

Limit leaves of a CMC lamination are stable

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Abstract

Suppose \mathcal{L} is a lamination of a Riemannian manifold by hypersurfaces with the same constant mean curvature. We prove that every limit leaf of \mathcal{L} is stable for the Jacobi operator. A simple but important consequence of this result is that the set of stable leaves of \mathcal{L} has the structure of a lamination.

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1 Introduction.

In this paper we prove that given a codimension one lamination \mathcal{L} in a Riemannian manifold N , whose leaves have a fixed constant mean curvature (minimality is included), then every limit leaf L of \mathcal{L} is stable with respect to the Jacobi operator. Our result is motivated by a partial result of Meeks and Rosenberg in Lemma A.1 in [6], where they proved the stability of L under the constraint that the holonomy representation on any compact subdomain $\Delta \subset L$ has subexponential growth (i.e., the covering space $\tilde{\Delta}$ of Δ corresponding to the kernel of the holonomy representation has subexponential area growth). In general, if we assume stability for a covering space \tilde{M} of a constant mean curvature hypersurface M in N and for any connected compact domain $\Delta \subset M$ the related restricted covering $\tilde{\Delta} \rightarrow \Delta$ has subexponential area growth, then M is also stable, see Lemma 6.2 in [4] for a proof using cutoff functions. However, if the area growth of the covering is exponential over some compact domain in M , then the stability of \tilde{M} does not imply the stability of M , as can be seen in the example described in the next paragraph. The existence of this example makes it clear that the application in [6] of cutoff

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functions used to prove the stability of a limit leaf L with holonomy of subexponential growth cannot be applied to case when the holonomy representation of L has exponential growth.

Consider a compact surface Σ endowed with a metric g of constant curvature -1 and a smooth function $f: \mathbb{R} \rightarrow (0, 1]$ with $f(0) = 1$ and $f''(0) < 0$. Then in the warped product metric $g + f^2 dt^2$ on $\Sigma \times \mathbb{R}$, the slice $M = \Sigma \times \{0\}$ is totally geodesic and unstable as a minimal surface. On the other hand, the universal cover \widetilde{M} of M is the hyperbolic plane and the Jacobi operator of \widetilde{M} reduces to its laplacian operator, which has positive first eigenvalue. Thus, as an immersed minimal surface, \widetilde{M} is stable. Similarly, for t sufficiently small the surface $M \times \{t\}$ has constant mean curvature and is unstable but its related universal cover is stable.

2 The statement and proof of the main theorem.

In order to help understand the results described in this paper, we make the following definitions.

Definition 1 Let M be a complete, embedded hypersurface in a manifold N . A point $p \in N$ is a *limit point* of M if there exists a sequence $\{p_n\}_n \subset M$ which diverges to infinity on M with respect to the intrinsic Riemannian topology on M but converges in N to p as $n \rightarrow \infty$. Let $L(M)$ denote the set of all limit points of M in N . In particular, $L(M)$ is a closed subset of N and $\overline{M} - M \subset L(M)$, where \overline{M} denotes the closure of M .

Definition 2 A *codimension one lamination* of a Riemannian manifold N^{n+1} is the union of a collection of pairwise disjoint, connected, injectively immersed hypersurfaces, with a certain local product structure. More precisely, it is a pair $(\mathcal{L}, \mathcal{A})$ satisfying:

1. \mathcal{L} is a closed subset of N ;
2. $\mathcal{A} = \{\varphi_\beta: \mathbb{D}^n \times (0, 1) \rightarrow U_\beta\}_\beta$ is a collection of coordinate charts of N (here \mathbb{D}^n is the open unit ball in \mathbb{R}^n , $(0, 1)$ the open unit interval and U_β an open subset of N);
3. For each β , there exists a closed subset C_β of $(0, 1)$ such that $\varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D}^n \times C_\beta$.

We will simply denote laminations by \mathcal{L} , omitting the charts φ_β in \mathcal{A} . A lamination \mathcal{L} is said to be a *foliation* of N if $\mathcal{L} = N$. Every lamination \mathcal{L} naturally decomposes into a collection of disjoint connected hypersurfaces, called the *leaves* of \mathcal{L} . As usual, the regularity of \mathcal{L} requires the corresponding regularity on the change of coordinate charts. Note that if $\Delta \subset \mathcal{L}$ is any collection of leaves of \mathcal{L} , then the closure of the union of these leaves has the structure of a lamination within \mathcal{L} , which we will call a *sublamination*.

Definition 3 A *codimension one CMC lamination* \mathcal{L} of a manifold N is a collection of immersed (not necessarily injectively) hypersurfaces $\{L_\alpha\}_{\alpha \in I}$ of constant mean curvature H (independent of α), called the *leaves* of \mathcal{L} , satisfying the following properties.

1. $\mathcal{L} = \bigcup_{\alpha \in I} \{L_\alpha\}$ is a closed subset of N .
2. If $H = 0$, then \mathcal{L} is a lamination of N .
3. If $H \neq 0$, then given a leaf L_α of \mathcal{L} , considered to be the zero section Z_α of its normal bundle L_α^\perp , there exists a one-sided neighborhood $U(Z_\alpha) \subset L_\alpha^\perp$ of Z_α such that:
 - (a) The exponential map $\exp: U(Z_\alpha) \rightarrow N$ is a submersion.
 - (b) With respect to the pullback metric on $U(Z_\alpha)$, $Z_\alpha \subset \partial U(Z_\alpha)$ is mean convex.
 - (c) The inverse image $\exp^{-1}(\mathcal{L}) \cap U(Z_\alpha)$ is a lamination of $U(Z_\alpha)$.

The reader not familiar with the subject of minimal or CMC laminations should think about a geodesic γ on a Riemannian surface. If γ is complete and embedded (a one-to-one immersion), then its closure is a geodesic lamination \mathcal{L} of the surface. When the geodesic γ has no accumulation points, then it is proper. Otherwise, there pass complete embedded geodesics in \mathcal{L} through the accumulation points of γ forming the leaves of \mathcal{L} . A similar result is true for a complete, embedded CMC hypersurface of locally bounded second fundamental form (bounded in compact extrinsic balls) in a Riemannian manifold, i.e., the closure of a complete, embedded CMC hypersurface of locally bounded second fundamental form has the structure of a CMC lamination. For the sake of completeness, we now give the proof of this elementary fact (see the beginning of Section 1 in [5] for the proof in the minimal case).

Consider a complete, embedded CMC hypersurface M of constant mean curvature H with locally bounded second fundamental form in a manifold N . Consider a limit point p of M , i.e., p is the limit of a sequence of divergent points p_n in M . Since M has bounded second fundamental form near p and M is embedded, then for some small $\varepsilon > 0$, a subsequence of the intrinsic ε -balls $B_M(p_n, \varepsilon)$ converges to an embedded CMC ball $B(p, \varepsilon) \subset N$ of intrinsic radius ε , centered at p and of constant mean curvature H . Since M is embedded, any two such limit balls, say $B(p, \varepsilon)$, $B'(p, \varepsilon)$, do not intersect transversally. By the maximum principle for CMC hypersurfaces, we conclude that if a second ball $B'(p, \varepsilon)$ exists, then $B(p, \varepsilon)$, $B'(p, \varepsilon)$ are the only such limit balls and they are oppositely oriented at p .

Now consider any sequence of embedded balls E_n of the form $B(q_n, \frac{\varepsilon}{4})$ such that q_n converges to a point in $B(p, \frac{\varepsilon}{2})$ and such that E_n locally lies on the mean convex side of $B(p, \varepsilon)$. For ε sufficiently small and for n, m large, E_n and E_m must be graphs over domains in $B(p, \varepsilon)$ such that when oriented as graphs, they have the same mean curvature. By the maximum principle, the graphs E_n and E_m are disjoint or equal. It follows that near p and on the mean convex side of $B(p, \varepsilon)$, \overline{M} has the structure of a lamination with leaves of the same constant mean curvature as M . This proves that \overline{M} has the structure of a CMC lamination of codimension one.

Definition 4 Let \mathcal{L} be a codimension one CMC lamination of a manifold N and L be a leaf of \mathcal{L} . We say that L is a *limit leaf* if L is contained in the closure of $\mathcal{L} - L$.

We claim that a leaf L of a codimension one CMC lamination \mathcal{L} is a limit leaf if and only if for any point $p \in L$ and any sufficiently small intrinsic ball $B \subset L$ centered at p , there exists a sequence of pairwise disjoint balls B_n in leaves L_n of \mathcal{L} which converges to B in N as $n \rightarrow \infty$, such that each B_n is disjoint from B . Furthermore, we also claim that the leaves L_n can be chosen different from L for all n . The implication where one assumes that L is a limit leaf of \mathcal{L} is clear. For the converse, it suffices to pick a point $p \in L$ and prove that p lies in the closure of $\mathcal{L} - L$. By hypothesis, there exists a small intrinsic ball $B \subset L$ centered at p which is the limit in N of pairwise disjoint balls B_n in leaves L_n of \mathcal{L} , as $n \rightarrow \infty$. If $L_n \neq L$ for all $n \in \mathbb{N}$, then we have done. Arguing by contradiction and after extracting a subsequence, assume $L_n = L$ for all $n \in \mathbb{N}$. Choosing points $p_n \in B_n$ and repeating the argument above with p_n instead of p , one finds pairwise disjoint balls $B_{n,m} \subset L$ which converge in N to B_n as $m \rightarrow \infty$. Note that for $(n_1, m_1) \neq (n_2, m_2)$, the related balls $B_{n_1, m_1}, B_{n_2, m_2}$ are disjoint. Iterating this process, we find an uncountable number of such disjoint balls on L , which contradicts that L admits a countable basis for its intrinsic topology.

Definition 5 A minimal hypersurface $M \subset N$ of dimension n is said to be *stable* if for every compactly supported normal variation of M , the second variation of area is non-negative. If M has constant mean curvature H , then M is said to be *stable* if the same variational property holds for the functional $A - nHV$, where A denotes area and V stands for oriented volume. A *Jacobi function* $f: M \rightarrow \mathbb{R}$ is a solution of the equation $\Delta f + |A|^2 f + \text{Ric}(\eta)f = 0$ on M ; if M is two-sided, then the stability of M is equivalent to the existence of a positive Jacobi function on M (see Fischer-Colbrie [1]).

The proof of the next theorem is motivated by a well-known application of the divergence theorem to prove that every compact domain in a leaf of an oriented, codimension one minimal foliation in a Riemannian manifold is area-minimizing in its relative \mathbb{Z} -homology class. For other related applications of the divergence theorem, see [8].

Theorem 1 *The limit leaves of a codimension one CMC lamination of a Riemannian manifold are stable.*

Proof. We will assume that the dimension of the ambient manifold N is three in this proof; the arguments below can be easily adapted to the n -dimensional setting. The first step in the proof is the following result.

Assertion 1 *Suppose $\overline{D}(p, r)$ is a compact, embedded CMC disk in N with constant mean curvature H (possibly negative), intrinsic diameter $r > 0$ and center p , such that there exist global normal coordinates (q, t) based at points $q \in \overline{D}(p, r)$, with $t \in [0, \varepsilon]$. Suppose that $T \subset [0, \varepsilon]$ is a closed set disconnected set with zero as a limit point and for each $t \in T$, there exists a function $f_t: \overline{D}(p, r) \rightarrow [0, \varepsilon]$ such that the normal graphs $q \mapsto \exp_q(f_t(q)\eta(q))$ define pairwise disjoint*

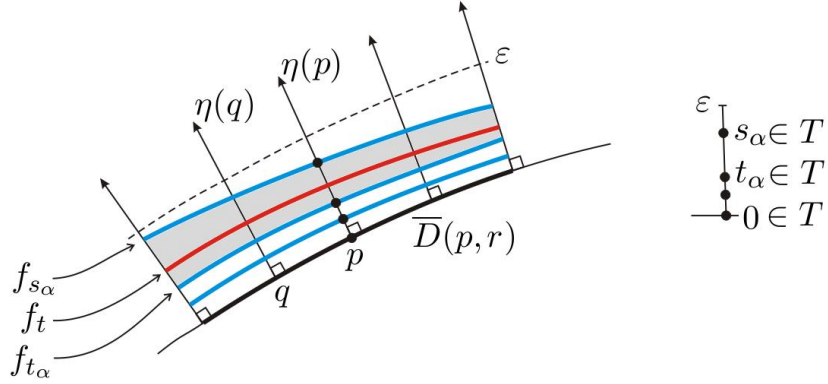


Figure 1: The interpolating graph of f_t between the CMC graphs of $f_{t_\alpha}, f_{s_\alpha}$.

surfaces of constant mean curvature H , with $f_t(p) = t$, where η stands for the oriented unit normal vector field to $\overline{D}(p, r)$. For each component (t_α, s_α) of $[0, \varepsilon] - T$ with $s_\alpha < \varepsilon$, consider the interpolating graphs $q \mapsto \exp_q(f_t(q)\eta(q))$, $t \in [t_\alpha, s_\alpha]$, where

$$f_t = f_{t_\alpha} + (t - t_\alpha) \frac{f_{s_\alpha} - f_{t_\alpha}}{s_\alpha - t_\alpha}.$$

(See Figure 1). Then, the mean curvature functions H_t of the graphs of f_t satisfy

$$\lim_{t \rightarrow 0^+} \frac{H_t(q) - H}{t} = 0 \quad \text{for all } q \in D(p, \varepsilon/2).$$

Proof of Assertion 1. Reasoning by contradiction, suppose there exists a sequence $t_n \in [0, \varepsilon] - T$, $t_n \searrow 0$, and points $q_n \in \overline{D}(p, r/2)$, such that $|H_{t_n}(q_n) - H| > Ct_n$ for some constant $C > 0$. Let $(t_{\alpha_n}, s_{\alpha_n})$ be the component of $[0, \varepsilon] - T$ which contains t_n . Then, we can rewrite f_{t_n} as

$$f_{t_n} = t_n \left[\frac{t_{\alpha_n}}{t_n} \frac{f_{t_{\alpha_n}}}{t_{\alpha_n}} + \left(1 - \frac{t_{\alpha_n}}{t_n}\right) \frac{f_{s_{\alpha_n}} - f_{t_{\alpha_n}}}{s_{\alpha_n} - t_{\alpha_n}} \right].$$

After extracting a subsequence, we may assume that as $n \rightarrow \infty$, the sequence of numbers $\frac{t_{\alpha_n}}{t_n}$ converges to some $A \in [0, 1]$, and the sequences of functions $\frac{f_{t_{\alpha_n}}}{t_{\alpha_n}}, \frac{f_{s_{\alpha_n}} - f_{t_{\alpha_n}}}{s_{\alpha_n} - t_{\alpha_n}}$ converge smoothly to Jacobi functions F_1, F_2 on $\overline{D}(p, r/2)$, respectively. Now consider the normal variation of $\overline{D}(p, r/2)$ given by

$$\tilde{\psi}_t(q) = \exp_q(t[AF_1 + (1 - A)F_2](q)\eta(q)),$$

for $t > 0$ small. Since $AF_1 + (1 - A)F_2$ is a Jacobi function, the mean curvature \tilde{H}_t of $\tilde{\psi}_t$ is $\tilde{H}_t = H + \mathcal{O}(t^2)$, where $\mathcal{O}(t^2)$ stands for a function satisfying $t\mathcal{O}(t^2) \rightarrow 0$ as $t \rightarrow 0^+$. On the

other hand, the normal graphs of f_{t_n} and of $t_n(AF_1 + (1 - A)F_2)$ over $\overline{D}(p, r/2)$ can be taken arbitrarily close in the C^4 -norm for n large enough, which implies that their mean curvatures H_{t_n}, \tilde{H}_{t_n} are C^2 -close. This is a contradiction with the assumed decay of H_{t_n} at q_n . \square

We now continue the proof of the theorem. Let L be a limit leaf of a CMC lamination \mathcal{L} of a manifold N by hypersurfaces. If L is one-sided, then we consider the two-sided 2:1 cover $\tilde{L} \rightarrow L$ and pullback the CMC lamination \mathcal{L} to a small neighborhood of the zero section \tilde{L}_0 of the normal bundle \tilde{L}^\perp to \tilde{L} (\tilde{L}_0 can be identified with \tilde{L} itself). In this case, we will prove that \tilde{L}_0 is stable, which in particular implies stability for L , see Remark 2. Hence, in the sequel we will assume L is two-sided.

Arguing by contradiction, suppose there exists an unstable compact subdomain $\Delta \subset L$ with non-empty smooth boundary $\partial\Delta$. Given a subset $A \subset \Delta$ and $\varepsilon > 0$ sufficiently small, we define

$$A^{\perp, \varepsilon} = \{\exp_q(t\eta(q)) \mid q \in A, t \in [0, \varepsilon]\}$$

to be the one-sided vertical ε -neighborhood of A , written in normal coordinates (q, t) (here we have picked the unit normal η to L such that L is a limit of leaves of \mathcal{L} at the side η points into). Since \mathcal{L} is a lamination and Δ is compact, there exists $\delta \in (0, \varepsilon)$ such that the following property holds:

(\star) *Given an intrinsic disk $D(p, \delta) \subset L$ centered at a point $p \in \Delta$ with radius δ , and given a point $x \in \mathcal{L}$ which lies in $D(p, \delta)^{\perp, \varepsilon/2}$, then there passes a disk $D_x \subset \mathcal{L}$ through x , which is entirely contained in $D(p, \delta)^{\perp, \varepsilon}$, and D_x is a normal graph over $D(p, \delta)$.*

Fix a point $p \in \Delta$ and let $x \in \mathcal{L} \cap \{p\}^{\perp, \varepsilon/2}$ be the point above p with greatest t -coordinate. Consider the disk D_x given by property (\star), which is the normal graph of a function f_x over $D(p, \delta)$. Since Δ is compact, ε can be assumed to be small enough so that the closed region given in normal coordinates by $U(p, \delta) = \{(q, t) \mid q \in D(p, \delta), 0 \leq t \leq f_x(q)\}$ intersects \mathcal{L} in a closed collection of disks $\{D(t) \mid t \in T\}$, each of which is the normal graph over $D(p, \delta)$ of a function $f_t: D(p, \delta) \rightarrow [0, \varepsilon)$ with $f_t(p) = t$, and T is a closed subset of $[0, \varepsilon/2]$, see Figure 2. We now foliate the region $U(p, \delta) - \bigcup_{t \in T} D(t)$ by interpolating the graphing functions as we did in Assertion 1. Consider the union of all these locally defined foliations \mathcal{F}_p with p varying in Δ . Since Δ is compact, we find $\varepsilon_1 \in (0, \varepsilon/2)$ such that the one-sided normal neighborhood $\Delta^{\perp, \varepsilon_1} \subset \bigcup_{p \in \Delta} \mathcal{F}_p$ of Δ is foliated by surfaces which are portions of disks in the locally defined foliations \mathcal{F}_p . Let $\mathcal{F}(\varepsilon_1)$ denote this foliation of $\Delta^{\perp, \varepsilon_1}$. By Assertion 1, the mean curvature function of the foliation $\mathcal{F}(\varepsilon_1)$ viewed locally as a function $H(p, t)$ with $p \in \Delta$ and $t \in [0, \varepsilon_1]$, satisfies

$$\lim_{t \rightarrow 0^+} \frac{H(p, t) - H}{t} = 0, \quad \text{for all } p \in \Delta. \quad (1)$$

On the other hand since Δ is unstable, the first eigenvalue λ_1 of the Jacobi operator J for the Dirichlet problem on Δ , is negative. Consider a positive eigenfunction h of J on Δ (note

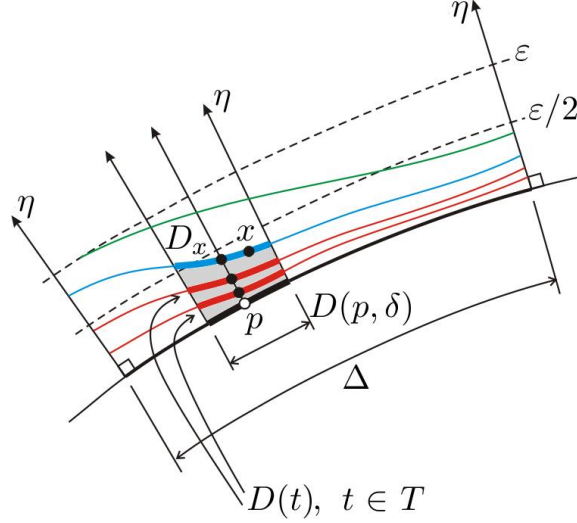


Figure 2: The shaded region between D_x and $D(p, \delta)$ corresponds to $U(p, \delta)$.

that $h = 0$ on $\partial\Delta$. For $t \geq 0$ small and $q \in \Delta$, $\exp_q(th(q)\eta(q))$ defines a family of surfaces $\{\Delta(t)\}_t$ with $\Delta(t) \subset \Delta^{\perp, \varepsilon}$ and the mean curvature \widehat{H}_t of $\Delta(t)$ satisfies

$$\left. \frac{d}{dt} \right|_{t=0} \widehat{H}_t = Jh = -\lambda_1 h > 0 \quad \text{on the interior of } \Delta. \quad (2)$$

Let $\Omega(t)$ be the compact region of N bounded by $\Delta \cup \Delta(t)$ and foliated away from $\partial\Delta$ by the surfaces $\Delta(s)$, $0 \leq s \leq t$. Consider the smooth unit vector field V defined at any point $x \in \Omega(t) - \partial\Delta$ to be the unit normal vector to the unique leaf $\Delta(s)$ which passes through x , see Figure 3. Since the divergence of V at $x \in \Delta(s) \subset \Omega(t)$ equals $-2\widehat{H}_s$ where \widehat{H}_s is the mean curvature of $\Delta(s)$ at x , then (2) gives

$$\operatorname{div}(V) = -2\widehat{H}_s = -2H + 2\lambda_1 sh + \mathcal{O}(s^2) \quad \text{on } \Delta(s)$$

for $s > 0$ small. It follows that there exists a positive constant C such that for t small,

$$\int_{\Omega(t)} \operatorname{div}(V) = -2H \operatorname{Vol}(\Omega(t)) + 2\lambda_1 \int_{\Omega(t)} sh + \mathcal{O}(t^2) < -2H \operatorname{Vol}(\Omega(t)) - Ct. \quad (3)$$

Since the foliation $\mathcal{F}(\varepsilon_1)$ has smooth leaves with uniformly bounded second fundamental form, then the unit normal vector field W to the leaves of $\mathcal{F}(\varepsilon_1)$ is Lipschitz on $\Delta^{\perp, \varepsilon_1}$ and hence, it is Lipschitz on $\Omega(t)$. Since W is Lipschitz, its divergence is defined almost everywhere in $\Omega(t)$ and the divergence theorem holds in this setting. Note that the divergence of W is

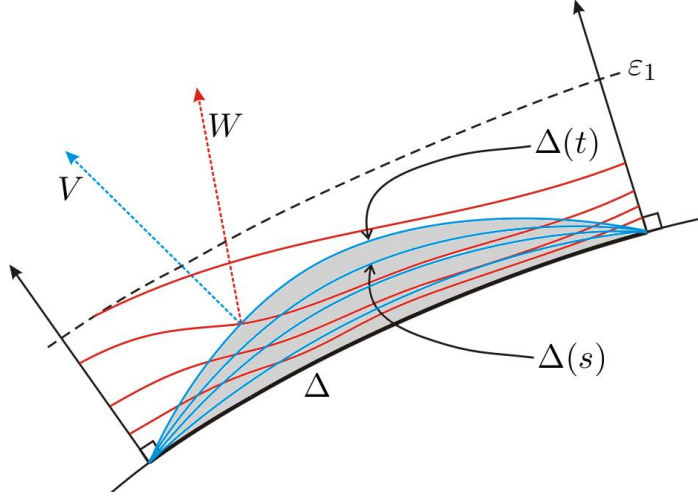


Figure 3: The divergence theorem is applied in the shaded region $\Omega(t)$ between Δ and $\Delta(t)$.

smooth in the regions of the form $U(p, \delta) - \bigcup_{t \in T} D(t)$ where it is equal to -2 times the mean curvature of the leaves of \mathcal{F}_p . Also, the mean curvature function of the foliation is continuous on $\mathcal{F}(\varepsilon_1)$ (see Assertion 1). Hence, the divergence of W can be seen to be a continuous function on $\Omega(t)$ which equals $-2H$ on the leaves $D(t)$, and by Assertion 1, $\text{div}(W)$ converges to the constant $-2H$ as $t \rightarrow 0$ to first order. Hence,

$$\int_{\Omega(t)} \text{div}(W) > -2H \text{Vol}(\Omega(t)) - Ct, \quad (4)$$

for any $t > 0$ sufficiently small.

Applying the divergence theorem to V and W in $\Omega(t)$ (note that $W = V$ on Δ), we obtain the following two inequalities:

$$\begin{aligned} \int_{\Omega(t)} \text{div}(V) &= \int_{\Delta(t)} \langle V, \eta(t) \rangle - \int_{\Delta} \langle V, \eta \rangle = \text{Area}(\Delta(t)) - \text{Area}(\Delta), \\ \int_{\Omega(t)} \text{div}(W) &= \int_{\Delta(t)} \langle W, \eta(t) \rangle - \int_{\Delta} \langle V, \eta \rangle < \text{Area}(\Delta(t)) - \text{Area}(\Delta), \end{aligned}$$

where $\eta(t)$ is the exterior unit vector field to $\Omega(t)$ on $\Delta(t)$. Hence, $\int_{\Omega(t)} \text{div}(W) < \int_{\Omega(t)} \text{div}(V)$. On the other hand, choosing t sufficiently small such that both inequalities (3) and (4) hold, we have $\int_{\Omega(t)} \text{div}(W) > \int_{\Omega(t)} \text{div}(V)$. This contradiction completes the proof of the theorem. \square

Remark 1 The proof of the theorem shows that given any two-sided cover \tilde{L} of a limit leaf L of \mathcal{L} as described in the statement of the theorem, then \tilde{L} is stable. This follows by lifting \mathcal{L} to a neighborhood $U(\tilde{L})$ of \tilde{L} in its normal bundle, considered to be the zero section in $U(\tilde{L})$. In the case of non-zero constant mean curvature hypersurfaces, L is already two-sided and then stability is equivalent to the existence of a positive Jacobi function. However, in the minimal case where a hypersurface L may be one-sided, this observation concerning stability of \tilde{L} is generally a stronger property; for example, the projective plane contained in projective three-space is a totally geodesic surface which is area minimizing in its \mathbb{Z}_2 -homology class but its oriented two-sided cover is unstable, see Ross [9] and also Ritor and Ros [7].

Next we give a useful and immediate consequence of Theorem 1. Let \mathcal{L} be a CMC codimension one lamination of a manifold N . We will denote by $\text{Stab}(\mathcal{L})$, $\text{Lim}(\mathcal{L})$ the collections of stable leaves and limit leaves of \mathcal{L} , respectively. Note that $\text{Lim}(\mathcal{L})$ is a closed set of leaves and so, it is a sublamination of \mathcal{L} .

Corollary 1 *Suppose that N is a not necessarily complete Riemannian manifold and \mathcal{L} is a CMC lamination of N with leaves of codimension one. Then, the closure of any collection of its stable leaves has the structure of a sublamination of \mathcal{L} , all of whose leaves are stable. Hence, $\text{Stab}(\mathcal{L})$ has the structure of a minimal lamination of N and $\text{Lim}(\mathcal{L}) \subset \text{Stab}(\mathcal{L})$ is a sublamination.*

Remark 2 Theorem 1 and Corollary 1 have many useful applications to the geometry of embedded minimal and constant mean curvature hypersurfaces in Riemannian manifolds. We refer the interested reader to the survey [2] by the first two authors and to our joint paper in [3] for some of these applications.

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