)$, such that $\{D_n\}_n$ converges as $n \to \infty$ to a singular minimal lamination $\mathcal{L}_1$ of the closed ball $\mathbb{B}(1)$ with an isolated singularity at the origin $\vec{0}$. The related lamination $\mathcal{L}_1(1)$ of $\mathbb{B}(1) - \{\vec{0}\}$ consists of three leaves, one being the horizontal punctured closed disk $[\mathbb{B}(1) \cap \{x_3 = 0\}] - \{\vec{0}\}$ (this is the unique limit leaf of $\mathcal{L}_1$), and the other two leaves being non-proper embedded minimal surfaces $L^+, L^-$ which spiral to $[\mathbb{B}(1) \cap \{x_3 = 0\}] - \{\vec{0}\}$ from opposite sides, say $L^+ \subset \{x_3 > 0\}$ and $L^- \subset \{x_3 < 0\}$. By the Local Removable Singularity Theorem (see Theorems 1.2 or 5.1 in [16] or see Theorem 2.1 below), the curvature function $K_{\mathcal{L}_1}$ of $\mathcal{L}_1$ satisfies that $|K_{\mathcal{L}_1}| R^2 \to \infty$ as $R \to 0$, where $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Then we can choose positive numbers $\lambda_n \to \infty$ such that the Gaussian curvature of the dilated disks $\lambda_n D_n$ blows up at $\vec{0}$. By Theorem 0.1 in Colding and Minicozzi [6], after passing to a subsequence, the $\lambda_n D_n$ converge to a foliation $\mathcal{F}$ of $\mathbb{R}^3$ by planes (which can be proved to be horizontal due to the properties of $D_n$ in [3]), with singular set of convergence $S(\mathcal{F})$ being the $x_3$-axis (here we are also using the Regularity Theorem for the singular set $S(\mathcal{F})$ by Meeks [12]). It then follows that $M_n = \lambda_n L^+ \subset \{x_3 > 0\}$ is a non-proper, embedded minimal disk, and the sequence $\{M_n\}_n$ has locally positive injectivity radius in $\mathbb{R}^3 - \{\vec{0}\}$ and converges to the minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{\vec{0}\}$ by all horizontal planes with non-negative height, with singular set of convergence $S(\mathcal{L})$ being the positive $x_3$-axis. In this case, $W = \{\vec{0}\}$ and $\mathcal{L}$ is the foliation of the closed upper half-space of $\mathbb{R}^3$ by horizontal planes.

We finally come to a rather surprising theorem, which describes the limits of certain sequences of compact minimal surfaces with boundary. The proof of this theorem uses the main result of Colding and Minicozzi in [2], together with several results from our previous papers.

**Theorem 1.6** Suppose $\{M_n\}_n$ is a sequence of compact, embedded minimal surfaces of finite genus $g$ in $\mathbb{R}^3$, with $\partial M_n \subset \partial \mathbb{B}(n)$ for each $n$. Suppose that some subsequence of disks $\{D_n \subset M_n\}_n$ converges $C^2$ to a non-flat minimal disk. Then, a subsequence of the $M_n$ converge smoothly on compact subsets of $\mathbb{R}^3$ with multiplicity one to a connected, properly embedded, non-flat minimal surface $M_\infty \subset \mathbb{R}^3$ of genus at most $g$, which is either a surface of finite total curvature, a helicoid with handles or a two limit end minimal surface. In particular, $M_\infty$ has bounded Gaussian curvature.

2 The proof of Theorem 1.5.

Conjecture 1.1 stated in the introduction has a global nature, because there exist interesting minimal laminations of the open unit ball in $\mathbb{R}^3$ punctured at the origin which do not extend across the origin, see Section 2 in [16]. Also in Section 2 of [16] we describe a rotationally invariant global minimal lamination of hyperbolic three-space $\mathbb{H}^3$, which has a similar unique isolated singularity. The existence of these global singular minimal laminations of $\mathbb{H}^3$ demonstrate that the validity of Conjecture 1.1 depends on the metric.
properties of $\mathbb{R}^3$. However, in [16], we obtained a remarkable local removable singularity result in any Riemannian three-manifold $N$ for certain possibly singular minimal laminations. Since we will apply this theorem and a related corollary repeatedly, we give their complete statements below.

Given a three-manifold $N$ and a point $p \in N$, we will denote by $B_N(p, r)$ the metric ball of center $p$ and radius $r > 0$.

**Theorem 2.1 (Local Removable Singularity Theorem)** A minimal lamination $\mathcal{L}$ of a punctured ball $B_N(p, r) - \{p\}$ in a Riemannian three-manifold $N$ extends to a minimal lamination of $B_N(p, r)$ if and only if there exists a positive constant $c$ such that $|K_L|d^2 < c$ in some subball, where $|K_L|$ is the absolute Gaussian curvature function of $\mathcal{L}$ and $d$ is the distance function in $N$ to $p$.

In Theorem**** of [16], we prove that the sublamination $\text{Lim}(\mathcal{L})$ of limit leaves of a minimal lamination $\mathcal{L}$ of a three-manifold $N$ are stable minimal surfaces. An immediate consequence of this result is that the set $\text{Stab}(\mathcal{L})$ of stable leaves of $\mathcal{L}$ is a sublamination of $\mathcal{L}$ with $\text{Lim}(\mathcal{L}) \subset \text{Stab}(\mathcal{L})$. These observations together with standard curvature estimates [23] for stable orientable minimal surfaces away from their boundaries give the following corollary from [16] to the above theorem.

**Corollary 2.2** Suppose that $N$ is a not necessarily complete Riemannian three-manifold. If $W \subset N$ is a closed countable subset and $\mathcal{L}$ is a minimal lamination of $N - W$, then the closure of any collection of its stable leaves extends across $W$ to a minimal lamination of $N$ consisting of stable minimal surfaces. In particular,

1. The closure $\overline{\text{Stab}(\mathcal{L})}$ of stable leaves of $\mathcal{L}$ is a minimal lamination of $N$ whose leaves are stable minimal surfaces.

2. The closure $\overline{\text{Lim}(\mathcal{L})}$ of the sublamination $\text{Lim}(\mathcal{L})$ of limit leaves of $\mathcal{L}$ extends across $W$ to a sublamination of $\overline{\text{Stab}(\mathcal{L})}$.

3. If $\mathcal{L}$ is a minimal foliation of $N - W$, then $\mathcal{L}$ extends across $W$ to a minimal foliation of $N$.

In this section and the next one, we shall prove two theorems on the structure of certain possibly singular minimal laminations of $\mathbb{R}^3$, which were stated in the introduction. In the lamination described in the first theorem, the singular set $\mathcal{S}$ of the lamination is countable; in the second theorem, the minimal singular lamination is obtained as a limit of a sequence of embedded minimal surfaces, which is locally simply-connected outside a countable set.

---

1 Equivalently by the Gauss theorem, for some positive constant $c'$, $|A_L|d < c'$, where $|A_L|$ is the norm of the second fundamental form of $\mathcal{L}$. 

---
In both theorems, the laminations can be expressed as a disjoint union of its possibly singular minimal leaves (see the last statement of item 6 of Theorem 1.5 and of item 5 of Theorem 1.3). The key result for proving this last property is Proposition 2.3 below; it gives a condition under which two different leaves of a singular minimal lamination cannot share a singular leaf point.

**Proposition 2.3** Let $\mathcal{L}^A$ be a singular lamination of an open set $A \subset \mathbb{R}^3$, with singular set $\mathcal{S}$ and related (regular) lamination $\mathcal{L}$ of $A - \mathcal{S}$. If $\mathcal{S}$ is countable, then any singular point is a singular leaf point of a unique leaf of $\mathcal{L}^A$.

**Proof.** Reasoning by contradiction, suppose $p \in \mathcal{S}$ is a singular leaf point of two leaves $\mathcal{L}^A(L_1), \mathcal{L}^A(L_2)$ of $\mathcal{L}^A$, associated to leaves $L_1, L_2$ of $\mathcal{L}$. By definition, $p \in \overline{L_1} \cap \overline{L_2}$ and there exists a ball $B(p, 2\varepsilon) \subset A$ such that $L_i \cap B(p, 2\varepsilon)$ is closed in $B(p, 2\varepsilon) - \mathcal{S}$, $i = 1, 2$. Since $\mathcal{S}$ is countable, we may assume that the sphere $\partial B(p, \varepsilon)$ is disjoint from $\mathcal{S}$. Let us define $L_i(\varepsilon) = L_i \cap B(p, \varepsilon)$, $i = 1, 2$. Then $L_1(\varepsilon), L_2(\varepsilon)$ are disjoint, properly embedded minimal surfaces in $B(p, \varepsilon) - \mathcal{S}$. Since $\mathcal{S}$ is countable, $L_1(\varepsilon)$ separates $\overline{B(p, \varepsilon)}$ into two connected components, and the same holds for $L_2(\varepsilon)$. Let $N$ be the closure of the component of $\overline{B(p, \varepsilon)} - (L_1(\varepsilon) \cup L_2(\varepsilon))$ which contains both $L_1(\varepsilon), L_1(\varepsilon)$ in its boundary.

Using $\partial N$ as a barrier for solving Plateau problems in $N$ and the fact that both $\overline{L_1(\varepsilon)}, \overline{L_2(\varepsilon)}$ are closed (by definition of singular leaf points), and hence compact, then from a compact exhaustion of $\overline{L_1(\varepsilon)} - \mathcal{S}$, we produce an embedded, area-minimizing varifold $\Sigma_1 \subset N - \mathcal{S}$ with $\partial \Sigma_1 = \partial L_1(\varepsilon)$ and with $p \in \Sigma_1$ (see Meeks and Yau [22] for similar type applications and a description of this barrier type construction). By regularity properties of area-minimizing varifolds, $\Sigma_1$ is regular except possibly at points in $\Sigma_1 \cap \mathcal{S}$. Now consider $\Sigma_1 - \mathcal{S}$ to lie in $\overline{B(p, \varepsilon)} - \mathcal{S}$ and so, $\Sigma_1 - \mathcal{S}$ represents a minimal lamination of $\overline{B(p, \varepsilon)} - \mathcal{S}$ with stable leaves. By Corollary 2.2, $\Sigma_1$ extends smoothly across $\mathcal{S} \cap \overline{B(p, \varepsilon)}$. Exchanging $\overline{L_1(\varepsilon)}$ by $\Sigma_1$ and reasoning analogously, we find an embedded, area-minimizing surface $\Sigma_2$ between $\Sigma_1$ and $\overline{L_2(\varepsilon)}$, with $\partial \Sigma_2 = \partial L_2(\varepsilon)$, such that $p \in \Sigma_2$ and $\Sigma_2$ is smooth. Clearly $\Sigma_1, \Sigma_2$ contradict the interior maximum principle at $p$, and the proposition is proved. □

Using similar arguments, we can extend Proposition 2.3 to the case of a general Riemannian three-manifold (for the proof to work and using the same notation as above, we also need the part of the boundary of $N$ coming from the boundary of $\overline{B(p, \varepsilon)}$ to be convex, which can be assumed by choosing $\varepsilon$ small enough). The following result is an interesting consequence of this generalization.

**Corollary 2.4** Let $\overline{B_N}(p, R)$ be a compact ball centered at a point $p$ in a Riemannian three-manifold $N$, with radius $R > 0$. Suppose $M_1, M_2 \subset \overline{B_N}(p, R) - \{p\}$ are two disjoint, properly embedded minimal surfaces with boundaries $\partial M_i \subset \partial \overline{B_N}(p, R)$, $i = 1, 2$. Then, at most one of these surfaces is not compact, and in this case it has just one end. Furthermore, if $M$ is a properly embedded, smooth minimal surface of finite genus in
We will first produce the possibly singular limit lamination $\overline{\mathcal{L}}$ of $\mathbb{B}_N(p, R) - \{p\}$. Let $\Lambda$ be the set and empty singular set of convergence (see for instance the arguments in the proof of Lemma 1.1 in [19]). In this case, we may assume, by passing to a smaller ball centered at $p$, we deduce directly that if $M_1$ is not compact, then $M_1$ has only one end. Finally, if $M$ is a properly embedded, smooth minimal surface of finite genus in $\mathbb{B}_N(p, R) - \{p\}$ with $\partial M \subset \partial \mathbb{B}_N(p, R)$ and $M$ is not compact, then we may assume, by passing to a smaller $R > 0$, that $M$ is an annulus. Consider the conformal change of metric $g_1 = \frac{1}{d^2}g$, where $g$ is the Riemannian metric of $N$ and $d$ denotes the distance function in $(N, g)$ to $p$. Since $M$ is properly embedded in $\mathbb{B}_N(p, R)$, a standard application of the divergence theorem gives that $(M, g)$ has finite area, which allows us to use the monotonicity formula on $M$, see the proof of Theorem 5.1 in [16] for a similar argument in the $\mathbb{R}^2$-setting. By the monotonicity formula, $(M, g)$ can be proven to have quadratic area growth, which in turn, implies that $(M, g_1) \subset (\mathbb{B}_N(p, R), g_1)$ is a complete annulus with linear area growth and compact boundary. Such a surface is conformally a punctured disk $\mathbb{D}^*$ (see Grigor’yan [9]). Thus, the related conformal harmonic map of $\mathbb{D}^*$ extends to a harmonic map on the whole disk $\mathbb{D}$, which gives rise to a conformal, branched minimal immersion $\mathbb{D}$. Since $M$ is embedded near $p$, then $p$ cannot be a branch point and the corollary holds.

Proof of Theorem 1.5. We will first produce the possibly singular limit lamination $\overline{\mathcal{L}}^A$. If the $M_n$ have uniformly locally bounded curvature in $A$, then it is a standard fact that a subsequence of the $M_n$ converge to a minimal lamination $\mathcal{L}$ of $A$ with empty singular set and empty singular set of convergence (see for instance the arguments in the proof of Lemma 1.1 in [19]). In this case, $\overline{\mathcal{L}}^A = \mathcal{L}$ and $\mathcal{S}^A = \emptyset$. Otherwise, there exists a point $p \in A$ such that, after replacing by a subsequence, the supremum of the absolute curvature of $\mathbb{B}(p, \frac{1}{k_n}) \cap M_n$ diverges to $\infty$ as $n \to \infty$, for any $k$. Since $A$ is open, we can assume $\mathbb{B}(p, \frac{1}{k}) \subset A$ and thus, the sequence of surfaces $\{\mathbb{B}(p, \frac{1}{k_n}) \cap M_n\}_n$ is locally simply connected in $\mathbb{B}(p, \frac{1}{k})$. By Proposition 1.1 in Colding and Minicozzi [1], for all $k$ and $n$ large $\mathbb{B}(p, \frac{1}{k}) \cap M_n$ consists of disks with boundary in $\partial \mathbb{B}(p, \frac{1}{k})$. By the one-sided curvature estimates and the main result in Colding-Minicozzi [4], for some $k_0$ sufficiently large, a subsequence of the surfaces $\{\mathbb{B}(p, \frac{1}{k_0}) \cap M_n\}_n$ (denoted with the same indexes $n$) converges to a possibly singular minimal lamination $\overline{\mathcal{L}}^p$ of $\mathbb{B}(p, \frac{1}{k_0})$ with singular set $\mathcal{S}_p \subset \mathbb{B}(p, \frac{1}{k_0})$, and the related (regular) minimal lamination $\mathcal{L}_p \subset \mathbb{B}(p, \frac{1}{k_0}) - \mathcal{S}_p$ contains a limit leaf $D_p$ which is a stable, minimal punctured disk with $\partial D_p \subset \partial \mathbb{B}(p, \frac{1}{k_0})$ and $\overline{\mathcal{D}}_p \cap \mathcal{S}_p \subseteq \{p\}$; furthermore, $D_p$ extends to the stable, embedded minimal disk $\overline{\mathcal{D}}_p$ in $\mathbb{B}(p, \frac{1}{k_0})$ (we can use
either Colding-Minicozzi theory here, or the Local Removable Singularity Theorem 2.1),
which is a leaf of \( \overline{L}_p \). By the one-sided curvature estimates in [6], there is a solid
double cone \( C_p \subset \mathbb{B}(p, \frac{1}{k_0}) \) with vertex at \( p \) and axis orthogonal to \( D_p \) at that point, that intersects
\( D_p \) only at the point \( p \) and such that the complement of \( C_p \) in \( \mathbb{B}(p, \frac{1}{k_0}) \) does not intersect \( S_p \).
Also, Colding-Minicozzi theory implies that for \( n \) large, \( \mathbb{B}(p, \frac{1}{k_0}) \cap M_n \) has the appearance
outside \( C_p \) of a highly-sheeted double multigraph around \( D_p \). Note that \( p \) might not lie in
\( S_p \) (i.e., \( \mathcal{L}_p \) might have an induced lamination structure in a neighborhood of \( p \)), but in
such case \( p \) would belong to the singular set of convergence \( S(\mathcal{L}_p) \) of \( \{ \mathbb{B}(p, \frac{1}{k_0}) \cap M_n \}_{n \to \infty} \) to \( \mathcal{L}_p \) since the Gaussian curvature of \( \mathbb{B}(p, \frac{1}{k_0}) \cap M_n \) blows up around \( p \) as \( n \to \infty \).

A standard diagonal argument implies, after extracting a subsequence, that the se-
quency \( \{ M_n \}_n \) converges to a possibly singular minimal lamination \( \mathcal{L}^A = \mathcal{L} \cup A^A \) of \( A \),
with related (regular) lamination \( \mathcal{L} \) of \( A - S^A \), singular set \( S^A \subset A \) and with singular
set of convergence \( S(\mathcal{L}) \subset A - S^A \) of the \( M_n \) to \( \mathcal{L} \). Furthermore, the above arguments
imply that in a neighborhood of every point \( p \in S^A \cup S(\mathcal{L}) \), \( \mathcal{L}^A \) has the appearance of the
singular minimal lamination \( \mathcal{L}_p \) described in the previous paragraph.

Next we describe the structure of the closure \( \overline{L} \) of \( \mathcal{L} \) in \( \mathbb{R}^3 \). The closure of \( \mathcal{L} \) in \( \mathbb{R}^3 \) can
be written as
\[
\overline{L} = \mathcal{L}^A \cup (W \cap \overline{L}) = (\mathcal{L} \cup S^A) \cup (W \cap \overline{L})^{\text{lam}} \cup (W \cap \overline{L})^{\text{sing}},
\]
where
\[
(W \cap \overline{L})^{\text{sing}} = \{ p \in W \cap \overline{L} \mid \overline{L} \text{ does not admit locally a lamination structure around } p \}
\]
\[
(W \cap \overline{L})^{\text{lam}} = (W \cap \overline{L}) - (W \cap \overline{L})^{\text{sing}}.
\]

Consider the closed set \( S = S^A \cup (W \cap \overline{L})^{\text{sing}} \). If we define \( L_1 = \mathcal{L} \cup (W \cap \overline{L})^{\text{lam}} \), then \( L_1 \)
can be endowed naturally with a structure of a (regular) minimal lamination of the open
set \( \mathbb{R}^3 - S \). Thus, the decomposition (1) gives that \( \overline{L} \) is a possibly singular lamination of
\( \mathbb{R}^3 \), with singular set \( S \) and related (regular) lamination \( L_1 \).

It remains to prove items 1, \ldots, 7 in the statement of Theorem 1.5. Item 1 of the
theorem holds since the limit of a convergent sequence of planes is a plane.

Next we show that item 2 holds. From the local picture of \( \mathcal{L}^A \) near a point of \( S^A \cup S(\mathcal{L}) \)
given above, each limit leaf \( L_1 \) of \( \mathcal{L} \) can be seen to extend smoothly across \( S^A \cup S(\mathcal{L}) \) to an embedded minimal surface \( \tilde{L}_1 \subset \mathbb{R}^3 - W \) whose universal cover is stable. Since \( \tilde{L}_1 \)
is smooth, has curvature estimates satisfied by stable minimal surfaces and is complete
outside the closed countable set \( W \) in \( \mathbb{R}^3 \), Corollary 2.2 implies that the closure of \( L_1 \) in \( \mathbb{R}^3 \) is a plane. Thus, the set \( \text{Lim}(\overline{L}) \) of limit leaves of \( \overline{L} \) is a collection of planes. Since the
set of limit leaves of \( \overline{L} \) forms a closed set in \( \mathbb{R}^3 \), the set of these planes forms a closed set
in \( \mathbb{R}^3 \). Now item 2 of the theorem holds.
Next we prove item 4 of the theorem. Again the local picture of $\overline{\mathcal{L}}^A$ around any point of $\mathcal{L}^A \cup S(\mathcal{L})$ given above implies that through each point $q \in \mathcal{L}^A \cup S(\mathcal{L})$ there passes a limit leaf of $\overline{\mathcal{L}}$ which, by item 2 of the theorem, must be a plane $P_q \in \text{Lim}(\overline{\mathcal{L}})$. Next we will prove that $P_q \cap (\mathcal{L}^A \cup S(\mathcal{L}) \cup W)$ is a countable closed set. By the local simply connected property of the sequence $\{M_n\}_n$, we have that $(\mathcal{L}^A \cup S(\mathcal{L})) \cap (P_q - W)$ is a discrete subset of $P_q - W$, which is clearly closed. Thus the limit points of $(\mathcal{L}^A \cup S(\mathcal{L})) \cap (P_q - W)$ lie in the countable closed set $P_q \cap W$. It then follows that $P_q \cap (\mathcal{L}^A \cup S(\mathcal{L}) \cup W)$ is a closed countable set of $\mathbb{R}^3$, and item 4 of the theorem now easily follows.

Suppose now that $p \in W$ satisfies the area hypothesis in item (5.A) in the theorem. Then it follows that either $p$ is in the closure of a limit leaf of $\overline{\mathcal{L}}$ (which must be a plane by item 2 and so, there passes a plane in $\mathcal{P}$ through $p$), or else condition (5.B) in the theorem holds, i.e., there exists a leaf $\Sigma$ of $\mathcal{L}$ having $p$ in its closure, such that the multiplicity of the convergence of portions of the $M_n$ to $\Sigma$ around $p$ is greater than one. This last property implies that the universal cover of $\Sigma$ is stable (see Lemma A.1 in [20]), and so the related curvature estimates imply that it extends across $p$, and that the universal cover of the leaf of $\overline{\mathcal{L}}$ that contains $\Sigma$ is stable as well. Again by the arguments above, an application of Corollary 2.2 gives that the closure of $\Sigma$ in $\mathbb{R}^3$ is a plane, thereby proving item 5 of the theorem.

In order to prove item 3, suppose now that $P$ is a plane in $\mathcal{P} - \text{Lim}(\overline{\mathcal{L}})$. Since $\text{Lim}(\overline{\mathcal{L}})$ is a closed set of planes, we can choose $\delta > 0$ such that the $2\delta$-neighborhood of $P$ is disjoint from $\text{Lim}(\overline{\mathcal{L}})$. By item 4, through every point in $S(\mathcal{L}) \cup \mathcal{L}^A$, there passes a plane in $\text{Lim}(\overline{\mathcal{L}})$. It then follows that $S(\mathcal{L}) \cup \mathcal{L}^A$ is at a positive distance from $P$. Now suppose that the intersection of $\overline{\mathcal{L}}$ with a closed ball $\overline{B}(p, \delta)$ centered at a point $p \in P$ has infinite area. Then a similar argument as in the last paragraph shows that we find a plane in $\text{Lim}(\overline{\mathcal{L}})$ that intersects $\overline{B}(p, \delta)$, which is impossible. It follows that the intersection of $\overline{\mathcal{L}}$ with every closed ball $\overline{B}(p, \delta)$ centered at a point $p \in P$ has finite area for some fixed positive sufficiently small $\delta$. If the $\delta$-neighborhood $P(\delta)$ of $P$ intersects $\mathcal{L}$ in a portion $L'$ of leaf different from $P$, then such a leaf, while it may have singularities in $W$, is proper in $P(\delta)$ (by the finite area property inside balls $\overline{B}(p, \delta)$ with $p \in P$). We now check that $L'$ is disjoint from $P$. Otherwise, $L'$ and $P$ intersect (any such intersection point must lie in $W$ by the maximum principle). Since $W$ is countable, Proposition 2.3 implies $L'$ does not intersect $P$. Now, a standard application of the proof of the Half-space Theorem [10] using catenoid barriers still works in this setting to obtain a contradiction to the existence of $L'$ (this standard application uses the maximum principle for minimal surfaces, which works as well for the singular minimal surfaces here via Proposition 2.3). Hence, $P(\delta) \cap \overline{\mathcal{L}} = P$, which proves item 3.

Suppose now that there exists a leaf $L$ of $\overline{\mathcal{L}}$ which is not a plane, and we will prove item 6 of the theorem. Note that $L$ might intersect $W$.

**Suppose that $L$ is proper in $\mathbb{R}^3$.** Then, the proof of the Half-space Theorem together
Thus, there exists a limit point $q$ of $L$. Since $L$ is not a limit leaf ($L$ is not flat), $q$ is not contained in $L$. We claim that through every limit point $q$ of $L$ there passes a plane $P \in \mathcal{P}$. If $q \in L_1$, then $q$ is a limit point of the regular lamination $L_1$. Thus, there passes a limit leaf $L_1$ of $L_1$ by $q$. By our previous arguments using Corollary 2.2, the closure $\overline{L}_1$ must be a plane in $\mathcal{P}$, and our claim holds. If $q \in S^A \cup S(L)$, then our claim also holds by item 4. Then, we may assume $q \in W$. Recall that $L$ is a leaf of $\overline{L}$, which means by Definition 1.2 that $L = \overline{L}^{\mathbb{R}^3} (L_1) = L_1 \cup S_{L_1}$, where $L_1$ is a leaf of the regular minimal lamination $L_1$ of $\mathbb{R}^3 - S$, and $S_{L_1}$ is the set of singular leaf points of $L_1$. As $q \notin L$, then $q$ is not a singular leaf point of $L_1$. Again by Definition 1.2, this implies that either $q$ does not lie in $\overline{L}_1 \cap S$ or for every open neighborhood $V$ of $q$ in $\mathbb{R}^3$, then $L_1 \cap V$ fails to be closed in $V - S$. Let us see that $q \notin \overline{L}_1 \cap S$. Since $q \in \overline{L} = \overline{L}_1 \cup S_{L_1} = L_1 \cup S_{L_1}$ and $q$ is not a singular leaf point of $L_1$, then $q \in \overline{L}_1$. On the other hand, $q \in S$ since $q \in W$ and $\overline{L}$ does not admit locally a lamination structure around $q$ (otherwise $q \in \text{Lim}(\overline{L})$, which by item 2 of the theorem implies $q \in \mathcal{P}$ and we are done). Thus $q \notin \overline{L}_1 \cap S$, which implies that for every open neighborhood $V$ of $q$ in $\mathbb{R}^3$, then $L_1 \cap V$ fails to be closed in $V - S$. Then one can find a sequence $\{V_k\}_k$ of open neighborhoods of $q$ and a sequence of points $x_k \in L_1 \cap V_k \setminus S = (L_1 \cap V_k)$, $k \in \mathbb{N}$. Without loss of generality, we can assume $V_k \to \{q\}$ as $k \to \infty$. Fix $k \in \mathbb{N}$. Since $x_k$ lies in the closure of $L_1 \cap V_k$ relative to $V_k - S$, there exists a sequence $\{y_k(m)\}_m \in L_1 \cap V_k$ with $y_k(m) \to x_k$ as $m \to \infty$. As $x_k \in (V_k - S) - (L_1 \cap V_k)$, then $x_k \notin L_1$. Thus $\{y_k(m)\}_m$ converges to $x_k$ in the topology of $\mathbb{R}^3$ but it does not converge to $x_k$ in the intrinsic topology of $L_1$ (otherwise $x_k$ would lie in $L_1$ since $x_k \notin S$). This gives that $x_k \in \text{Lim}(L_1)$, and our previous arguments imply that there passes a plane in $\mathcal{P}$ through $x_k$. Since this happens for all $k$, $x_k \to q$ as $k \to \infty$ and $\mathcal{P}$ is a closed set of planes, then there also passes a plane in $\mathcal{P}$ through $q$ and our claim is proved.

Since $L$ is not proper and through any limit point of $L$ there passes a plane in $\mathcal{P}$, a straightforward connectedness argument shows that $\overline{L} = L \cup \mathcal{P}(L)$ with $\mathcal{P}(L)$ consisting of one or two planes. In particular, $L$ must be proper in the component $C(L)$ of $\mathbb{R}^3 - \mathcal{P}(L)$ that contains $L$, and item 6.2 of Theorem 1.5 is also proved.

We now prove item 7. Suppose from now on that the hypotheses of this item hold, i.e., the surfaces $M_n$ have uniformly bounded genus and $S \cup S(L) \neq \emptyset$. 

13
Assertion 2.5  Through every point \( p \in S \cup S(\mathcal{L}) \), there passes a plane of \( \mathcal{P} \) (in particular, \( \mathcal{P} \neq \emptyset \)).

Proof of Assertion 2.5. Fix a point \( p \in S \cup S(\mathcal{L}) = S^A \cup S(\mathcal{L}) \cup (W \cap \overline{\mathcal{L}})^{\text{sing}} \). We will discuss three possibilities for \( p \).

Assume \( p \in S^A \cup S(\mathcal{L}) \). In this case, item 4 implies that there exists a plane \( P \in \text{Lim}(\overline{\mathcal{L}}) \subset \mathcal{P} \) passing through \( p \).

Assume \( p \) is an isolated point of \( S \cap W \). Arguing by contradiction, suppose no plane of \( \mathcal{P} \) passes through \( p \). By item 5, neither of the conditions (5.A), (5.B) hold. Since (5.A) does not occur, we may assume that there is a small closed ball \( \overline{B}(p, \varepsilon) \) such that \( \mathcal{L} \cap \overline{B}(p, \varepsilon) \) contains a finite number of compact, smooth minimal surfaces with boundary on \( \partial \overline{B}(p, \varepsilon) \) and a finite number of non-compact, smooth, properly embedded minimal surfaces \( \{\Sigma_1, \ldots, \Sigma_m\} \) in \( \overline{B}(p, \varepsilon) - \{p\} \) (otherwise there would be a limit leaf of \( \mathcal{L} \cap (\overline{B}(p, \varepsilon) - \{p\}) \)), contradicting (5.A). By Corollary 2.4 there is exactly one such non-compact surface (i.e., \( m = 1 \)) and that this surface \( \Sigma_1 \) has just one end.

Since the surfaces \( M_n \) have uniformly bounded genus and converge with multiplicity one to \( \Sigma_1 \) (this last property follows from the fact that (5.B) does not occur at \( p \)), then \( \Sigma_1 \) has finite genus at most equal to the uniform bound on the genus of the surfaces in \( \{M_n\}_n \). By the last statement in Corollary 2.4, \( \Sigma_1 \) extends smoothly across \( p \), contradicting that \( p \in S \).

Assume that \( p \in S \cap W \) is not an isolated point. Since \( S \cap W \) is a countable closed set of \( \mathbb{R}^3 \), \( p \) must be a limit of isolated points \( p_k \in S \cap W \), so by the preceding case, there pass planes in \( \mathcal{P} \) through the points \( p_k, k \in \mathbb{N} \). Our assertion holds in this case by taking limits of these planes. This finishes the proof of Assertion 2.5.

Assertion 2.6  \( \overline{\mathcal{L}} = \mathcal{P} \).

Proof of Assertion 2.6. Arguing by contradiction, we can choose a leaf \( \mathcal{L} \) of \( \overline{\mathcal{L}} \) in \( \overline{\mathcal{L}} - \mathcal{P} \). Since \( S \cup S(\mathcal{L}) \neq \emptyset \), Assertion 2.5 implies that \( \mathcal{P} \neq \emptyset \). By item 6 in Theorem 1.5, \( \mathcal{L} \) does not intersect \( S^A \cup S(\mathcal{L}) \) and there exists a subcollection \( \mathcal{P}(L) \) of one of two planes in \( \mathcal{P} \) such that \( \overline{\mathcal{L}} = L \cup \mathcal{P}(L) \) and \( L \) is proper in the open slab or half-space component \( C(L) \) of \( \mathbb{R}^3 - \mathcal{P}(L) \) which contains \( \mathcal{L} \). Since the convergence of portions of the \( M_n \) to \( L \) has multiplicity one (otherwise \( L \) would be flat), then \( L \) has finite genus. Since \( L \) is connected, Assertion 2.5 implies that \( L \cap \mathcal{P}(L) \) has no singularities, thus \( L \cup \mathcal{P}(L) \) is a possibly singular minimal lamination of \( \mathbb{R}^3 \) (it is a sublamination of \( \overline{\mathcal{L}} \)) with singular set \( S_1 \) contained in \( S \cap \mathcal{P}(L) \). Since \( S \cap \mathcal{P}(L) = (S^A \cup (W \cap \overline{\mathcal{L}})^{\text{sing}}) \cap \mathcal{P}(L) \), the countability of \( W \) together with item 4 of Theorem 1.5 give that \( S \cap \mathcal{P}(L) \) is a countable set of \( \mathbb{R}^3 \). In particular, the singular set \( S_1 \) of \( L \cup \mathcal{P}(L) \) is also countable. Applying Theorem 1.3 to the singular minimal lamination \( L \cup \mathcal{P}(L) \), its item 6 gives that the finite genus leaf
L is the only leaf of $L \cup \mathcal{P}(L)$, which is absurd. This contradiction finishes the proof of Assertion 2.6.

By Assertion 2.6, $\overline{\mathcal{L}} = \mathcal{P}$, which means that $\mathcal{S} = \emptyset$. Since by hypothesis $\mathcal{S} \cup \mathcal{S}(\mathcal{L}) \neq \emptyset$, it follows that $\mathcal{S}(\mathcal{L}) \neq \emptyset$. Let $P \in \mathcal{P}$ be a plane with $P \cap \mathcal{S}(\mathcal{L}) \neq \emptyset$. We claim that $P \cap \mathcal{S}(\mathcal{L})$ contains at most two points. Otherwise, since $P \cap \mathcal{S}(\mathcal{L})$ is discrete in $P - W$, it would contain three isolated points $p_1, p_2, p_3$. Let $\Gamma \subset P - W$ be a smooth, embedded compact arc joining $p_1$ to $p_2$ and disjoint from other points in $\mathcal{S}(\mathcal{L})$. Then the corresponding forming double multigraphs in the surfaces $M_n$ around the points $p_1, p_2$ are oppositely oriented (otherwise, for $n$ large in a fixed size small neighborhood of $\Gamma$ in $\mathbb{R}^3 - W$, the surfaces $M_n$ have unbounded genus, see for example Lemma 1 in [13] for this argument).

Using an analogous local picture near the point $p_3$, one sees that the handedness of the multigraph in $M_n$ near $p_3$ must be opposite to the handedness of the multigraphs in $M_n$ near $p_1$ and near $p_2$, which is impossible since they have opposite handedness. Hence, $P \cap \mathcal{S}(\mathcal{L})$ contains at most two points.

We now complete the proof of item 7 of Theorem 1.5. If $P \cap (\mathcal{S}(\mathcal{L}) \cup \mathcal{S}^A)$ contains exactly two points, then we just proved that the double multigraphs forming in $M_n$ near these two points are oppositely oriented. By the curvature estimates in [6] and the earlier described local picture of $\mathcal{L}$ in a small neighborhood of a point $p \in \mathcal{S}(\mathcal{L})$, it follows that $\mathcal{S}(\mathcal{L})$ is a transverse arc to a local foliation of disks in the planes in $\mathcal{P}$. Meeks’ regularity theorem [12] implies that $\mathcal{S}(\mathcal{L})$ consists of straight line segments orthogonal to $\mathcal{P}$ with end points in $W$. Now choose $P$ so that it is disjoint from $W$, which is always possible since $W$ is countable. Thus, there is a related limiting minimal parking garage structure $\mathcal{F}$ in some $\varepsilon$-neighborhood of $P$, thereby proving that the first two statements of item 7 hold.

The proof of the last statement in item 7 (assuming that the surfaces $M_n$ are compact with boundary) is standard once one has $\overline{\mathcal{L}} = \mathcal{P}$; recall that the hypothesis that $\{M_n\}_n$ has locally positive injectivity radius in $\mathbb{R}^3 - W$ implies that the components of $\partial M_n$ must eventually diverge in space or converge to a subset of $W$. This completes the proof of Theorem 1.5.

3 The proof of Theorem 1.3.

Let $\overline{\mathcal{L}} = \mathcal{L} \cup \mathcal{S}$ be a possibly singular minimal lamination of $\mathbb{R}^3$ with countable closed singular set $\mathcal{S}$ and related (regular) lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$. The set of planes $\mathcal{P}$ in $\overline{\mathcal{L}}$ clearly forms a closed set of $\mathbb{R}^3$, and the set of limit planes $\operatorname{Lim}(\overline{\mathcal{L}})$ of $\overline{\mathcal{L}}$ consists of planes, since if $L$ is a limit leaf of $\mathcal{L}$, then it extends across the countable set $\mathcal{S}$ to a plane by Corollary 2.2. Since the set of limit leaves of a minimal lamination forms a closed set, then $\operatorname{Lim}(\overline{\mathcal{L}})$ forms a closed set of $\mathbb{R}^3$. These observations prove the first two items in the statement of Theorem 1.3. Item 3 follows from the same arguments we used in the proof of the similar item 3 of Theorem 1.5. Item 4 of Theorem 1.3 follows with little modification.
from the arguments given in the proofs of Proposition 2.3 and Corollary 2.4, see also the proof of Assertion 2.5 for a related argument.

Assume from now on that there exists a non-planar leaf $L$ of $\mathcal{L}$. Then, the arguments in the proof of item 6 of Theorem 1.5 apply to prove that item 5 of Theorem 1.3 hold, except for the properties contained in the following assertion.

**Assertion 3.1** Suppose that $\mathcal{L} \neq \{L\}$ and let $\mathcal{L} = \mathcal{L}^{\mathbb{R}^3}(L) \cup \mathcal{P}(L)$ be the sublaminination of $\mathcal{L}$ given by the closure of $L$ in $\mathbb{R}^3$, where $\mathcal{L}^{\mathbb{R}^3}(L)$ is the leaf of $\mathcal{L}$ associated to $L$ and $\mathcal{P}(L)$ is a collection of one or two planes in $\mathcal{P}$. Let $C(L)$ be the component of $\mathbb{R}^3 - \mathcal{P}(L)$ in which $\mathcal{L}^{\mathbb{R}^3}(L)$ is properly embedded. Then:

A. $C(L) \cap L = L$.

B. Given $P \in \mathcal{P}(L)$ and $\varepsilon > 0$, let $P(\varepsilon) = \{x \in C(L) \cup P \mid \text{dist}(x, P) < \varepsilon\}$, where $\text{dist}(\cdot, P)$ denotes extrinsic distance to the plane $P$. Then, $P(\varepsilon)$ intersects $\mathcal{L}^{\mathbb{R}^3}(L)$ in a (possibly singular) connected minimal surface with infinite genus and unbounded Gaussian curvature, for all $\varepsilon > 0$.

**Proof of Assertion 3.1.** If the singular set $S$ is empty, then Assertion 3.1 would follow from the statements of Theorem 1.6 in [19] and from the proof of Corollary 1 in [17], which states that a non-flat finite genus leaf of a minimal lamination of $\mathbb{R}^3$ is a properly embedded minimal surface and the only leaf of the lamination. We will need to check that the proofs presented in these papers can be generalized to the case where $S \neq \emptyset$ and countable. This verification will be more difficult here but it is still possible to carry out because the main tool in these proofs is to produce, via barrier constructions, complete properly embedded stable minimal surfaces which are planes in the complement of a given leaf; in our case, we can similarly construct properly embedded stable minimal surfaces (not necessarily complete) which by Theorem 2.1 and Corollary 2.2 can be extended through the countable set $S$ to complete stable minimal surfaces which are planes.

Let $S_L$ be the set of singular leaf points of $\mathcal{L}^{\mathbb{R}^3}(L)$. Since $\mathcal{L}^{\mathbb{R}^3}(L)$ is connected and not flat, there are no planar leaves of $\mathcal{L}$ in $C(L)$. Now suppose that $L'$ is a non-flat leaf of $\mathcal{L}$ that is different from $L$ and which intersects $C(L)$. Reversing the roles of $L$ and $L'$ one can easily check that $\mathcal{P}(L) = \mathcal{P}(L')$ and $C(L) = C(L')$. As both $L$ and $L'$ are properly embedded in the simply connected region $C(L)$, then $L \cup L'$ bounds a closed region $X$ in $C(L)$; since the two boundary components of $X$ are good barriers for solving Plateau problems in $X$ (in spite of being singular by using Proposition 2.3), a standard argument (see Meeks, Simon and Yau [21]) shows that there exists a properly embedded, least-area surface $\Sigma \subset X$ which separates $L$ from $L'$. However, since $X$ is not necessarily complete, the surface $\Sigma$ may not be complete. On the other hand, it is clear that when considered to be a surface in $\mathbb{R}^3$, $\Sigma$ is complete outside the countable closed set $S \cap \mathcal{P}(L)$. Using our
Local Removable Singularity Theorem 2.1 and Corollary 2.2, we deduce that \( \Sigma \) extends to a complete, stable minimal surface \( \Sigma \) in \( \mathbb{R}^3 \). Since \( \Sigma \) is a plane, clearly \( \Sigma = \Sigma \) is also a plane which is impossible. This proves item \( A \) of Assertion 3.1.

We next prove item \( B \) of the assertion. Fix a plane \( P \in \mathcal{P}(L) \). After a rotation, we may assume that \( P = \{ x_3 = 0 \} \) and \( L \) limits to \( P \) from above. Now take \( \varepsilon > 0 \) and consider the smooth minimal surface \( L_\varepsilon = L \cap \{ 0 < x_3 < \varepsilon \} \). Note that \( L_\varepsilon \) is possibly non-complete (completeness of \( L_\varepsilon \) may fail at the set \( S \cap \{ 0 \leq x_3 \leq \varepsilon \} \)). Since \( S \) is countable, we may also assume that the closure \( \overline{L_\varepsilon} \) in \( \mathbb{R}^3 \) of \( L_\varepsilon \) does not have singularities in the plane \( \{ x_3 = \varepsilon \} \). In a similar way as in the last paragraph, applying the proof of Theorem 1.6 in [19] and using the local extendability of a stable minimal surface in \( \overline{C(L)} \) which is complete outside of \( S \cap \mathcal{P}(L) \) and has its boundary in a plane in \( C(L) \), one sees that \( \{ 0 \leq x_3 \leq \varepsilon \} \) intersects \( \overline{L}^{\mathbb{R}^3}(L) \) in a connected set.

We next prove that the Gaussian curvature of \( L_\varepsilon \) is unbounded. Reasoning by contradiction, assume \( L_\varepsilon \) has bounded Gaussian curvature. In this case, Theorem 2.1 and the proof of Corollary 2.2 imply that \( L_\varepsilon \cup P \) is a minimal lamination of \( \{ 0 \leq x_3 < \varepsilon \} \). In this situation, one can apply Lemma 1.4 in [19] to deduce that \( L_\varepsilon \) is a graph over its projection into \( P \), in particular it is proper in the closed slab \( \{ 0 \leq x_3 \leq \varepsilon \} \), which contradicts the proof of the Half-space Theorem. Hence, \( L_\varepsilon \) has unbounded Gaussian curvature.

To finish the proof of Assertion 3.1, it remains to demonstrate that the smooth, connected, possibly non-complete minimal surface \( L_\varepsilon \) has infinite genus. If this property were to fail, then we can first choose \( \varepsilon \) sufficiently small so that \( L_\varepsilon \) has genus zero. Since \( S \) is countable, we may also assume that the closure \( \overline{L_\varepsilon} \) of \( L_\varepsilon \) does not have singularities in the plane \( \{ x_3 = \varepsilon \} \). Since there are non-planar leaves of \( \mathcal{L} \) in \( C(L) \) and \( L_\varepsilon \) has genus zero, then item 4.2 of Theorem 1.3 implies that \( S \cap \{ 0 < x_3 < \varepsilon \} = \emptyset \) and thus, \( L_\varepsilon \) is a smooth, connected minimal surface with genus zero, which is complete outside a countable number of points in \( P \) and has boundary on the plane \( \{ x_3 = \varepsilon \} \). In [17], we considered a related easier situation where \( L \) is a (complete) leaf of finite genus in a non-singular minimal lamination of \( \mathbb{R}^3 \) with more than one leaf. In that paper, we obtained a contradiction to the existence of such a minimal leaf by applying a blow-up argument on the scale of topology together with a variant of the López-Ros deformation argument (see [11] for details on this argument). Instead of using the López-Ros argument here to obtain a contradiction to the existence of \( L_\varepsilon \), we will need a different approach to obtain a contradiction.

It remains to prove that the set \( \mathcal{S}_\varepsilon \subset P \) of singularities of \( \overline{L_\varepsilon} \) is empty, since the same argument would show that \( \mathcal{L} \) is a minimal lamination of \( \mathbb{R}^3 \) which we have already shown to be impossible in this case. Arguing by contradiction, assume that \( \mathcal{S}_\varepsilon \) is non-empty. Since this set is closed and countable, there exists an isolated point \( p \in \mathcal{S}_\varepsilon \). After a translation and homothety, assume \( p = \hat{0} \) and \( \mathcal{B}(2\delta) \cap \mathcal{S}_\varepsilon = \{ \hat{0} \} \) for some positive \( \delta < \frac{\varepsilon}{2} \).

Let \( I_L \) be the injectivity radius function of \( L \). We will find the desired contradiction by discussing whether or not \( I_L|_{L \cap \mathcal{B}(\delta)} \) decays faster than linearly in terms of the extrinsic
distance $| \cdot | = d(\cdot, \bar{0})$ from $L$ to $\bar{0}$.

Case I: Suppose that $I_L/| \cdot |$ is not bounded away from zero in $L \cap \mathbb{B}(\delta)$.

By the proof of the local picture theorem on the scale of topology (Theorem ** in [15]), there exists a sequence $\{p_n\}_n \subset L$ of blow-up points on the scale of topology such that the following properties hold:

- $\lim_{n \to \infty} p_n = \bar{0}$ and $\lim_{n \to \infty} \frac{I_L(p_n)}{|p_n|} = 0$.
- For all $n$ large, there exists an Euclidean ball $\mathbb{B}(p_n, \varepsilon_n)$ with $0 < \varepsilon_n < |p_n|$ and $\varepsilon_n \to 0$, such that the component $L(n)$ of $L \cap \mathbb{B}(p_n, \varepsilon_n)$ containing $p_n$ is compact and has its boundary in $\partial \mathbb{B}(p_n, \varepsilon_n)$.
- Defining $\lambda_n = 1/I_L(p_n)$, then the sequence $\{M_n = \lambda_n [L(n) - p_n]\}_n$ converges as $n \to \infty$ to either a non-simply connected, properly embedded minimal surface $M_{\infty} \subset \mathbb{R}^3$ of genus zero or to a parking garage structure with two oppositely oriented columns.

We first consider the case where the limit object of the sequence $\{M_n\}_n$ is a catenoid $M_{\infty}$. We refer the reader to our paper [13] for more details in the following construction. In this case, there exists a plane $Q$ in $\mathbb{R}^3$ which intersects $M_{\infty}$ orthogonally along its waist circle $\Gamma$. Let $\Gamma_n \subset M_n \cap Q$ be the nearby simple closed curves in $M_n$ for $n$ large and let $\gamma_n = p_n + \frac{1}{\lambda_n} \Gamma_n$ be the related simple closed planar curves in $L(n)$. In particular, the sequence of simple closed curves $\gamma_n \subset L(n)$ near $p_n$ have lengths converging to zero as $n \to \infty$.

Let $W$ be the closure of a component of $C(L) - L$ in which $\gamma_n$ fails to bound a disk (after extracting a subsequence, we can assume that $W$ does not depend upon $n$). Our previous arguments using $L$ as a barrier imply that $\gamma_n$ is the boundary of an area-minimizing, non-compact, orientable, properly embedded minimal surface $\Sigma_n \subset W$. The surfaces $\Sigma_n$ are complete in $\mathbb{R}^3$ outside of the countable set $P(L) \cap S$. Since the $\Sigma_n$ are stable, our previous arguments show that each $\Sigma_n$ extends to a complete, orientable, properly embedded, stable minimal surface $\Sigma_n \subset W$ with boundary $\gamma_n$. By Fischer-Colbrie [8], each $\Sigma_n$ has finite total curvature. Curvature estimates for stable minimal surfaces and the flat geometry of the non-simply complement $X$ of the catenoid $M_{\infty}$ imply that for $n$ large and outside of a small neighborhood $N(p_n) \subset \mathbb{B}(p_n, \varepsilon_n)$ of $p_n$ intersected with the region $X_n \subset \mathbb{B}(p_n, \varepsilon_n)$ defined by $\lambda_n(X_n - p_n) \subset X$ and $\lambda_n(\partial X_n - p_n) \subset \partial X$, where the catenoidal region of $M_n$ is well-formed, the stable, finite total curvature surface $\Omega_n = \Sigma_n - N(p_n)$ is connected with connected boundary and the Gauss map $G_n$ of $\Omega_n$ along $\partial \Omega_n$ is almost constant and equal to the normal vector of one of the ends of $M_{\infty}$. On the other hand, the ends of $\Sigma_n$ must be parallel to the plane $P$ since $\Omega_n \subset \bar{\Sigma_n} \subset W$. The openness of the Gauss map $G_n$ implies that the spherical image $G_n(\Omega_n) \subset S^2$ is contained in arbitrarily small neighborhoods of the north or south pole of $S^2$ as $n \to \infty$. It follows that $M_{\infty}$ is a vertical catenoid, that $\Omega_n$
is a graph over the complement of a small disk in $P$ and that this graph has non-negative logarithmic growth. Using $\Omega_n$, it is straightforward to construct a topological plane $P_n$ which is a graph over $P$, as the union of $\Omega_n$ together with an annulus in $X_n$ and the horizontal disk bounded by $\gamma_n$. Also, it is not difficult to construct the planes $P_n$ so that they are pairwise disjoint (see [13] for this type of argument).

After choosing a subsequence and re-indexing, the ordering of the indices of the planes $\{P_n\}_n$ agrees with the inverse ordering of the relative heights of these topological planes over $P$. Let $W_n$ be the closed region of $C(L)$ between $P_n$ and $P_{n+1}$. By Lemma 2.2 in [7], the surface $L \cap W_n$ has a finite number of genus zero ends, which therefore must be either planar or catenoidal ends of non-negative logarithmic growth. It follows that the norms of the non-zero vertical components of the fluxes of $L$ along the curves $\gamma_n$ are non-decreasing as $n$ increases. Since these fluxes are not greater than the lengths of the $\gamma_n$, which are converging to zero as $n \to \infty$, we obtain a contradiction.

The above arguments show that the limit object $M_\infty$ of the $M_n$ is not a catenoid. Thus, either the limit object is a properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ of genus zero, which must have two limit ends by Theorem 1 in [18], or else the limit object of the sequence $\{M_n\}_n$ is a parking garage structure on $\mathbb{R}^3$ with two oppositely oriented columns. In either of these cases, the arguments in the previously considered case where $M_\infty$ was a catenoid can be adapted in a straightforward manner to obtain a contradiction to the non-zero flux of $L$ (see [13] for similar adaptations). Thus, Case I does not occur at any isolated point in $S_\varepsilon$.

**Case II: Suppose that there is a constant $c > 0$ such that $I_L > c \cdot |\cdot|$ in $P(\varepsilon)$.**

Since $L_\varepsilon$ does not extend across $\{0\}$, Theorem 2.1 implies that there exists a sequence of points $\{p_n\}_n \subset L_\varepsilon$ converging to $0$ with $|K_L(p_n)|p_n| \geq n$. Consider the sequence of related minimal surfaces $M'_n = \frac{1}{|p_n|}L_\varepsilon$ and note that by letting $W = \{0\}$, these surfaces satisfy the hypothesis of Theorem 1.5. Since these surfaces have genus zero, Theorem 1.5 gives that a subsequence converges to a possibly singular minimal lamination $\Lambda$ of $\mathbb{R}^3$. Since the curvatures of these surfaces are unbounded on the unit sphere $S^2$, then the singular set of convergence $S(\Lambda)$ of $M'_n$ to $\Lambda$ is non-empty.

By the curvature estimate for embedded minimal disks in [1], the injectivity radius function of the surfaces $M_n$ at the points $\frac{p_n}{|p_n|}$ must converge to zero if $x_3(\frac{p_n}{|p_n|}) \to 0$ as $n \to \infty$. By our hypothesis in Case II, the injectivity radius function of the surfaces $M_n$ at these points is bounded away from zero. Hence, $S(\Lambda) \cap S^2$ lies above a vertical cone based at $0$, and more generally, this argument shows that $S(\Lambda)$ lies above some cone $C$.

We claim that: $\Lambda$ is a foliation of the closed upper half-space of $\mathbb{R}^3$ by horizontal planes and with singular set of convergence being the non-negative $x_3$-axis. This claim would follow directly from item 7 of Theorem 1.5 but we cannot apply this item because its proof depends on item 6 of Theorem 1.3. However, we can and will apply other items in the statement of Theorem 1.5 to prove our claim. By construction, there exists a
point $x \in S(\Lambda) \cap S^2$ which lies above the cone $C$. It follows from curvature estimates for embedded minimal disks [6] and Meeks’ $C^{1,1}$-regularity theorem [12] that if the leaves of $\Lambda$ are planes, then the claim would hold. So suppose $L_\Lambda$ is a non-flat leaf of $\Lambda$ and we will find a contradiction.

By item 4 of Theorem 1.5, there exists a horizontal planar $P_x$ passing through $x$. Item 6.2 of Theorem 1.5 implies that $L_\Lambda$ lies above the limit plane $P_1$. Assume for the sake of concreteness that $L_\Lambda$ lies above the plane $P_1$. Since the singular set of convergence $S(\Lambda)$ lies above $C$ and the sequence $\{M_n\}_n$ is locally simply connected in a fixed size neighborhood $P_1(\delta)$ of $P_1$, then $L_\Lambda$ contains a finite number of singularities on $P_1$ and in a neighborhood of each such singularity, $L_\Lambda$ has the appearance of a disk which has the geometry of a spiraling double staircase which limits to a disk in $P_1$ centered at the singularity. Since this set of singularities is finite, the arguments at the end of the proof of the local picture theorem on the scale of topology in [15] give a contradiction. Hence, $L_\Lambda$ contains no singularities on $P_1$. Note that $L_\Lambda$ has injectivity radius bounded away from zero in $P_1(\delta)$ for some $\delta > 0$, because this can be assumed to be true outside of the singular set of $\Lambda$, and so, $L_\Lambda$ is a lamination near $P_1$.

By curvature estimates in [1], $L_\Lambda$ has bounded curvature in some smaller neighborhood $P_1(\delta')$ of $P_1$. This last observation contradicts Lemma 1.3 in [19]. This completes the proof of our claim. It also follows from these arguments that $L \cap B(\delta)$ is a disk for some sufficiently small $\delta > 0$ and that this disk has the geometry of a spiraling double staircase limiting to the disk $B(\delta) \cap P$.

Since $L_\varepsilon$ is proper in the half-open slab $P \times (0, \varepsilon]$, the above arguments imply that for given $k$ isolated points $\{p_1, p_2, ..., p_k\} \subset S(T_\varepsilon) \subset P$, there exists disjoint disks $D(p_k, \varepsilon_k) \subset P$ such that the $\partial D(p_k, \varepsilon_k) \times (0, \varepsilon]$ intersects $L_\varepsilon$ in two spiraling curves that limit to the circle $\partial D(p_k, \varepsilon_k) \times \{0\}$. Straightforward modifications of the topological and flux-type arguments near the end of the proof of the local picture theorem on the scale of topology in [15], show that there must exist exactly two singular points of $S(T_\varepsilon)$ and connecting loops $\gamma_n \subset L_\varepsilon$ along which $L_\varepsilon$ has non-decreasing, non-negative scalar flux of $\nabla x_3$ (because $\gamma_n \cup \gamma_{n+1}$ bounds a proper subdomain of $L_\varepsilon$ with a finite number of horizontal ends of finite total curvature, which therefore must be either planar or catenoidal ends of positive logarithmic growth). As $n \to \infty$, these loops are becoming almost-horizontal with uniformly bounded length, and so, the third component of the flux of $L_\varepsilon$ along $\gamma_n$ must converge to zero. This contradiction proves that CASE II cannot occur, and thus, the singular set $S_\varepsilon$ of $L_\varepsilon$ is empty, thereby finishing the proof of Assertion 3.1, which in turn demonstrates the validity of item 5 of Theorem 1.3. Item 6 of Theorem 1.3 follows immediately from item 5. The theorem now follows. \qed
4 The proof of Theorem 1.6.

We now prove Theorem 1.6. Suppose \( \{M_n\}_n \) is a sequence of compact, embedded minimal surfaces of finite genus \( g \) in \( \mathbb{R}^3 \) with \( \partial M_n \subset \partial B(n) \) for each \( n \). By Theorem 0.14 in [2] (see also Footnote 8 in the statement of that theorem), a subsequence of these compact minimal surfaces converges to a possibly singular minimal lamination \( \mathcal{L} \) of \( \mathbb{R}^3 \). Replace \( \{M_n\}_n \) by this convergent subsequence. If either the singular set of \( \mathcal{L} \) or the singular set of convergence of the \( M_n \) to \( \mathcal{L} \) is non-empty, then Lemmas VI.2.1 and VI.3.1 in [2] imply that every leaf of \( \mathcal{L} \) is contained in a plane. Our hypothesis that a sequence of disks \( D_n \subset M_n \) converges \( C^2 \) to a non-flat minimal disk implies that there is a non-flat leaf \( L \) in \( \mathcal{L} \). It then follows that \( \mathcal{L} \) has no singularities and the singular set of convergence of \( \{M_n\}_n \) to \( \mathcal{L} \) is empty. By the Structure Theorem for (regular) minimal laminations of \( \mathbb{R}^3 \) (see Theorem 1.6 in [19]), the collection \( \mathcal{P} \) of planes in \( \mathcal{L} \) forms a possibly empty, closed set of \( \mathbb{R}^3 \), each of the components \( W \) of \( \mathbb{R}^3 - \mathcal{P} \) contains at most one leaf \( L_W \) of \( \mathcal{L} \), and such a leaf \( L_W \) is proper in \( W \). By standard lifting arguments, one can lift any handle on a leaf \( L' \) of \( \mathcal{L} \) to a nearby handle on an approximating surface \( M_n \) for \( n \) large, such that any fixed finite collection of pairwise disjoint handles in \( L' \) lift to disjoint handles on the nearby \( M_n \). Since the genus of each \( M_n \) is \( g \), then \( L' \) has genus at most \( g \). By Corollary 1 in [17], \( L \) is the only leaf in \( \mathcal{L} \) and \( L \) is properly embedded in \( \mathbb{R}^3 \). This proves the first item in Theorem 1.6 with \( M_\infty \) being \( L \).

By Theorem 1 in [18], the surface \( M_\infty \) is either a surface of finite total curvature, a helicoid with handles or a two limit end minimal surface. The same theorem states that \( M_\infty \) has bounded curvature, which completes the proof of Theorem 1.6.

William H. Meeks, III at bill@math.umass.edu
Mathematics Department, University of Massachusetts, Amherst, MA 01003

Joaquín Pérez at jperez@ugr.es
Department of Geometry and Topology, University of Granada, Granada, Spain

Antonio Ros at aros@ugr.es
Department of Geometry and Topology, University of Granada, Granada, Spain

References


