A survey on classical minimal surface theory

BY

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Preface.

We present in this monograph a survey of recent spectacular successes in classical minimal surface theory. Many of these successes were covered in our survey article "The classical theory of minimal surfaces" that appeared in the Bulletin of the AMS [125]. The focus of our BAMS article was to describe the work that led up to the theorem that the plane, the helicoid, the catenoid and the one-parameter family of Riemann minimal examples are the only properly embedded, minimal planar domains in $\mathbb{R}^3$. The proof of this result depends primarily on work of Colding and Minicozzi [37, 34], Collin [38], López and Ros [107], Meeks, Pérez and Ros [136] and Meeks and Rosenberg [148]. Partly due to limitations of length of our BAMS article, we omitted the discussion of many other important recent advances in the theory. The central topics missing in [125] and which appear here include the following ones:

1. The topological classification of minimal surfaces in $\mathbb{R}^3$ (Frohman and Meeks [63]).

2. The uniqueness of Scherk's singly-periodic minimal surfaces (Meeks and Wolf [155]).

3. The Calabi-Yau problem for minimal surfaces based on work by Nadirashvili [166] and Ferrer, Martín and Meeks [56].

4. Colding-Minicozzi theory for minimal surfaces of finite genus [24].

5. The asymptotic behavior of minimal annular ends with infinite total curvature (Meeks and Pérez [126]).

6. The local removable singularity theorem for minimal laminations and its applications: quadratic decay of curvature theorem, dynamics theorem and the local picture theorem on the scale of topology (Meeks, Pérez and Ros [134]).

Besides the above items, every topic that is in [125] appears here as well, with small modifications and additions. Another purpose of this monograph is to provide a more complete reference for the general reader of our BAMS article where he/she can find further discussion on related topics covered in [125], as well as the proofs of some of the results stated there.

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1 Introduction.

The theory of minimal surfaces in three-dimensional Euclidean space has its roots in the calculus of variations developed by Euler and Lagrange in the 18-th century and in later investigations by Enneper, Scherk, Schwarz, Riemann and Weierstrass in the 19-th century. During the years, many great mathematicians have contributed to this theory. Besides the above mentioned names that belong to the nineteenth century, we find fundamental contributions by Bernstein, Courant, Douglas, Morrey, Morse, Radó and Shiffman in the first half of the last century. Paraphrasing Osserman, most of the activity in minimal surface theory in those days was focused almost exclusively on either Plateau's problem or PDE questions, and the only global result was the negative one of Bernstein's theorem[1].

Much of the modern global theory of complete minimal surfaces in three-dimensional Euclidean space has been influenced by the pioneering work of Osserman during the 1960's. Many of the global questions and conjectures that arose in this classical subject have only recently been addressed. These questions concern analytic and conformal properties, the geometry and asymptotic behavior, and the topology and classification of the images of certain injective minimal immersions $f\colon M \to \mathbb{R}^3$ which are complete in the induced Riemannian metric; we call the image of such a complete, injective, minimal immersion a *complete, embedded minimal surface* in $\mathbb{R}^3$.

The following classification results solve two of these long standing conjectures[2].

[1]This celebrated result by Bernstein [6] asserts that the only minimal graphs over the entire plane are graphs given by affine functions, which means that the graphs are planes.

[2]Several authors pointed out to us that Osserman seems to be the first to ask the question about whether the plane and the helicoid were the only embedded, simply-connected, complete minimal surfaces. He described this question as potentially the most beautiful

Theorem 1.0.1 *A complete, embedded, simply-connected minimal surface in $\mathbb{R}^3$ is a plane or a helicoid.*

Theorem 1.0.2 *Up to scaling and rigid motion, any connected, properly embedded, minimal planar domain in $\mathbb{R}^3$ is a plane, a helicoid, a catenoid or one of the Riemann minimal examples*[3]*. In particular, for every such surface there exists a foliation of $\mathbb{R}^3$ by parallel planes, each of which intersects the surface transversely in a connected curve which is a circle or a line.*

The proofs of the above theorems depend on a series of new results and theory that have been developed over the past decade. The purpose of this monograph is two fold. The first is to explain these new global results in a manner accessible to the general mathematical public, and the second is to explain how these results transcend their application to the proofs of Theorems 1.0.1 and 1.0.2 and enhance dramatically our current understanding of the theory, giving rise to new theorems and conjectures, some of which were considered to be almost unapproachable dreams just 15 years ago. The interested reader can also find more detailed history and further results in the following surveys, reports and popular science articles [3, 12, 36, 30, 44, 45, 72, 73, 75, 77, 82, 115, 122, 129, 173, 188, 198]. We refer the reader to the excellent graduate texts on minimal surfaces by Dierkes et al [46, 47], Lawson [102] and Osserman [178], and especially see Nitsche's book [175] for a fascinating account of the history and foundations of the subject.

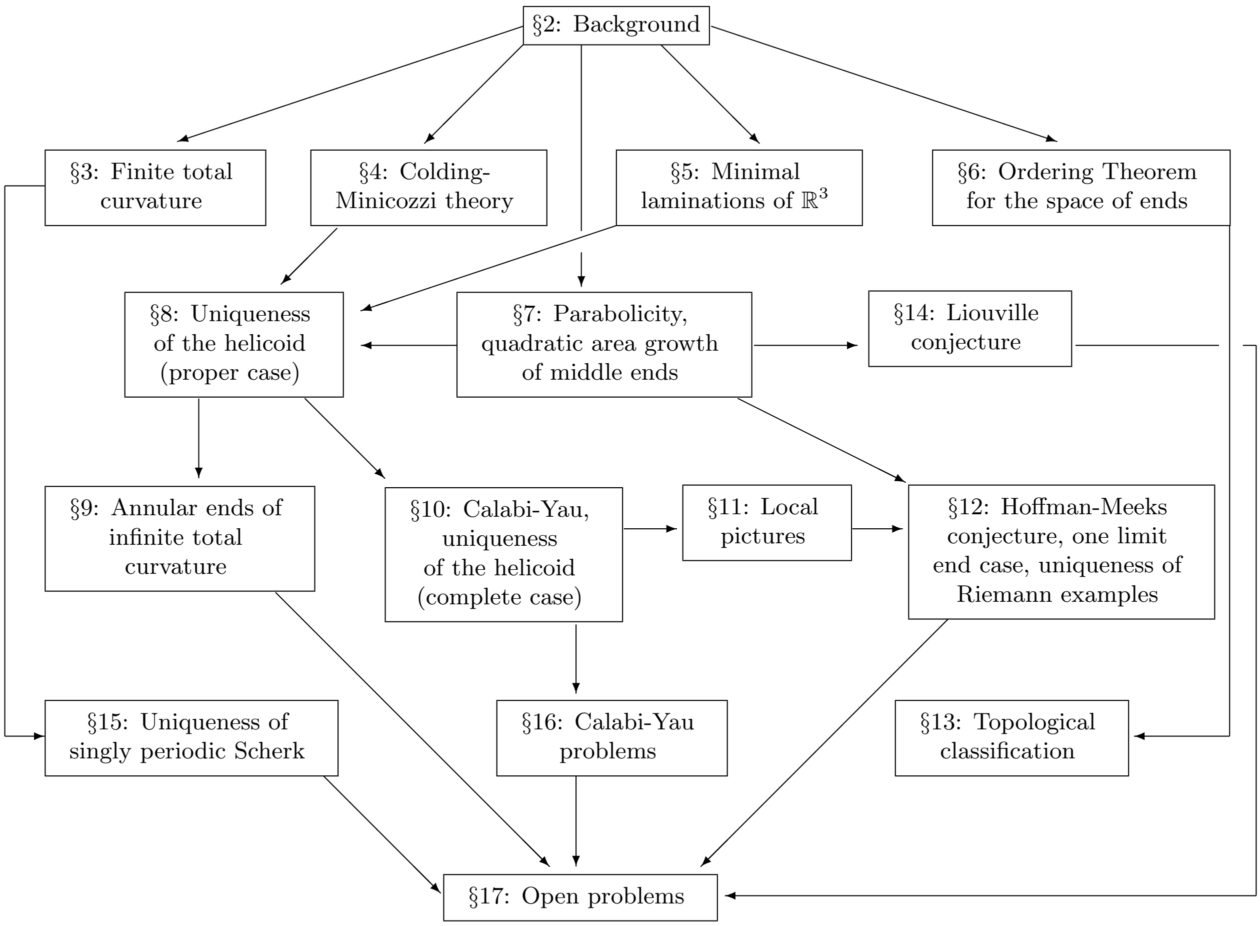
Before proceeding, we make a few general comments on the proof of Theorem 1.0.1 which we feel can suggest to the reader a visual idea of what is going on. The most natural motivation for understanding this theorem, Theorem 1.0.2 and other results presented in this survey is to try to answer the following question: *What are the possible shapes of surfaces which satisfy a variational principle and have a given topology?* For instance, if the variational equation expresses the critical points of the area functional, and if the requested topology is the simplest one of a disk, then Theorem 1.0.1 says that the possible shapes for complete examples are the trivial one given by a plane and (after a rotation) an infinite double spiral staircase, which is a visual description of a vertical helicoid. A more precise description of the double spiral staircase nature of a vertical helicoid is that this surface is the union of two infinite-sheeted multigraphs (see Definition 4.1.1 for the notion of a multigraph), which are glued along a vertical axis. Crucial in the proof of Theorem 1.0.1 are local results of Colding and Minicozzi which describe the structure of compact, embedded minimal disks, as essentially being modeled by one of the above two examples, i.e. either they are graphs

extension and explanation of Bernstein's Theorem.

[3]The *Riemann minimal examples* referred to here were discovered by Riemann around 1860. See Chapter 2.5 for further discussion of these surfaces and images of them.

or pairs of finitely sheeted multigraphs glued along an "axis". A last key ingredient in the proof of Theorem 1.0.1 is a result on the global aspects of limits of these shapes, which is given in the Lamination Theorem for Disks by Colding and Minicozzi, see Theorem 4.1.2 below.

For the reader's convenience, we next include a guide of how chapters depend on each other, see below for a more detailed explanation of their contents.



Our survey is organized as follows. We present the main definitions and background material in the introductory Chapter 2. In that chapter we also briefly describe geometrically, as well as analytically, many of the important classical examples of properly embedded minimal surfaces in $\mathbb{R}^3$. As in many other areas in mathematics, understanding key examples in this subject is crucial in obtaining a feeling for the subject, making important theoretical advances and asking the right questions. We believe that before going further, the unacquainted reader to this subject will benefit by taking a few minutes to view and identify the computer graphics images of these surfaces, which appear near the end of Chapter 2.5, and to read the brief historical and descriptive comments related to the individual images.

In Chapter 3, we describe the best understood family of complete embedded minimal surfaces: those with finite topology and more than one end. Recall that a compact, orientable surface is homeomorphic to a connected sum of tori, and the number of these tori is called the genus of the surface. An (orientable) surface M of finite topology is one which is homeomorphic to a surface of genus $g \in \mathbb{N} \cup \{0\}$ with finitely many points removed, called the *ends* of M. These punctures can be naturally identified with different ways to escape to infinity on M, and also can be identified with punctured disk neighborhoods of these points; these punctured disk neighborhoods are clearly annuli, hence they are called *annular ends.* The crucial result for minimal surfaces with finite topology and more that one end is Collin's Theorem, valid under the additional assumption of properness (a surface in $\mathbb{R}^3$ is *proper* if each closed ball in $\mathbb{R}^3$ contains a compact portion of the surface with respect to its intrinsic topology). Note that properness implies completeness.

Theorem 1.0.3 (Collin [38]) *If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with more than one end, then each annular end of M is asymptotic to the end of a plane or a catenoid. In particular, if M has finite topology and more than one end, then M has finite total Gaussian curvature*[4].

Collin's Theorem reduces the analysis of properly embedded minimal surfaces of finite topology and more than one end in $\mathbb{R}^3$ to complex function theory on compact Riemann surfaces. This reduction then leads to classification results and to interesting topological obstructions, which we include in Chapter 3 as well. At the end of Chapter 3, we discuss several different methods for constructing properly embedded minimal surfaces of finite topology and for describing their moduli spaces.

In Chapter 4, we present an overview of some results concerning the geometry, compactness and regularity of limits of what are called *locally simply-connected* sequences of minimal surfaces. These results are central in the proofs of Theorems 1.0.1 and 1.0.2 and are taken from a series of papers by Colding and Minicozzi [27, 31, 32, 33, 34, 37].

In Chapters 4 and 5, we define and develop the notion of *minimal lamination,* which is a natural limit object for a locally simply-connected sequence of embedded minimal surfaces. The reader not familiar with the subject of minimal laminations should think about the closure of an embedded, non-compact geodesic γ on a complete Riemannian surface, a topic which has been widely covered in the literature (see e.g. Bonahon [9]). The closure of such a geodesic γ is a geodesic lamination $\mathcal{L}$ of the surface. When γ has no accumulation points, then it is proper and it is the unique leaf of

[4]See equation (2.3) for the definition of total curvature.

$\mathcal{L}$. Otherwise, there pass complete, embedded, pairwise disjoint geodesics through the accumulation points, and these geodesics together with γ form the leaves of the geodesic lamination $\mathcal{L}$. The similar result is true for a complete, embedded minimal surface of locally bounded curvature (i.e. whose Gaussian curvature is bounded in compact extrinsic balls) in a Riemannian three-manifold [148]. We include in Chapter 5 a discussion of the *Structure Theorem for Minimal Laminations of* $\mathbb{R}^3$ (Meeks-Rosenberg [148] and Meeks-Pérez-Ros [139]).

In Chapter 6 we explain the *Ordering Theorem* of Frohman and Meeks [61] for the linear ordering of the ends of any properly embedded minimal surface with more than one end; this is a fundamental result for our purposes of classifying embedded, minimal planar domains in Theorem 1.0.2. We note that the results in this chapter play an important role in proving Theorems 1.0.1 and 1.0.2. For instance, they are useful when demonstrating that every properly embedded, simply-connected minimal surface in $\mathbb{R}^3$ is conformally $\mathbb{C}$; this conformal property is a crucial ingredient in the proof of the uniqueness of the helicoid. Also, a key point in the proof of Theorem 1.0.2 is the fact that a properly embedded minimal surface in $\mathbb{R}^3$ cannot have more than two limit ends (see Definition 2.7.2 for the concept of limit end), which is a consequence of the results in Chapter 7.

Chapter 7 is devoted to conformal questions on minimal surfaces. Roughly speaking, this means to study the conformal type of a given minimal surface (rather than its Riemannian geometry) considered as a Riemann surface, i.e. an orientable surface in which one can find an atlas by charts with holomorphic change of coordinates. To do this, we first define the notion of *universal superharmonic function* for domains in $\mathbb{R}^3$ and give examples. Next we explain how to use these functions to understand the conformal structure of properly immersed minimal surfaces in $\mathbb{R}^3$ which fail to have finite topology. We then follow the work of Collin, Kusner, Meeks and Rosenberg in [39] to analyze the asymptotic geometry and conformal structure of properly embedded minimal surfaces of infinite topology.

In Chapter 8, we apply the results in the previous chapters to explain the main steps in the proof of Theorem 1.0.1 after replacing completeness by the stronger hypothesis of properness. This theorem together with Theorem 1.0.3 above, Theorem 1.0.5 below and with results by Bernstein and Breiner [4] or by Meeks and Pérez [126] lead to a complete understanding of the asymptotic geometry of any annular end of a complete, embedded minimal surface with finite topology in $\mathbb{R}^3$; namely, the annular end must be asymptotic to an end of a plane, catenoid or helicoid.

A discussion of the proof of Theorem 1.0.4 in the case of positive genus and a more general classification result of complete embedded minimal annular ends with compact boundary and infinite total curvature in $\mathbb{R}^3$ can be found in Chapter 9.

Theorem 1.0.4 *Every properly embedded, non-planar minimal surface in $\mathbb{R}^3$ with finite genus and one end has the conformal structure of a compact Riemann surface Σ minus one point, can be analytically represented by meromorphic data on Σ and is asymptotic to a helicoid. Furthermore, when the genus of Σ is zero, the only possible examples are helicoids.*

In Chapter 10, we complete our sketch of the proof of Theorem 1.0.1 by allowing the surface to be complete rather than proper. The problem of understanding the relation between the intrinsic notion of completeness and the extrinsic one of properness is known as the *embedded Calabi-Yau problem* in minimal surface theory, see [13], [21], [229], [230] and [166] for the original Calabi-Yau problem in the complete immersed setting. Along these lines, we also describe the powerful *Minimal Lamination Closure Theorem* (Meeks-Rosenberg [149], Theorem 10.1.2 below), the *$C^{1,1}$-Regularity Theorem* for the singular set of convergence of a Colding-Minicozzi limit minimal lamination (Meeks [121] and Meeks-Weber [152]), the *Finite Genus Closure Theorem* (Meeks-Pérez-Ros [134]) and the *Lamination Metric Theorem* (Meeks [123]). Theorem 10.1.2 is a refinement of the results and techniques used by Colding and Minicozzi [37] to prove the following deep result (see Chapter 10 where we deduce Theorem 1.0.5 from Theorem 10.1.2).

Theorem 1.0.5 (Colding, Minicozzi [37]) *A complete, embedded minimal surface of finite topology in $\mathbb{R}^3$ is properly embedded.*

With Theorem 1.0.5 in hand, we note that the hypothesis of properness for M in the statements of Theorem 1.0.4, of Collin's Theorem 1.0.3 and of the results in Chapter 9, can be replaced by the weaker hypothesis of completeness for M. Hence, Theorem 1.0.1 follows from Theorems 1.0.4 and 1.0.5.

In Chapter 11, we examine how the new theoretical results in the previous chapters lead to deep global results in the classical theory and to a general understanding of the local geometry of any complete, embedded minimal surface M in a homogeneously regular three-manifold (see Definition 2.8.5 below for the concept of homogeneously regular three-manifold). This local description is given in two local picture theorems by Meeks, Pérez and Ros [134, 133], each of which describes the local extrinsic geometry of M near points of concentrated curvature or of concentrated topology. The first of these results is called the *Local Picture Theorem on the Scale of Curvature,* and the second is known as the *Local Picture Theorem on the Scale of Topology.* In order to understand the second local picture theorem, we define and develop in Chapter 10 the important notion of a *parking garage structure* on $\mathbb{R}^3$, which is one of the possible limiting local pictures in the topological setting. Also discussed here is the *Fundamental Singularity Conjecture* for minimal laminations and the *Local Removable Singularity Theorem* of Meeks, Pérez and Ros for certain possibly singular minimal laminations of a three-manifold [134]. Global applications of the Local Removable Singularity Theorem to the classical theory are also discussed here. The most important of these applications are the *Quadratic Curvature Decay Theorem* and the *Dynamics Theorem* for properly embedded minimal surfaces in $\mathbb{R}^3$ by Meeks, Pérez and Ros [134]. An important consequence of the Dynamics Theorem and the Local Picture Theorem on the Scale of Curvature is that every complete, embedded minimal surface in $\mathbb{R}^3$ with infinite topology has a surprising amount of internal dynamical periodicity.

In Chapter 12, we present some of the results of Meeks, Pérez and Ros [130, 131, 136, 139, 140] on the geometry of complete, embedded minimal surfaces of finite genus with possibly an infinite number of ends, including the proof of Theorem 1.0.2 stated above. We first explain in Theorem 1.0.6 the existence of a bound on the number of ends of a complete, embedded minimal surface with finite total curvature solely in terms of the genus. So far, this is the best result towards the solution of the so called *Hoffman-Meeks conjecture:*

> A connected surface of finite topology with genus g and r ends, $r > 2$, can be properly minimally embedded in $\mathbb{R}^3$ if and only if $r \leq g + 2$.

By Theorem 1.0.3 and the Jorge-Meeks formula displayed in equation (2.9), a complete, embedded minimal surface M with genus $g \in \mathbb{N} \cup \{0\}$ and a finite number $r \geq 2$ of ends, has total finite total curvature $-4\pi(g+r-1)$ and so the next theorem also gives a lower bound estimate on the total curvature in terms of its genus g. Then, by a theorem of Tysk [220], the index of stability of complete of a complete embedded minimal surface in $\mathbb{R}^3$ can be estimated in terms of the total curvature of the surface; for the definition of index of stability, see Definition 2.8.2. Closely related to Tysk's theorem is the beautiful result of Fischer-Colbrie [57] that a complete, orientable minimal surface with compact (possibly empty) boundary in $\mathbb{R}^3$ has finite index of stability if and only if it has finite total curvature.

Theorem 1.0.6 (Meeks, Pérez, Ros [130]) *For every non-negative integer g, there exists an integer $e(g)$ such that if $M \subset \mathbb{R}^3$ is a complete, embedded minimal surface of finite topology with genus g, then the number of ends of M is at most $e(g)$. Furthermore, if M has more than one end, then its finite index of stability is bounded solely as a function of the finite genus of M.*

In Chapter 12, we also describe the essential ingredients of the proof of Theorem 1.0.2 on the classification of properly embedded minimal planar domains in $\mathbb{R}^3$. At the end of this chapter, we briefly explain a structure theorem by Colding and Minicozzi for non-simply-connected embedded minimal surfaces of prescribed genus. This result deals with limits of sequences of compact, embedded minimal surfaces M_n with diverging boundaries and uniformly bounded genus, and roughly asserts that there are only three allowed shapes for the surfaces M_n: either they converge smoothly (after choosing a subsequence) to a properly embedded, non-flat minimal surface of bounded Gaussian curvature and restricted geometry, or the Gaussian curvatures of the M_n blow up in a neighborhood of some points in space; in this case, the M_n converge to a lamination $\mathcal{L}$ of $\mathbb{R}^3$ by planes away from a non-empty, closed singular set of convergence $S(\mathcal{L})$, and the limiting geometry of the M_n consists either of a parking garage structure on $\mathbb{R}^3$ with two oppositely handed columns (see Chapter 10.2 for this notion), or small necks form in M_n around every point in $S(\mathcal{L})$.

The next goal of Chapter 12 is to focus on the proof of Theorem 1.0.2. In this setting of infinitely many ends, the set of ends has a topological structure which makes it a compact, total disconnected metric space with infinitely many points, see Definition 2.7.1 and the paragraph below it. This set has accumulation points, which produce the new notion of *limit end.* The first step in the proof of Theorem 1.0.2 is to notice that the number of limit ends of a properly embedded minimal surface in $\mathbb{R}^3$ is at most 2; this result appears as Theorem 7.3.1. Then one rules out the existence of a properly embedded minimal planar domain with just one limit end: this is the purpose of Theorem 12.2.1. At that point, we are ready to finish the proof of Theorem 1.0.2. This breaks into two parts, the first of which is a quasiperiodicity property coming from curvature estimates for any surface satisfying the hypotheses of Theorem 1.0.2 (see [139]), and the second one is based on the Shiffman function and its relation to integrable systems theory through the Korteweg-de Vries (KdV) equation, see Theorem 12.3.1 below.

In Chapter 13, we explain how to topologically classify all properly embedded minimal surfaces in $\mathbb{R}^3$, through Theorem 1.0.7 below by Frohman and Meeks [63]. This result depends on their Ordering Theorem for the ends of a properly embedded minimal surface $M \subset \mathbb{R}^3$, which is discussed in Chapter 6. The Ordering Theorem states that there is a natural linear ordering on the set $\mathcal{E}(M)$ of ends of M. The ends of M which are not extremal in this ordering are called *middle ends* and have a parity which is even or odd (see Theorem 7.3.1 for the definition of this parity).

Theorem 1.0.7 (Topological Classification Theorem, Frohman, Meeks) *Two properly embedded minimal surfaces in $\mathbb{R}^3$ are ambiently isotopic*[5] *if and only if there exists a homeomorphism between the surfaces that preserves both the ordering of their ends and the parity of their middle ends.*

At the end of Chapter 13, we present an explicit cookbook-type recipe for constructing a smooth (non-minimal) surface $\widehat{M}$ that represents the ambi-

[5]See Definition 13.0.1.

ent isotopy class of a properly embedded minimal surface M in $\mathbb{R}^3$ with prescribed topological invariants.

Most of the classical periodic, properly embedded minimal surfaces $M \subset \mathbb{R}^3$ have infinite genus and one end, and so, the topological classification of non-compact surfaces shows that any two such surfaces are homeomorphic. For example, all non-planar doubly and triply-periodic minimal surfaces have one end and infinite genus [16], as do many singly-periodic minimal examples such as any of the classical singly-periodic Scherk minimal surfaces $\mathcal{S}_\theta$, for any $\theta \in (0, \frac{\pi}{2}]$ (see Chapter 2.5 for a description of these surfaces and a picture for $S_{\frac{\pi}{2}}$). Since there are no middle ends for these surfaces and they are homeomorphic, Theorem 1.0.7 demonstrates that there exists an orientation preserving diffeomorphism $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ with $f(\mathcal{S}_{\frac{\pi}{2}}) = M$, where M is an arbitrarily chosen, properly embedded minimal surface with infinite genus and one end. Also, given two homeomorphic, properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$, we can always choose a diffeomorphism between the surfaces that preserves the ordering of the ends. Thus, since the parity of an annular end is always odd, one obtains the following classical unknottedness theorem of Meeks and Yau as a corollary.

Corollary 1.0.8 (Meeks, Yau [158]) *Two homeomorphic, properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$ are ambiently isotopic.*

One of the important open questions in classical minimal surface theory asks whether a positive harmonic function on a properly embedded minimal surface in $\mathbb{R}^3$ must be constant; this question was formulated as a conjecture by Meeks [122] and is called the *Liouville Conjecture* for properly embedded minimal surfaces. The Liouville Conjecture is known to hold for many classes of minimal surfaces, which include all properly embedded minimal surfaces of finite genus and all of the classical examples discussed in Chapter 2.5. In Chapter 14, we present some recent positive results on this conjecture and on the related questions of *recurrence and transience of Brownian motion* for properly embedded minimal surfaces.

In Chapter 15, we explain a recent classification result, which is closely related to the procedure of *minimal surgery;* in certain cases, Kapouleas [92, 93] has been able to approximate two embedded, transversely intersecting, minimal surfaces $\Sigma_1, \Sigma_2 \subset \mathbb{R}^3$ by a sequence of embedded minimal surfaces $\{M_n\}_n$ that converges to $\Sigma_1 \cup \Sigma_2$. He obtains each M_n by first sewing in necklaces of singly-periodic Scherk minimal surfaces along $\Sigma_1 \cap \Sigma_2$ and then applies the implicit function theorem to get a nearby embedded minimal surface (see Chapter 3.2 for a description of the desingularization procedure of Kapouleas). The *Scherk Uniqueness Conjecture* (Meeks [122]) states that the only connected, properly embedded minimal surfaces in $\mathbb{R}^3$ with quadratic area growth[6] constant 2π (equal to the quadratic area growth

[6]A properly immersed minimal surface M in $\mathbb{R}^3$ has *quadratic area growth* if

constant of two planes) are the catenoid and the family $\mathcal{S}_\theta$, $\theta \in (0, \frac{\pi}{2}]$, of singly-periodic Scherk minimal surfaces. The validity of this conjecture would essentially imply that the only way to desingularize two embedded, transversely intersecting, minimal surfaces is by way of the *Kapouleas minimal surgery construction.* In Chapter 15, we outline the proof of the following important partial result on this conjecture.

Theorem 1.0.9 (Meeks, Wolf [155]) *A connected, properly embedded minimal surface in $\mathbb{R}^3$ with infinite symmetry group and quadratic area growth constant 2π must be a catenoid or one of the singly-periodic Scherk minimal surfaces.*

In Chapter 16, we discuss what are usually referred to as the *Calabi-Yau problems* for complete minimal surfaces in $\mathbb{R}^3$. These problems arose from questions asked by Calabi [14] and by Yau (see page 212 in [21] and problem 91 in [229]) concerning the existence of complete, immersed minimal surfaces which are constrained to lie in a given region of $\mathbb{R}^3$, such as in a bounded domain. Various aspects of the Calabi-Yau problems constitute an active field of research with an interesting mix of positive and negative results. We include here a few recent fundamental advances on this problem.

The final chapter of this survey is devoted to a discussion of the main outstanding conjectures in the subject. Many of these problems are motivated by the recent advances in classical minimal surface theory reported on in previous chapters. Research mathematicians, not necessarily schooled in differential geometry, are likely to find some of these problems accessible to attack with methods familiar to them. We invite anyone with an inquisitive mind and strong geometrical intuition to join in the game of trying to solve these intriguing open problems.

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$\lim_{R\to\infty} R^{-2}\mathrm{Area}(M \cap \mathbb{B}(R)) := A_M < \infty$, where $\mathbb{B}(R) = \{x \in \mathbb{R}^3 \mid \|x\| < R\}$. We call the number A_M the *quadratic area growth constant* of M in this case.

Basic results in classical minimal surface theory.

2

We will devote this chapter to give a fast tour through the foundations of the theory, enough to understand the results to be explained in future chapters.

2.1 Eight equivalent definitions of minimality.

One can define a minimal surface from different points of view. The equivalences between these starting points give insight into the richness of the classical theory of minimal surfaces and its connections with other branches of mathematics.

Definition 2.1.1 Let $X = (x_1, x_2, x_3)\colon M \to \mathbb{R}^3$ be an isometric immersion of a Riemannian surface into space. X is said to be *minimal* if x_i is a harmonic function on M for each i. In other words, $\Delta x_i = 0$, where Δ is the Riemannian Laplacian on M.

Very often, it is useful to identify a Riemannian surface M with its image under an isometric embedding. Since harmonicity is a local concept, the notion of minimality can be applied to an immersed surface $M \subset \mathbb{R}^3$ (with the underlying induced Riemannian structure by the inclusion). Let H be the *mean curvature* function of X, which at every point is the average normal curvature, and let $N\colon M \to \mathbb{S}^2 \subset \mathbb{R}^3$ be its unit normal or Gauss map[1]. The well-known vector-valued formula $\Delta X = 2HN$, valid for an isometric immersion $X\colon M \to \mathbb{R}^3$, leads us to the following equivalent definition of minimality.

[1]Throughout the paper, all surfaces will be assumed to be *orientable* unless otherwise stated.

Definition 2.1.2 A surface $M \subset \mathbb{R}^3$ is *minimal* if and only if its mean curvature vanishes identically.

After rotation, any (regular) surface $M \subset \mathbb{R}^3$ can be locally expressed as the graph of a function $u = u(x, y)$. In 1776, Meusnier [159] discovered that the condition on the mean curvature to vanish identically can be expressed as the following quasilinear, second order, elliptic partial differential equation, found in 1762 by Lagrange[2] [99]:

$$(1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx} = 0, \tag{2.1}$$

which admits a divergence form version:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0.$$

Definition 2.1.3 A surface $M \subset \mathbb{R}^3$ is *minimal* if and only if it can be locally expressed as the graph of a solution of the equation (2.1).

Let Ω be a subdomain with compact closure in a surface $M \subset \mathbb{R}^3$. If we perturb normally the inclusion map X on Ω by a compactly supported smooth function $u \in C_0^\infty(\Omega)$, then $X + tuN$ is again an immersion whenever $|t| < \varepsilon$, with ε sufficiently small. The mean curvature function H of M relates to the infinitesimal variation of the area functional $A(t) = \text{Area}((X + tuN)(\Omega))$ for compactly supported normal variations by means of the *first variation of area* (see for instance [175]):

$$A'(0) = -2 \int_\Omega uH\, dA, \tag{2.2}$$

where dA stands for the area element of M. This variational formula lets us state a fourth equivalent definition of minimality.

Definition 2.1.4 A surface $M \subset \mathbb{R}^3$ is *minimal* if and only if it is a critical point of the *area functional* for all compactly supported variations.

In fact, a consequence of the *second variation of area* (Chapter 2.8) is that any point in a minimal surface has a neighborhood with least-area relative to its boundary. This property justifies the word “minimal” for these surfaces. It should be noted that the global minimization of area on every compact subdomain is a strong condition for a complete minimal surface to satisfy; in fact, it forces the surface to be a plane (Theorem 2.8.4).

Definition 2.1.5 A surface $M \subset \mathbb{R}^3$ is *minimal* if and only if every point $p \in M$ has a neighborhood with least-area relative to its boundary.

[2] In reality, Lagrange arrived to a slightly different formulation, and equation (2.1) was derived five years later by Borda [11].

Definitions 2.1.4 and 2.1.5 establish minimal surfaces as the 2-dimensional analog to geodesics in Riemannian geometry, and connect the theory of minimal surfaces with one of the most important classical branches of mathematics: the calculus of variations. Besides the area functional A, another well-known functional in the calculus of variations is the *Dirichlet energy,*

$$E = \int_{\Omega} |\nabla X|^2 dA,$$

where again $X\colon M \to \mathbb{R}^3$ is an isometric immersion and $\Omega \subset M$ is a subdomain with compact closure. These functionals are related by the inequality $E \geq 2A$, with equality if and only if the immersion $X\colon M \to \mathbb{R}^3$ is conformal. The classical formula $K - e^{2u}\overline{K} = \Delta u$ that relates the Gaussian curvature functions $K, \overline{K}$ for two conformally related metrics g, $\overline{g} = e^{2u}g$ on a 2-dimensional manifold (Δ stands for the Laplace operator with respect to g) together with the existence of solutions of the Laplace equation $\Delta u = K$ for any subdomain with compact closure in a Riemannian manifold, guarantee the existence of local *isothermal* or *conformal coordinates* for any 2-dimensional Riemannian manifold, modeled on domains of $\mathbb{C}$. The relation between area and energy together with the existence of isothermal coordinates, allow us to give two further characterizations of minimality.

Definition 2.1.6 A conformal immersion $X\colon M \to \mathbb{R}^3$ is *minimal* if and only if it is a critical point of the Dirichlet energy for all compactly supported variations, or equivalently if any point $p \in M$ has a neighborhood with least energy relative to its boundary.

From a physical point of view, the mean curvature function of a homogeneous membrane separating two media is equal, up to a non-zero multiplicative constant, to the difference between the pressures at the two sides of the surface. When this pressure difference is zero, then the membrane has zero mean curvature. Therefore, soap films (i.e. not bubbles) in space are physical realizations of the ideal concept of a minimal surface.

Definition 2.1.7 A surface $M \subset \mathbb{R}^3$ is *minimal* if and only if every point $p \in M$ has a neighborhood D_p which is equal to the unique idealized soap film with boundary ∂D_p.

Consider again the Gauss map $N\colon M \to \mathbb{S}^2$ of M. Then, the tangent space T_pM of M at $p \in M$ can be identified as subspace of $\mathbb{R}^3$ under parallel translation with the tangent space $T_{N(p)}\mathbb{S}^2$ to the unit sphere at $N(p)$. Hence, one can view the differential $A_p = -dN_p$ as an endomorphism of T_pM, called the *shape operator*. A_p is a symmetric linear transformation, whose orthogonal eigenvectors are called the *principal directions* of M at p, and the corresponding eigenvalues are the *principal curvatures* of M at p.

Since the mean curvature function H of M equals the arithmetic mean of such principal curvatures, minimality reduces to the following expression

$$A_p = -dN_p = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

in an orthonormal tangent basis. After identification of N with its stereographic projection, the Cauchy-Riemann equations give the next and last characterization of minimality.

Definition 2.1.8 A surface $M \subset \mathbb{R}^3$ is *minimal* if and only if its stereographically projected Gauss map $g\colon M \to \mathbb{C} \cup \{\infty\}$ is meromorphic with respect to the underlying Riemann surface structure.

This concludes our discussion of the equivalent definitions of minimality. The connection between minimal surface theory and complex analysis made possible the advances in the so-called *first golden age* of classical minimal surface theory (approximately 1855-1890). In this period, many great mathematicians took part, such as Beltrami, Bonnet, Catalan, Darboux, Enneper, Lie, Riemann, Schoenflies, Schwarz, Serret, Weierstrass, Weingarten, etc. (these historical references and many others can be found in the excellent book by Nitsche [175]). A *second golden age* of classical minimal surface theory occurred in the decade 1930-1940, with pioneering works by Courant, Douglas, Morrey, Morse, Radó, Shiffman and others. Among the greatest achievements of this period, we mention that Douglas won the first Fields medal (jointly with Ahlfors) for his complete solution to the classical *Plateau problem*[3] for disks.

Many geometers believe that since the early 1980's, we are living in a *third golden age* of classical minimal surface theory. A vast number of new embedded examples have been found in this period, very often with the aid of modern computers, which allow us to visualize beautiful models such as those appearing in the figures of this text. In these years, geometric measure theory, conformal geometry, functional analysis, integrable systems and other branches of mathematics have joined the classical methods, contributing fruitful new techniques and results, some of which will be treated in this monograph. At the same time, the theory of minimal surfaces has diversified and expanded its frontiers. Minimal submanifolds in more general ambient geometries have been studied, and subsequent applications of minimal submanifolds have helped lead to solutions of some fundamental problems in other branches of mathematics, including the Positive Mass Conjecture (Schoen, Yau) and the Penrose Conjecture (Bray) in mathematical physics, and the Smith Conjecture (Meeks, Yau), the Poincaré Conjecture (Colding,

[3]In its simplest formulation, this problem asks whether any smooth Jordan curve in $\mathbb{R}^3$ bounds a disk of least area, a problem proposed by the Belgian physicist Plateau in 1870.

Minicozzi) and the Thurston Geometrization Conjecture in three-manifold theory.

Returning to our background discussion, we note that Definition 2.1.1 and the maximum principle for harmonic functions imply that no compact minimal surfaces in $\mathbb{R}^3$ exist. Although the study of compact minimal surfaces with boundary has been extensively developed and dates back to the well-known Plateau problem (see Footnote 3), in this survey we will focus on the study of *complete* minimal surfaces (possibly with boundary), in the sense that all geodesics can be indefinitely extended up to the boundary of the surface. Note that with respect to the natural Riemannian distance function between points on a surface, the property of being "geodesically complete" is equivalent to the surface being a complete metric space. A stronger global hypothesis, whose relationship with completeness is an active field of research in minimal surface theory, is presented in the following definition.

Definition 2.1.9 A map $f\colon X \to Y$ between topological spaces is *proper* if $f^{-1}(C)$ is compact in X for any compact set $C \subset Y$. A minimal surface $M \subset \mathbb{R}^3$ is proper when the inclusion map is proper.

The Gaussian curvature function K of a surface $M \subset \mathbb{R}^3$ is the product of its principal curvatures. If M is minimal, then its principal curvatures are oppositely signed and thus, K is non-positive. Another interpretation of K is the determinant of the shape operator A, thus $|K|$ is the absolute value of the Jacobian for the Gauss map N. Therefore, after integrating $|K|$ on M (note that this integral may be $-\infty$ or a non-positive number), we obtain the same quantity as when computing the negative of the spherical area of M through its Gauss map, counting multiplicities. This quantity is called the *total curvature* of the minimal surface:

$$C(M) = \int_M K\, dA = -\mathrm{Area}(N\colon M \to \mathbb{S}^2). \tag{2.3}$$

2.2 Weierstrass representation.

Recall that the Gauss map of a minimal surface M can be viewed as a meromorphic function $g\colon M \to \mathbb{C}\cup\{\infty\}$ on the underlying Riemann surface. Furthermore, the harmonicity of the third coordinate function x_3 of M lets us define (at least locally) its harmonic conjugate function x_3^*; hence, the so-called *height differential*[4] $dh = dx_3 + i dx_3^*$ is a holomorphic differential on M. The pair (g, dh) is usually referred to as the *Weierstrass data*[5] of the

[4]Note that the height differential might not be exact since x_3^* need not be globally well-defined on M. Nevertheless, the notation dh is commonly accepted and we will also make use of it here.

[5]This representation was derived locally by Weierstrass in 1866.

minimal surface, and the minimal immersion $X\colon M \to \mathbb{R}^3$ can be expressed up to translation by $X(p_0)$, $p_0 \in M$, solely in terms of this data as

$$X(p) = \Re \int_{p_0}^{p} \left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{i}{2}\left(\frac{1}{g} + g\right), 1 \right) dh, \tag{2.4}$$

where $\Re$ stands for real part [102, 178]. The pair (g, dh) satisfies certain compatibility conditions, stated in assertions *i), ii)* of Theorem 2.2.1 below. The key point is that this procedure has the following converse, which gives a cookbook-type recipe for analytically defining a minimal surface.

Theorem 2.2.1 (Osserman [176]) *Let M be a Riemann surface, $g\colon M \to \mathbb{C} \cup \{\infty\}$ a meromorphic function and dh a holomorphic one-form on M. Assume that:*

i) The zeros of dh coincide with the poles and zeros of g, with the same order.

ii) For any closed curve $\gamma \subset M$,

$$\overline{\int_\gamma g\,dh} = \int_\gamma \frac{dh}{g}, \qquad \Re \int_\gamma dh = 0, \tag{2.5}$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. Then, the map $X\colon M \to \mathbb{R}^3$ given by (2.4) *is a conformal minimal immersion with Weierstrass data (g, dh).*

Condition *i)* in Theorem 2.2.1 expresses the non-degeneracy of the induced metric by X on M, so by weakening it to the condition that the zeros and poles of g coincide with the zeros of dh with at most the same order, we allow the conformal X to be a *branched minimal surface.* Condition *ii*) deals with the independence of (2.4) on the integration path, and it is usually called the *period problem*[6]. By the Divergence Theorem, it suffices to consider the period problem on homology classes in M.

All local geometric invariants of a minimal surface M can be expressed in terms of its Weierstrass data. For instance, the first and second fundamental forms are respectively (see [75, 178]):

$$ds^2 = \left(\frac{1}{2}(|g| + |g|^{-1})|dh| \right)^2, \qquad II(v, v) = \Re \left(\frac{dg}{g}(v) \cdot dh(v) \right), \tag{2.6}$$

where v is a tangent vector to M, and the Gaussian curvature is

$$K = -\left(\frac{4\,|dg/g|}{(|g| + |g|^{-1})^2 |dh|} \right)^2. \tag{2.7}$$

[6]The reason for this name is that the failure of (2.5) implies that equation (2.4) only defines X as an additively multivalued map. The translation vectors given by this multivaluation are called the *periods* of the pair (g, dh).

If (g, dh) is the Weierstrass data of a minimal surface $X\colon M \to \mathbb{R}^3$, then for each $\lambda > 0$ the pair $(\lambda g, dh)$ satisfies condition *i)* of Theorem 2.2.1 and the second equation in (2.5). The first equation in (2.5) holds for this new Weierstrass data if and only if $\int_\gamma g dh = \int_\gamma \frac{dh}{g} = 0$ for all homology classes γ in M, a condition which can be stated in terms of the notion of flux, which we now define. Given a minimal surface M with Weierstrass data (g, dh), the *flux vector* along a closed curve $\gamma \subset M$ is defined as

$$F(\gamma) = \int_\gamma \mathrm{Rot}_{90^\circ}(\gamma') = \Im \int_\gamma \left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{i}{2}\left(\frac{1}{g} + g\right), 1 \right) dh \in \mathbb{R}^3, \quad (2.8)$$

where Rot_{90° denotes the rotation by angle $\pi/2$ in the tangent plane of M at any point, and $\Im$ stands for imaginary part.

Coming back to our Weierstrass data $(\lambda g, dh)$, the first equation in (2.5) holds for this new pair if and only if the flux of M along γ is a vertical vector for all closed curves $\gamma \subset M$. Thus, for a minimal surface X with vertical flux and for λ a positive real number, the Weierstrass data $(\lambda g, dh)$ produces a well-defined minimal surface $X_\lambda\colon M \to \mathbb{R}^3$. The family $\{X_\lambda\}_\lambda$ is a smooth deformation of $X_1 = X$ by minimal surfaces, called the *López-Ros deformation*[7]. Clearly, the conformal structure, height differential and the set of points in M with vertical normal vector are preserved throughout this deformation. Another important property of the López-Ros deformation is that if a component of a horizontal section $X(M) \cap \{x_3 = \text{constant}\}$ is convex, then the same property holds for the related component at the same height for any X_λ, $\lambda > 0$. For details, see [107, 186].

2.3 Minimal surfaces of finite total curvature.

Among the family of complete minimal surfaces in space, those with *finite total curvature* have been the most extensively studied. The principal reason for this is that they can be thought of as compact algebraic objects in a natural sense, which opens tremendously the number and depth of tools that can be applied to study these kinds of surfaces. We can illustrate this point of view with the example of the *catenoid* (see Chapter 2.5 for a computer image and discussion of this surface, as well as other examples of complete, embedded minimal surfaces of finite total curvature). The vertical catenoid C is obtained as the surface of revolution of the graph of $\cosh x_3$ around the x_3-axis. It is straightforward to check that its Gauss map $N\colon C \to \mathbb{S}^2$ is a conformal diffeomorphism of C with its image, which consists of $\mathbb{S}^2$ punctured in the north and south poles. Hence, the conformal

[7]This deformation is well-known since López and Ros used it as the main ingredient in their proof of Theorem 3.1.2 below, although it had already been previously used by other authors, see Nayatani [167].

compactification $\overline{C}$ of C is conformally the sphere and the Gauss map extends holomorphically to the compactification $\overline{C}$. More generally, we have the following result.

Theorem 2.3.1 (Huber [88], Osserman [178]) *Let $M \subset \mathbb{R}^3$ be a complete (oriented), immersed minimal surface with finite total curvature. Then,*

i) M is conformally a compact Riemann surface $\overline{M}$ with a finite number of points removed (called the ends of M).

ii) The Weierstrass data (g, dh) extends meromorphically to $\overline{M}$. In particular, the total curvature of M is a multiple of -4π.

In this setting, the Gauss map g has a well-defined finite degree on $\overline{M}$. A direct consequence of (2.3) is that the total curvature of an M as in Theorem 2.3.1 is -4π times the degree of its Gauss map g. It turns out that this degree can be computed in terms of the genus of the compactification $\overline{M}$ and the number of ends, by means of the *Jorge-Meeks formula*[8] [90]. Rather than stating here this general formula for an immersed surface M as in Theorem 2.3.1, we will emphasize the particular case when all the ends of M are embedded:

$$\deg(g) = \text{genus}(\overline{M}) + \#(\text{ends}) - 1. \tag{2.9}$$

The asymptotic behavior of a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature is well-understood. Schoen [206] demonstrated that, after a rotation, each embedded end of a complete minimal surface with finite total curvature can be parameterized as a graph over the exterior of a disk in the (x_1, x_2)-plane with height function

$$x_3(x_1, x_2) = a \log r + b + \frac{c_1 x_1 + c_2 x_2}{r^2} + \mathcal{O}(r^{-2}), \tag{2.10}$$

where $r = \sqrt{x_1^2 + x_2^2}$, $a, b, c_1, c_2 \in \mathbb{R}$ and $\mathcal{O}(r^{-2})$ denotes a function such that $r^2\mathcal{O}(r^{-2})$ is bounded as $r \to \infty$. The coefficient a in (2.10) is called the *logarithmic growth* of the end. When $a \neq 0$, the end is called a *catenoidal end;* if $a = 0$, we have a *planar end.* We use this language since a catenoidal end is asymptotic to one of the ends of a catenoid (see Chapter 2.5 for a description of the catenoid) and a planar end is asymptotic to the end of a plane. In particular, complete embedded minimal surfaces with finite total curvature are always proper; in fact, an elementary analysis of the asymptotic behavior shows that the equivalence between completeness and properness still holds for immersed minimal surfaces with finite total curvature.

[8]This is an application of the classical Gauss-Bonnet formula. Different authors, such as Gackstatter [65] and Osserman [178], have obtained related formulas.

As explained above, minimal surfaces with finite total curvature have been widely studied and their comprehension is the starting point to deal with more general questions about complete embedded minimal surfaces. In Chapter 2.5 we will briefly describe some examples in this family. Chapter 3 contains short explanations of some of the main results and constructions for finite total curvature minimal surfaces. Beyond these two short incursions, we will not develop extensively the theory of minimal surfaces with finite total curvature, since this is not the purpose this monograph, as explained in the Introduction. We refer the interested reader to the treatises by Hoffman and Karcher [75], López and Martín [106] and Pérez and Ros [186] for a more in depth treatment of these special surfaces.

2.4 Periodic minimal surfaces.

A properly embedded minimal surface $M \subset \mathbb{R}^3$ is called *singly, doubly* or *triply-periodic* when it is invariant by an infinite group G of isometries of $\mathbb{R}^3$ of rank $1, 2, 3$ (respectively) that acts properly and discontinuously. Very often, it is useful to study such an M as a minimal surface in the complete, flat three-manifold $\mathbb{R}^3/G$. Up to finite coverings and after composing by a fixed rotation in $\mathbb{R}^3$, these three-manifolds reduce to $\mathbb{R}^3/T$, $\mathbb{R}^3/S_\theta$, $\mathbb{T}^2 \times \mathbb{R}$ and $\mathbb{T}^3$, where T denotes a non-trivial translation, S_θ is the screw motion symmetry resulting from the composition of a rotation of angle θ around the x_3-axis with a translation in the direction of this axis, and $\mathbb{T}^2$, $\mathbb{T}^3$ are flat tori obtained as quotients of $\mathbb{R}^2$, $\mathbb{R}^3$ by 2 or 3 linearly independent translations, respectively.

Meeks and Rosenberg [143, 146] developed the theory of periodic minimal surfaces. For instance, they obtained in this setting similar conclusions as those given in Theorem 2.3.1, except that the Gauss map g of a minimal surface in $\mathbb{R}^3/G$ is not necessarily well-defined (the Gauss map g does not descend to the quotient for surfaces in $\mathbb{R}^3/S_\theta$, $\theta \in (0, 2\pi)$, and in this case the role of g is played by the well-defined meromorphic differential form dg/g). An important fact, due to Meeks and Rosenberg [143, 146], is that for properly embedded minimal surfaces in $\mathbb{R}^3/G$, $G \neq \{\text{identity}\}$, the conditions of finite total curvature and finite topology are equivalent[9].

Theorem 2.4.1 (Meeks, Rosenberg [143, 146, 147, 149]) *A complete, embedded minimal surface in a non-simply-connected, complete, flat three-manifold has finite topology if and only if it has finite total curvature. Furthermore, if Σ denotes a compact surface of non-positive curvature and $M \subset \Sigma \times \mathbb{R}$ is a properly embedded minimal surface of finite genus, then*

[9] This equivalence does not hold for properly embedded minimal surfaces in $\mathbb{R}^3$, as demonstrated by the helicoid.

M has finite topology, finite index of stability[10] *and its finite total curvature is equal to 2π times the Euler characteristic of M.*

The second statement in the above theorem was motivated by a result of Meeks [118], who proved that every properly embedded minimal surface in $\mathbb{T}^2 \times \mathbb{R}$ has a finite number of ends; hence, in this setting finite genus implies finite total curvature.

Meeks and Rosenberg [143, 146] also studied the asymptotic behavior of complete, embedded minimal surfaces with finite total curvature in $\mathbb{R}^3/G$. Under this condition, there are three possibilities for the ends of the quotient surface: all ends must be simultaneously asymptotic to planes (as in the Riemann minimal examples, see Chapter 2.5; such ends are called *planar ends*), to ends of helicoids (called *helicoidal ends*), or to flat annuli (as in the singly or doubly-periodic Scherk minimal surfaces; for this reason, such ends are called *Scherk-type ends*).

2.5 Some interesting examples of complete minimal surfaces.

We will now use the Weierstrass representation for introducing some of the most celebrated complete minimal surfaces. Throughout the presentation of these examples, we will freely use Collin's Theorem 1.0.3 and Colding-Minicozzi's Theorem 1.0.5.

The plane. $M = \mathbb{C}$, $g(z) = 1$, $dh = dz$. It is the only complete, flat minimal surface in $\mathbb{R}^3$.

The catenoid. $M = \mathbb{C} - \{0\}$, $g(z) = z$, $dh = \frac{dz}{z}$. In 1741, Euler [52] discovered that when a catenary $x_1 = \cosh x_3$ is rotated around the x_3-axis, one obtains a surface which minimizes area among surfaces of revolution after prescribing boundary values for the generating curves. This surface was called the *alysseid* or since Plateau, the catenoid. In 1776, Meusnier verified that the catenoid is a solution of Lagrange's equation (2.1). This surface has genus zero, two ends and total curvature -4π. Together with the plane, the catenoid is the only minimal surface of revolution (Bonnet [10]) and the unique complete, embedded minimal surface with genus zero, finite topology and more than one end (López and Ros [107]). Also, the catenoid is characterized as being the unique complete, embedded minimal surface with finite topology and two ends (Schoen [206]). See Figure 2.1 Left.

The helicoid. $M = \mathbb{C}$, $g(z) = e^z$, $dh = i\,dz$. This surface was first proved to be minimal by Meusnier in 1776 [159]. When viewed in $\mathbb{R}^3$, the helicoid has genus zero, one end and infinite total curvature. Together with the plane, the helicoid is the only ruled minimal surface (Catalan [17]) and the

[10] See Definition 2.8.2 for the definition of index of stability.

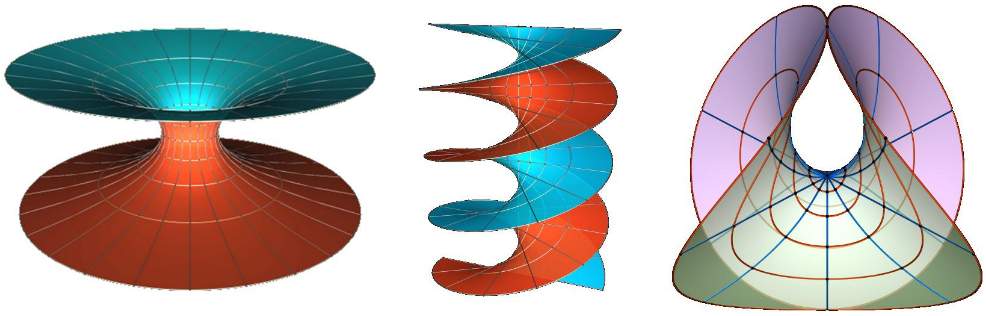

Figure 2.1: Left: The catenoid. Center: The helicoid. Right: The Enneper surface. Images courtesy of M. Weber.

unique simply-connected, complete, embedded minimal surface (Meeks and Rosenberg [148]). The vertical helicoid also can be viewed as a genus-zero surface with two ends in a quotient of $\mathbb{R}^3$ by a vertical translation or by a screw motion. See Figure 2.1 Center. The catenoid and the helicoid are *conjugate* minimal surfaces, in the sense of the following definition.

Definition 2.5.1 *Two minimal surfaces in $\mathbb{R}^3$ are said to be conjugate if the coordinate functions of one of them are the harmonic conjugates of the coordinate functions of the other one.*

Note that in the case of the helicoid and catenoid, we consider the catenoid to be defined on its universal cover $e^z : \mathbb{C} \to \mathbb{C} - \{0\}$ in order for the harmonic conjugate of x_3 to be well-defined. Equivalently, both surfaces share the Gauss map e^z and their height differentials differ by multiplication by $i = \sqrt{-1}$.

The Enneper surface. $M = \mathbb{C}$, $g(z) = z$, $dh = z\,dz$. This surface was discovered by Enneper [51] in 1864, using his newly formulated analytic representation of minimal surfaces in terms of holomorphic data, equivalent to the Weierstrass representation[11]. This surface is non-embedded, has genus zero, one end and total curvature -4π. It contains two horizontal orthogonal lines (after cutting the surface along either of these lines, one divides it into two embedded pieces bounded by a line) and the entire surface has two vertical planes of reflective symmetry. See Figure 2.1 Right. Every rotation of the coordinate z around the origin in $\mathbb{C}$ is an (intrinsic) isometry of the Enneper surface, but most of these isometries do not extend to ambient isometries. The catenoid and Enneper's surface are the unique complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature -4π (Osserman [178]).

[11] For this reason, the Weierstrass representation is also referred to in the literature as the *Enneper-Weierstrass representation.*

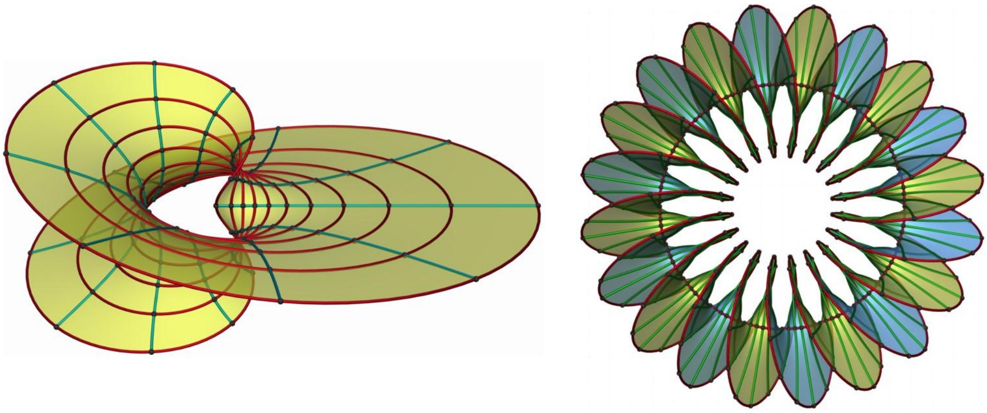

Figure 2.2: Left: The Meeks minimal Möbius strip. Right: A bent helicoid near the circle $\mathbb{S}^1$, which is viewed from above along the x_3-axis. Images courtesy of M. Weber.

The implicit form of Enneper's surface is

$$\left(\frac{y^2-x^2}{2z}+\frac{2}{9}z^2+\frac{2}{3}\right)^3-6\left(\frac{y^2-x^2}{4z}-\frac{1}{4}(x^2+y^2+\frac{8}{9}z^2)+\frac{2}{9}\right)^2=0.$$

The Meeks minimal Möbius strip. $M=\mathbb{C}-\{0\}$, $g(z)=z^2\left(\frac{z+1}{z-1}\right)$, $dh=i\left(\frac{z^2-1}{z^2}\right)dz$. Found by Meeks [114], the minimal surface defined by this Weierstrass data double covers a complete, immersed minimal surface $M_1\subset\mathbb{R}^3$ which is topologically a Möbius strip. This is the unique complete, minimally immersed surface in $\mathbb{R}^3$ of finite total curvature -6π. It contains a unique closed geodesic which is a planar circle, and also contains a line bisecting the circle, see Figure 2.2 Left.

The bent helicoids. $M=\mathbb{C}-\{0\}$, $g(z)=-z\frac{z^n+i}{iz^n+i}$, $dh=\frac{z^n+z^{-n}}{2z}dz$. Discovered by Meeks and Weber [152] and independently by Mira [162], these are complete, immersed minimal annuli $\widetilde{H}_n\subset\mathbb{R}^3$ with two non-embedded ends and finite total curvature; each of the surfaces $\widetilde{H}_n$ contains the unit circle $\mathbb{S}^1$ in the (x_1,x_2)-plane, and a neighborhood of $\mathbb{S}^1$ in $\widetilde{H}_n$ contains an embedded annulus H_n which approximates, for n large, a highly spinning helicoid whose usual straight axis has been periodically bent into the unit circle $\mathbb{S}^1$ (thus the name of bent helicoids), see Figure 2.2 Right. Furthermore, the H_n converge as $n\to\infty$ to the foliation of $\mathbb{R}^3$ minus the x_3-axis by vertical half-planes with boundary the x_3-axis, and with $\mathbb{S}^1$ as the singular set of C^1-convergence. The method applied by Meeks, Weber and Mira to find the bent helicoids is the classical *Björling formula* [175] with an orthogonal unit field along $\mathbb{S}^1$ that spins an arbitrary number n of times around the circle. This construction also makes sense when n is half an integer; in the case $n = \frac{1}{2}$, $\widetilde{H}_{1/2}$ is the double cover of the Meeks minimal Möbius strip described in the previous example. The bent helicoids H_n play an important role in proving the converse of Meeks' $C^{1,1}$-Regularity Theorem (see Meeks and Weber [152] and also Theorems 10.2.1 and 10.2.3 below) for the singular set of convergence in a Colding-Minicozzi limit minimal lamination.

For the next group of examples, we need to introduce some notation. Given $k \in \mathbb{N}$, $k \geq 1$ and $a \in \mathbb{R} - \{0, -1\}$, we define the compact genus-k surface $\overline{M}_{k,a} = \{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^{k+1} = \frac{(z+1)(z-a)}{z}\}$. Let $M_{k,a} = \overline{M}_{k,a} - \{(-1,0), (\infty,\infty), (a,0)\}$ and

$$g_{k,a,m,A}(z,w) = A\frac{zw}{mz+1}, \qquad dh_{k,a,m} = \frac{mz+1}{(z+1)(z-a)}dz,$$

where $A \in \mathbb{R} - \{0\}$. Given $k \in \mathbb{N}$ and $a \in (0, \infty)$, there exist $m = m(a) \in \mathbb{R}$ and $A = A(a) \in \mathbb{R} - \{0\}$ such that the pair $(g_{k,a,m(a),A(a)}, dh_{k,a,m(a)})$ is the Weierstrass data of a well-defined minimal surface $X \colon M_{k,a} \to \mathbb{R}^3$ with genus k and three ends (Hoffman, Karcher [75]). Moreover, $m(1) = 0$ for any $k \in \mathbb{N}$. With this notation, we have the following examples.

The Costa torus. $M = M_{1,1}$, $g = g_{1,1,0,A(1)}$, $dh = dh_{1,1,0}$. Perhaps the most celebrated complete minimal surface in $\mathbb{R}^3$ since the classical examples from the nineteenth century[12], was discovered in 1982 by Costa [40, 41]. This is a thrice punctured torus with total curvature -12π, two catenoidal ends and one planar middle end. Costa [41] demonstrated the existence of this surface but only proved its embeddedness outside a ball in $\mathbb{R}^3$. Hoffman and Meeks [79] demonstrated its global embeddedness, thereby disproving a longstanding conjecture that the only complete, embedded minimal surfaces in $\mathbb{R}^3$ of finite topological type are the plane, catenoid and helicoid. The Costa surface contains two horizontal straight lines l_1, l_2 that intersect orthogonally, and has vertical planes of symmetry bisecting the right angles made by l_1, l_2, see Figure 2.3 Left.

The Costa-Hoffman-Meeks surfaces. For any $k \geq 2$, take $M = M_{k,1}$, $g = g_{1,1,0,A(1)}$, $dh = dh_{1,1,0}$. These examples generalize the Costa torus (given by $k = 1$), and are complete, embedded, genus-k minimal surfaces with two catenoidal ends and one planar middle end. Both existence and embeddedness were given by Hoffman and Meeks [80]. The symmetry group of the genus-k example is generated by 180°-rotations about $k+1$ horizontal lines contained in the surface that intersect at a single point, together with the reflective symmetries in vertical planes that bisect those lines. As

[12]It also should be mentioned that Chen and Gackstatter [19, 20] found in 1981 a complete minimal immersion (not embedded) of a once punctured torus in $\mathbb{R}^3$, obtained by adding a handle to the Enneper surface. This surface was really a direct ancestor of the Costa torus and the Costa-Hoffman-Meeks surfaces, see Hoffman [74] for geometric and historical connections between these surfaces.

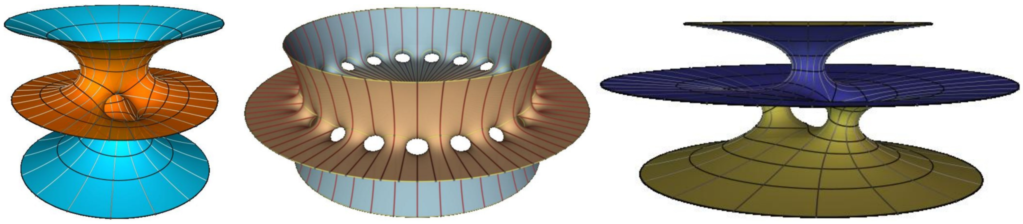

Figure 2.3: Left: The Costa torus. Center: A Costa-Hoffman-Meeks surface of genus 20. Right: Deformed Costa. Images courtesy of M. Weber.

$k \to \infty$, suitable scalings of the $M_{k,1}$ converge either to the singular configuration given by a vertical catenoid and a horizontal plane passing through its waist circle, or to the singly-periodic Scherk minimal surface for $\theta = \pi/2$ (Hoffman-Meeks [81]). Both of these limits are suggested by Figure 2.3 Center.

The deformation of the Costa torus. The Costa surface is defined on a square torus $M_{1,1}$, and admits a deformation (found by Hoffman and Meeks, unpublished) where the planar end becomes catenoidal. For any $a \in (0,\infty)$, take $M = M_{1,a}$ (which varies on arbitrary rectangular tori), $g = g_{1,a,m(a),A(a)}$, $dh = dh_{1,a,m(a)}$, see Figure 2.3 Right. Thus, $a = 1$ gives the Costa torus. The deformed surfaces are easily seen to be embedded for a close to 1. Hoffman and Karcher [75] proved the existence and embeddedness of these surfaces for all values of a, see also the survey by López and Martín [106]. Costa [42, 43] showed that any complete, embedded minimal torus with three ends must lie in this family.

The deformation of the Costa-Hoffman-Meeks surfaces. For any $k \geq 2$ and $a \in (0,\infty)$, take $M = M_{k,a}$, $g = g_{k,a,m(a),A(a)}$, $dh = dh_{k,a,m(a)}$. When $a = 1$, we find the Costa-Hoffman-Meeks surface of genus k and three ends. As in the case of genus 1, Hoffman and Meeks discovered this deformation for values of a close to 1. A complete proof of existence and embeddedness for these surfaces is given in [75] by Hoffman and Karcher. These surfaces are conjectured to be the unique complete, embedded minimal surfaces in $\mathbb{R}^3$ with genus k and three ends.

The genus-one helicoid. Now M is conformally a certain rhombic torus $\mathbb{T}$ minus one point E. If we view $\mathbb{T}$ as a rhombus with edges identified in the usual manner, then E corresponds to the vertices of the rhombus, and the diagonals of $\mathbb{T}$ are mapped into perpendicular straight lines contained in the surface, intersecting at a single point in space. The unique end of M is asymptotic to a helicoid, so that one of the two lines contained in the surface is an *axis* (like in the genuine helicoid). The Gauss map g is a meromorphic function on $\mathbb{T} - \{E\}$ with an essential singularity at E, and

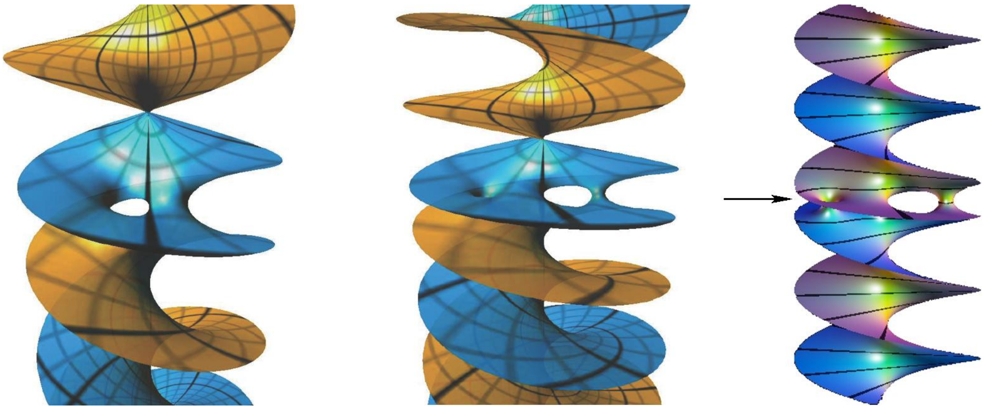

Figure 2.4: Left: The genus-one helicoid. Center and Right: Two views of the (possibly existing) genus-two helicoid. The arrow in the figure at the right points to the second handle. Images courtesy of M. Schmies (Left, Center) and M. Traizet (Right).

both dg/g and dh extend meromorphically to $\mathbb{T}$, see Figure 2.4 Left. This surface was discovered in 1993 by Hoffman, Karcher and Wei [76, 77]. Using flat structures[13], Hoffman, Weber and Wolf [84] proved the embeddedness of a genus-one helicoid, obtained as a limit of singly-periodic "genus-one" helicoids invariant by screw motions of arbitrarily large angles[14]. Later Hoffman and White [86] gave a variational proof of the existence of a genus-one helicoid. There is computational evidence pointing to the existence of a unique complete, embedded minimal surface in $\mathbb{R}^3$ with one helicoidal end for any positive genus (Traizet —unpublished—, Bobenko [7], Bobenko and Schmies [8], Schmies [204], also see Figure 2.4), but both the existence and the uniqueness questions remain unsolved (see Conjecture 17.0.20).

The singly-periodic Scherk surfaces. $M = (\mathbb{C} \cup \{\infty\}) - \{\pm e^{\pm i\theta/2}\}$, $g(z) = z$, $dh = \frac{iz\,dz}{\prod(z \pm e^{\pm i\theta/2})}$, for fixed $\theta \in (0, \pi/2]$. Discovered by Scherk [203] in 1835, these surfaces denoted by $\mathcal{S}_\theta$ form a 1-parameter family of complete, embedded, genus-zero minimal surfaces in a quotient of $\mathbb{R}^3$ by a translation, and have four annular ends. Viewed in $\mathbb{R}^3$, each surface $\mathcal{S}_\theta$ is invariant under reflection in the (x_1, x_3) and (x_2, x_3)-planes and in horizontal planes at integer heights, and can be thought of geometrically as a desingularization of two vertical planes forming an angle of θ. The special case $\mathcal{S}_{\theta=\pi/2}$ also contains pairs of orthogonal lines at planes of half-integer heights, and has implicit equation $\sin z = \sinh x \sinh y$, see Figure 2.5 Left. Together with the

[13]See Chapter 3.2 (the method of Weber and Wolf) for the definition of flat structure.

[14]These singly-periodic "genus-one" helicoids have genus one in their quotient spaces and were discovered earlier by Hoffman, Karcher and Wei [78] (case of translation invariance) and by Hoffman and Wei [85] (case of screw-motion invariance).

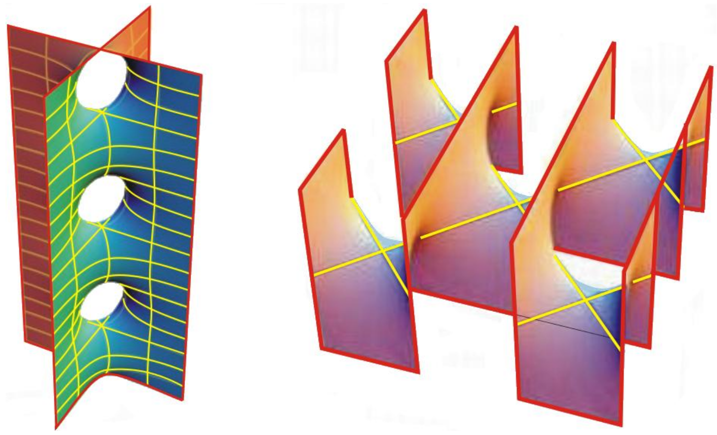

Figure 2.5: Singly-periodic Scherk surface with angle $\theta = \frac{\pi}{2}$ (left), and its conjugate surface, the doubly-periodic Scherk surface (right). Images courtesy of M. Weber.

plane and catenoid, the surfaces $\mathcal{S}_\theta$ are conjectured to be the only connected, complete, immersed, minimal surfaces in $\mathbb{R}^3$ whose area in balls of radius R is less than $2\pi R^2$ (Conjecture 17.0.27). See Chapter 15 for a solution on this conjecture under the additional hypothesis of infinite symmetry.

The doubly-periodic Scherk surfaces. $M = (\mathbb{C} \cup \{\infty\}) - \{\pm e^{\pm i\theta/2}\}$, $g(z) = z$, $dh = \frac{z\,dz}{\prod(z \pm e^{\pm i\theta/2})}$, where $\theta \in (0, \pi/2]$ (the case $\theta = \frac{\pi}{2}$ has implicit equation $e^z \cos y = \cos x$). These surfaces, also discovered by Scherk [203] in 1835, are the conjugate surfaces of the singly-periodic Scherk surfaces, and can be thought of geometrically as the desingularization of two families of equally spaced vertical parallel half-planes in opposite half-spaces, with the half-planes in the upper family making an angle of θ with the half-planes in the lower family, see Figure 2.5 Right. These surfaces are doubly-periodic with genus zero in their corresponding quotient $\mathbb{T}^2 \times \mathbb{R}$, and were characterized by Lazard-Holly and Meeks [103] as being the unique properly embedded minimal surfaces with genus zero in any $\mathbb{T}^2 \times \mathbb{R}$. It has been conjectured by Meeks, Pérez and Ros [134] that the singly and doubly-periodic Scherk minimal surfaces are the only complete, embedded minimal surfaces in $\mathbb{R}^3$ whose Gauss maps miss four points on $\mathbb{S}^2$ (Conjecture 17.0.33). They also conjecture that the singly and doubly-periodic Scherk minimal surfaces, together with the catenoid and helicoid, are the only complete, embedded minimal surfaces of negative curvature (Conjecture 17.0.32).

The Schwarz Primitive triply-periodic surface. $M = \{(z, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = z^8 - 14z^4 + 1\}$, $g(z, w) = z$, $dh = \frac{z\,dz}{w}$. Discovered by Schwarz in the 1880's, this minimal surface has a rank-three symmetry group and is invariant by translations in $\mathbb{Z}^3$. Such a structure, common to any triply-periodic minimal surface (TPMS), is also known as a *crystallographic cell* or *space tiling*. Embedded TPMS divide $\mathbb{R}^3$ into two connected components (called *labyrinths* in crystallography), sharing M as boundary (or *interface*) and interweaving each other. This property makes TPMS objects of interest to neighboring sciences as material sciences, crystallography, biology and others. For example, the interface between single calcite crystals and amorphous organic matter in the skeletal element in sea urchins is approximately described by the Schwarz Primitive surface [3, 49, 171]. The piece of a TPMS that lies inside a crystallographic cell of the tiling is called a *fundamental domain*, see Figure 2.6.

In the case of the Schwarz Primitive surface, one can choose a fundamental domain that intersects the faces of a cube in closed geodesics which are almost circles. In fact, the Schwarz Primitive surface has many more symmetries than those coming from the spatial tiling: some of them are produced by rotation around straight lines contained in the surface, which by the *Schwarz reflection principle*[15] divide the surface into congruent graphs with piecewise linear quadrilateral boundaries. Other interesting properties of this surface are that it divides space into two congruent three-dimensional regions (as do many others TPMS), and the quotient surface in the three-torus associated to the above crystallographic cells is a compact hyperelliptic Riemann surface with genus three (in fact, any TPMS Σ with genus three is hyperelliptic, since by application of the Gauss-Bonnet formula, the corresponding Gauss map $g\colon \Sigma \to \mathbb{S}^2$ is holomorphic with degree two). Its conjugate surface, also discovered by Schwarz, is another famous example of an embedded TPMS, called the *Schwarz Diamond surface.* In the 1960's, Schoen [205] made a surprising discovery: another associate surface[16] of the Primitive and Diamond surface is an embedded TPMS, and named this surface the *Gyroid.*

The Primitive, Diamond and Gyroid surfaces play important roles as surface interfaces in material sciences, in part since they are stable in their quotient tori under volume preserving variations (see Ross [201]). Furthermore, these surfaces have index of stability one, and Ros [196] has shown that any orientable, embedded minimal surface of index one in a flat three-torus must have genus three. He conjectures that the Primitive, Diamond and Gyroid are the unique index-one minimal surfaces in their tori, and furthermore, that any flat three-torus can have at most one embedded, ori-

[15] A minimal surface that contains a straight line segment l (resp. a planar geodesic) is invariant under the rotation of angle 180° about l (resp. under reflection in the plane that contains l). Both assertions follow directly from the general Schwarz reflection principle for harmonic functions.

[16] The family of *associate surfaces* of a simply-connected minimal surface with Weierstrass data (g, dh) are those with the same Gauss map and height differential $e^{i\theta}dh$, $\theta \in [0, 2\pi)$. In particular, the case $\theta = \pi/2$ is the conjugate surface. This notion can be generalized to non-simply-connected surfaces, although in that case the associate surfaces may have periods.

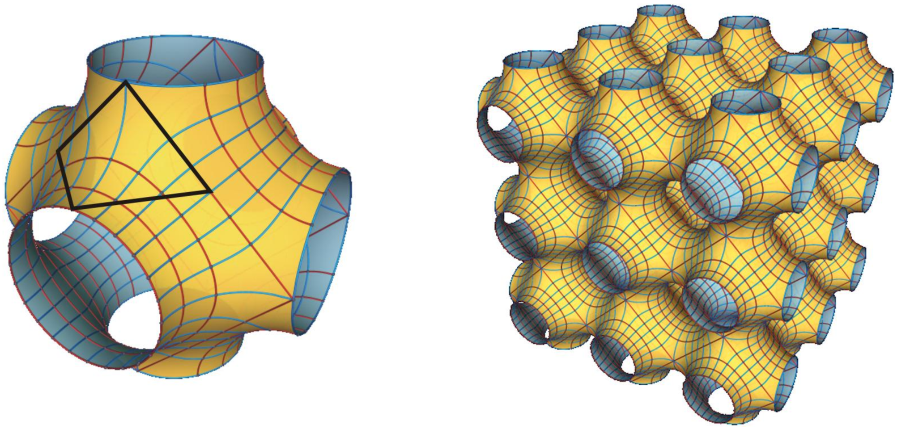

Figure 2.6: Left: A fundamental domain of the Schwarz P-surface, with a graphical quadrilateral. Right: A larger piece of the corresponding space tiling. Images courtesy of M. Weber.

entable minimal surface of index one. We refer the interested reader to Karcher [95, 96], Meeks [117] and Ros [195] for further examples and properties of triply-periodic minimal surfaces. We remark that Traizet [217] has shown that every flat three-torus contains an infinite number of embedded, genus-g, $g \geq 3$, minimal surfaces which are prime in the sense that they do not descend to minimal surfaces in another three-torus.

The Riemann minimal examples. This is a one-parameter family of surfaces, defined in terms of a parameter $\lambda > 0$. Let $M_\lambda = \{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = z(z-\lambda)(\lambda z + 1)\} - \{(0,0),(\infty,\infty)\}$, $g(z,w) = z$, $dh = A_\lambda \frac{dz}{w}$, for each $\lambda > 0$, where A_λ is a non-zero complex number satisfying $A_\lambda^2 \in \mathbb{R}$. Discovered by Riemann (and posthumously published, Hattendorf and Riemann [190, 191]), these examples are invariant under reflection in the (x_1, x_3)-plane and by a translation T_λ. The induced surfaces M_λ/T_λ in the quotient spaces $\mathbb{R}^3/T_\lambda$ have genus one and two planar ends, see [136] for a more precise description. The Riemann minimal examples have the amazing property that every horizontal plane intersects each of these surfaces in a circle or in a line, see Figure 2.7. The conjugate minimal surface of the Riemann minimal example for a given $\lambda > 0$ is the Riemann minimal example for the parameter value $1/\lambda$ (the case $\lambda = 1$ gives the only self-conjugate surface in the family). Meeks, Pérez and Ros [136] showed that these surfaces are the only properly embedded minimal surfaces in $\mathbb{R}^3$ of genus zero and infinite topology (Theorem 12.3.1).

The KMR doubly-periodic tori. Given $(\theta, \alpha, \beta) \in (0, \frac{\pi}{2}) \times [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ with $(\alpha, \beta) \neq (0, \theta)$, we consider the rectangular torus $\Sigma_\theta = \{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = (z^2 + \lambda^2)(z^2 + \lambda^{-2})\}$, where $\lambda = \cot \frac{\theta}{2}$. Then $M = \Sigma_\theta -$

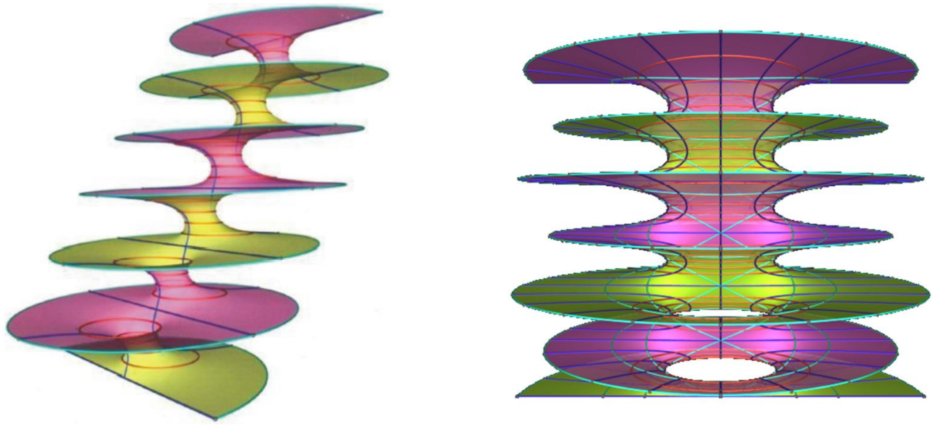

Figure 2.7: Two Riemann minimal examples (for different values of the parameter λ). Images courtesy of M. Weber.

$g^{-1}(\{0,\infty\})$,

$$g(z,w)=\frac{z\left(i\cos(\frac{\alpha-\beta}{2})+\cos(\frac{\alpha+\beta}{2})\right)+\sin(\frac{\alpha-\beta}{2})+i\sin(\frac{\alpha+\beta}{2})}{\cos(\frac{\alpha-\beta}{2})+i\cos(\frac{\alpha+\beta}{2})-z\left(i\sin(\frac{\alpha-\beta}{2})+\sin(\frac{\alpha+\beta}{2})\right)},\quad \text{and }\ dh=\mu\frac{dz}{w},$$

where $\mu=1$ or $\sqrt{-1}$. This Weierstrass data gives rise to a three-dimensional family of doubly-periodic minimal surfaces in $\mathbb{R}^3$, that in the smallest quotient in some $\mathbb{T}^2\times\mathbb{R}$ have four parallel Scherk type ends and total curvature -8π. Furthermore, the conjugate surface of any KMR surface also lies in this family. The first KMR surfaces were found by Karcher [94] in 1988 (he found three subfamilies, each one with dimension one, and named them *toroidal half-plane layers*). One year later, Meeks and Rosenberg [143] found examples of the same type as Karcher's, although the different nature of their approach made it unclear what the relationship was between their examples and those by Karcher. In 2005, Pérez, Rodríguez and Traizet [184] gave a general construction that produces all possible complete, embedded minimal tori with parallel ends in any $\mathbb{T}^2\times\mathbb{R}$, and proved that this moduli space reduces to the three-dimensional family of surfaces defined by the Weierstrass data given above, see Figure 2.8. It is conjectured that the only complete, embedded minimal surfaces in $\mathbb{R}^3$ whose Gauss map misses exactly 2 points on $\mathbb{S}^2$ are the catenoid, helicoid, Riemann examples, and these KMR examples.

The singly-periodic Callahan-Hoffman-Meeks surfaces. In 1989, Callahan, Hoffman and Meeks [15] generalized the Riemann minimal examples by constructing for any integer $k\geq 1$ a singly-periodic, properly embedded minimal surface $M_k\subset\mathbb{R}^3$ with infinite genus and an infinite

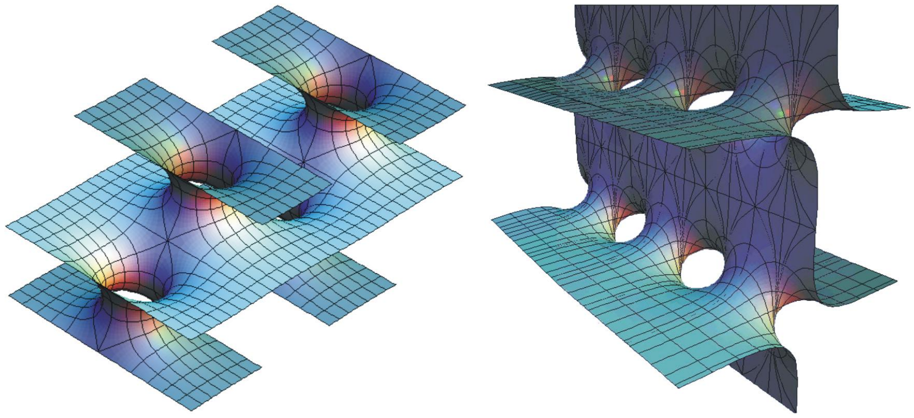

Figure 2.8: Two examples of doubly-periodic KMR surfaces. Images taken from the 3D-XplorMath Surface Gallery.

number of horizontal planar ends at integer heights, invariant under the orientation preserving translation by vector $T = (0,0,2)$, such that M_k/T has genus $2k+1$ and two ends. They not only produced the Weierstrass data of the surface (see [15] for details), but also gave an alternative method for finding this surface, based on blowing-up a limiting singularity of a sequence of compact minimal annuli with boundaries. This rescaling process was a prelude to the crucial role that rescaling methods play nowadays in minimal surface theory, and that we will treat in some detail in this survey. Other properties of the surfaces M_k are the following ones.

1. Every horizontal plane at a non-integer height intersects M_k in a simple closed curve.

2. Every horizontal plane at an integer height intersects M_k in $k+1$ straight lines that meet at equal angles along the x_3-axis.

3. Every horizontal plane at half-integer heights $n+\frac{1}{2}$ is a plane of symmetry of M_k, and any vertical plane whose reflection leaves invariant the horizontal lines on M_k described in item 2, is also a plane of symmetry. For pictures of M_k with $k = 1, 2$, see Figure 2.9.

2.6 Monotonicity formula and classical maximum principles.

As we will see in Chapter 7.3, the conformal type of a minimal surface in $\mathbb{R}^3$ is strongly influenced by its area growth in extrinsic balls. The first result along these lines comes from the coarea formula applied to the distance

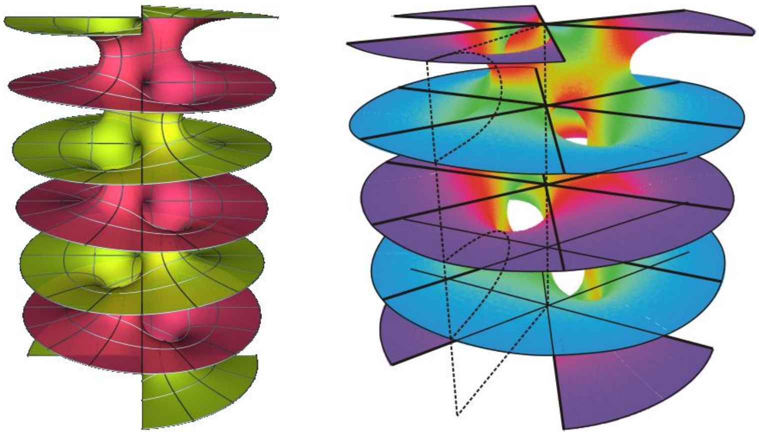

Figure 2.9: Callahan-Hoffman-Meeks surfaces M_k, for $k = 1$ (left, image courtesy of M. Weber) and $k = 2$ (right, from the Scientific Graphics Project at http://www.msri.org/about/sgp/SGP/index.html). The dotted lines correspond to a vertical plane of symmetry of M_2.

function to a given point $p \in \mathbb{R}^3$. The following statement of the coarea formula appears in [18], see [55] for a more general version.

Proposition 2.6.1 (Coarea Formula) *Let Ω be a domain with compact closure in a Riemannian n-manifold M and $f\colon \overline{\Omega} \to \mathbb{R}$ a function in $C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ with $f|_{\partial\Omega} = 0$. For any regular value t of $|f|$, we let $\Gamma(t) = |f|^{-1}(t)$ and let $A(t)$ denote the $(n-1)$-dimensional area of $\Gamma(t)$. Then, for any function $\phi \in L^1(\Omega)$, we have*

$$\int_\Omega \phi |\nabla f| \, dV = \int_0^\infty \left(\int_{\Gamma(t)} \phi \, dA_t \right) dt,$$

where ∇f is the gradient of f in M and dV, dA_t are respectively the volume elements in M and $\Gamma(t)$.

Theorem 2.6.2 (Monotonicity Formula [25, 98]) *Let $X\colon M \to \mathbb{R}^3$ be a connected, properly immersed minimal surface. Given $p \in \mathbb{R}^3$, let $A(R)$ be the area of the portion of $X(M)$ inside a ball of radius $R > 0$ centered at p. Then, $A(R)R^{-2}$ is non-decreasing. In particular, $\lim_{R\to\infty} A(R)R^{-2} \geq \pi$ with equality if and only if M is a plane.*

One of the consequences of the fact that minimal surfaces can be viewed locally as solutions of the partial differential equation (2.1) is that they satisfy certain maximum principles. We will state them for minimal surfaces in $\mathbb{R}^3$, but they also hold when the ambient space is any Riemannian three-manifold.

Theorem 2.6.3 (Interior Maximum Principle [206]) *Let M_1, M_2 be connected minimal surfaces in $\mathbb{R}^3$ and p an interior point to both surfaces, such that $T_pM_1 = T_pM_2 = \{x_3 = 0\}$. If M_1, M_2 are locally expressed as the graphs of functions u_1, u_2 around p and $u_1 \leq u_2$ in a neighborhood of p, then $M_1 = M_2$ in a neighborhood of p.*

A beautiful application of Theorem 2.6.3 is the following result by Hoffman and Meeks.

Theorem 2.6.4 (Half-space Theorem [83]) *Let $M \subset \mathbb{R}^3$ be a proper, connected, possibly branched, non-planar minimal surface without boundary. Then, M cannot be contained in a half-space.*

More generally, one has the following result of Meeks and Rosenberg based on earlier partial results in [22, 83, 100, 144, 210].

Theorem 2.6.5 (Maximum Principle at Infinity [150]) *Let $M_1, M_2 \subset N$ be disjoint, connected, properly immersed minimal surfaces with (possibly empty) boundary in a complete flat three-manifold N.*

i) If $\partial M_1 \neq \emptyset$ or $\partial M_2 \neq \emptyset$, then after possibly reindexing, the distance between M_1 and M_2 (as subsets of N) is equal to $\inf\{\mathrm{dist}(p,q) \mid p \in \partial M_1,\, q \in M_2\}$.

ii) If $\partial M_1 = \partial M_2 = \emptyset$, then M_1 and M_2 are flat.

We now come to a beautiful and deep application of the general maximum principle at infinity. The next corollary appears in [150] and a slightly weaker variant of it in Soret [210].

Corollary 2.6.6 (Regular Neighborhood Theorem) *Suppose $M \subset N$ is a properly embedded minimal surface in a complete flat three-manifold N, with the absolute value of the Gaussian curvature of M at most 1. Let $N_1(M)$ be the open unit interval bundle of the normal bundle of M given by the normal vectors of length strictly less that 1. Then, the corresponding exponential map* $\exp\colon N_1(M) \to N$ *is a smooth embedding. In particular:*

1. *M has an open, embedded tubular neighborhood of radius 1.*

2. *There exists a constant $C > 0$ such that for all balls $B \subset N$ of radius 1, the area of $M \cap B$ is at most C times the volume of B.*

2.7 Ends of properly embedded minimal surfaces.

One of the fundamental problems in classical minimal surface theory is to describe the behavior of a properly embedded minimal surface $M \subset \mathbb{R}^3$ outside a large compact set in space. This problem is well-understood if M has finite total curvature (see Theorem 2.3.1), because in this case, each of the ends of M is asymptotic to an end of a plane or a catenoid. Theorem 1.0.4 states that if M has finite topology but infinite total curvature (thus M has exactly one end by Collin's Theorem 1.0.3), then M is asymptotic to a helicoid. More complicated asymptotic behaviors can be found in periodic minimal surfaces in $\mathbb{R}^3$, although this asymptotic behavior is completely understood when the periodic minimal surface has finite topology in the corresponding quotient ambient space (thus the quotient surface has finite total curvature by Theorem 2.4.1); in this setting, only planar, helicoidal or Scherk-type ends can occur, see Chapter 2.4.

We next consider the question of understanding the asymptotics of a general properly embedded minimal surface in $\mathbb{R}^3$. A crucial point is the notion of *topological end,* which we now explain. Let M be a non-compact connected manifold. We define an equivalence relation in the set $\mathcal{A} = \{\alpha \colon [0,\infty) \to M \mid \alpha \text{ is a proper arc}\}$, by setting $\alpha_1 \sim \alpha_2$ if for every compact set $C \subset M$, α_1, α_2 lie eventually[17] in the same component of $M - C$.

Definition 2.7.1 Each equivalence class in $\mathcal{E}(M) = \mathcal{A}/_\sim$ is called an *end* of M. If $e \in \mathcal{E}(M)$, $\alpha \in e$ is a representative proper arc and $\Omega \subset M$ is a proper subdomain with compact boundary such that $\alpha \subset \Omega$, then we say that the domain Ω *represents* the end e.

The space $\mathcal{E}(M)$ has the following natural Hausdorff topology. For each proper domain $\Omega \subset M$ with compact boundary, we define the basis open set $B(\Omega) \subset \mathcal{E}(M)$ to be those equivalence classes in $\mathcal{E}(M)$ which have representatives contained in Ω. With this topology, $\mathcal{E}(M)$ is a totally disconnected compact space which embeds topologically as a subspace of $[0,1] \subset \mathbb{R}$. On pages 288-289 of [129], we gave a short proof of this embedding result for $\mathcal{E}(M)$, which works even in the more general case where M is a manifold or a finite dimensional simplicial complex. In the case that M is a properly embedded minimal surface in $\mathbb{R}^3$ with more than one end, there is also a natural topological embedding of $\mathcal{E}(M)$ into $[0,1]$ which uses the relative heights of the ends of M, see the Ordering Theorem 6.0.11.

Definition 2.7.2 Any isolated point $e \in \mathcal{E}(M)$ is called a *simple end of* M. If $e \in \mathcal{E}(M)$ is not a simple end (equivalently, if it is a limit point of $\mathcal{E}(M) \subset [0,1]$), we will call it a *limit end* of M.

[17]Throughout the paper, the word *eventually* for proper arcs means outside a compact subset of the parameter domain $[0,\infty)$.

When M has dimension 2, then an elementary topological analysis using compact exhaustions shows that an end $e \in \mathcal{E}(M)$ is simple if and only if it can be represented by a proper subdomain $\Omega \subset M$ with compact boundary which is homeomorphic to one of the following models:

(a) $\mathbb{S}^1 \times [0, \infty)$ (this case is called an *annular end*).

(b) $\mathbb{S}^1 \times [0, \infty)$ connected sum with an infinite number of tori where the n-th connected sum occurs at the point $(1, n) \in \mathbb{S}^1 \times [0, \infty)$, $n \in \mathbb{N}$ (this case is called a *simple end of infinite genus*).

For limit ends there are similar notions: a limit end $e \in \mathcal{E}(M)$ is said to have *genus zero* if it can be represented by a proper subdomain $\Omega \subset M$ with compact boundary and genus zero. If a limit end e does not have genus zero, then we say that it has *infinite genus*; in this case every proper subdomain with compact boundary representing e has infinite genus.

We will devote Chapter 6 to the Ordering Theorem for ends of properly embedded minimal surfaces in $\mathbb{R}^3$; this theorem is the starting point for the theory of properly embedded minimal surfaces with more than one end. Concerning one-ended minimal surfaces, the classical example in this family is the helicoid. Also, one has the newer examples of helicoids with handles (rigorously proven to exist only for the case of one handle), see Figure 2.4. Theoretically, Theorem 1.0.4 insures that any non-planar, properly embedded, one-ended, minimal surface with finite topology must be necessarily asymptotic to a helicoid with finitely many handles, and it can be described analytically by meromorphic data $(dg/g, dh)$ on a compact Riemann surface by means of the classical Weierstrass representation. Regarding one-ended surfaces with infinite topology, Callahan, Hoffman and Meeks [16] showed that any non-flat, doubly or triply-periodic minimal surface in $\mathbb{R}^3$ must have infinite genus and only one end.

In the preceding paragraphs, our discussion dealt with ends of properly embedded minimal surfaces in $\mathbb{R}^3$. The definitions above can be directly extended to properly embedded minimal surfaces in quotient spaces as $\mathbb{R}^3/T$, $\mathbb{R}^3/S_\theta$ or $\mathbb{T}^2 \times \mathbb{R}$ (with the notation of Chapter 2.4). In particular, it makes sense to consider simple and limit ends of minimal surfaces in these ambient spaces. Meeks proved [118] that a properly embedded minimal surface in $\mathbb{T}^2 \times \mathbb{R}$ always has a finite number of ends. Later on [120], he extended this result to minimal surfaces in $\mathbb{R}^3/S_\theta$, $\theta \neq 0, \pi$. In these cases, limit ends are then not possible for properly embedded minimal surfaces. The situation is different when the ambient space is $\mathbb{R}^3/T$; an example in this setting is any doubly-periodic Scherk minimal surface appropriately viewed as a singly-periodic surface. This example gives a quotient surface in $\mathbb{R}^3/T$ with genus zero and exactly one limit end (in particular, the first statement in Theorem 12.2.1 below does not extend to $\mathbb{R}^3/T$). Especially interesting is the existence of an infinite dimensional family of properly embedded minimal surfaces with genus zero and exactly one limit end in $\mathbb{R}^3/T$, but which do not lift to doubly-periodic minimal surfaces in $\mathbb{R}^3$ (Mazet, Rodríguez, Traizet [112], see also the discussion before Theorem 11.0.13).

2.8 Second variation of area, index of stability and Jacobi functions.

Let $M \subset \mathbb{R}^3$ be a minimal surface and $\Omega \subset M$ a subdomain with compact closure. Any smooth normal deformation of the inclusion $X: M \to \mathbb{R}^3$ which is compactly supported in Ω can be written as $X + tuN$, where N is the Gauss map of M and $u \in C_0^\infty(\Omega)$. By equation (2.2), the area functional $A = A(t)$ for this deformation has $A'(0) = 0$. The *second variation of area* can be easily shown to be (see [175])

$$A''(0) = -\int_\Omega u(\Delta u - 2Ku)\, dA, \tag{2.11}$$

where K is the Gaussian curvature function of M and Δ its Laplace operator. Formula (2.11) can be viewed as the bilinear form associated to the linear elliptic L^2-selfadjoint operator $L = \Delta - 2K = \Delta + |\nabla N|^2$, which is usually called the *Jacobi operator* or *stability operator.*

Definition 2.8.1 A C^2-function $u: M \to \mathbb{R}$ satisfying $\Delta u - 2Ku = 0$ on M is called a *Jacobi function.* We will let $\mathcal{J}(M)$ denote the linear space of Jacobi functions on M.

Classical elliptic theory implies that given a subdomain $\Omega \subset M$ with compact closure, the Dirichlet problem for the Jacobi operator in Ω has an infinite discrete spectrum $\{\lambda_k\}_{k\in\mathbb{N}\cup\{0\}}$ of eigenvalues with $\lambda_k \nearrow +\infty$ as k goes to infinity, and each eigenspace is a finite dimensional linear subspace of $C^\infty(\Omega) \cap H_0^1(\Omega)$, where $H_0^1(\Omega)$ denotes the usual Sobolev space of L^2-functions with L^2 weak partial derivatives and trace zero. Since any normal variation by minimal surfaces has vanishing second derivative of the area functional, it follows that the normal parts of variational fields coming from Killing or dilatation vector fields of $\mathbb{R}^3$ produce elements in $\mathcal{J}(M)$. For instance, when the Killing field is a translation by a vector $v \in \mathbb{R}^3$, then the corresponding Jacobi function is $\langle N, v\rangle$ (called a *linear Jacobi function*). Similarly, rotations around an axis of direction $v \in \mathbb{R}^3$ produce the Jacobi function $\det(p, N, v)$ (here p denotes the position vector) and homotheties give rise to the *support function* $\langle p, N\rangle \in \mathcal{J}(M)$.

A particularly interesting Jacobi function, which can be defined when the minimal surface is transverse to a family of horizontal planes, is the *Shiffman function*, which is proportional to the derivative of the curvature of each planar section with respect to a parameter of such a section (this parameter is not the arclength of the section, see equation (12.1)). Thus, the condition of vanishing identically for the Shiffman function means that the surface is foliated my planar curves of constant curvature, i.e. lines or circles. This argument can be used to prove that every compact minimal annulus A in $\mathbb{R}^3$ whose boundary consists of circles in parallel planes Π_1, Π_2, is foliated by circles which lie in planes parallel to Π_1 (see Shiffman [208]). By classical results of Riemann [190, 191], it follows that such an annulus A must be contained in a catenoid or in one of the Riemann minimal examples discussed in Chapter 2.5. The Shiffman function and its connection with the Korteweg-de Vries equation plays a fundamental role in the characterization by Meeks, Pérez and Ros [136] of the Riemann minimal examples as the only properly embedded minimal surfaces in $\mathbb{R}^3$ with their topology (see Theorem 1.0.2 and Chapter 12.3). For more details about the Shiffman function, see [53, 54, 129, 179, 182].

Definition 2.8.2 Let $\Omega \subset M$ be a subdomain with compact closure. The *index of stability* of Ω is the number of negative eigenvalues of the Dirichlet problem associated to L in Ω. The *nullity* of Ω is the dimension of $\mathcal{J}(\Omega) \cap H_0^1(\Omega)$. Ω is called *stable* if its index of stability is zero, and *strictly stable* if both its index and nullity are zero.

Elliptic theory also implies that Ω is strictly stable provided that it is sufficiently small, which justifies the Definition 2.1.5 of minimal surface as a local minimum of area. Another consequence of elliptic theory is that Ω is stable if and only if it carries a positive Jacobi function. Since the Gauss map N of a graph defined on a domain in a plane Π has image set contained in an open half-sphere, the inner product of N with the unit normal to Π provides a positive Jacobi function, from where we conclude that any minimal graph is stable. Stability makes sense for non-compact minimal surfaces, as we next explain.

Definition 2.8.3 A minimal surface $M \subset \mathbb{R}^3$ is called *stable* if any subdomain $\Omega \subset M$ with compact closure is stable. For orientable minimal surfaces, stability is equivalent to the existence of a positive Jacobi function on M (Proposition 1 in Fischer-Colbrie [57]). M is said to have *finite index* if outside of a compact subset it is stable. The *index of stability* of M is the supremum of the indices of stability of subdomains with compact closure in M.

By definition, stable surfaces have index zero. The following theorem explains how restrictive is the property of stability for complete minimal surfaces. It was proved independently by Fischer-Colbrie and Schoen [58], do Carmo and Peng [48], and Pogorelov [189] for orientable surfaces. Later, Ros [196] proved that a complete, non-orientable minimal surface in $\mathbb{R}^3$ is never stable.

Theorem 2.8.4 *If $M \subset \mathbb{R}^3$ is a complete, immersed, stable minimal surface, then M is a plane.*

A short elementary proof of Theorem 2.8.4 in the orientable case is given in Section 4 of [129] and also in [124]; also see the proof of Lemma 12.2.2 below.

A crucial fact in minimal surface theory is that stable, minimally immersed surfaces with boundary in $\mathbb{R}^3$ have curvature estimates up to their boundary. These curvature estimates were first obtained by Schoen for two-sided surfaces in homogeneously regular three-manifolds (see Definition 2.8.5 below) and later improved by Ros to the one-sided case, and are a simple consequence of Theorem 2.8.4 after a rescaling argument.

Definition 2.8.5 A Riemannian three-manifold N is *homogeneously regular* if there exists an $\varepsilon > 0$ such that ε-balls in N are uniformly close to ε-balls in $\mathbb{R}^3$ in the C^2-norm. In particular, if N is compact, then N is homogeneously regular.

Theorem 2.8.6 (Schoen [206], Ros [196]) *For any homogeneously regular three-manifold N there exists a universal constant $c > 0$ such that for any stable, minimally immersed surface*[18] *M in N, the absolute value $|K|$ of the Gaussian curvature of M satisfies*

$$|K(p)| \operatorname{dist}_N(p, \partial M)^2 \leq c \quad \textit{for all } p \in M,$$

where dist_N *denotes distance in N and ∂M is the boundary of M.*

We remark that Rosenberg, Souam and Toubiana [199] have obtained a version of Theorem 2.8.6 valid in the two-sided case when the ambient three-manifold has a bound on its sectional curvature.

If we weaken the stability hypothesis in Theorem 2.8.4 to finite index of stability and we allow compact boundary, then completeness and orientability also lead to a well-known family of minimal surfaces.

Theorem 2.8.7 (Fischer-Colbrie [57]) *Let $M \subset \mathbb{R}^3$ be a complete, orientable, minimally immersed surface in $\mathbb{R}^3$, with possibly empty compact boundary. Then, M has finite index of stability if and only if it has finite*

[18] For a two-sided minimal surface M in a general three-manifold N with unit normal vector $\nu\colon M \to TN$, the second derivative of the area functional along a compactly supported normal variation with variational field $u\nu$, is $A''(0) = -\int_M u(\Delta u + (\operatorname{Ric}(\nu) + |\sigma|^2)u)\, dA$, where Ric denotes the Ricci curvature of N and $|\sigma|^2$ is the squared length of the second fundamental form of M in N. M is called *stable* if $A''(0) \geq 0$ for all $u \in C_0^\infty(M)$, and *unstable* otherwise. If M is one-sided (i.e. has a non-trivial normal bundle), then stability imposes $A''(0) \geq 0$ on the two-sided covering $\widetilde{M}$ of M, for all functions $u \in C_0^\infty(\widetilde{M})$ which anti-commute with the natural $2:1$ covering map $\widetilde{M} \to M$. When the ambient manifold N is orientable, two-sidedness is equivalent to orientability of M.

total curvature. In this case, the index and nullity of M coincide with the index and nullity of the meromorphic extension of its Gauss map to the compactification $\overline{M}$ obtained from M after attaching its ends, see Theorem 2.3.1.

The similar result to Theorem 2.8.7 for non-orientable, complete minimal surfaces is not known to hold, see Conjecture 17.0.41.

By the conformal invariance of the Dirichlet integral, both the index and nullity of the Jacobi operator $L = \Delta + |\nabla N|^2$ remain constant under a conformal change of metric. Recall that the Huber-Osserman Theorem 2.3.1 asserts that every complete, orientable, immersed minimal surface $M \subset \mathbb{R}^3$ with finite total curvature is conformally equivalent to a finitely punctured compact Riemann surface $\overline{M}$, and the Gauss map N of M extends meromorphically to $\overline{M}$. In this case, it can be shown (Pérez and Ros [185]) that there exists a smooth metric $\overline{ds}^2$ on $\overline{M}$ such that the metric ds^2 on M induced by the inner product of $\mathbb{R}^3$ can be expressed as $ds^2 = \mu\,\overline{ds}^2$, where μ is a positive smooth function which blows up at the ends of M. Furthermore, both the index and nullity of L can be computed as the index and nullity of the operator $\overline{L} = \overline{\Delta} + |\overline{\nabla} N|^2$ on $\overline{M}$, where a bar means that the corresponding object refers to $\overline{ds}^2$. Note that $\overline{L}$ is nothing more than the classical Schrödinger operator associated to the meromorphic extension of N to $\overline{M}$. The subspace $\mathcal{K}(M)$ of *bounded* Jacobi functions on M can be identified with the eigenspace associated to the eigenvalue 0 of the operator $\overline{L}$. Inside $\mathcal{K}(M)$, we have the subspace of linear functions $\mathcal{L}(M) = \{\langle N, v\rangle \mid v \in \mathbb{R}^3\}$ (which are the normal parts of variational fields of deformations by translations). If additionally all the ends of M are parallel[19], say horizontal, then the function $\det(p, N, e_3) \in \mathcal{K}(M)$, where $e_3 = (0, 0, 1)$, is a bounded Jacobi function (it is the normal part of the variational field of the deformation of M by rotations around the x_3-axis). In particular, $\mathcal{K}(M)$ has dimension at least 4 for any complete, embedded minimal surface of finite total curvature in $\mathbb{R}^3$ (except for the catenoid and the plane, where $\det(p, N, e_3)$ vanishes).

Montiel and Ros found a beautiful relationship between bounded Jacobi functions and branched minimal immersions with prescribed Gauss map. For a complete minimal surface $M \subset \mathbb{R}^3$ with finite total curvature and conformal compactification $\overline{M}$, let $B(N) \subset \overline{M}$ be the set of branch points of the extended Gauss map and $\mathcal{M}(N)$ the linear space of all complete, branched minimal immersions (including the constant maps) $X : \overline{M} - B(N) \to \mathbb{R}^3$ with the same Gauss map N as M.

Theorem 2.8.8 (Montiel, Ros [163]) *Let $M \subset \mathbb{R}^3$ be a complete, immersed minimal surface with finite total curvature*[20]*. Then, there exists a*

[19] This occurs, for instance, when M is embedded.

[20] Theorems 2.8.8 and 2.8.9 remain valid for complete minimal surfaces in any quotient of $\mathbb{R}^3$ where the Gauss map makes sense, and which have finite total curvature in the quotient.

linear map $u \in \mathcal{K}(M) \mapsto X_u \in \mathcal{M}(N)$ such that the support function[21] *of X_u is u, and $u \in \mathcal{L}(M)$ if and only if X_u is constant. Furthermore, this linear map gives rise to an isomorphism between the quotient spaces $\mathcal{K}(M)/\mathcal{L}(M)$ and $\mathcal{M}(N)/\{constants\}$.*

Among the admissible conformal metrics which can be used to express questions related with the Jacobi operator, a particularly interesting choice comes from consideration of the pullback metric ds_N^2 through the Gauss map $N\colon M \to \mathbb{S}^2$ from the standard spherical metric on $\mathbb{S}^2$. The metric ds_N^2 has singularities at the branch points of N, and the Jacobi operator transforms into $L_N = \Delta_N + 2$, where Δ_N is the Laplace operator of ds_N^2. Eigenvalues and eigenfunctions of L_N are well-defined by a variational approach (Tysk [220]). In particular, the index of stability of a subdomain $\Omega \subset M$ with compact closure is equal to the number of eigenvalues of Δ_N which are strictly less than 2, and the nullity of Ω is the multiplicity of 2 as an eigenvalue of Δ_N. Using these ideas, Montiel and Ros gave some estimates for the index and nullity under different geometrical assumptions, among which we emphasize the following one.

Theorem 2.8.9 (Montiel, Ros [163]) *Let $M \subset \mathbb{R}^3$ be a complete, immersed minimal surface with finite total curvature*[20]*. If all the branch values of the Gauss map of M lie on an equator of $\mathbb{S}^2$, then the dimension of $\mathcal{K}(M)$ is 3 and M has index $2d-1$, where d is the degree of the extended Gauss map.*

2.9 Barrier constructions.

Barrier constructions allow one to construct compact and non-compact stable minimal surfaces in $\mathbb{R}^3$, that are constrained to lie in subdomains of $\mathbb{R}^3$ whose boundaries have non-negative mean curvature. To illustrate this construction, we next give an example. Consider two disjoint, connected, properly embedded minimal surfaces M_1, M_2 in $\mathbb{R}^3$ and the closed connected region W of $\mathbb{R}^3$ with $\partial W = M_1 \cup M_2$. We now show how to produce compact, stable, embedded minimal surfaces in W. First note that W is a complete flat three-manifold with boundary ∂W, and ∂W has mean curvature zero. Meeks and Yau [157] proved that W embeds isometrically in a homogeneously regular Riemannian three-manifold $(\widetilde{W}, \widetilde{g})$, with $\widetilde{W}$ diffeomorphic to the interior of W. Morrey [165] proved that in a homogeneously regular manifold, one can solve the classical Plateau problem and other area-minimizing problems. In particular, if Γ is an embedded 1-cycle in $\widetilde{W}$ which

[21]The *support function* u of an immersion $X\colon M \to \mathbb{R}^3$ is $u(p) = \langle p, N(p)\rangle$, where N is the normal unit vector field of X; note that u is the normal part of the variational field of X by homotheties.

bounds an orientable chain[22] in $\widetilde{W}$, then by standard results in geometric measure theory [55], Γ is the boundary of a compact, least-area embedded surface $\Sigma_\Gamma(\widetilde{g}) \subset \widetilde{W}$. Meeks and Yau proved that the metric $\widetilde{g}$ on $\widetilde{W}$ can be approximated by a family of homogeneously regular metrics $\{g_n\}_{n\in\mathbb{N}}$ on $\widetilde{W}$, which converges smoothly on compact subsets of $\widetilde{W}$ to $\widetilde{g}$, and each g_n satisfies a convexity condition outside of $W \subset \widetilde{W}$, which forces the least-area surface $\Sigma_\Gamma(g_n)$ to lie in W if Γ lies in W. A subsequence of the $\Sigma_\Gamma(g_n)$ converges to a smooth minimal surface Σ_Γ of least-area in W with respect to the original flat metric, thereby finishing our description of the barrier construction.

We now use this barrier construction to prove the following theorem due to Hoffman and Meeks [83], from which Theorem 2.6.4 follows directly.

Theorem 2.9.1 (Strong Half-space Theorem [83]) *If M_1 and M_2 are two disjoint, proper, possibly branched minimal surfaces in $\mathbb{R}^3$, then M_1 and M_2 are parallel planes.*

Proof. Let W be the closed complement of $M_1 \cup M_2$ in $\mathbb{R}^3$ that has portions of both M_1 and M_2 on its boundary. As explained above, the surface ∂W is a good barrier for solving Plateau-type problems in W. Let $M_1(1) \subset \ldots \subset M_1(n) \subset \ldots$ be a compact exhaustion of M_1 and let $\Sigma_1(n)$ be a least-area surface in W with boundary $\partial M_1(n)$. Let α be a compact arc in W which joins a point in $M_1(1)$ to a point in $\partial W \cap M_2$. By elementary intersection theory, α intersects every least-area surface $\Sigma_1(n)$. By compactness of least-area surfaces, a subsequence of the surfaces $\Sigma_1(n)$ converges to a properly embedded area-minimizing surface Σ in W with a component Σ_0 which intersects α; this proof of the existence of Σ is due to Meeks, Simon and Yau [151]. Since Σ_0 separates $\mathbb{R}^3$, Σ_0 is orientable and so by Theorem 2.8.4, Σ_0 is a plane. Hence, M_1 and M_2 lie in closed half-spaces of $\mathbb{R}^3$. Then, the height function of M_1, M_2 over their separating plane is a positive harmonic function, which must be constant provided that M_1 and M_2 are recurrent (see Definition 7.1.2 and Proposition 7.1.4). The fact that M_1 and M_2 are recurrent will be proved in Theorem 7.2.3 below. Hence, M_1 and M_2 must be planes[23]. $\square$

Another useful application of the barrier construction is the following. Suppose Γ is an *extremal* simple closed curve in $\mathbb{R}^3$, i.e. Γ lies on the boundary of its convex hull B. We first assume that ∂B is smooth. By the Jordan Curve Theorem, Γ is the boundary of two disks $D_1, D_2 \subset \partial B$. Suppose Γ

[22] We are considering here a finite sum of differentiable simplices.

[23] The original proof of the Strong Half-space Theorem uses the Half-space Theorem (Theorem 2.6.4) to conclude that M_1, M_2 are planes, once one knows that Σ_0 is a separating plane. Here we have used Proposition 7.1.4 since we wanted to deduce the Half-space Theorem 2.6.4 as a consequence of Theorem 2.9.1.

bounds two different compact, branched minimal immersions and let Σ denote their union. By the convex hull property[24], $\Sigma \subset B$. Let W_1, W_2 be the geodesic completions[25] of the two components of $B - \Sigma$ which contain the disks D_1, D_2. In this case, ∂W_1 and ∂W_2 consist of smooth pieces with non-negative mean curvature and convex corners. Meeks and Yau [157] proved that such boundaries are good barriers for solving least-area problems. In fact, in this case they proved that Γ bounds a least-area embedded disk $\widetilde{D}_1 \subset W_1$ and a different least-area embedded disk $\widetilde{D}_2 \subset W_2$. Similarly, if Γ bounds a unique, branched minimal surface which is not an embedded stable minimal disk, then with this barrier argument, we can produce two different embedded minimal disks with boundary Γ. If ∂B is not assumed to be smooth, then one can use an approximation argument by convex smooth boundaries to obtain the same conclusion (see e.g. [156]).

On the other hand, Nitsche [174] proved that a regular, analytic Jordan curve in $\mathbb{R}^3$ whose total curvature is at most 4π bounds a unique minimal disk. The hypothesis of analyticity for the boundary curve in Nitsche's Theorem comes from consideration of boundary branch points. When Γ is of class C^2 and extremal, there are never boundary branch points as shown in [157]; one uses here Hildebrandt's boundary regularity results [71]. The last two paragraphs can be summarized in the following statement.

Theorem 2.9.2 (Meeks, Yau [157]) *If $\Gamma \subset \mathbb{R}^3$ is a C^2-extremal curve with total curvature at most 4π, then Γ is the boundary of a unique, compact, branched minimal surface and this surface is an embedded minimal disk of least-area.*

[24] This property states that any compact minimal surface with boundary lies in the convex hull of its boundary. It works in much more generality than for minimal surfaces; in fact, Osserman [177] characterized the surfaces in $\mathbb{R}^3$ all whose precompact subdomains lie in the convex hull of their boundaries, as those surfaces with non-positive Gaussian curvature.

[25] The *geodesic completion* of a Riemannian manifold is the completion of the underlying intrinsic metric space structure defined by infimum of length of curves in that manifold.

Minimal surfaces with finite topology and more than one end. 3

Collin's Theorem 1.0.3 stated in the Introduction is a key result which reduces the study of properly embedded minimal surfaces in $\mathbb{R}^3$ with finite topology and at least two ends (see Definition 2.7.1 for the general definition of end) to the family of surfaces with finite total curvature. As mentioned in Chapter 2.3, the powerful theory of compact Riemann surfaces applies to these last surfaces, which has helped to make possible a rather good understanding of them. In this chapter we have two goals: to mention the main construction methods that are used to produce complete, embedded minimal surfaces of finite total curvature, and to state uniqueness and non-existence results for these kinds of surfaces. We will start with this last goal.

3.1 Classification results for embedded minimal surfaces of finite total curvature.

The Jorge-Meeks formula (equation (2.9)) relates the total curvature, genus and number of ends of a complete, embedded minimal surface with finite total curvature. Consequently, it is natural to look for classification and non-existence results fixing two of these numbers, or even better, fixing only one of the three. The first deep result along these lines is due to Schoen, who proved the following theorem in 1983 as an application of the Alexandrov reflection technique (see Alexandrov [2] and Hopf [87] for a description of this classical technique).

Theorem 3.1.1 (Schoen [206]) *The catenoid is the unique complete, immersed minimal surface in $\mathbb{R}^3$ with finite total curvature and two embedded* *ends.*

Eight years later, López and Ros classified the embedded genus-zero examples of finite total curvature. The main ingredients in the proof of the result below are the López-Ros deformation explained in Chapter 2.2 together with the maximum principle for minimal surfaces.

Theorem 3.1.2 (López, Ros [107]) *The plane and the catenoid are the only complete, embedded minimal surfaces in* $\mathbb{R}^3$ *with genus zero and finite total curvature.*

Costa [42, 43] showed that any complete, embedded minimal surface in $\mathbb{R}^3$ with genus one and three ends either is the Costa surface or lies in the Hoffman-Meeks deformation discussed in Chapter 2.5, with the moduli space of complete, embedded, minimal, thrice punctured tori being diffeomorphic to an open interval. This was the first time that a space of complete, embedded minimal surfaces with a prescribed topology had an explicit description as a manifold of finite positive dimension. Concerning moduli spaces of minimal surfaces with finite total curvature and prescribed topology, Pérez and Ros [185] gave general conditions on the space $\mathcal{M}(g,r)$ whose elements are the complete, embedded minimal surfaces in $\mathbb{R}^3$ with finite total curvature, genus g and r ends, to have a structure of real analytic manifold of dimension $r-2$ around a given minimal surface $M \in \mathcal{M}(g,r)$, and they called such conditions on M the *non-degeneracy* of the surface. This term comes from the fact that the manifold structure on $\mathcal{M}(g,r)$ follows from an application of the Implicit Function Theorem, which needs a certain derivative to be surjective (also sometimes called non-degenerate). The non-degeneracy condition is expressed in terms of the bounded Jacobi functions on M (see Chapter 2.8 for the definition of Jacobi function). They also identified the tangent space to $\mathcal{M}(g,r)$ at a non-degenerate minimal surface M, with the set of Jacobi functions on M which have at most logarithmic singularities at the ends. Other compactness results for moduli spaces of complete, embedded minimal surfaces with finite total curvature have been given by Ros [193], Traizet [216] and by Meeks, Pérez and Ros [130, 142]. We will explain in Chapter 12 some further advances in this area, see specifically Theorem 1.0.6 stated in the Introduction.

3.2 Constructing embedded minimal surfaces of finite total curvature.

The Weierstrass representation gives a direct method for constructing minimal surfaces of finite total curvature, starting from an input which is a compact Riemann surface $\overline{M}$ and three meromorphic 1-forms ϕ_1, ϕ_2, ϕ_3 on $\overline{M}$ satisfying $\sum_i \phi_i^2 = 0$. One then defines the (stereographically projected)

Gauss map as $g = \frac{\phi_3}{\phi_1 - i\phi_2}$, and the height differential as $dh = \phi_3$. The degrees of freedom to work with come from the moduli space of conformal structures for a given genus, together with the points which correspond to the ends (poles of ϕ_i, $i = 1, 2, 3$). Avoiding branch points for the induced metric (condition *i)* in Theorem 2.2.1) amounts to a rather simple problem of adjusting the canonical divisors of the 1-forms ϕ_1, ϕ_2, ϕ_3. The first main task is to solve the period problem (condition *ii)* in Theorem 2.2.1). Since the extrinsic symmetries of a minimal surface simplify the period calculations with the Weierstrass data, the period problem is sometimes tackled by assuming enough symmetries; in this way, the constraint equations in (2.5) reduce to a sufficiently small number of closed curves in $M = \overline{M} - \{\text{ends}\}$, so that one can adjust the remaining free parameters, often by continuity or degree-type arguments, and solve the period problem. A variant of this idea is to express the period problem as a Teichmüller theory issue (we will explain more about this below, see the *method of Weber and Wolf*). An even more serious difficulty appears when we try to obtain an embedded example, since the Weierstrass data gives no direct information on this matter. A useful technique for proving embeddedness is to decompose M into congruent pieces (which are fundamental domains of the action of the symmetry group of M, so we need again to assume enough symmetries here), and to prove that each piece is embedded.

Another remarkable method to produce examples is to perturb singular configurations (i.e. minimal surfaces with singularities, modeled on Riemann surfaces with nodes) via the Implicit Function Theorem, which is the main idea of the *method of Traizet,* or to desingularize configurations of smooth minimal surfaces, first by constructing a surface of mean curvature approximately zero, and then by applying a fixed point theorem to obtain a nearby minimal surface (*method of Kapouleas*). Next we will explain a bit more about these three different methods. If we were to follow the historical order, then Kapouleas's method would be first; nevertheless, we will leave it to the end of this section since the other two methods rely on the Weierstrass representation, while Kapouleas's method uses PDE and functional analysis arguments.

The method of Weber and Wolf.
The goal is to produce a surface with prescribed topology and symmetries via the Weierstrass representation, although this method has interesting features that differ from the general presentation given above. The key idea is an original viewpoint to solve the period problem, based on *flat structures.* If α is a meromorphic 1-form on a Riemann surface $\mathcal{R}$, then $|\alpha|^2$ defines a flat metric on $\mathcal{R} - \alpha^{-1}(\{0, \infty\})$. In particular, we can develop locally $\mathcal{R}$ as a domain Ω_α in $\mathbb{C}$ (so that the usual holomorphic differential dz pulls back to α), except around a zero or pole of α, where the developing map produces a *cone point* (i.e. a generalized sector with edges identified, whose aperture

angle is neither necessarily less than 2π nor a rational multiple of π), whose cone angle depends explicitly on the order of the zero or pole of α. The line element $|\alpha|$ associated to the metric $|\alpha|^2$ is called a *flat structure* on $\mathcal{R}$.

Once we understand the concept of flat structure, the method is easy to follow. One starts by assuming the desired minimal surface M exists, and computes the divisors of the 1-forms $g\,dh$ and $g^{-1}dh$ (as usual, g is the Gauss map and dh the height differential of M; this divisor information can be read from the expected shape of the surface). Assuming enough symmetry, one supposes that a fundamental piece of the action of the symmetry group of M is a disk D with boundary, and develops D as two domains $\Omega_{g\,dh}, \Omega_{g^{-1}dh}$ with piecewise smooth boundary in $\mathbb{C}$ as we explained above for α. The periods of $g\,dh$ and $g^{-1}\,dh$ transform into integrals of dz along paths in $\mathbb{C}$, and the complex period condition in equation (2.5)-left is equivalent to the fact that the developing domains in $\mathbb{C}$ associated to $g\,dh$ and $g^{-1}\,dh$ are conjugate in a natural sense. This period condition is, thus, automatically satisfied by taking appropriate pairs of developing domains. The real period condition in equation (2.5)-right is also easy to fulfill, since it reduces to an algebraic condition on the sum of the cone angles. The reader could think that the period problem has been magically solved, but in reality we have only exchanged it by another problem in Teichmüller theory: find a pair of conjugate developing domains $\Omega_{g\,dh}, \Omega_{g^{-1}dh}$ so that they match up to give a well-defined Riemann surface on which $g\,dh$, $g^{-1}dh$ both exist (once this difficulty is solved, the Weierstrass data (g, dh) is easily recovered from $g\,dh$, $g^{-1}dh$). This task is done by studying the moduli space of pairs of conjugate domains and using specific tools of Teichmüller theory, such as extremal length.

Weber and Wolf have used this method in several different situations [221, 222]. For instance, they produced for all odd genera g, a complete minimal surface $M(g) \subset \mathbb{R}^3$ with finite total curvature, genus g, two catenoidal ends and g planar horizontal ends. Furthermore, $M(g)$ is embedded outside a compact set[1] and the symmetry group of $M(g)$ is generated by reflective symmetries about a pair of orthogonal vertical planes and a rotational symmetry about a horizontal line. The surface $M(g)$ has also theoretical importance, since if it were proved to be embedded, then it would represent the borderline case for Conjecture 17.0.19 below. Also, the surfaces $M(g)$ might be used to produce as a limit as $g \to \infty$, a properly embedded minimal surface in $\mathbb{R}^3$ with exactly one limit end and infinite genus, see the discussion at the end of Chapter 12.

The method of Traizet.

Again this method is based on the Weierstrass representation, but with a different flavor from the preceding one. The technique also applies to construct

[1]The conjecture that the surfaces $M(g)$ are all embedded is supported by graphical and numerical evidence.

minimal surfaces with infinite total curvature, but we will focus here on the finite total curvature case. The key point is that certain continuous families of embedded minimal surfaces $M(\theta)$ (here θ is a multi-parameter) with the same topology and total curvature, have a *weak limit* as θ approaches to a point θ_∞ in the boundary of the range of allowed parameters. A weak limit is a finite collection of minimal surfaces $\{M_{1,\infty},\ldots,M_{k,\infty}\}$ such that

- For each i, $M_{i,\infty}$ is a smooth limit of suitable rescalings of the $M(\theta)$ as $\theta \to \theta_\infty$.
- The topology and total curvature of the $M(\theta)$ can be recovered as a sum of the ones of all the $M_{i,\infty}$, $i = 1,\ldots,k$.

This phenomenon allows one to model the weak limit as being a compact *Riemann surface with nodes*[2]. Now the method go backwards: starting from a finite configuration of minimal surfaces $M_{1,\infty},\ldots,M_{k,\infty}$ with finite total curvature modeled over a compact Riemann surface with nodes $\mathcal{R}(0)$, one studies how to deform $\mathcal{R}(0)$ by opening slightly its nodes, creating in this way an analytic family $\mathcal{R}(t)$ of compact Riemann surfaces without nodes, where t is a vector-valued complex parameter around $t = 0$. At the same time, the collection of Weierstrass representations $(g_{i,\infty}, dh_{i,\infty})$ of the $M_{i,\infty}$ on $\mathcal{R}(0)$ deform analytically to a global Weierstrass representation (g_t, dh_t) on $\mathcal{R}(t)$, and we want to find some subfamily of parameters t such that the period problem for the triple $(\mathcal{R}(t), g_t, dh_t)$ is solved. To solve this period problem, one writes its solutions as the zeros of an analytic map P from the t-domain into certain $\mathbb{C}^m$. Furthermore, $P(0) = 0$ since the $M_{1,\infty},\ldots,M_{k,\infty}$ are real minimal surfaces. The desired family of zeros of P passing through $t = 0$ is then obtained by application of the Implicit Function Theorem, provided that certain non-degeneracy condition holds for the limit configuration (namely, the differential of P at $t = 0$ must be surjective).

It is remarkable that Traizet's method makes no restrictions *a priori* either on the genus of the surfaces to be constructed, or on their symmetry groups. In fact, Traizet applied this method to produce the first example of a complete, embedded minimal surface with finite total curvature in $\mathbb{R}^3$ and trivial symmetry group, see [215]. For this result, Traizet desingularized a configuration of planes and tiny catenoids. One limitation of the method is that, by its nature, it only produces examples which are close to the boundary of their corresponding moduli spaces. As said before, the method also works for surfaces of infinite total curvature. For instance, Traizet and Weber [218] have used these ideas to perturb weak limits of planes and helicoids, thereby producing the first non-classical examples of *minimal parking garage surfaces* (see the comments after Theorems 4.1.5 and 4.1.7; also see Chapter 10.2 for a more detailed explanation of this concept).

[2]See [89] page 245 for the definition of a Riemann surface with nodes.

We also remark that according to Traizet [215], his technique was inspired on a clever modification of an argument due to Meeks, Pérez and Ros [138], used in proving the uniqueness of the Riemann minimal examples in their natural periodic moduli space, around a weak limit given by infinitely many catenoids and planes.

The method of Kapouleas.

This method differs in an essential manner from the preceding ones, since it is not based on the Weierstrass representation but on PDE theory and functional analysis techniques. One starts by considering a finite collection $\mathcal{F}$ of horizontal planes and catenoids with axis the x_3-axis. The goal is to desingularize $\mathcal{F}$ to obtain a complete, embedded minimal surface $M \subset \mathbb{R}^3$ of finite total curvature. By desingularization, we mean here that each circle C of intersection of the planes and/or catenoids in $\mathcal{F}$ is replaced in a first stage by a suitably bent, slightly deformed and appropriately scaled down singly-periodic Scherk minimal surface $\mathcal{S}_\theta$, in the same way that $\mathcal{S}_\theta$ can be viewed as the desingularization of a pair of planes intersecting at angle θ (see Figure 2.3 Center for an indication of the geometry of the replaced neighborhood near C when $\theta = \pi/2$). The fact that one desingularizes a circle instead of a straight line can be overcome by thinking of a line as the limit of a circle with arbitrarily large radius, or equivalently, by scaling down and slightly bending the Scherk connection piece. The removing of all the intersection circles decomposes the catenoids and planes in $\mathcal{F}$ into bounded pieces (namely discs and annuli) and annular ends, all of which, after being slightly perturbed, glue to the wings of the bent Scherk pieces. In this way, one constructs a smooth (non-minimal) surface M_1, which is the base of the second stage of the method: perturb M_1 by small normal graphs, to find a minimal surface M which is the final goal of the method. Since the surface M_1 has a large number of handles (depending on the precision of the desingularization in the first stage), one does not have a control on a lower bound for the genus of the final minimal surface M. The annular ends of the original configuration $\mathcal{F}$, viewed as graphs of logarithmic growth over the (x_1, x_2)-plane, change after the perturbation in the second stage, to give ends of the minimal surface M with slightly different logarithmic growths and positions from the original ones. In particular, planar ends may become asymptotic to catenoids of small logarithmic growths. But this change is not explicit, since the second stage is solved by application of a fixed point theorem in a suitable space of functions. In order to insure embeddedness of M, one must then start from a configuration $\mathcal{F}$ having all its ends with *different* logarithmic growths (in particular, $\mathcal{F}$ contains at most one plane). For details, see Kapouleas [92].

A good example for understanding the essence of this method is to consider the Costa-Hoffman-Meeks surfaces $M_{k,1}$ described in Chapter 2.5. Hoffman and Meeks [81] proved that after suitable normalizations, the $M_{k,1}$

converge as $k \to \infty$ to the following two limits:

- The union of a catenoid and a plane passing through its waist circle (this limit is attained by fixing the logarithmic growth of the catenoidal ends of $M_{k,1}$).
- The singly-periodic Scherk minimal surface $\mathcal{S}_{\theta=\pi/2}$ (after normalizing the maximal absolute curvature of $M_{k,1}$ to be some fixed positive number).

This phenomenon for the surfaces $M_{k,1}$ with k large motivated the question of whether is possible to desingularize the intersection set I of two transversely intersecting minimal surfaces, by replacing neighborhoods of I with necklaces of singly-periodic Scherk minimal surfaces[3], see also the discussion in Chapter 15. Kapouleas used his method in [92] to perform this desingularization under the assumption of coaxial symmetry. In [93], he announced a general theorem for intersecting minimal surfaces in Riemannian three-manifolds, without symmetry assumptions, based on the same method.

[3] A related interesting question is if one could use a model different from a singly-periodic Scherk minimal surface, to desingularize two intersecting embedded minimal surfaces. This open problem is closely related to Conjecture 17.0.27 below.

4 Sequences of embedded minimal surfaces with no local area bounds.

Two central problems in minimal surface theory are to understand the possible geometries or shapes of embedded minimal surfaces in $\mathbb{R}^3$ of finite genus, as well as the structure of limits of sequences of embedded minimal surfaces with fixed genus. The classical theory deals with these limits when the sequence has uniform local area and curvature bounds, since in this case one can reduce the problem of taking limits of minimal surfaces to taking limits of solutions of the minimal surface equation (for this reduction, one uses the local curvature bound in order to express the surfaces as local graphs of uniform size, and the local area bound to constrain locally the number of such graphs to a fixed finite number), which follows from the classical Arzelà-Ascoli theorem, see e.g. [186]. Hence, we will concentrate here on the case where we do not have such estimates. The understanding of these problems starts with the analysis of the local structure in a fixed extrinsic ball, an issue tackled by Colding and Minicozzi in a series of papers where they study the structure of a sequence of compact, embedded minimal surfaces M_n with fixed genus and no area bounds in balls $\mathbb{B}_n \subset \mathbb{R}^3$, whose radii tend to infinity as $n \to \infty$ and with boundaries $\partial M_n \subset \partial \mathbb{B}_n$, $n \in \mathbb{N}$.

4.1 Colding-Minicozzi theory (locally simply-connected).

As we said above, a main goal of the Colding-Minicozzi theory is to understand the limit objects for a sequence of compact, embedded minimal surfaces M_n with fixed genus but not *a priori* area bounds, each one with boundary contained in the boundary sphere of a Euclidean ball centered at the origin, say with radius $R_n > 0$. Typically, one finds *minimal lami-*

nations[1] as limits of subsequences of the M_n. Nevertheless, we will see in Theorems 4.1.2 and 4.1.7, versus Example II of Chapter 4.2, that the behavior of the limit lamination changes dramatically depending on whether R_n diverges to ∞ or stays bounded, in the sense that the limit lamination can develop removable or essential singularities. This phenomenon connects with another main problem in the current state of the theory of minimal surfaces: finding removable singularity theorems for minimal laminations, or equivalently, finding extension theorems for minimal laminations defined outside of a small set. The interested reader can find details about removable singularity results in Meeks, Pérez and Ros [134], see also Chapter 11 and Conjectures 17.0.16 and 17.0.17 below.

Coming back to the Colding-Minicozzi results, the most important structure theorem in this theory is when the M_n are disks whose Gaussian curvature blows up near the origin. The basic example in this setting is a sequence of rescaled helicoids $M_n = \lambda_n H = \{\lambda_n x \mid x \in H\}$, where H is a fixed vertical helicoid with axis the x_3-axis and $\lambda_n \in \mathbb{R}^+$, $\lambda_n \searrow 0$. The curvature of the sequence $\{M_n\}_n$ blows up along the x_3-axis and the M_n converge away from the axis to the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes. The x_3-axis is the singular set of C^1-convergence $S(\mathcal{L})$ of M_n to $\mathcal{L}$ (i.e. the M_n do not converge C^1 to the leaves of $\mathcal{L}$ along the x_3-axis), but each leaf L of $\mathcal{L}$ extends smoothly across $L \cap S(\mathcal{L})$ (i.e. $S(\mathcal{L})$ consists of removable singularities of $\mathcal{L}$).

Definition 4.1.1 In polar coordinates (ρ, θ) on $\mathbb{R}^2 - \{0\}$ with $\rho > 0$ and $\theta \in \mathbb{R}$, a *k-valued graph on an annulus of inner radius r and outer radius R,* is a single-valued graph of a function $u(\rho, \theta)$ defined over $\{(\rho, \theta) \mid r \leq \rho \leq R,\ |\theta| \leq k\pi\}$, k being a positive integer. The *separation* between consecutive sheets is $w(\rho, \theta) = u(\rho, \theta + 2\pi) - u(\rho, \theta) \in \mathbb{R}$.

With this notation at hand, the statement of the so-called *Limit Lamination Theorem for Disks* (Theorem 0.1 of [34]) can be easily understood. Given $p \in \mathbb{R}^3$ and $R > 0$, we denote by $\mathbb{B}(p, R) = \{x \in \mathbb{R}^3 \mid \|x - p\| < R\}$, $\mathbb{B}(R) = \mathbb{B}(\vec{0}, R)$ and K_M the Gaussian curvature function of a surface M.

Theorem 4.1.2 (Limit Lamination Theorem for Disks, Colding, Minicozzi [34]) *Let $M_n \subset \mathbb{B}(R_n)$ be a sequence of embedded minimal disks with $\partial M_n \subset \partial \mathbb{B}(R_n)$ and $R_n \to \infty$. If $\sup |K_{M_n \cap \mathbb{B}(1)}| \to \infty$, then there exists a subsequence of the M_n (denoted in the same way) and a Lipschitz curve $S \colon \mathbb{R} \to \mathbb{R}^3$ such that up to a rotation of $\mathbb{R}^3$,*

1. *$x_3(S(t)) = t$ for all $t \in \mathbb{R}$.*
2. *Each M_n consists of exactly two multigraphs away from $S(\mathbb{R})$ (which spiral together).*

[1] See Definition 4.1.4 for the concept of minimal lamination.

3. *For each $\alpha \in (0,1)$, the surfaces $M_n - S(\mathbb{R})$ converge in the C^α-topology to the foliation $\mathcal{L} = \{x_3 = t\}_{t\in\mathbb{R}}$ by horizontal planes.*

4. $\sup |K_{M_n \cap \mathbb{B}(S(t),r)}| \to \infty$ *as $n \to \infty$, for any $t \in \mathbb{R}$ and $r > 0$.*

Theorem 4.1.2 has two main ingredients in its proof, which we explain very roughly. The first ingredient is that the embedded minimal disk M_n with large curvature at some interior point can be divided into building blocks, each one being a multivalued graph $u_n(\rho, \theta)$ defined on an annulus, and that these basic pieces fit together properly. In particular, Colding and Minicozzi proved that the number of sheets of $u_n(\rho, \theta)$ rapidly grows as the curvature blows up as n goes to infinity and at the same time, the sheets of u_n do not accumulate in a half-space. This is obtained by means of sublinear and logarithmic bounds for the separation $w_n(\rho, \theta)$ (see Definition 4.1.1) as a function of $\rho \to \infty$; by these bounds we mean the following inequalities, see Corollary 11.2 in Colding and Minicozzi [30] and Theorem 0.8 in [27]:

$$|w_n|(\rho_2, 0) \leq |w_n|(\rho_1, 0) \left(\frac{\rho_2}{\rho_1}\right)^\beta, \qquad \frac{1}{C \log \rho} \leq \frac{w_n(\rho, 0)}{w_n(1, 0)} \leq C \log \rho,$$

where $\beta \in (0,1)$, $C > 0$ are constants independent of n.

Another consequence of these bounds is that by allowing the inner radius r of the annulus where the multigraph is defined to go to zero, the sheets of the multigraphs u_n collapse (i.e. $|w_n(\rho, \theta)| \to 0$ as $n \to \infty$ for ρ, θ fixed); thus a subsequence of the u_n converges to a smooth minimal graph through $r = 0$. The fact that the R_n go to ∞ then implies this limit graph is entire and, by Bernstein's Theorem [6], it is a plane.

The second main ingredient in the proof of Theorem 4.1.2 is the so-called *one-sided curvature estimate,* a scale invariant bound for the Gaussian curvature of any embedded minimal disk in a half-space.

Theorem 4.1.3 (Colding, Minicozzi [34]) *There exists an $\varepsilon > 0$ such that the following holds. Given $r > 0$ and an embedded minimal disk $M \subset \mathbb{B}(2r) \cap \{x_3 > 0\}$ with $\partial M \subset \partial \mathbb{B}(2r)$, then for any component M' of $M \cap \mathbb{B}(r)$ which intersects $\mathbb{B}(\varepsilon r)$,*

$$\sup_{M'} |K_M| \leq r^{-2}.$$

(See Figure 4.1).

The hypothesis on M to be simply-connected in Theorem 4.1.3 is necessary, as the catenoid demonstrates (see Figure 4.4 Left). Actually the 1-sided curvature estimate is equivalent to the property that every component of $M \cap \mathbb{B}(r)$ which intersects $\mathbb{B}(\varepsilon r)$ is a graph of small gradient over its projection on the (x_1, x_2)-plane. This result is needed in the proof of Theorem 4.1.2 in the following manner: once it has been proven that an embedded minimal

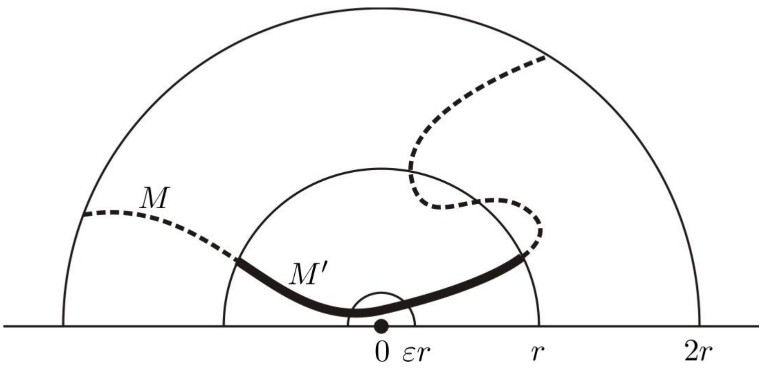

Figure 4.1: The one-sided curvature estimate.

disk M contains a highly sheeted double multigraph $\widetilde{M}$, then $\widetilde{M}$ plays the role of the plane in the one-sided curvature estimate, which implies that reasonably large pieces of M consist of multigraphs away from a cone with axis "orthogonal" to the double multigraph. The proofs of Theorems 4.1.2 and 4.1.3 are long and delicate. References [28, 29, 30, 35, 36] by Colding and Minicozzi are reading guides for the complete proofs of these results, which go through various papers [31, 32, 34].

Theorems 4.1.2 and 4.1.3 have been applied to obtain a number of results, among which we highlight three: Meeks and Rosenberg [148] proved that the helicoid and the plane are the unique properly embedded, simply-connected minimal surfaces in $\mathbb{R}^3$ (see Theorem 1.0.4); Meeks, Pérez and Ros used Colding-Minicozzi theory to study properly embedded minimal surfaces in $\mathbb{R}^3$ with finite genus and infinitely many ends, proving that such a surface M has necessarily two limit ends (Theorem 12.2.1 below); finally, the same authors proved (Theorem 1 in [139]) that if the genus of M is zero, then its absolute Gaussian curvature function is bounded. These are key properties in the classification of the properly embedded, minimal planar domains in $\mathbb{R}^3$ (Theorem 1.0.2). We will discuss other applications of the Colding-Minicozzi results, such as Theorem 1.0.6.

We have mentioned the importance of understanding limits of sequences of embedded minimal surfaces with fixed (or bounded) genus but no *a priori* area bounds in a three-manifold N, a situation of which Theorem 4.1.2 is a particular case. This result shows that one must consider limit objects other than minimal surfaces, such as minimal foliations or more generally, minimal laminations of N. We next define these objects in the case N is an open subset of $\mathbb{R}^3$, although the reader can easily extend the following definition to other ambient spaces.

Definition 4.1.4 A *lamination* of an open subset $U \subset \mathbb{R}^3$ is the union of a collection of pairwise disjoint, connected, injectively immersed surfaces, with a certain local product structure. More precisely, it is a pair $(\mathcal{L}, \mathcal{A})$ satisfying:

1. $\mathcal{L}$ is a closed subset of U;
2. $\mathcal{A} = \{\varphi_\beta \colon \mathbb{D} \times (0,1) \to U_\beta\}_\beta$ is a collection of coordinate charts of $\mathbb{R}^3$ (here $\mathbb{D}$ is the open unit disk, $(0,1)$ the open unit interval and U_β an open subset of U);
3. For each β, there exists a closed subset C_β of $(0,1)$ such that $\varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D} \times C_\beta$.

We will simply denote laminations by $\mathcal{L}$, omitting the charts φ_β in $\mathcal{A}$. A lamination $\mathcal{L}$ is said to be a *foliation of* U if $\mathcal{L} = U$. Every lamination $\mathcal{L}$ naturally decomposes into a union of disjoint connected surfaces, called the *leaves* of $\mathcal{L}$. As usual, the regularity of $\mathcal{L}$ requires the corresponding regularity on the change of coordinate charts. A lamination is *minimal* if all its leaves are minimal surfaces.

Each leaf of a minimal lamination $\mathcal{L}$ is smooth, and if C is a compact subset of a limit leaf[2] $L \in \mathcal{L}$, then the leaves of $\mathcal{L}$ converge smoothly to L over C (i.e. they converge uniformly in the C^k-topology on C for any k). In general, a codimension-one minimal lamination of a Riemannian manifold is of class $C^{0,1}$ (see Proposition B.1 in Colding and Minicozzi [34] for the proof of this result when the leaves are two-dimensional). In particular, a codimension-one minimal foliation of a Riemannian manifold is of class $C^{0,1}$ and furthermore, the unit normal vector field is $C^{0,1}$ as well (see Solomon [209]).

The following result concerns the behavior of limit leaves for a minimal lamination, which generalizes some statements in Lemma A.1 in Meeks and Rosenberg [149].

Theorem 4.1.5 (Meeks, Pérez, Ros [132]) *Any limit leaf L of a codimension-one minimal lamination of a Riemannian manifold is stable for the Jacobi operator*[3] *. More strongly, every two-sided cover of such a limit leaf L is stable. Therefore, the collection of stable leaves of a minimal lamination $\mathcal{L}$ has the structure of a sublamination containing all the limit leaves of $\mathcal{L}$.*

We next return to our discussion about Theorem 4.1.2. Using the regularity Theorem 10.2.1 below, one can replace the Lipschitz curve in Theorem 4.1.2 by a vertical line, which on large balls, yields what we refer to as

[2] A point $p \in \mathcal{L}$ is said to be a *limit point* if for some β, $\varphi_\beta^{-1}(p) = (x, y) \in \mathbb{D} \times C_\beta$, with y being a limit point of C_β (this definition does not depend on the local chart). A leaf L of $\mathcal{L}$ is called a *limit leaf* if it contains a limit point, which is equivalent to consisting entirely of limit points.

[3] The Jacobi operator of a two-sided minimal surface M in a Riemannian three-manifold N is $L = \Delta + (\mathrm{Ric}(\nu) + |\sigma|^2)$, where we are using the same notation as in Footnote 18.

a limiting *parking garage structure* on $\mathbb{R}^3$ with one column (see the next to last paragraph before Theorem 10.2.1 for a description of the notion of minimal parking garage structure). We will find again a limiting parking garage structure in Theorem 4.1.7 below, but with two columns instead of one. In a parking garage structure one can travel quickly up and down the horizontal levels of the limiting surfaces only along the (helicoidal[4]) columns, in much the same way that some parking garages are configured for traffic flow; hence, the name parking garage structure. We will study these structures in Chapter 10.2.

Theorem 4.1.2 does not hold if we exchange the hypothesis that "the radii R_n of the balls go to ∞" by "R_n equals a constant", as demonstrated by a counterexample due to Colding and Minicozzi [29], see also Example II in Chapter 4.2 below; they constructed a sequence of embedded minimal disks $M_n \subset \mathbb{B}(1)$ with $\partial M_n \subset \partial\mathbb{B}(1)$, all passing through the origin $\vec{0}$, and with Gaussian curvature blowing up only at $\vec{0}$. This sequence produces a limit lamination of $\mathbb{B}(1) - \{\vec{0}\}$ with an isolated singularity at the origin. The limit lamination consists of three leaves, one of them being the flat horizontal punctured disk (which extends through $\vec{0}$), and the other two being non-proper multigraphs with this flat disk as limit set (see Figure 4.4 Right for a similar example). In particular, both smoothness and properness of the leaves of the limit lamination fail for this local example.

Theorem 4.1.2 deals with limits of sequences of disks, but it is also useful when studying more general situations, as for instance, locally simply-connected sequences of minimal surfaces, a notion which we now define.

Definition 4.1.6 Suppose that $\{M_n\}_n$ is a sequence of embedded minimal surfaces (possibly with boundary) in an open set U of $\mathbb{R}^3$. Given $p \in U$ and $n \in \mathbb{N}$, we let $r_n(p) \in (0, \infty)$ be the supremum of the set

$$\{r > 0 \mid \mathbb{B}(p, r) \subset U \text{ intersects } M_n \text{ in simply-connected components}\}.$$

If for any $p \in U$ there exists a number $r(p) > 0$ such that $\mathbb{B}(p, r(p)) \subset U$ and for n sufficiently large, M_n intersects $\mathbb{B}(p, r(p))$ in compact disks whose boundaries lie on $\partial\mathbb{B}(p, r(p))$, then we say that $\{M_n\}_n$ is *locally simply-connected in U*. If $\{M_n\}_n$ is a locally simply-connected sequence in $U = \mathbb{R}^3$ and the positive number $r(p)$ can be chosen independently of $p \in \mathbb{R}^3$, then we say that $\{M_n\}_n$ is *uniformly locally simply-connected.*

There is a subtle difference between our definition of uniformly locally simply-connected and that of Colding and Minicozzi [24], which may lead to some confusion. Colding and Minicozzi define a sequence $\{M_n\}_n$ to be uniformly locally simply-connected in an open set $U \subset \mathbb{R}^3$ if for any compact set $K \subset U$, the number $r(p)$ in Definition 4.1.6 can be chosen independently

[4]See Remark 10.2.2.

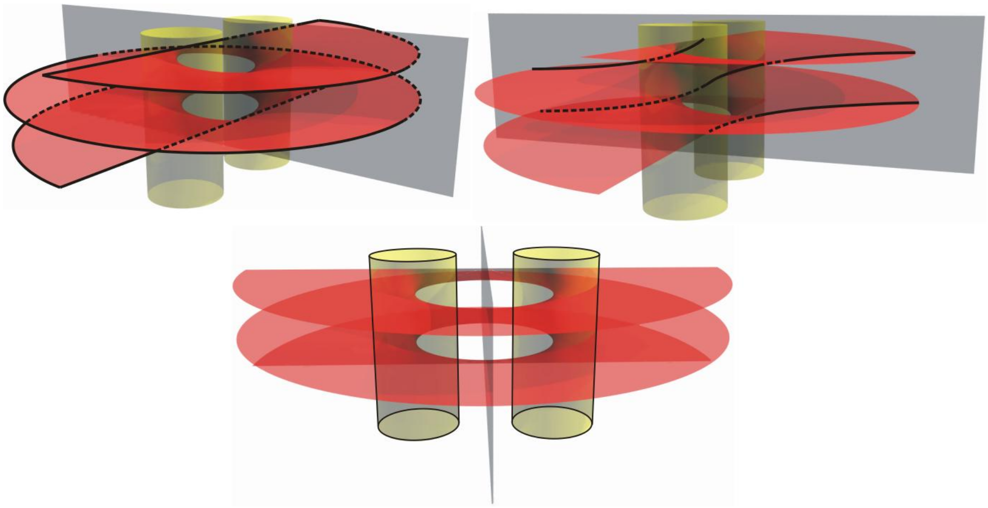

Figure 4.2: Three views of the same Riemann minimal example, with large horizontal flux and two oppositely handed vertical helicoids forming inside solid almost vertical cylinders, one at each side of the vertical plane of symmetry.

of $p \in K$. It is not difficult to check that this concept coincides with our definition of a locally simply-connected sequence in U.

The Limit Lamination Theorem for Disks (Theorem 4.1.2) admits a generalization to a locally simply-connected sequence of non-simply-connected planar domains, which we now explain since it will be useful for our goal of classifying minimal planar domains. Instead of the scaled down limit of the helicoid, the basic example in this case is an appropriate scaled down limit of Riemann minimal examples M_λ, $\lambda > 0$, defined in Chapter 2.5. To understand this limit, one considers the *flux* $F(\lambda)$ of M_λ, which is the flux vector along any compact horizontal section, $M_\lambda \cap \{x_3 = \text{constant}\}$, normalized so that the third component of this vector equals one. The fact that M_λ is invariant by reflection in the (x_1, x_3)-plane Π forces $F(\lambda)$ to be contained in Π for each λ. Furthermore, $\lambda \mapsto F = F(\lambda)$ is a bijection that parameterizes the whole family of Riemann minimal examples, with F running from horizontal to vertical (with monotonically increasing slope function). When F tends to vertical, then it can be proved that $M_{\lambda(F)}$ converges to a vertical catenoid with waist circle of radius $\frac{1}{2\pi}$. When F tends to horizontal, the one can shrink $M_{\lambda(F)}$ so that F tends to $(4, 0, 0)$, and in that case the $M_{\lambda(F)}$ converge to the foliation of $\mathbb{R}^3$ by horizontal planes, outside of the two vertical lines $\{(0, \pm 1, x_3) \mid x_3 \in \mathbb{R}\}$, along which the surface $M_{\lambda(F)}$ approximates two oppositely handed, highly sheeted, scaled down vertical helicoids, see Figures 4.2 and 4.3. With this basic family of examples in mind, we state

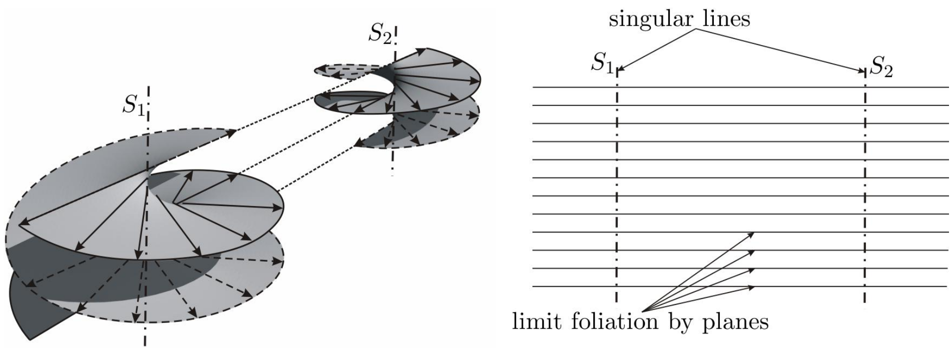

Figure 4.3: Left: Two oppositely handed double spiral staircases. Right: The limit foliation by parallel planes and the singular set of convergence $S_1 \cup S_2$.

the following result by Colding and Minicozzi.

Theorem 4.1.7 (Limit Lamination Theorem for Planar Domains [24], [121]) *Let $M_n \subset \mathbb{B}(R_n)$ be a locally simply-connected sequence of embedded minimal planar domains with $\partial M_n \subset \partial \mathbb{B}(R_n)$, $R_n \to \infty$, such that $M_n \cap \mathbb{B}(2)$ contains a component which is not a disk for any n. If $\sup |K_{M_n \cap \mathbb{B}(1)}| \to \infty$, then there exists a subsequence of the M_n (denoted in the same way) and two vertical lines S_1, S_2, such that*

(a) M_n converges away from $S_1 \cup S_2$ to the foliation $\mathcal{F}$ of $\mathbb{R}^3$ by horizontal planes.

(b) Away from $S_1 \cup S_2$, each M_n consists of exactly two multivalued graphs spiraling together. Near S_1 and S_2, the pair of multivalued graphs form double spiral staircases with opposite handedness at S_1 and S_2. Thus, circling only S_1 or only S_2 results in going either up or down, while a path circling both S_1 and S_2 closes up, see Figure 4.3.

Theorem 4.1.7 shows a second example of a limiting parking garage structure on $\mathbb{R}^3$ (we obtained the first example in Theorem 4.1.2 above), now with two columns which are $(+,-)$-handed[5], just like in the case of the Riemann minimal examples M_λ discussed before the statement of Theorem 4.1.7. We refer the reader to Chapter 10.2 for more details about parking garage structures on $\mathbb{R}^3$, to Chapter 12.4 for a generalization of Theorems 4.1.2 and 4.1.7 to the case of fixed genus, and to Theorem 11.0.9 for a generalization to the case where there is not a bound on the genus of the surfaces M_n.

[5]Here, $+$, (resp. $-$) means that the corresponding forming helicoid or multigraph is right-handed.

To finish this section, we want to mention that the one-sided curvature estimate of Colding and Minicozzi (Theorem 4.1.3) can be used to solve an important problem in classical minimal surface theory, known as the *Generalized Nitsche Conjecture.* In 1962, Nitsche [172] conjectured that if a minimal surface in $\mathbb{R}^3$ meets every horizontal plane in a Jordan curve, then it must be a catenoid (he also proved this conjecture with the additional assumption that every horizontal section of the surface is a star-shaped curve). In 1993, Meeks and Rosenberg [145] showed that if a properly embedded minimal surface $M \subset \mathbb{R}^3$ has at least two ends, then any annular end $E \subset M$ either has finite total curvature or it satisfies the hypotheses of the following conjecture.

Conjecture 4.1.8 (Generalized Nitsche Conjecture) *Let $E \subset \{x_3 \geq 0\}$ be a properly embedded minimal annulus with $\partial E \subset \{x_3 = 0\}$, such that E intersects each plane $\{x_3 = t\}$, $t > 0$, in a simple closed curve. Then, E has finite total curvature, and so, E is asymptotic to a plane or to an end of a catenoid.*

This problem was originally solved by Collin [38] before the Colding-Minicozzi results, with a beautiful and long proof. Later on, Colding and Minicozzi gave an alternative proof of Conjecture 4.1.8, as a short application of Theorem 4.1.3 that can be found in detail in [26] and in a survey by Rosenberg [198]. We will only state here a consequence of the one-sided curvature estimate by Colding and Minicozzi, which implies directly the Generalized Nitsche Conjecture. Given $\varepsilon \in \mathbb{R}$, we denote by $\mathcal{C}_\varepsilon$ the conical region $\{x_3 > \varepsilon\sqrt{x_1^2 + x_2^2}\}$.

Theorem 4.1.9 (Colding, Minicozzi [26]) *There exists a $\delta > 0$ such that any properly embedded minimal annular end $E \subset \mathcal{C}_{-\delta}$ has finite total curvature.*

4.2 Minimal laminations with isolated singularities.

A fundamental theoretical question deals with finding appropriate conditions under which a minimal lamination of a three-manifold N minus a closed subset $A \subset N$, extends across A. In this section, we first describe examples in $\mathbb{B}(1) = \mathbb{B}(\vec{0}, 1) \subset \mathbb{R}^3$ with the origin as the unique non-removable singularity. We then show how these examples lead to related singular minimal laminations in the homogeneous spaces $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ (here $\mathbb{H}^n$ denotes the n-dimensional hyperbolic space).

Example I. *Catenoid type laminations.* Consider the sequence of horizontal circles $C_n = \mathbb{S}^2 \cap \{x_3 = \frac{1}{n}\}$, $n \geq 2$. Note that each pair

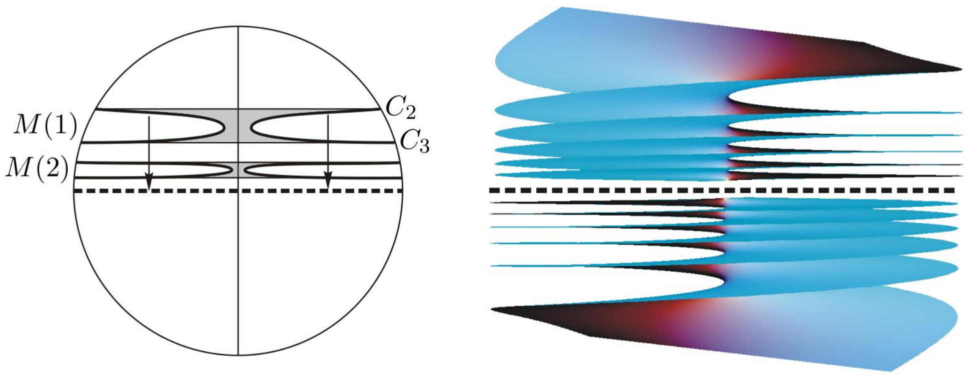

Figure 4.4: Left: A catenoid type lamination, which explains why the hypothesis of simply-connectedness of Theorem 4.1.3 is necessary. Right: A Colding-Minicozzi type lamination in a cylinder.

C_{2k}, C_{2k+1} bounds a compact unstable piece of a catenoid $M(k)$, and that $M(k) \cap M(k') = \emptyset$ if $k \neq k'$. The sequence $\{M(k)\}_k$ converges with multiplicity two outside the origin $\vec{0}$ to the closed horizontal disk $\overline{\mathbb{D}}$ of radius 1 centered at $\vec{0}$. Thus, $\{M(k) - \partial M(k)\}_k \cup \{\mathbb{D} - \{\vec{0}\}\}$ is a minimal lamination of $\mathbb{B}(1) - \{\vec{0}\}$, which does not extend through the origin, see Figure 4.4 Left.

Example II. *Colding-Minicozzi examples.* In their paper [29], Colding and Minicozzi constructed a sequence of compact, embedded minimal disks $D_n \subset \mathbb{B}(1)$ with boundary in $\mathbb{S}^2$, that converges to a singular minimal lamination $\overline{\mathcal{L}}$ of the closed ball $\overline{\mathbb{B}}(\vec{0}, 1)$ which has an isolated singularity at $\vec{0}$. The related lamination $\mathcal{L}$ of $\mathbb{B}(1) - \{\vec{0}\}$ consists of a unique limit leaf which is the punctured open disk $\mathbb{D} - \{\vec{0}\}$, together with two nonproper leaves that spiral into $\mathbb{D} - \{\vec{0}\}$ from opposite sides, see Figure 4.4 Right.

Consider the exhaustion of $\mathbb{H}^3$ (naturally identified with $\mathbb{B}(1)$ through the Poincaré model) by hyperbolic balls of hyperbolic radius n centered at the origin, together with compact minimal disks with boundaries on the boundaries of these balls, similar to the compact Colding-Minicozzi disks. We conjecture that these examples produce a similar limit lamination of $\mathbb{H}^3 - \{\vec{0}\}$ with three leaves, one which is totally geodesic and the other two which are not proper and that spiral into the first one. We remark that one of the main results of the Colding-Minicozzi theory, Theorem 4.1.2 above, insures that such an example cannot be constructed in $\mathbb{R}^3$.

Example III. *Catenoid type laminations in $\mathbb{H}^3$ and in $\mathbb{H}^2 \times \mathbb{R}$.* Consider the same circles C_n as in Example I, where $\mathbb{S}^2$ is now viewed as the

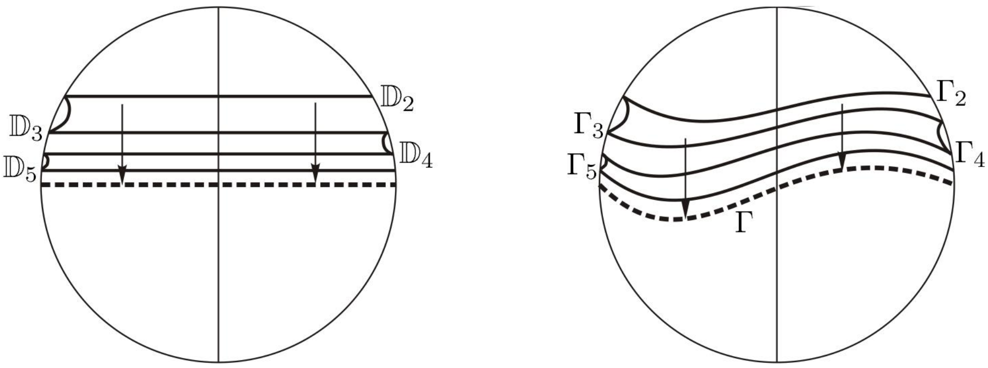

Figure 4.5: Left: Almost flat minimal disks joined by small bridges. Right: A similar example with a non-flat limit leaf.

boundary at infinity of $\mathbb{H}^3$. Then each pair of circles C_{2k}, C_{2k+1} is the asymptotic boundary of a properly embedded, unstable minimal annulus $M(k)$, which is a surface of revolution called a catenoid. The sequence $\{M(k)\}_k$ converges with multiplicity two outside of $\vec{0}$ to the horizontal, totally geodesic subspace $\mathbb{D}$ at height zero. Thus, $\{M(k) - \partial M(k)\}_k \cup \{\mathbb{D} - \{\vec{0}\}\}$ is a minimal lamination of $\mathbb{H}^3 - \{\vec{0}\}$, which does not extend through the origin. A similar catenoidal construction can be done in $\mathbb{H}^2 \times \mathbb{R}$, where we consider $\mathbb{H}^2$ with the Poincaré disk model of the hyperbolic plane. Note that the Half-space Theorem 2.6.4 excludes this type of singular minimal lamination in $\mathbb{R}^3$.

Example IV. *Simply-connected bridged examples.* Coming back to the Euclidean closed unit ball $\overline{\mathbb{B}}(1)$, consider the sequence of horizontal closed disks $\mathbb{D}_n = \overline{\mathbb{B}}(1) \cap \{x_3 = \frac{1}{n}\}$, $n \geq 2$. Connect each pair $\mathbb{D}_n, \mathbb{D}_{n+1}$ by a thin, almost vertical, minimal bridge (in opposite sides for consecutive disks, as in Figure 4.5 left), and perturb slightly this non-minimal surface to obtain an embedded, stable minimal surface with boundary in $\overline{\mathbb{B}}(1)$ (this is possible by the *bridge principle* [157, 225, 226]). We denote by M the intersection of this surface with $\mathbb{B}(1)$. Then, the closure of M in $\mathbb{B}(1)$ is a minimal lamination of $\mathbb{B}(1)$ with two leaves, both being stable, one of which is $\mathbb{D}$ (this is a limit leaf) and the other one is not flat and not proper.

A similar example with a non-flat limit leaf can be constructed by exchanging the horizontal circles by suitable curves in $\mathbb{S}^2$. Consider a non-planar smooth Jordan curve $\Gamma \subset \mathbb{S}^2$ which admits a one-to-one projection onto a convex planar curve in a plane Π. Let Γ_n be a sequence of smooth Jordan curves in $\mathbb{S}^2$ converging to Γ, so that each Γ_n also projects injectively onto a convex planar curve in Π and $\{\Gamma_n\}_n \cup \{\Gamma\}$ is a lamination on $\mathbb{S}^2$. An elementary application of the maximum principle implies that each of the Γ_n is the boundary of a unique minimal surface $\overline{M_n}$, which is a graph over its projection to Π. Now join slight perturbations of the $\overline{M_n}$ by thin bridges as in the preceding paragraph, to obtain a simply-connected minimal surface in the closed unit ball. Let M be the intersection of this surface with $\mathbb{B}(1)$. Then, the closure of M in $\mathbb{B}(1)$ is a minimal lamination of $\mathbb{B}(1)$ with two leaves, both being non-flat and stable, and exactly one of them is properly embedded in $\mathbb{B}(1)$ and is a limit leaf (see Figure 4.5 right).

EXAMPLE V. *Simply-connected bridged examples in $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$.* As in the previous Example III, the minimal laminations in Example IV give rise to minimal laminations of $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ consisting of two stable, complete, simply-connected minimal surfaces, one of which is proper and the other one which is not proper in the space, and either one is not totally geodesic or both of them are not totally geodesic, depending on the choice of the Euclidean model surface in Figure 4.5. In both cases, the proper leaf is the unique limit leaf of the minimal lamination. More generally, Theorem 13 in [149] implies that the closure of any complete, embedded minimal surface of finite topology in $\mathbb{H}^3$ or $\mathbb{H}^2 \times \mathbb{R}$ has the structure of a minimal lamination.

5 The structure of minimal laminations of $\mathbb{R}^3$.

The goal of this chapter is to give a description of the structure of a general minimal lamination of $\mathbb{R}^3$. This description is a crucial tool to study limits of embedded minimal surfaces, and it will be used in the proofs of Theorems 1.0.1 and 1.0.2.

In his beautiful survey on minimal surfaces, Rosenberg [198] introduced the subject of his paper (and of this chapter) through a question asked to him by Haefliger about twenty years ago: *Is there a foliation of $\mathbb{R}^3$ by minimal surfaces, other than a foliation by parallel planes?*

Any leaf L of a minimal foliation of $\mathbb{R}^3$ is a complete limit leaf, and by Theorem 4.1.5, it is stable. Now Theorem 2.8.4 implies L is a plane. Thus, the answer to Haefliger's question is *no*. Immediately one is tempted to extend this question to minimal laminations.

Question 5.0.1 *What are the minimal laminations of $\mathbb{R}^3$?*

There are only two known types of minimal laminations of $\mathbb{R}^3$, both rather simple: a lamination with exactly one leaf which is a properly embedded minimal surface, or a lamination consisting of a closed set of parallel planes. Rosenberg [148] has conjectured that these are the unique possible examples. To the contrary, Meeks has conjectured that there exists a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with two planar leaves and a non-flat leaf which is proper in the open slab between these two planes in $\mathcal{L}$. Since every leaf of a minimal lamination of $\mathbb{R}^3$ is complete, the above question is closely related to the following one, known as the *embedded Calabi-Yau problem* in $\mathbb{R}^3$ (see also Chapter 16 and Conjecture 17.0.35):

Question 5.0.2 *When is a complete, embedded minimal surface $M \subset \mathbb{R}^3$ proper?*

Given a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$, the function that assigns to each point p in $\mathcal{L}$ the Gaussian curvature of the leaf $L \in \mathcal{L}$ passing through p is continuous in the subspace topology. Since the intersection of $\mathcal{L}$ with any closed ball is compact, we conclude that the intersection of any leaf $L \in \mathcal{L}$ with a ball has Gaussian curvature bounded from below by a constant that only depends on the ball (in other words, $\mathcal{L}$ has *locally bounded Gaussian curvature*). Reciprocally, if M is a complete, embedded minimal surface in $\mathbb{R}^3$ with locally bounded Gaussian curvature, then the closure $\overline{M}$ of M is a minimal lamination of $\mathbb{R}^3$ (Lemma 1.1 in [148]). With this perspective, it is natural to study complete, embedded minimal surfaces $M \subset \mathbb{R}^3$ with locally bounded Gaussian curvature, as a first stage for answering Questions 5.0.1 and 5.0.2. If M is such a minimal surface and it is not proper, then $\overline{M} - M$ may or may not be non-empty; but since M has locally bounded curvature, $\mathcal{L} = \overline{M}$ is a non-trivial minimal lamination of $\mathbb{R}^3$. Since M is not proper, then some leaf $L \in \mathcal{L}$ must be a limit leaf, and hence stable by Theorem 4.1.5. Now an argument similar to the one we used to answer Haefliger's question at the beginning of this chapter insures that L is a plane; so in this case, $\overline{M} - M$ is always non-empty. This can be stated as follows.

Lemma 5.0.3 (Meeks, Rosenberg [148]) *Let $M \subset \mathbb{R}^3$ be a connected, complete, embedded minimal surface with locally bounded Gaussian curvature. Then, exactly one of the following statements holds:*

1. *M is properly embedded in $\mathbb{R}^3$.*
2. *M is properly embedded in an open half-space, with limit set consisting of the boundary plane of this half-space.*
3. *M is properly embedded in an open slab, with limit set consisting of the boundary planes of this slab.*

It should be mentioned that in a previous work, Xavier [228] proved that a complete, immersed, non-flat minimal surface of bounded curvature in $\mathbb{R}^3$ cannot be contained in a half-space. This result together with Lemma 5.0.3 gives a partial answer to Question 5.0.2.

Corollary 5.0.4 (Meeks, Rosenberg [148]) *If $M \subset \mathbb{R}^3$ is a connected, complete, embedded minimal surface with bounded Gaussian curvature, then M is proper.*

The next step in the study of complete, embedded, non-proper minimal surfaces consists of understanding how they accumulate to the limit set described in Lemma 5.0.3.

Lemma 5.0.5 (Meeks, Rosenberg [148]) *Let $M \subset \mathbb{R}^3$ be a connected, complete, embedded minimal surface with locally bounded Gaussian curvature. Suppose that M is not proper and let Π be a limit plane of M. Then,* *for any $\varepsilon > 0$, the closed ε-neighborhood of Π intersects M in a path connected set.*

Outline of the proof. The argument is by contradiction. Without loss of generality, we may assume that $\Pi = \{x_3 = 0\}$ and that M limits to Π from above. Assuming that the statement fails, one can find an $\varepsilon > 0$ and a stable minimal surface Σ between two components of the intersection of M with the slab $\{0 < x_3 \leq \varepsilon\}$ by the usual barrier construction argument discussed in Chapter 2.9. Since Σ satisfies curvature estimates away from its boundary (Theorem 2.8.6), we conclude that for sufficiently small $\delta \in (0, \varepsilon)$, the orthogonal projection π to Π, restricted to a component $\Sigma(\delta)$ of $\Sigma \cap \{0 < x_3 < \delta\}$, is a local diffeomorphism. A topological argument shows that $\pi|_{\Sigma(\delta)}$ is in fact a diffeomorphism between $\Sigma(\delta)$ and its image. This implies $\Sigma(\delta)$ is a graph, and so, it is properly embedded in the slab $\{0 \leq x_3 \leq \delta\}$. By Theorem 7.2.3 below, $\Sigma(\delta)$ is a parabolic surface but $x_3|_{\Sigma(\delta)}$ is a non-constant, bounded harmonic function with constant boundary value δ, which gives a contradiction and proves Lemma 5.0.5. □

A refinement of the argument in the previous paragraph gives the following result.

Lemma 5.0.6 (Meeks, Rosenberg [148]) *Let $M \subset \mathbb{R}^3$ be a connected, complete, non-proper, embedded minimal surface with locally bounded Gaussian curvature, which limits to the plane $\Pi = \{x_3 = 0\}$ from above. Then, for any $\varepsilon > 0$, the Gaussian curvature of $M \cap \{0 < x_3 \leq \varepsilon\}$ is not bounded from below.*

Let $M \subset \mathbb{R}^3$ satisfy the hypotheses of Lemma 5.0.6. Then, there exists a sequence $\{p_n\}_n \subset M$ such that $x_3(p_n) \searrow 0$ and the curvature function K_M of M satisfies $|K_M(p_n)| \to \infty$ as n goes to infinity. Such a sequence must diverge in space because K_M is locally bounded. If we additionally assume M has finite topology, then an application of the Colding-Minicozzi one-sided curvature estimate (Theorem 4.1.3) contradicts that $|K_M(p_n)| \to \infty$. This is a rough sketch of the proof of the following statement.

Theorem 5.0.7 (Meeks, Rosenberg [148]) *If $M \subset \mathbb{R}^3$ is a connected, complete, embedded minimal surface in $\mathbb{R}^3$ with finite topology and locally bounded Gaussian curvature, then M is proper.*

Meeks, Pérez and Ros (Theorem 5 in [139]) have combined the last statement with deeper arguments using the results of Colding and Minicozzi, which let us exchange the finite topology assumption by the weaker hypothesis of finite genus.

Theorem 5.0.8 (Meeks, Pérez, Ros [139]) *If $M \subset \mathbb{R}^3$ is a connected, complete, embedded minimal surface in $\mathbb{R}^3$ with finite genus and locally bounded Gaussian curvature, then M is proper.*

In conclusion, we state the following descriptive result for minimal laminations of $\mathbb{R}^3$, by Meeks and Rosenberg [148] as generalized by Meeks, Pérez and Ros [139].

Theorem 5.0.9 (Structure Theorem for Minimal Laminations of $\mathbb{R}^3$) *For a given minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$, one of the following possibilities holds.*

i) *$\mathcal{L}$ has exactly one leaf, which consists of a properly embedded minimal surface in $\mathbb{R}^3$.*

ii) *$\mathcal{L}$ has more that one leaf, and consists of the disjoint union of a non-empty closed set of parallel planes $\mathcal{P} \subset \mathcal{L}$ together with a collection of complete minimal surfaces of unbounded Gaussian curvature and infinite genus, which are properly embedded in the open slabs and half-spaces of $\mathbb{R}^3 - \mathcal{P}$. Furthermore, each of the open slabs and half-spaces in $\mathbb{R}^3 - \mathcal{P}$ contains at most one leaf of $\mathcal{L}$, and every plane parallel to but different from the planes in $\mathcal{P}$ intersects at most one of the leaves of $\mathcal{L}$ and separates such an intersecting leaf into exactly two components.*

There are no known examples of laminations of $\mathbb{R}^3$ as described in item *ii)* of the last theorem, see item *1* in Conjecture 17.0.35.

6 The Ordering Theorem for the space of ends.

The study of the ends of a properly embedded minimal surface $M \subset \mathbb{R}^3$ with more than one end has been extensively developed. Callahan, Hoffman and Meeks [16] showed that in one of the closed complements of M in $\mathbb{R}^3$, there exists a non-compact, properly embedded minimal surface Σ with compact boundary and finite total curvature. By the discussion following Theorem 2.3.1, the ends of Σ are of catenoidal or planar type, and the embeddedness of Σ forces its ends to have parallel normal vectors at infinity.

Definition 6.0.10 In the above situation, the *limit tangent plane at infinity* of M is the plane in $\mathbb{R}^3$ passing through the origin, whose normal vector equals (up to sign) the limiting normal vector at the ends of Σ. Such a plane is unique [16], in the sense that it does not depend on the finite total curvature minimal surface $\Sigma \subset \mathbb{R}^3 - M$.

The limit tangent plane at infinity is a key notion for studying the way in which a minimal surface with more than one end embeds properly in space.

Theorem 6.0.11 (Ordering Theorem, Frohman, Meeks [61]) *Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with more than one end and horizontal limit tangent plane at infinity. Then, the space $\mathcal{E}(M)$ of ends of M is linearly ordered geometrically by the relative heights of the ends over the (x_1, x_2)-plane, and embeds topologically in $[0, 1]$ in an ordering preserving way. Furthermore, this ordering has a topological nature in the following sense: If M is properly isotopic to a properly embedded minimal surface M' with horizontal limit tangent plane at infinity, then the associated ordering of the ends of M' either agrees with or is opposite to the ordering coming from M.*

Given a minimal surface $M \subset \mathbb{R}^3$ satisfying the hypotheses of Theorem 6.0.11, we define the *top end* e_T of M as the unique maximal element in $\mathcal{E}(M)$ for the ordering given in this theorem (recall from Chapter 2.7 that $\mathcal{E}(M) \subset [0,1]$ is compact, hence e_T exists). Analogously, the *bottom end* e_B of M is the unique minimal element in $\mathcal{E}(M)$. If $e \in \mathcal{E}(M)$ is neither the top nor the bottom end of M, then it is called a *middle end* of M.

Rather than sketching the proof of the Ordering Theorem, we will be content to explain how one obtains the linear ordering. Suppose $M \subset \mathbb{R}^3$ is a minimal surface in the hypotheses of Theorem 6.0.11 and let $\mathcal{A} \subset \mathcal{E}(M)$ be the set of annular ends of M. In this setting, Collin's Theorem 1.0.3 insures that every proper annular representative of an end $e \in \mathcal{A}$ has finite total curvature and thus, it is asymptotic to a horizontal plane or to a half-catenoid. Since the ends in $\mathcal{A}$ are all graphs over complements of compact subdomains in the (x_1, x_2)-plane, we see that $\mathcal{A}$ has a natural linear ordering by relative heights of its ends over the (x_1, x_2)-plane. Hence the Ordering Theorem is proved when $\mathcal{A} = \mathcal{E}(M)$.

We next indicate how to extend the above linear ordering to $\mathcal{E}(M) - \mathcal{A}$. By Theorem 2.8.7, any end of M which can be represented by a proper subdomain that is stable, can be also represented by a surface of finite total curvature and so, it can be represented by an annulus. Let $e_1 = [\alpha_1] \in \mathcal{E}(M)$ be an end which is not annular. Such an end can always be represented by a proper subdomain $E_1 \subset M$ with the following two properties (recall we are assuming M has at least two ends):

- E_1 is unstable, and ∂E_1 is compact and connected.
- $\overline{M - E_1}$ is unstable and non-compact.

Let W_1, W_2 be the two closed complements of M in $\mathbb{R}^3$. Note that we can consider E_1 to lie on the boundary of both of these complete flat three-manifolds W_1, W_2, and that their boundaries $\partial W_1, \partial W_2$ are good barriers for solving Plateau-type problems. Since E_1 and $\overline{M - E_1}$ are both non-compact, elementary separation properties for proper surfaces in $\mathbb{R}^3$ imply that ∂E_1 is not homologous to zero in one of the domains W_1, W_2; suppose that ∂E_1 is not homologous to zero in W_1. Since ∂E_1 bounds the locally finite 2-chain E_1 in ∂W_1, the barrier argument in Chapter 2.9 shows that ∂E_1 is the boundary of a properly embedded, orientable, least-area surface Σ_1 in W_1, which is non-compact since ∂E_1 is not homologous to zero in W_1. Similarly, ∂E_1 is the boundary of a least-area (possibly compact) surface Σ_2 in W_2. Since E_1 and $M - E_1$ are unstable, the maximum principle implies that $(\Sigma_1 \cup \Sigma_2) \cap M = \partial E_1$, see Figure 6.1.

Let R_1 be the closed complement of $\Sigma_1 \cup \Sigma_2$ in $\mathbb{R}^3$ which contains E_1 and let R_2 be the other closed complement. Since E_1 and $\overline{M - E_1}$ are both non-compact and M is properly embedded in $\mathbb{R}^3$, then R_1 and R_2 are both non-compact. It follows from Theorem 2.8.7 that outside a large

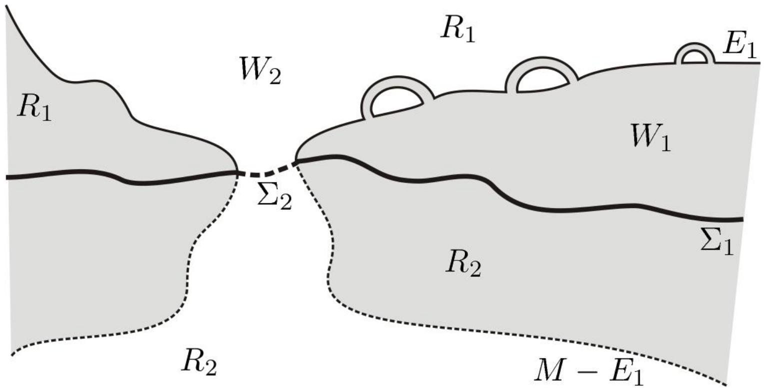

Figure 6.1: The least-area surfaces Σ_1, Σ_2. W_1 is the shaded region, and R_1 lies above $\Sigma_1 \cup \Sigma_2$.

ball containing ∂E_1, the boundary of R_1, which equals $\Sigma_1 \cup \Sigma_2 = \partial R_2$, consists of a finite positive number of graphical ends which are asymptotic to ends of horizontal planes and vertical catenoids. Let $e_2 = [\alpha_2] \in \mathcal{E}(M)$ be an end with a proper subdomain representative $E_2 \subset M$ which is disjoint from E_1 (note that any two distinct ends can be chosen to have disjoint representatives). Next we define when $[\alpha_1] < [\alpha_2]$ or vice versa in the linear ordering of Theorem 6.0.11. The proper arcs α_1, α_2 which represent e_1, e_2 eventually lie in R_1, R_2 and so, exactly one of the following cases holds:

- α_1 eventually lies in the open region between two successive graphical ends of ∂R_1, and α_2 eventually lies above or below this region.
- α_1 eventually lies in the open region above the top graphical end of ∂R_1, and α_2 eventually lies below this region.
- α_1 eventually lies in the open region below the bottom graphical end of ∂R_1, and α_2 eventually lies above this region.

In particular, there is a topological plane P which is a graph over the (x_1, x_2)-plane and whose end is one of the ends of ∂R_1, such that eventually α_1 and α_2 lie at opposite sides of P. If α_1 eventually lies below P and α_2 eventually lies above P, then $[\alpha_1] < [\alpha_2]$ in the linear ordering given by the Ordering Theorem; otherwise, $[\alpha_2] < [\alpha_1]$. The ordering we have just described can be proven to be a well-defined linear ordering, see [61] for more details.

Conformal structure of minimal surfaces. 7

7.1 Recurrence and parabolicity for manifolds.

The conformal structure of a complete minimal surface has a strong influence on its global properties. For this reason, an important conformal question is to decide the so-called *type problem* for a minimal surface M in $\mathbb{R}^3$, in the sense of classical Riemann surfaces: i.e. whether M is hyperbolic or parabolic[1] (the elliptic or compact case is impossible for such a minimal surface by the maximum principle for harmonic functions). It turns out that parabolicity for Riemann surfaces without boundary is equivalent to recurrence of Brownian motion on such surfaces. This field lies in the borderline between several branches of mathematics such as Riemannian geometry, stochastic analysis, partial differential equations and potential theory. A particularly interesting source, where the reader can find an excellent introduction to these questions, is the survey of recurrence and Brownian motion on Riemannian manifolds by Grigor'yan [67].

The goal of this chapter is to introduce some key concepts which are useful when dealing with these conformal questions. Instead of using concepts related to probability (such as random walks or Brownian motion), in this paper we will follow an alternative way to define recurrence and parabolicity that is slightly different from Grigor'yan's approach; our approach is well-known and is explained in greater detail in the notes by the second

[1]Classically, a Riemann surface without boundary is called *hyperbolic* if it carries a non-constant positive superharmonic function, and *parabolic* if it is neither elliptic (i.e. compact) nor hyperbolic. The reader should be aware that we will use the concept of parabolicity for Riemannian manifolds with boundary (see Definition 7.1.1) and reserve the word *recurrent* for manifolds without boundary (Definition 7.1.2). For Riemannian manifolds, the relationship between parabolicity and recurrence will become clear soon.

author [181]. However, we will briefly explain the connection between these two approaches. Most of the results stated in this chapter are proved in [67] or [181].

Definition 7.1.1 Let (M^n, g) be an n-dimensional Riemannian manifold with non-empty boundary. M is *parabolic* if every bounded harmonic function on M is determined by its boundary values.

Definition 7.1.2 Let (M^n, g) be a n-dimensional Riemannian manifold without boundary. M is *recurrent* if for any non-empty open set $U \subset M$ $(U \neq M)$ with smooth boundary, $M - U$ is parabolic. M is called *transient* if it is not recurrent.

Given a Riemannian manifold (M, g) with boundary $\partial M \neq \emptyset$ and a point $p \in \mathrm{Int}(M)$, the *harmonic measure* μ_p *with respect to* p can be defined as follows. Let $I \subset \partial M$ be a non-empty open set with smooth boundary. Consider a compact exhaustion $M_1 \subset M_2 \subset \ldots$ of M. Given $k \in \mathbb{N}$, let $h_k \colon M_k \to [0,1]$ be the (bounded) harmonic function on M_k with boundary values 1 on the interior of $I \cap M_k$ and 0 on $\partial M_k - \overline{I}$. After extending h_k by zero to M, we can view $\{h_k\}_k$ as an increasing sequence of subharmonic functions, bounded from above by 1. The functions h_k limit to a unique bounded harmonic function $h_I \colon M \to [0,1]$ (defined except at countably many points in $\partial I \subset \partial M$). In this situation, we define $\mu_p(I) = h_I(p)$. It turns out that μ_p extends to a Borel measure μ_p on ∂M. Another interpretation of μ_p related to Brownian motion, developed in [67], is that $\mu_p(I)$ is the probability of a Brownian path beginning at p, of hitting ∂M the first time somewhere on the interval I, and for this reason the harmonic measure of M is also called the *hitting measure with respect to* p. We now explain how to computationally calculate the hitting measure μ_p at an interval I contained in the boundary of a smooth domain $\Omega \subset \mathbb{R}^2$, where $p \in \mathrm{Int}(\Omega)$. For $n \in \mathbb{N}$ and $\varepsilon > 0$, consider the set $\Gamma(p, n, \varepsilon)$ of all n-step orthogonal random ε-walks starting at p, i.e. continuous mappings $\sigma \colon [0, n\varepsilon] \to \mathbb{R}^2$ which begin at $\sigma(0) = p$ and for any integer $k = 0, \ldots, n-1$,

$$\left(\sigma|_{[k\varepsilon,(k+1)\varepsilon]}\right)(t) = \sigma(k\varepsilon) \pm te_i,$$

where e_i is one of the unit vectors $(1,0), (0,1)$. We define $\mu_p(n,\varepsilon)(I)$ to be the probability that some $\sigma \in \Gamma(p,n,\varepsilon)$ crosses $\partial\Omega$ a first time in I. As $n \to \infty$, $\mu_p(n,\varepsilon)(I)$ converges to a number $\mu_p(\varepsilon)(I) \in [0,1]$. Similarly, as $\varepsilon \to 0$, the $\mu_p(\varepsilon)$ converge to a measure μ_p on ∂M, which is equal to the hitting measure obtained from Brownian motion starting at p.

For an open interval $I \subset \partial\Omega$, consider the function $P_I(n,\varepsilon) \colon \mathrm{Int}(\Omega) \to [0,1]$ defined as $P_I(n,\varepsilon)(p) = \mu_p(n,\varepsilon)(I)$ for $p \in \mathrm{Int}(\Omega)$. Note that for any $p \in \mathrm{Int}(\Omega)$ and for ε smaller that the distance from p to $\partial\Omega$, the following formula holds

$$P_I(n,\varepsilon)(p) = \frac{1}{4}\left(\sum_{i=1}^{2} P_I(n-1,\varepsilon)(p+e_i) + \sum_{i=1}^{2} P_I(n-1,\varepsilon)(p-e_i)\right),$$

and so, the limiting function $P_I(\varepsilon) = \lim_{n\to\infty} P_I(n,\varepsilon)$ satisfies an infinitesimal 4 point mean value property. As $\varepsilon \to 0$, $P_I(\varepsilon)$ converges to a function P_I on $\mathrm{Int}(\Omega)$ which satisfies the usual mean value property. Therefore, the function $p \mapsto P_I(p)$, which is the probability of a Brownian path starting at p of exiting Ω a first time on I, is a harmonic function which takes its values in $[0,1]$. Note that P_I has limiting values 1 on I and 0 on the interior of $\partial\Omega - I$ and so, it is the unique bounded harmonic function on $\mathrm{Int}(\Omega)$ whose boundary values correspond almost everywhere to the characteristic function of $I \subset \partial\Omega$. By definition of P_I, the hitting measure is $\mu_p(I) = P_I(p)$, which gives the desired equivalence between hitting and harmonic measure for planar domains.

The above discussion generalizes easily to a Riemannian manifold M with boundary. We will briefly explain this generalization in the case of dimension 2 and when M lies in the interior of a larger complete manifold $\overline{M}$. Let M be an oriented Riemannian surface with boundary and $p \in \mathrm{Int}(M)$. Given a unit tangent vector $v_p \in T_pM$, $n \in \mathbb{N}$ and $\varepsilon > 0$, we let $\Gamma(v_p, n, \varepsilon)$ denote the set of n-step orthogonal random ε-walks $\sigma\colon [0, n\varepsilon] \to \overline{M}$ such that $\sigma|_{[0,\varepsilon]}$ is the unit speed geodesic beginning at p in one of the directions $\pm v_p, \pm J v_p$, where J is the almost complex structure for M, and $\sigma|_{[k\varepsilon,(k+1)\varepsilon]}$ is the unit speed geodesic in $\overline{M}$ beginning at $\sigma(k\varepsilon)$ in one of the directions $\pm\sigma'(k\varepsilon), \pm J\sigma'(k\varepsilon)$, $0 \le k \le n-1$. These sets of random walks produce, as in the planar domain case, a limiting hitting measure on ∂M, which is independent of the initial choice of v_p, and by the previous arguments, is equal to the harmonic measure μ_p.

From the above discussion, it easily follows that a Riemannian manifold without boundary is recurrent precisely when almost all Brownian paths are dense in the manifold. Also, parabolicity and harmonic measure are closely related, as states the following elementary result.

Proposition 7.1.3 *Let (M,g) be a Riemannian manifold with $\partial M \neq \emptyset$. Then, the following statements are equivalent:*

1. *M is parabolic.*
2. *There exists a point $p \in \mathrm{Int}(M)$ such that the harmonic measure μ_p is full, i.e. $\int_{\partial M} \mu_p = 1$.*
3. *Given any $p \in \mathrm{Int}(M)$ and any bounded harmonic function $f\colon M \to \mathbb{R}$, then $f(p) = \int_{\partial M} f\, \mu_p$.*
4. *The universal covering of M is parabolic.*

Furthermore, if there exists a proper, non-negative superharmonic function on M, then M is parabolic. When M is simply-connected and two-dimensional, then the existence of such a function is equivalent to being parabolic.

We note that the parabolicity of a Riemannian manifold with boundary is not affected by adding compact sets or by removing interiors of compact sets, and if a manifold M can be decomposed as the union of two parabolic domains with compact intersection, then M is parabolic (or recurrent, depending on whether ∂M is empty or not). Since the closed unit disk $\overline{\mathbb{D}}$ is parabolic and $x \in \mathbb{R}^2 - \mathbb{D} \mapsto \log |x|$ is a proper, non-negative harmonic function, then it follows that $\mathbb{R}^2 - \mathbb{D}$ is parabolic and $\mathbb{R}^2$ is recurrent. On the other hand, for $n \in \mathbb{N}$, $n \geq 3$, the function $x \in \mathbb{R}^n - \{|x| < 1\} \mapsto |x|^{2-n}$ is bounded and harmonic with constant boundary values. So, $\mathbb{R}^n - \{|x| < 1\}$ is not parabolic and $\mathbb{R}^n$ is transient if $n \geq 3$.

Note that if $h\colon M \to \mathbb{R}$ is a non-constant, positive harmonic function on a recurrent Riemannian manifold, then for any positive regular value $t \in \mathbb{R}$ of h, the closed subset $M_t = h^{-1}((0,t])$ is parabolic and $h|_{M_t}$ is a bounded harmonic function with constant boundary value t. By Proposition 7.1.3, $h|_{M_t}$ is the constant function t, which contradicts that t is a regular value of h. This contradiction completes the proof of the following well-known result.

Proposition 7.1.4 (Liouville Theorem) *Every positive harmonic function on a recurrent Riemannian manifold is constant.*

7.2 Universal superharmonic functions.

As we mentioned before, the knowledge of the conformal type of a minimal surface M is crucial when tackling uniqueness questions. Sometimes it is useful to decompose M in pieces, and study the conformal structure of each piece as a Riemann surface with boundary. For instance, the proof by Meeks and Rosenberg of the uniqueness of the helicoid rests of the non-trivial fact[2] that a simply-connected, properly embedded minimal surface $M \subset \mathbb{R}^3$ must admit a plane which intersects M transversely in a single proper arc γ. Each of the two closed complements of γ in M is contained in a closed half-space. Hence, from Theorem 7.2.3 below, both halves are parabolic, and one then uses this information to prove that M is conformally $\mathbb{C}$. A key ingredient in the proof of Theorem 7.2.3 is the construction of a function defined on a certain region of space, whose restriction to any minimal surface contained in that region is superharmonic. This construction leads us to the following definition.

[2]See the third paragraph in the sketch of the proof of Theorem 1.0.1 in Chapter 8.

Definition 7.2.1 Given a region $W \subset \mathbb{R}^3$, a function $h\colon W \to \mathbb{R}$ is said to be a *universal superharmonic function on* W if its restriction to any minimal surface $M \subset W$ is superharmonic.

Examples of universal superharmonic functions on all of $\mathbb{R}^3$ include coordinate functions such as x_1 or the function $-x_1^2$. Collin, Kusner, Meeks and Rosenberg proved the following useful inequality (Lemma 2.2 in [39]) valid for any immersed minimal surface in $\mathbb{R}^3$:

$$|\Delta \ln r| \leq \frac{|\nabla x_3|^2}{r^2} \qquad \text{in } M - (x_3\text{-axis}), \tag{7.1}$$

where $r = \sqrt{x_1^2 + x_2^2}$ and ∇, Δ denote the intrinsic gradient and laplacian on M. Using the estimate (7.1), it is easy to check that the following statements hold.

Lemma 7.2.2

i) The function $\ln r - x_3^2$ *is a universal superharmonic function in the region* $\{r^2 \geq \frac{1}{2}\}$.

ii) The function $\ln r - x_3 \arctan x_3 + \frac{1}{2}\ln(x_3^2 + 1)$ *is a universal superharmonic function in the region* $\{r^2 \geq x_3^2 + 1\}$.

With the above lemma, we now prove that any properly immersed minimal surface M in a closed half-space of $\mathbb{R}^3$ is either recurrent (if M has no boundary) or a parabolic surface with boundary. Let M be a properly immersed minimal surface in a closed half-space. If $\partial M = \emptyset$, then M is planar by the Half-space Theorem 2.6.4 (and hence, M is recurrent); if $\partial M \neq \emptyset$, then M is parabolic by a direct application of the following general theorem. Also note that in the case $\partial M = \emptyset$, the planarity of M also follows from the next result, since on recurrent surfaces positive harmonic functions are constant (Proposition 7.1.4).

Theorem 7.2.3 (Collin, Kusner, Meeks, Rosenberg [39]) *Let M be a connected, non-planar, properly immersed minimal surface in $\mathbb{R}^3$, possibly with boundary. Then, every component of the intersection of M with a closed half-space is a parabolic surface with boundary. In particular, if M has empty boundary and intersects some plane in a compact set, then M is recurrent.*

Proof. Up to a rotation, it suffices to check that any component C of $M(+) = M \cap \{x_3 \geq 0\}$ is parabolic. For fixed $n \in \mathbb{N}$, let $C_n = C \cap x_3^{-1}([0, n])$. By part *i)* of Lemma 7.2.2, the function $h = \ln r - x_3^2$ is superharmonic and proper when restricted to $C_n \cap \{r^2 \geq \frac{1}{2}\}$. Furthermore, h is positive outside a compact domain on C_n, which by Proposition 7.1.3 implies that

$C_n \cap \{r^2 \geq \frac{1}{2}\}$ is parabolic. Since M is proper and $\{r^2 \leq \frac{1}{2}\} \cap \{0 \leq x_3 \leq n\}$ is compact, we deduce that $C_n - \{r^2 > \frac{1}{2}\}$ is a compact subset of C_n. Since parabolicity is not affected by adding compact subsets, it follows that C_n is parabolic.

We now check that C is parabolic. Fix a point $p \in C$ with $x_3(p) > 0$ and let μ_p^C be the harmonic measure of C with respect to p. For n large enough, p lies in the interior of C_n. Since x_3 is a bounded harmonic function on the parabolic surface C_n, part *3* of Proposition 7.1.3 insures that

$$x_3(p) = \int_{\partial C_n} x_3 \, \mu_p^n \geq n \int_{\partial C_n \cap x_3^{-1}(n)} \mu_p^n,$$

where μ_p^n is the harmonic measure of C_n with respect to p. Since μ_p^n is full on ∂C_n, it follows that

$$\int_{\partial C_n - x_3^{-1}(n)} \mu_p^n = 1 - \int_{\partial C_n \cap x_3^{-1}(n)} \mu_p^n \geq 1 - \frac{x_3(p)}{n} \stackrel{(n\to\infty)}{\longrightarrow} 1.$$

Suppose now that M and N are Riemannian manifolds with $M \subset N$, ∂ is a component of $\partial M \cap \partial N, p \in \mathrm{Int}(M)$ and μ_p^M and μ_p^N are the respective harmonic measures. Then it follows immediately from the definition of harmonic measure that $\int_\partial \mu_p^M \leq \int_\partial \mu_p^N \leq 1$. By letting $M = C_n$, $N = C$ and $\partial = \partial C_n - x_3^{-1}(n)$, the above displayed inequality implies that $\lim_n \int_{\partial C_n - x_3^{-1}(n)} \mu_p^C \geq 1$, from where we conclude that $\int_{\partial C} \mu_p^C = 1$ and the proof is complete. $\square$

There are several problems and results about parabolicity of properly immersed minimal surfaces with boundary in $\mathbb{R}^3$ (see Conjecture 17.0.41 below). Concerning these problems, it is worth mentioning the following ones due to Neel.

Theorem 7.2.4

1. *(Neel [169]) Let M be a properly embedded minimal surface of bounded curvature in $\mathbb{R}^3$. Then, M has no non-constant bounded harmonic functions.*

2. *(Neel [168]) Let M be a non-flat complete, embedded minimal surface with boundary in $\mathbb{R}^3$. Assume that, outside of a compact subset of M, the Gauss map image of M is contained in an open, hyperbolic subset of $\mathbb{S}^2$. Then, M is parabolic. In particular, every minimal graph with boundary over a domain in a plane is parabolic.*

7.3 Quadratic area growth and middle ends.

In this section, we will sketch the proof of a theorem by Collin, Kusner, Meeks and Rosenberg that constrains both the geometry and the topology of any properly embedded minimal surface in $\mathbb{R}^3$ with more than one end. This theorem has been used in an essential way by Meeks, Pérez and Ros in the proofs of their classification results in Theorems 1.0.2, 1.0.6 and 12.2.1, as well as by Frohman and Meeks in their proof of the Topological Classification Theorem 1.0.7.

The Ordering Theorem in Chapter 6 represents the first step in trying to understand the geometry of a properly embedded minimal surface in $\mathbb{R}^3$ with more than one end. By the proof of the Ordering Theorem, every middle end of such a surface $M \subset \mathbb{R}^3$ with horizontal limit tangent plane at infinity can be represented by a proper subdomain $E \subset M$ with compact boundary such that E "lies between two half-catenoids". This means that E is contained in a neighborhood S of the (x_1, x_2)-plane, S being topologically a slab, whose width grows at most logarithmically with the distance from the origin. Next we explain how this constraint on a middle end representative gives information about its area growth.

Suppose for the moment that E is contained in the region $W = \{(x_1, x_2, x_3) \mid r \geq 1,\ 0 \leq x_3 \leq 1\}$, where $r = \sqrt{x_1^2 + x_2^2}$. In Chapter 7.2, we defined the universal superharmonic function $\ln r - x_3^2$ in W. In particular, its restriction $f\colon E \to \mathbb{R}$ is superharmonic and proper. Suppose $f(\partial E) \subset [-1, c]$ for some $c > 0$. Replace E by $f^{-1}[c, \infty)$ and let $E(t) = f^{-1}[c, t]$ for $t > c$. Assuming that both c, t are regular values of f, the Divergence Theorem gives

$$\int_{E(t)} \Delta f \, dA = -\int_{f^{-1}(c)} |\nabla f| \, ds + \int_{f^{-1}(t)} |\nabla f| \, ds,$$

where ∇, Δ are the intrinsic gradient and laplacian on M, and dA, ds denote the corresponding area and length elements.

Since f is superharmonic, the function $t \mapsto \int_{E(t)} \Delta f \, dA$ is monotonically decreasing and bounded from below by $-\int_{f^{-1}(c)} |\nabla f| \, ds$. In particular, Δf lies in $L^1(E)$. Furthermore, $|\Delta f| = |\Delta \ln r - 2|\nabla x_3|^2| \geq -|\Delta \ln r| + 2|\nabla x_3|^2$. By estimate (7.1) in Chapter 7.2, we have $|\Delta f| \geq \left(2 - \frac{1}{r^2}\right) |\nabla x_3|^2$. Since $r^2 \geq 1$ in W, it follows $|\Delta f| \geq |\nabla x_3|^2$ and thus, both $|\nabla x_3|^2$ and $\Delta \ln r$ are in $L^1(E)$. This implies that outside of a subdomain of E of finite area, E can be assumed to be as close to being horizontal as one desires, and in particular, for the radial function r on this horizontal part of E, $|\nabla r|$ is almost equal to 1.

Let $r_0 = \max r|_{\partial E}$. With a slight abuse of notation, redefine $E(t)$ to be the subdomain of E that lies inside the region $\{r_0^2 \leq x_1^2 + x_2^2 \leq t^2\}$. Since

$$\int_{E(t)} \Delta \ln r \, dA = -\int_{r=r_0} \frac{|\nabla r|}{r} ds + \int_{r=t} \frac{|\nabla r|}{r} ds = \text{ const. } + \frac{1}{t} \int_{r=t} |\nabla r| \, ds$$

and $\Delta \ln r \in L^1(E)$, then the following limit exists:

$$\lim_{t \to \infty} \frac{1}{t} \int_{r=t} |\nabla r| \, ds = C \tag{7.2}$$

for some positive constant C. Thus, $t \mapsto \int_{r=t} |\nabla r|\, ds$ grows at most linearly as $t \to \infty$. By the coarea formula, for t_1 fixed and large,

$$\int_{E\cap\{t_1\leq r\leq t\}} |\nabla r|^2\, dA = \int_{t_1}^{t} \left(\int_{r=\tau} |\nabla r|\, ds \right) d\tau; \tag{7.3}$$

hence, $t \mapsto \int_{E\cap\{t_1\leq r\leq t\}} |\nabla r|^2\, dA$ grows at most quadratically as $t \to \infty$. Finally, since outside of a domain of finite area E is arbitrarily close to horizontal and $|\nabla r|$ is almost equal to one, we conclude that the area of $E \cap \{r \leq t\}$ grows at most quadratically as $t \to \infty$. In fact, from (7.2) and (7.3) it follows that

$$\int_{E\cap\{r\leq t\}} dA = \frac{C}{2} t^2 + o(t^2),$$

where $t^{-2}o(t^2) \to 0$ as $t \to \infty$.

We now check that the constant C must be an integer multiple of 2π. Since the area of $E \cap \{r \leq t\}$ grows at most quadratically in t and $E \cap \{r \leq t\}$ is contained in a fixed size horizontal slab, geometric measure theory insures that the locally finite, minimal integral varifolds associated to the homothetically shrunk surfaces $\frac{1}{n}E$ converge as $n \to \infty$ to a locally finite, minimal integral varifold V with empty boundary, which is supported on the (x_1, x_2)-plane. Since V is an integer multiple of the (x_1, x_2)-plane, C must be an integer multiple of 2π.

In the case that the end E "lies between half-catenoids", a similar analysis (see [39] for details) using the universal superharmonic function $\ln r - c(x_3 \arctan(x_3) - \frac{1}{2}\ln(x_3^2+1))$, for some $c > 0$, shows that $E \cap \{r \leq t\}$ has area growth of the order $m\pi t^2$ for some $m \in \mathbb{N}$. This in turn implies that E has area growth $m\pi R^2$ where $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$, in the sense that

$$\lim_{R\to\infty} \frac{\text{Area}(E \cap \mathbb{B}(R))}{\pi R^2} = m,$$

where as usual, $\mathbb{B}(R) = \{x \in \mathbb{R}^3 \mid \|x\| < R\}$. This concludes our sketch of the proof that the middle ends of M have representatives with quadratic area growth.

By the Monotonicity Formula for area (Theorem 2.6.2), every end representative of a minimal surface must have asymptotic area growth at least equal to half of the area growth of a plane. Since we have just checked that the middle ends of a properly embedded minimal surface have representatives with at most quadratic area growth, then we can find representatives for middle ends which have exactly one end, which means they are never limit ends. This discussion gives the next theorem and we refer the reader to [39] for further details.

Theorem 7.3.1 (Collin, Kusner, Meeks, Rosenberg [39]) *Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with more than one end and horizontal limit tangent plane at infinity. Then, any limit end of M must be a top or bottom end of M. In particular, M can have at most two limit ends, each middle end is simple and the number of ends of M is countable. Furthermore, for each middle end E of M there exists a positive integer $m(E)$ such that*

$$\lim_{R\to\infty} \frac{\mathrm{Area}(E \cap \mathbb{B}(R))}{\pi R^2} = m(E).$$

The parity of $m(E)$ is called the parity of the middle end E.

Remark 7.3.2

1. Theorem 7.3.1 and the Ordering Theorem (Theorem 6.0.11) are crucial tools in the topological classification of properly embedded minimal surfaces in $\mathbb{R}^3$ by Frohman and Meeks [63], see Chapter 13.

2. Embeddedness is crucial in showing that the surface M in the last theorem has a countable number of ends, as follows from the construction by Alarcón, Ferrer and Martín [1] of a genus-zero, properly immersed minimal surface in $\mathbb{R}^3$ whose space of ends is a Cantor set. In particular, every end of their surface is a limit end.

Collin, Kusner, Meeks and Rosenberg [39] were also able to use universal superharmonic functions to control the geometry of properly embedded minimal surfaces with exactly two limit ends. Their proof of the following theorem is motivated by the proof of a similar theorem by Callahan, Hoffman and Meeks [16] in the classical singly-periodic setting.

Theorem 7.3.3 (Collin, Kusner, Meeks, Rosenberg [39]) *Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with two limit ends and horizontal limit tangent plane at infinity. Then, there exists a proper collection $\{P_n \mid n \in \mathbb{Z}\}$ of horizontal planes in $\mathbb{R}^3$, ordered by heights, such that each plane P_n intersects M transversely in a finite number of simple closed curves. Furthermore, the closed slab S_n bounded by $P_n \cup P_{n+1}$ intersects M in a non-compact domain which represents the n-th end of M. In particular, by Theorem 7.2.3, M is recurrent.*

8 Uniqueness of the helicoid I: proper case.

The goal of this chapter is to outline the arguments by Meeks and Rosenberg [148] in their proof of the uniqueness of the helicoid, which is Theorem 1.0.4 stated in the Introduction. We remark that this result is stated under the assumption that the surface in question is proper, instead of merely complete. We will postpone the proof of this more general version (i.e. in the complete setting, see Theorem 1.0.1) to Chapter 10.1.

Sketch of the proof of Theorem 1.0.1, assuming properness.

Let M be a properly embedded, non-flat, simply-connected, minimal surface in $\mathbb{R}^3$. Consider any sequence of positive numbers $\{\lambda_n\}_n$ which decays to zero and let $M(n) = \lambda_n M$ be the surface scaled by λ_n. By the Limit Lamination Theorem for Disks (Theorem 4.1.2), a subsequence of these surfaces converges on compact subsets of $\mathbb{R}^3$ to a minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by parallel planes, with singular set of C^1-convergence $S(\mathcal{L})$ being a Lipschitz curve that can be parameterized by the height over the planes in $\mathcal{L}$. Part of the proof of Theorem 4.1.2 depends on a unique extension result for the forming multigraphs, which in our case implies that for n large, the almost flat multigraph which starts to form on $M(n)$ near the origin extends all the way to infinity. From this extension result, one can deduce that the limit foliation $\mathcal{L}$ is independent of the sequence $\{\lambda_n\}_n$. After a rotation of M and replacement of the $M(n)$ by a subsequence, we can suppose that the $M(n)$ converge C^1 to the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes, on compact subsets outside of the singular set of convergence given by a Lipschitz curve $S(\mathcal{L})$ parameterized by its x_3-coordinate. Note that this last property implies that $S(\mathcal{L})$ intersects each horizontal plane exactly once.

The next sketched argument shows that M intersects transversely each of the planes in $\mathcal{L}$. After a translation, we may assume that the origin lies in M. Since the origin is a singular point of convergence of the $M(n)$ to $\mathcal{L}$ and $S(\mathcal{L})$ is Lipschitz, it follows that $S(\mathcal{L})$ passes through the origin and is contained in the solid cone $\mathcal{C}_\varepsilon = \{x_3^2 \geq \varepsilon^2(x_1^2 + x_2^2)\}$, where $\varepsilon > 0$ only depends on the one-sided curvature estimate in Theorem 4.1.3. Let Δ be the solid cylinder $\{x_1^2 + x_2^2 \leq 1,\ |x_3| \leq \varepsilon\}$. The two flat horizontal multigraphs $M_1(n), M_2(n)$ referred to in item 2 of Theorem 4.1.2 intersect the cylindrical sides of $\partial\Delta$ almost orthogonally along two long spiraling arcs which are multigraphs over the unit circle $\mathbb{S}^1$ in the (x_1, x_2)-plane, possibly together with a finite number of open arcs, starting and finishing at the top (resp. bottom) planar disks of $\partial\Delta$, which are graphs over their projections in $\mathbb{S}^1$. Both spirals lie on the compact component $D(n)$ of $M(n) \cap \Delta$ which contains the origin. After a slight perturbation $\Delta(n)$ of Δ near the top and bottom boundary disks of $\partial\Delta$ and replacing Δ by $\Delta(n)$, it can be shown that the boundary of $D(n)$ consists of the two spiraling arcs on the boundary of the cylinder together with two arcs which connect them, one on each of the boundary disks in $\partial\Delta(n)$; in this replacement the new top and bottom disks in $\partial\Delta(n)$ are minimal although not necessarily flat. Without much difficulty, one can extend the top and bottom disks of $\Delta(n)$ to an almost-horizontal, product minimal foliation of $\Delta(n)$ by graphical minimal disks, such that every boundary circle of these (not necessarily planar) disks intersects each of the two spiraling curves in $\partial D(n)$ at a single point. Since at a point of tangency, two minimal surfaces intersect hyperbolically (negative index), Morse theory implies that each leaf of the minimal disk foliation of $\Delta(n)$ intersects $D(n)$ transversely in a simple arc. When $n \to \infty$, these foliations converge to the restricted foliation $\mathcal{L} \cap \Delta$ by flat horizontal disks. This last statement, together with the openness of the Gauss map of the original surface M, implies that M is transverse to $\mathcal{L}$, as desired. Therefore, the stereographical projection of the Gauss map $g\colon M \to \mathbb{C} \cup \{\infty\}$ can be expressed as $g(z) = e^{f(z)}$ for some holomorphic function $f\colon M \to \mathbb{C}$.

The next step in the proof is to check that the conformal structure of M is $\mathbb{C}$, and that its height differential can be written as $dh = dz$, $z \in \mathbb{C}$. This part of the proof is longer and delicate, and depends on a finiteness result for the number of components of a minimal graph over a possibly disconnected, proper domain in $\mathbb{R}^2$ with zero boundary values[1]. Note that the non-existence of points in M with vertical normal vector implies that the intrinsic gradient of $x_3\colon M \to \mathbb{R}$ has no zeros on M. Through a series of geometric and analytic arguments that use both the double multigraph convergence of the $M(n)$ to $\mathcal{L}$ outside the cone $\mathcal{C}_\varepsilon$ and the above finiteness result for graphs over proper domains of $\mathbb{R}^2$, one eventually proves that none of the integral curves of ∇x_3 are asymptotic to a plane in $\mathcal{L}$, and

[1] We will discuss in more detail this finiteness property in Chapter 17, see Conjecture 17.0.18.

that every such horizontal plane intersects M transversely in a single proper arc. Then a straightforward argument using Theorem 7.2.3 or applying Corollary 9.1.4 below implies M is recurrent, and thus M is conformally $\mathbb{C}$. The non-existence of points in M with vertical normal vector and the connectedness of its horizontal sections force the height differential to be $dh = dx_3 + i\,dx_3^* = dz$ in a conformal parametrization of M. In particular, the third coordinate $x_3 \colon \mathbb{C} \to \mathbb{R}$ is $x_3(z) = \Re(z)$, the real part of z.

To finish the proof, it only remains to determine the Gauss map g of M. Recall that we have already indicated how to show that $g(z) = e^{f(z)}$. If the holomorphic function $f(z)$ is a linear function of the form $az + b$, then the Weierstrass data $(e^{f(z)}, dz)$ for M shows that M is an associate surface to the helicoid. Since none of the non-trivial associate surfaces to the helicoid are injective as mappings, then M must be the helicoid itself when $f(z)$ is linear. Thus, it remains to show that $f(z)$ is linear. The formula (2.7) in Chapter 2.2 for the Gaussian curvature K and an application of Picard's Theorem imply $f(z)$ is linear if and only if M has bounded curvature. This fact completes the sketch of proof of the theorem in the special case that K is bounded. However, Theorem 4.1.2 and a clever blow-up argument on the scale of curvature reduces the proof that $f(z)$ is linear in the general case to the case where K is bounded, and so M is a helicoid. For further details, see [148]. $\square$

Remark 8.0.4 The first two main steps in the above outlined proof (the second and third paragraphs) can be simplified using the fact that any embedded minimal multigraph with a large number of sheets contains a sub-multigraph which can be approximated by the multigraph of a helicoid with an additional logarithmic term, an approximation result which appears as Corollary 14.3 in Colding and Minicozzi [30]. Bernstein and Breiner [5], as well as Meeks and Pérez [126], verified that for the double multigraph appearing in the above sketched proof, this approximation property together with estimates by Colding and Minicozzi for the separation of the sheets of the multigraph, is sufficient to prove that the surface M intersects each horizontal plane transversely in a single proper arc, which as we observed above, is a delicate step in the proof of Theorem 1.0.1. A important detail is that this simplification does not need the hypothesis of simply-connectedness and works in the case of proper minimal annuli with one compact boundary curve and infinite total curvature, giving that such an annulus intersects transversely some horizontal plane in a single proper arc. For details, see [5, 4, 126] and see also Chapter 9.2 below.

Theorem 1.0.1 solves a long standing conjecture about the uniqueness of the helicoid among properly embedded, simply-connected minimal surfaces in $\mathbb{R}^3$. Very little is known about genus-g helicoids with $g \geq 1$. We have already mentioned Theorem 1.0.4 on the asymptotic behavior and conformal structure of every such minimal surface. It is conjectured that the genus-one helicoid of Hoffman, Karcher and Wei [76, 77], discussed in Chapter 2.5, is the unique complete, embedded minimal example with this topology. Also, Hoffman and White [86] have given a new approach to the proof of the existence of the genus-one helicoid, which holds the promise of proving the existence of a genus-g helicoid for any $g \in \mathbb{N}$. It is expected (Conjecture 17.0.20) that for every non-negative integer g, there exists a unique non-planar, properly embedded minimal surface in $\mathbb{R}^3$ with genus g and one end.

Embedded minimal annular ends with infinite total curvature. 9

In this chapter we will explain the proof of Theorem 1.0.4 stated in the Introduction, and analyze the more general problem of describing the asymptotic geometry of properly embedded minimal annular ends with compact boundary and infinite total curvature in $\mathbb{R}^3$. Crucial in this description is the understanding of the conformal structure of such annular ends, a goal which will be tackled in the next section.

9.1 Harmonic functions on annuli.

A detailed exposition of the results in this section can be found in the paper by Meeks and Pérez [127].

Definition 9.1.1 Given a non-constant harmonic function $f\colon M \to \mathbb{R}$ on a non-compact Riemann surface with compact boundary, we say that an annular end E of M has *finite type for* f if for some $t_0 \in \mathbb{R}$, the one-complex $f^{-1}(t_0) \cap E$ has a finite number of ends.

Note that $f^{-1}(t_0)$ might fail to be smooth: at each critical point p of f lying in $f^{-1}(t_0)$, this level set consists locally of an equiangular system of curves crossing at p; also note that $f^{-1}(t_0)$ cannot bound a compact subdomain in $E - \partial E$ by the maximum principle. These observations imply that $f^{-1}(t_0) \cap E$ has a finite number of ends if and only if $f^{-1}(t_0) \cap E$ has a finite number of components and a finite number of crossing points.

When E is an annular end of finite type for f, we will state several results on the level sets of $f|_E$ depending on the conformal type of E. In order to describe these results, we first fix some notation. For $R \in [0, 1)$,

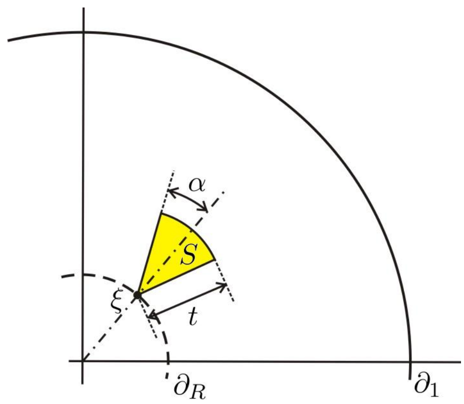

Figure 9.1: Angular sector in $A(R,1)$, centered at $\xi \in \partial_R$.

let $A(R,1) = \{z \in \mathbb{C} \mid R < |z| \leq 1\}$, $\overline{A}(R,1) = \{z \in \mathbb{C} \mid R \leq |z| \leq 1\}$, $\partial_R = \{t \in \mathbb{C} \mid |z| = R\}$ and $\partial_1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Thus, the closure $\overline{A}(0,1)$ is the closed unit disk $\overline{\mathbb{D}}$ and $A(0,1) = \overline{\mathbb{D}} - \{0\}$.

Definition 9.1.2 If $0 < R < 1$ and $F\colon A(R,1) \to \mathbb{C}$ is a holomorphic function, then F is said to have *angular limit* $F(\xi) \in \mathbb{C} \cup \{\infty\}$ *at* $\xi \in \partial_R$ if $\lim_{z\to\xi, z\in S} F(z)$ exists and equals $F(\xi)$ for every angular sector $S \subset A(R,1)$ centered at ξ, whose median line is the radial arc $[1,2]\xi$, with small radius $t \in (0, 1-R)$ and total angle 2α, $0 < \alpha < \frac{\pi}{2}$, see Figure 9.1. This definition of having angular limit can be directly extended to (real valued) harmonic functions.

We can now state the main result of this section.

Theorem 9.1.3 *Suppose $f\colon M \to \mathbb{R}$ is a non-constant harmonic function and E is an annular end of M of finite type for f. Then:*

(1) If E is conformally $A(0,1)$, then the holomorphic one-form $\omega = df + i\,df^*$ on $A(0,1)$ extends to a meromorphic one-form $\widetilde{\omega}$ on the closed unit disk $\overline{\mathbb{D}}$ (here f^* denotes the locally defined conjugate harmonic function of f). Furthermore, if for some $t_0 \in \mathbb{R}$ the one-complex $f^{-1}(t_0) \cap E$ has $2k$ ends[1] and $\widetilde{\omega}$ has a pole at $z = 0$, then this pole is of order $k+1$. In particular, for every $t \in \mathbb{R}$, the level set $f^{-1}(t)$ has exactly $2k$ ends.

(2) If E is conformally $A(R,1)$ for some $R \in (0,1)$, then:

[1]Note that the number of ends of $f^{-1}(t_0) \cap E$ is always even, since f alternates values greater and smaller than t_0 in consecutive components of the complement of $f^{-1}(t_0) \cap E$ around $z = 0$.

(a) *f has angular limits almost everywhere on ∂_R.*

(b) *Given $t_0 \in \mathbb{R}$ such that $f^{-1}(t_0) \cap E$ has a finite number of ends, then the limit set of each end of $f^{-1}(t_0) \cap E$ consists of a single point in ∂_R. In particular, the closure in $A(R,1)$ of at least one component of $\{z \in A(R,1) \mid f(z) \neq t_0\}$ is hyperbolic.*

In the special case that the flux $\int_{\partial_1} \frac{\partial f}{\partial r}\, ds$ vanishes, then item 2 of Theorem 9.1.3 was proven earlier by Meeks and Rosenberg, see Theorem 7.1 in [150]. This special case occurs when the harmonic conjugate function f^* of f is well-defined on E. In fact, the proof of Theorem 9.1.3 is a refinement of the ideas in this special case in [150], and is based on an analysis of the possibly multivalued holomorphic function $f + if^*$ using methods from the classical theory of holomorphic functions of one complex variable.

The following corollary contains the information needed to control the conformal structure of a properly embedded minimal end of infinite total curvature and compact boundary in $\mathbb{R}^3$, whose classification will be the goal of Chapter 9.2 below.

Corollary 9.1.4 *Suppose $X = (x_1, x_2, x_3)\colon A(R,1) \to \mathbb{R}^3$ is a proper, conformal minimal immersion. If for some horizontal plane $P \subset \mathbb{R}^3$, the one-complex $X^{-1}(P)$ has a finite number $2k$ of ends, then $R = 0$ and the holomorphic height differential $dx_3 + idx_3^*$ extends meromorphically to $\overline{\mathbb{D}}$ with a pole of order $k+1$. In particular, for every horizontal plane $P' \subset \mathbb{R}^3$, the level set $X^{-1}(P')$ has exactly $2k$ ends.*

Furthermore, after replacing $A(0,1)$ by $A(0,R') = \{z \in \mathbb{C} \mid 0 < |z| \leq R'\}$ for some $R' \in (0,1)$, the Gauss map of the induced minimal immersion $X|_{A(0,R')}$ is never vertical. Hence, on $A(0,R')$, the meromorphic Gauss map of X can be expressed as $g(z) = z^k e^{H(z)}$ for some $k \in \mathbb{Z}$ and for some holomorphic function $H\colon A(0,R') \to \mathbb{C}$.

Regarding the proof of Corollary 9.1.4, we will only mention that the hyperbolic case appearing in item 2 of Theorem 9.1.3 cannot hold under the hypotheses of the corollary, by direct application of Theorem 7.2.3.

Remark 9.1.5 A direct consequence of Corollary 9.1.4 and of Picard's Theorem is that a proper conformal minimal immersion satisfying the hypotheses of Corollary 9.1.4 has finite total curvature if and only if there exist two non-parallel planes P_1, P_2 such that each of these planes individually intersects the surface transversely in a finite number of immersed curves.

9.2 Annular minimal ends of infinite total curvature.

In Chapter 8 we outlined the proof by Meeks and Rosenberg of the uniqueness of the helicoid in the proper case, and in Remark 8.0.4 we mentioned how some difficult parts of the proof could be simplified using results of Colding and Minicozzi in [30], which also can be used to obtain the asymptotic behavior and conformal structure properties stated in Theorem 1.0.4. In this section we will give some ideas about the proof of these properties and we will also consider the more general problem of describing the asymptotic behavior, conformal structure and analytic representation of an annular end of any properly embedded minimal surface M in $\mathbb{R}^3$ with compact boundary and finite topology. We remark that the results obtained in this section will be improved in Chapter 10 by replacing the hypothesis of properness by the one of completeness, see Remark 10.1.5. For detailed proofs of the results in this section, see Meeks and Pérez [126].

Definition 9.2.1 For $R > 0$, let $D(\infty, R) = \{z \in \mathbb{C} \mid |z| \geq R\}$. Note that $D(\infty, R)$ is conformally equivalent to the domain $A(0,1)$ defined in the previous section.

Next we outline how to construct examples $E_{a,b}$ of complete embedded minimal annuli depending on parameters $a \in [0, \infty)$, $b \in \mathbb{R}$ so that their flux vector is $(a, 0, -b)$ and their total curvature is infinite. Such examples will serve as models for the asymptotic geometry of every properly embedded minimal end with infinite total curvature and compact boundary. This construction will be performed by using the Weierstrass representation (g, dh) where g is the Gauss map and dh the height differential. We first note that after an isometry in $\mathbb{R}^3$ and a possible change of orientation, we can assume that $b \geq 0$. We consider three separate cases.

1. If $a = b = 0$, then define $g(z) = e^{iz}$, $dh = dz$, which produces the end of a vertical helicoid.
2. If $a \neq 0$ and $b \geq 0$ (i.e. the flux vector is not vertical), we choose
$$g(z) = t\, e^{iz} \frac{z - A}{z}, \qquad dh = \left(1 + \frac{B}{z}\right) dz, \qquad z \in D(\infty, R), \quad (9.1)$$
where $B = \frac{b}{2\pi}$, and the parameters $t > 0$ and $A \in \mathbb{C} - \{0\}$ are to be determined (here $R > |A|$). Note that with this choice of B, the imaginary part of $\int_{\{|z|=R\}} dh$ is $-b$ because we use the orientation of $\{|z| = R\}$ as the boundary of $D(\infty, R)$.
3. In the case of vertical flux, i.e. $a = 0$ and $b > 0$, we take
$$g(z) = e^{iz} \frac{z - A}{z - \overline{A}}, \qquad dh = \left(1 + \frac{B}{z}\right) dz, \quad z \in D(\infty, R), \qquad (9.2)$$
where $B = \frac{b}{2\pi}$ and $A \in \mathbb{C} - \{0\}$ is to be determined (again $R > |A|$).

Note that in each of the three cases above, g can rewritten as $g(z) = t\, e^{iz+f(z)}$ where $f(z)$ is a well-defined holomorphic function in $D(\infty, R)$ with $f(\infty) = 0$. In particular, the differential $\frac{dg}{g}$ extends meromorphically through the puncture at ∞. The same extendability holds for dh. These properties will be collected in the next definition.

Definition 9.2.2 *A complete minimal surface $M \subset \mathbb{R}^3$ with Weierstrass data (g, dh) is said to be of finite type if M is conformally diffeomorphic to a finitely punctured, compact Riemann surface $\overline{M}$ and after a possible rotation, both dg/g, dh extend meromorphically to $\overline{M}$.*

This notion was first introduced by Rosenberg [197] and later studied by Hauswirth, Pérez and Romon [70].

Coming back to our discussion about the canonical ends $E_{a,b}$, it turns out that in any of the cases 2, 3 above, given $B = \frac{b}{2\pi}$, there exist parameters $t > 0$, $A \in \mathbb{C} - \{0\}$ (actually $t = 1$ in case 3) so that the Weierstrass data given by (9.1), (9.2) solve the corresponding period problem (the only period to be killed is the first equation in (2.5) along $\{|z| = R\}$). This period problem can be solved by explicitly computing the period of (g, dh) along $\{|z| = R\}$ and applying an intermediate value argument. At the same time, one can calculate the flux vector F of the resulting minimal immersion and prove that its horizontal component covers all possible values. This defines for each $a, b \in [0, \infty)$ a complete immersed minimal annulus with infinite total curvature and flux vector $(a, 0, -b)$.

Definition 9.2.3 With the notation above, we will call the end $E_{a,b}$ a *canonical end*[2].

Remark 9.2.4 In case 3 above, it is easy to check that the conformal map $z \overset{\Phi}{\mapsto} \overline{z}$ in the parameter domain $D(\infty, R)$ of $E_{0,b}$ satisfies $g \circ \Phi = 1/\overline{g}$, $\Phi^* dh = \overline{dh}$. Hence, after translating the surface so that the image of the point $R \in D(\infty, R)$ lies on the x_3-axis, we deduce that Φ produces an isometry of $E_{0,b}$ which extends to a 180-rotation of $\mathbb{R}^3$ around the x_3-axis; in particular, $E_{0,b} \cap (x_3\text{-axis})$ contains two infinite rays.

To understand the geometry of the canonical end $E_{a,b}$ and in particular prove that it is embedded if the radius R is taken large enough, it will be convenient to analyze its multigraph structure, which is the purpose of the next result. The image in Figure 9.2 describes how the flux vector $(a, 0, -b)$ of $E_{a,b}$ influences its geometry. In the sequel we will use the notation $u(\rho, \theta)$ for multigraphs as introduced in Definition 4.1.1. Note that the separation function $w(\rho, \theta)$ used below refers to the vertical separation between the two disjoint multigraphs Σ_1, Σ_2 appearing in the next result (versus the

[2]In spite of the name "canonical end", we note that the choice of $E_{a,b}$ depends on the explicit parameters t, A in equations (9.1) and (9.2).

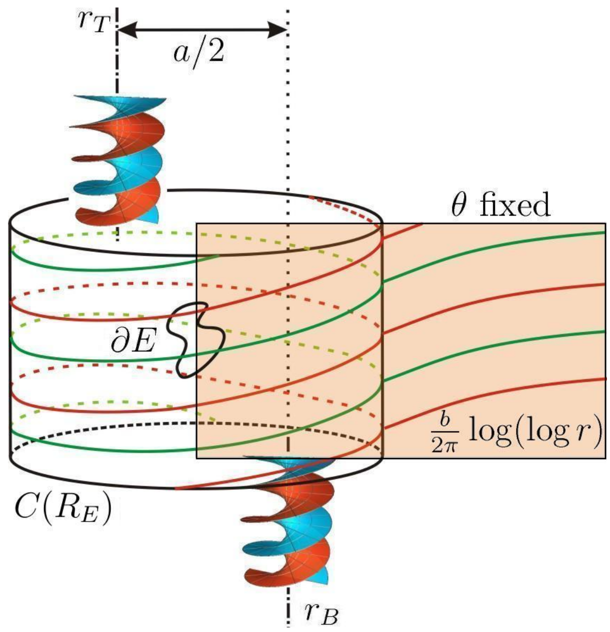

Figure 9.2: A schematic image of the embedded canonical end $E = E_{a,b}$ with flux vector $(a, 0, -b)$ (see Theorem 9.2.5). The half-lines r_T, r_B refer to the axes of the vertical helicoids H_T, H_B.

separation used in Definition 4.1.1, which measured the vertical distance between two consecutive sheets of the *same* multigraph). We also use the notation $\widetilde{D}(\infty, R) = \{(\rho, \theta) \mid \rho \geq R, \theta \in \mathbb{R}\}$ and $C(R) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq R^2\}$.

Theorem 9.2.5 (Asymptotics of the canonical end $E_{a,b}$) *Given $a \geq 0$ and $b \in \mathbb{R}$, the canonical end $E = E_{a,b}$ satisfies the following properties.*

(1) There exists $R = R_E > 0$ large such that $E_{a,b} - C(R)$ consists of two disjoint multigraphs Σ_1, Σ_2 over $D(\infty, R)$ of smooth functions $u_1, u_2 \colon \widetilde{D}(\infty, R) \to \mathbb{R}$ such that their gradients satisfy $\nabla u_i(\rho, \theta) \to 0$ as $\rho \to \infty$ and the separation function $w(\rho, \theta) = u_1(\rho, \theta) - u_2(\rho, \theta)$ between both multigraphs converges to π as $\rho + |\theta| \to \infty$. Furthermore for θ fixed[3] and $i = 1, 2$,

$$\lim_{\rho\to\infty} \frac{u_i(\rho,\theta)}{\log(\log \rho)} = \frac{b}{2\pi}. \tag{9.3}$$

(2) The translated surfaces $E_{a,b} + (0, 0, -2\pi n - \frac{b}{2\pi}\log n)$ (resp. $E_{a,b} +$

[3]This condition expresses the intersection of $E - C(R_E)$ with a vertical half-plane bounded by the x_3-axis, of polar angle θ, see Figure 9.2.

$(0, 0, 2\pi n - \frac{b}{2\pi}\log n))$ converge as $n \to \infty$ to a vertical helicoid H_T (resp. H_B) such that[4] *$H_B = H_T + (0, a/2, 0)$.*

The proof of Theorem 9.2.5 is based on a careful analysis of the horizontal and vertical projections of $E = E_{a,b}$ in terms of the Weierstrass representation. In fact, the explicit formulation of g, dh in equations (9.1), (9.2) is not used, but only that those choices of g, dh have the structure

$$g(z) = e^{iz+f(z)}, \quad dh = \left(1 + \frac{\lambda}{z - \mu}\right) dz, \tag{9.4}$$

where $\lambda \in \mathbb{R}$ and $f\colon D(\infty, R) \cup \{\infty\} \to \mathbb{C}$ is holomorphic with $f(\infty) = 0$, which is common to both (9.1) and (9.2) (the multiplicative constant t appearing in equation (9.1) can be absorbed by μ after an appropriate change of variables in the parameter domain).

We are now ready to state the main result of this section.

Theorem 9.2.6 *Let $E \subset \mathbb{R}^3$ be a properly embedded minimal annulus with infinite total curvature and compact boundary. Then, E is conformally diffeomorphic to a punctured disk, its Gaussian curvature function is bounded, and after replacing E by a subend and applying a suitable homothety and rigid motion to E, we have:*

(1) The Weierstrass data of E is of the form (9.4) defined on $D(\infty, R) = \{z \in \mathbb{C} \mid R \leq |z|\}$ for some $R > 0$, where f is a holomorphic function in $D(\infty, R)$ with $f(\infty) = 0$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{C}$. In particular, dh extends meromorphically across infinity with a double pole.

(2) E is asymptotic to (up to a rotation and homothety) to the canonical end $E_{a,b}$ determined by the equality $F = (a, 0, -b)$, where F is the flux vector of E along its boundary. In particular, E is asymptotic to the end of a helicoid if and only if it has zero flux.

Remark 9.2.7 Note that Theorem 1.0.4 stated in the Introduction is a direct consequence of Theorem 9.2.6, since the divergence theorem insures that the flux vector of a one-ended minimal surface along a loop that winds around its end vanishes. By Corollary 10.1.4 and Remark 10.1.5 below, the hypothesis "properly embedded" for E in the above theorem can be replaced by "complete embedded".

The proof of Theorem 9.2.6 can be divided into four steps. In the first one shows that after a rotation and a replacement of E by a subend, E is conformally diffeomorphic to $D(\infty, 1)$, its height differential dh extends

[4]The equality $H_B = H_T + (0, a/2, 0)$ together with item 1 of Theorem 9.2.5 imply that different values of a, b, the related canonical ends $E_{a,b}$ are not asymptotic after a rigid motion and homothety.

meromorphically across the puncture ∞ with a double pole, and the meromorphic Gauss map $g\colon D(\infty,1)\to\mathbb{C}\cup\{\infty\}$ of M has the form

$$g(z)=z^k e^{H(z)}, \tag{9.5}$$

where $k\in\mathbb{Z}$ and H is holomorphic in $D(\infty,1)$ (in particular, g misses the vertical directions in $D(\infty,1)$). This is a direct consequence of Corollary 9.1.4 once we know that there exists a horizontal plane $P\subset\mathbb{R}^3$ which intersects transversely E in exactly one proper arc. The existence of such a plane is a consequence of the next proposition, which in turn is an application of Colding-Minicozzi theory, where some care is needed since we must apply it in this context with compact boundary.

Proposition 9.2.8 *After suitable rotation of E in $\mathbb{R}^3$ there exists a solid cylinder $C(r,h)=C(r)\cap\{-h\leq x_3\leq h\}$ with $\partial E\subset \mathrm{Int}(C(r,h))$, whose top and bottom disks each intersects E transversely in a compact arc and whose side $\{x_1^2+x_2^2=r^2, |x_3|\leq h\}$ intersects E transversely in two compact spiraling arcs $\alpha^1(\theta)$, $\alpha^2(\theta)$ with $\frac{d\alpha^i}{d\theta}>0$, $i=1,2$ (after a possible change of orientation of E), where θ is the multi-valued angle parameter over the circle $\partial C(r,h)\cap\{x_3=0\}$.*

The second step in the proof of Theorem 9.2.6 consists of demonstrating that the number k in equation (9.5) is zero. Since k is the winding number of $g|_{\partial D(\infty,1)}$ in $\mathbb{C}-\{0\}$, then k will vanish provided that there exists a simple closed curve $\Gamma\subset D(\infty,1)$ which is the boundary of an end representative of $D(\infty,1)$, such that $g|_\Gamma$ has winding number 0. The loop Γ is defined as the intersection of E with the boundary of a compact vertical "cylinder" $\widetilde{C}(r_1)$ of large radius $r_1>0$; here the quotes refer to the fact that the "top" and "bottom" arcs of $E\cap\widetilde{C}(r_1)$ do not really lie on horizontal planes, but rather in the intersection of E with two suitably chosen vertical planes[5], and the radius r_1 of $\widetilde{C}(r_1)$ is chosen large enough so that the argument of g along each of the "top" and "bottom" arcs of $E\cap\widetilde{C}(r_1)$ is contained in an interval length at most π in the real line, while the argument of g along each the two long spiraling arcs β_1, β_2 obtained after intersecting E with the sides of $\widetilde{C}(r_1)$ gives the same number of turns along both spiraling arcs, which is possible thanks to the condition $\frac{d\alpha^i}{\partial\theta}(r,\theta)>0$, $i=1,2$, see Proposition 9.2.8.

The third step in the proof of Theorem 9.2.6 is to decompose the function H in equation (9.5) as $H(z)=P(z)+f(z)$, where $P(z)$ is a polynomial of degree one and $f(z)$ extends to a holomorphic function on $D(\infty,1)\cup\{\infty\}$ with $f(\infty)=0$. Here the argument is very similar to the one in the last

[5]Note that the intersection of a vertical helicoid with a vertical plane passing through its axis is a horizontal line; also note that the already proven uniqueness of the helicoid in the proper case implies that under every sequence of vertical translations, E converges (up to subsequence) to a vertical helicoid (which might depend on the subsequence). These two facts allow us to find the desired "top" and "bottom" arcs of $E\cap\widetilde{C}(r_1)$.

paragraph of the sketch of proof of the uniqueness of the helicoid in the proper case, see Chapter 8.

Once we know that the Gauss map of our properly embedded end E is $g(z) = e^{P(z)+f(z)}$ and its height differential dh extends meromorphically to ∞ with a double pole there, a simple change of variables allows us to arrive to a Weierstrass pair as the one in equation (9.4). The fourth and last step in the proof of Theorem 9.2.6 consists of showing that the multigraph structure of an end E given by the Weierstrass data (9.4) is actually independent of $f(z)$ and μ, which is the only possible difference between the Weierstrass data of E and that of the canonical end $E_{a,b}$ with the same flux vector as E. This property was already observed when sketching the proof of Theorem 9.2.5 and finishes our outline of the proof of Theorem 9.2.6.

The embedded Calabi-Yau problem. 10

In the previous chapters we have seen that minimal laminations constitute a key tool in the understanding of the global behavior of embedded minimal surfaces in $\mathbb{R}^3$. In this chapter, we will present some results about the existence and structure of minimal laminations, which have deep consequences in various aspects of minimal surface theory, including the general three-manifold setting. For example, we will give a natural condition under which the closure of a complete, embedded minimal surface in a Riemannian three-manifold has the structure of a minimal lamination. From this analysis, we will deduce, among other things, that certain complete, embedded minimal surfaces are proper, a result which is used to prove the uniqueness of the helicoid in the complete setting (Theorem 1.0.1) by reducing it to the corresponding characterization assuming properness (Theorem 1.0.4).

10.1 Uniqueness of the helicoid II: complete case.

In the Introduction we mentioned the result, proved in [148] by Meeks and Rosenberg, that the closure of a complete, embedded minimal surface M with locally bounded Gaussian curvature[1] in a Riemannian three-manifold N, has the structure of a minimal lamination. The same authors have demonstrated that this result still holds true if we substitute the locally bounded curvature assumption by a weaker hypothesis, namely that the injectivity radius of M is positive[2].

[1]This means that for every $p \in N$, there exists a neighborhood of p in N where M has bounded Gaussian curvature.

[2]More precisely, the hypothesis is that for every $p \in N$, there exists a neighborhood D_p of p in N where the injectivity radius function I_M restricted to $M \cap D_p$ is bounded

Definition 10.1.1 Let Σ be a complete Riemannian manifold. The *injectivity radius function* $I_\Sigma\colon \Sigma \to (0,\infty]$ is defined at a point $p \in \Sigma$, to be the supremum of the radii of disks centered at the origin in $T_p\Sigma$, such that the exponential map at p, restricted to each of these disks, is a diffeomorphism onto its image. The *injectivity radius of* Σ is the infimum of I_Σ.

If M is an immersed surface in a three-manifold N, then $p \in N$ is a *limit point* of M if either $p \in \overline{M} - M$ or $p \in M$ and there is a sequence $\{p_n\}_n \subset M$ which converges to p in N but does not converge to p in the intrinsic topology on M. We let $L(M)$ denote the (closed) set of limit points of M in N.

The next theorem was proved by Meeks and Rosenberg [149].

Theorem 10.1.2 (Minimal Lamination Closure Theorem) *Let M be a complete, embedded minimal surface of positive injectivity radius in a Riemannian three-manifold N (not necessarily complete). Then, the closure $\overline{M}$ of M in N has the structure of a $C^{1,\alpha}$-minimal lamination $\mathcal{L}$, some of whose leaves are the connected components of M.*

Furthermore:

1. *If N is homogeneously regular, then there exist positive constants C and ε depending on N and on the injectivity radius of M, such that the absolute Gaussian curvature of M in the ε-neighborhood of any limit leaf of $\overline{M}$ is less than C.*

2. *If M is connected, then exactly one of the following three statements holds:*

   (a) M is properly embedded in N (hence, $L(M) = \emptyset$).

   (b) The limit set $L(M) \subset \mathcal{L}$ is non-empty and disjoint from M, and M is properly embedded in the open set $N - L(M)$.

   (c) $L(M) = \mathcal{L}$ and $\mathcal{L}$ contains an uncountable number of leaves.

In the particular case $N = \mathbb{R}^3$, more can be said. Suppose $M \subset \mathbb{R}^3$ is a connected, complete, embedded minimal surface with positive injectivity radius. By Theorem 10.1.2, the closure of M has the structure of a minimal lamination of $\mathbb{R}^3$. If item 2.*(a)* in Theorem 10.1.2 does not hold for M, then the sublamination of limit points $L(M) \subset \overline{M}$ contains some leaf L. By Theorem 4.1.5 L is stable, hence L is a plane by Theorem 2.8.4. Now Theorem 10.1.2 insures that M has bounded curvature in some ε-neighborhood of the plane L, which contradicts Lemma 5.0.6. This contradiction proves the following result.

away from zero.

Corollary 10.1.3 (Meeks, Rosenberg [149]) *Every connected, complete, embedded minimal surface in* $\mathbb{R}^3$ *with positive injectivity radius is properly embedded.*

Suppose M is a complete, embedded minimal surface of finite topology in $\mathbb{R}^3$. If the injectivity radius of M is zero, then there exists a divergent sequence of embedded geodesic loops $\gamma_n \subset M$ (i.e. closed geodesics with at most one corner) with lengths going to zero. Since M has finite topology, we may assume the γ_n are all contained in a fixed annular end E of M. By the Gauss-Bonnet formula, each γ_n is homotopically non-trivial, and so, the cycles $\gamma_n \cup \gamma_1$, $n \geq 2$, bound compact annular subdomains in E, whose union is a subend of E. However, the Gauss-Bonnet formula implies that the total curvature of this union is finite (greater than -4π). Hence, E is asymptotic to an end of a plane or of a half-catenoid, which is absurd. This argument proves that the following result holds.

Corollary 10.1.4 (Colding-Minicozzi [37]) *A connected, complete, embedded minimal surface of finite topology in* $\mathbb{R}^3$ *is properly embedded.*

The above corollary fails to hold if the surface is allowed to have self-intersections, since there exists a complete minimal disk contained in the unit ball (Nadirashvili [166]). Colding and Minicozzi obtained Corollary 10.1.4 from a deep analysis of the relation between intrinsic and extrinsic distances for an embedded minimal disk in $\mathbb{R}^3$ (see [37]), and the proof of Theorem 10.1.2 is inspired by this previous work of Colding and Minicozzi.

The proof of the main assertion in Theorem 10.1.2 is based on the fact that the closure of a complete, embedded minimal surface M with locally bounded curvature in a Riemannian three-manifold N, is a minimal lamination of N. Thus, it remains to explain why the positive injectivity radius hypothesis implies that the Gaussian curvature of M is locally bounded. This implication follows by contradiction: if M does not have locally bounded curvature in N, then there exists a point $p \in N$ such that in arbitrarily small *extrinsic* balls centered at p, M does not have bounded curvature (in particular, $p \in L(M)$). This fact allows us to find a sequence $p_n \in M$ converging to p where the absolute Gaussian curvature of M blows up. Since the injectivity radius of M is positive, there exists a positive number ε such that the *intrinsic* geodesic balls $B_M(p_n, \varepsilon)$ of center p_n and radius ε are topologically disks. Furthermore, the $B_M(p_n, \varepsilon)$ can be assumed pairwise disjoint since p is a limit point of M. But although the intrinsic radii of these disks is a fixed positive constant, their *extrinsic size* could be arbitrarily small, in the sense that $\partial B_M(p_n, \varepsilon)$ could be contained in extrinsic balls $\mathbb{B}_N(p, \varepsilon_n)$ with $0 < \varepsilon_n \searrow 0$. Using the number $\delta > 0$ in the next theorem, one deduces that for n large, the component Σ_n of $B_M(p_n, \varepsilon) \cap \mathbb{B}_N(p, \frac{\delta}{2})$ containing p_n is a disk, whose boundary lies on the boundary of $\mathbb{B}_N(p, \frac{\delta}{2})$.

Using the barrier construction arguments in Chapter 2.9, one can find a stable minimal disk Δ_n between Σ_n and Σ_{n+1}, with $\partial\Delta_n \subset \partial\mathbb{B}_N(p, \frac{\delta}{2})$. By stability, Δ_n has uniform curvature estimates near p. By the one-sided curvature estimate (Theorem 4.1.3 with Δ_n playing the role of a piece of the plane), one obtains uniform curvature estimates for the disks Σ_n near p for all n sufficiently large, which is a contradiction. The existence of the number δ is a subtle and crucial point in the proof of Theorem 10.1.2, and it is based on the following comparison result between intrinsic and extrinsic distances for an embedded minimal disk. In the case N is flat, the next theorem was earlier proved by Colding and Minicozzi [37].

Remark 10.1.5 The arguments in the proof by Colding and Minicozzi of Corollary 10.1.4 can be extended to the case of compact boundary. This extension allows us to replace the hypothesis of properness in Theorem 9.2.6 by completeness.

Theorem 10.1.6 (Weak δ-chord Arc Property, Meeks, Rosenberg [149]) *Let N be a homogeneously regular three-manifold. Then, there exist numbers $\delta \in (0, \frac{1}{2})$ and $R_0 > 0$ such that the following statement holds.*

Let $\Sigma \subset N$ be a compact, embedded minimal disk with boundary, whose injectivity radius function $I_\Sigma \colon \Sigma \to [0, \infty)$ equals the intrinsic distance to the boundary function $d_\Sigma(\cdot, \partial\Sigma)$. If $B_\Sigma(x, R) = \{y \in \Sigma \mid d_\Sigma(x, y) < R\} \subset \Sigma - \partial\Sigma$ and $R \leq R_0$, then

$$\Sigma(x, \delta R) \subset B_\Sigma(x, R/2),$$

where $\Sigma(x, \delta R)$ stands for the component of $\Sigma \cap \mathbb{B}_N(x, \delta R)$ passing through x. Furthermore, $\Sigma(x, \delta R)$ is a compact, embedded minimal disk in $\mathbb{B}_N(x, \delta R)$ with $\partial\Sigma(x, \delta R) \subset \partial\mathbb{B}_N(x, \delta R)$.

To finish this section, we remark that by direct application of Corollary 10.1.4, several results stated in this monograph for properly embedded minimal surfaces of finite topology remain valid after replacing properness by completeness. This applies, for instance, to one of the central results in this survey, Theorem 1.0.1 stated in the Introduction.

10.2 Regularity of the singular sets of convergence of minimal laminations.

An important technique which is used when dealing with a sequence of embedded minimal surfaces M_n is to rescale each surface in the sequence to obtain a well-defined limit after rescaling, from where one deduces information about the original sequence. An important case in these rescaling processes is that of blowing-up a sequence of minimal surfaces *on the scale of curvature* (for details, see e.g. the proofs of Theorem 15 in [129] or of

Corollary 2.2 in [124]). When the surfaces in the sequence are complete and embedded in $\mathbb{R}^3$, this blowing-up process produces a limit which is a properly embedded, non-flat minimal surface with bounded Gaussian curvature, whose genus and rank of homology groups are bounded above by the ones for the M_n. For example, if each M_n is a planar domain, then the same holds for the limit.

Recall that we defined in Chapter 4.1 the concept of a locally simply-connected sequence of properly embedded minimal surfaces in $\mathbb{R}^3$. This concept can be easily generalized to a sequence of properly embedded, non-simply-connected minimal surfaces in a Riemannian three-manifold[3] N. For useful applications of the notion of locally simply-connected sequence, it is essential to consider sequences of properly embedded minimal surfaces which *a priori* may not satisfy the locally simply-connected condition, and then modify them to produce a new sequence which satisfies that condition. We accomplish this by considering a blow-up argument on a scale which, in general, is different from blowing-up on the scale of curvature. We call this procedure *blowing-up on the scale of topology.* This scale was defined and used in [139, 140] to prove that any properly embedded minimal surface in $\mathbb{R}^3$ of finite genus and infinitely many ends has bounded curvature and is recurrent. We now explain the elements of this new scale.

Suppose $\{M_n\}_n$ is a sequence of non-simply-connected, properly embedded minimal surfaces in $\mathbb{R}^3$ which is not uniformly locally simply-connected. Note that the Gaussian curvature of the collection M_n is not uniformly bounded, and so, one could blow-up these surfaces on the scale of curvature to obtain a properly embedded, non-flat minimal surface which may or may not be simply-connected. Also note that, after choosing a subsequence, there exist points $p_n \in \mathbb{R}^3$ such that by defining

$$r_n(p_n) = \sup\{r > 0 \mid \mathbb{B}(p_n, r) \text{ intersects } M_n \text{ in disks}\},$$

then $r_n(p_n) \to 0$ as $n \to \infty$. Let $\widetilde{p}_n$ be a point in $\mathbb{B}(p_n, 1) = \{x \in \mathbb{R}^3 \mid \|x - p_n\| < 1\}$ where the function $x \mapsto d(x, \partial\mathbb{B}(p_n, 1))/r_n(x)$ attains its maximum (here d denotes extrinsic distance). Then, the translated and rescaled surfaces $\widetilde{M}_n = \frac{1}{r_n(\widetilde{p}_n)}(M_n - \widetilde{p}_n)$ intersect for all n the closed unit ball $\overline{\mathbb{B}}(\vec{0}, 1)$ in at least one component which is not simply-connected, and for n large they intersect any ball of radius less than $1/2$ in simply-connected components, in particular the sequence $\{\widetilde{M}_n\}_n$ is uniformly locally simply-connected (see [139] for details).

For the sake of clarity, we now illustrate this blow-up procedure on certain sequences of Riemann minimal examples, defined in Chapter 2.5. Each of these surfaces is foliated by circles and straight lines in horizontal planes, with the (x_1, x_3)-plane as a plane of reflective symmetry. After a translation

[3]Exchange the Euclidean balls $\mathbb{B}(p, r(p))$ in Definition 4.1.6 by extrinsic balls $\mathbb{B}_N(p, r(p))$ relative to the Riemannian distance function on N.

and a homothety, we can also assume that these surfaces are normalized so that any ball of radius less than 1 intersects these surfaces in compact disks, and the closed unit ball $\overline{\mathbb{B}}(\vec{0},1)$ intersects every Riemann example in at least one component which is not a disk. Under this normalization, any sequence of Riemann minimal examples is uniformly locally simply-connected. The flux of each Riemann minimal example along a compact horizontal section has horizontal and vertical components which are not zero; hence it makes sense to consider the ratio V of the norm of its horizontal component over the vertical one. It turns out that V parameterizes the family $\{M(V)\}_V$ of Riemann minimal examples, with $V \in (0,\infty)$. When $V \to 0$, the surfaces $M(V)$ converge smoothly to the vertical catenoid centered at the origin with waist circle of radius 1. But for our current purposes, we are more interested in the limit object of $M(V)$ as $V \to \infty$. In this case, the Riemann minimal examples $M(V)$ converge smoothly to a foliation of $\mathbb{R}^3$ by horizontal planes away from the two vertical lines passing through $(0,-1,0),(0,1,0)$. Given a horizontal slab $S \subset \mathbb{R}^3$ of finite width, the description of $M(V)\cap S$ for V large is as follows, see Figure 4.2.

(a) If C_1, C_2 are disjoint vertical cylinders in S with axes $S\cap[\{(0,-1)\}\times\mathbb{R}]$, $S\cap[\{(0,1)\}\times\mathbb{R}]$, then $M(V)\cap C_i$ is arbitrarily close to the intersection of S with a highly sheeted vertical helicoid with axis the axis of C_i, $i=1,2$. Furthermore, the fact that the (x_1,x_3)-plane is a plane of reflective symmetry of $M(V)$ implies that these limit helicoids are oppositely handed.

(b) In $S-(C_1\cup C_2)$, the surface $M(V)$ consists of two almost flat, almost horizontal multigraphs, with number of sheets increasing to ∞ as $V \to \infty$.

(c) If C is a vertical cylinder in S containing $C_1\cup C_2$, then $M(V)$ intersects $S-C$ in a finite number $n(V)$ of univalent graphs, each one representing a planar end of $M(V)$. Furthermore, $n(V)\to\infty$ as $V\to\infty$.

This description depicts one particular case of what we call a parking garage structure on $\mathbb{R}^3$ for the limit of a sequence of minimal surfaces (we mentioned this notion before Definition 4.1.6). Roughly speaking, a *parking garage surface with n columns* is a smooth embedded surface in $\mathbb{R}^3$ (not necessarily minimal), which in any fixed finite width horizontal slab $S\subset\mathbb{R}^3$, can be decomposed into 2 disjoint, almost flat horizontal multigraphs over the exterior of n disjoint disks $D_1,\ldots,D_n$ in the (x_1,x_2)-plane, together n topological strips each one contained in one of the solid cylinders $S\cap(D_i\times\mathbb{R})$ (these are the columns), such that each strip lies in a small regular neighborhood of the intersection of a vertical helicoid H_i with $S\cap(D_i\times\mathbb{R})$. One can associate to each column a $+$ or $-$ sign, depending on the handedness of the corresponding helicoid H_i. Note that a vertical helicoid is the basic example of a parking garage surface with 1 column, and the Riemann minimal examples $M(V)$ with $V \to \infty$ have the structure of a parking garage with two oppositely handed columns. Other limiting parking garage structures with varying numbers of columns (including the case where $n = \infty$) and associated signs can be found in Traizet and Weber [218] and in Meeks, Pérez and Ros [133].

There are interesting cases where the locally simply-connected condition guarantees the convergence of a sequence of minimal surfaces in $\mathbb{R}^3$ to a limiting parking garage structure. Typically, one proves that the sequence converges (up to a subsequence and a rotation) to a foliation of $\mathbb{R}^3$ by horizontal planes, with singular set of convergence consisting of a locally finite set of Lipschitz curves parameterized by heights. In fact, these Lipschitz curves are vertical lines, and locally around the lines the surfaces in the sequence can be arbitrarily approximated by highly sheeted vertical helicoids. The first assertion in this last sentence can be deduced from the next result, and the second one follows from application of the uniqueness of the helicoid (Theorem 1.0.4) after a blowing-up process on the scale of curvature.

Theorem 10.2.1 ($C^{1,1}$-regularity of $S(\mathcal{L})$, Meeks [121]) *Suppose $\{M_n\}_n$ is a locally simply-connected sequence of properly embedded minimal surfaces in a three-manifold N, that converges C^α, $\alpha \in (0,1)$, to a minimal lamination $\mathcal{L}$ of N, outside a locally finite collection of Lipschitz curves $S(\mathcal{L}) \subset N$ transverse to $\mathcal{L}$ (along which the convergence fails to be C^α). Then, $\mathcal{L}$ is a $C^{1,1}$-minimal foliation in a neighborhood of $S(\mathcal{L})$, and $S(\mathcal{L})$ consists of $C^{1,1}$-curves orthogonal to the leaves of $\mathcal{L}$.*

To give an idea of the proof of Theorem 10.2.1, first note that the nature of this theorem is purely local, so it suffices to consider the case of a sequence of properly embedded minimal disks M_n in the unit ball $\mathbb{B}(1) = \mathbb{B}(\vec{0}, 1)$ of $\mathbb{R}^3$ (the general case follows from adapting the arguments to a three-manifold). After passing to a subsequence, one can also assume that the surfaces M_n converge to a $C^{0,1}$-minimal foliation[4] $\mathcal{L}$ of $\mathbb{B}(1)$ and the convergence is C^α, $\alpha \in (0,1)$, outside of a transverse Lipschitz curve $S(\mathcal{L})$ that passes through the origin. Since unit normal vector field $N_{\mathcal{L}}$ to $\mathcal{L}$ is Lipschitz (Solomon [209]), then the integral curves of $N_{\mathcal{L}}$ are of class $C^{1,1}$. Then, the proof consists of demonstrating that $S(\mathcal{L})$ is the integral curve of $N_{\mathcal{L}}$ passing through the origin. To do this, one first proves that $S(\mathcal{L})$ is of class C^1, hence it admits a continuous tangent field T, and then one shows that T is orthogonal to the leaves of $\mathcal{L}$. These properties rely on a local analysis of the singular set $S(\mathcal{L})$ as a limit of minimizing geodesics γ_n in M_n that join pairs of appropriately chosen *points of almost maximal curvature,* (in a sense similar to the points p_n in Theorem 11.0.8 below), together with the fact that these minimizing geodesics γ_n converge C^1 as $n \to \infty$ to the

[4]Any codimension-one minimal foliation is of class $C^{0,1}$ and its unit normal vector field is $C^{0,1}$ as well, see Solomon [209].

integral curve of $N_{\mathcal{L}}$ passing through the origin. Crucial in this proof is the uniqueness of the helicoid (Theorem 1.0.4), since it gives the local picture of the minimal disks M_n near the origin as being closely approximated by portions of a highly sheeted helicoid near its axis.

Remark 10.2.2 The local structure of the surfaces M_n for n large near a point in $S(\mathcal{L})$ as an approximation of a highly sheeted helicoid is the reason for the parenthetical comment of the columns being helicoidal for the surfaces limiting to a parking garage structure, see the paragraph just after Theorem 4.1.5.

Meeks and Weber [152] have shown that the $C^{1,1}$-regularity of $S(\mathcal{L})$ given by Theorem 10.2.1 is the best possible result. They do this by proving the following theorem:

Theorem 10.2.3 (Meeks, Weber [152]) *Let Γ be a properly embedded $C^{1,1}$-curve in an open set U of $\mathbb{R}^3$. Then, Γ is the singular set of convergence for some Colding-Minicozzi type limit foliation of some neighborhood of Γ in U.*

In the special case that Γ is the unit circle in the (x_1, x_2)-plane, Meeks and Weber defined for any $n \in \mathbb{N}$ a complete, immersed minimal annulus $\widetilde{H}_n \subset \mathbb{R}^3$ of finite total curvature which contains the circle Γ. Like in the helicoid along its straight axis, the Gauss map of $\widetilde{H}_n$ turns at a constant rate $2\pi n$ along its "circle axis" Γ. Meeks and Weber called the surfaces $\widetilde{H}_n$ *bent helicoids*, which are described analytically in Chapter 2.5, see also Figure 2.2 Right. Moreover, they prove that there are compact annuli $H_n \subset \widetilde{H}_n$ which are embedded and which converge to the foliation $\mathcal{L}$ of $\mathbb{R}^3 - (x_3\text{-axis})$ by vertical half-planes with boundary the x_3-axis and with singular set of convergence $S(\mathcal{L}) = \Gamma$. Returning to the statement of Theorem 10.2.3, the embeddedness of the compact bent helicoids H_n plays a key role in constructing a locally simply-connected sequence of embedded, compact minimal surfaces in the open set U, which converges to a foliation of a neighborhood of the curve Γ, with singular set of C^1-convergence being Γ, thereby proving Theorem 10.2.3.

Theorem 10.2.1 and its proof, which we have just outlined, have as a natural corollary the Lamination Metric Theorem by Meeks in [121]. This result describes how the intrinsic metrics on minimal surfaces in Theorem 10.2.1 converge to a natural metric space structure on the limit lamination. For simplicity, we only state the following result in the simpler case where the ambient space is $\mathbb{R}^3$; this simpler case plays an important role in the proof of the Limit Lamination Closure Theorem 10.1.2 above.

Theorem 10.2.4 (Lamination Metric Theorem in $\mathbb{R}^3$) *Let $\{M_n\}_n$ be a locally simply-connected sequence of connected, properly embedded minimal* *surfaces in $\mathbb{R}^3$. Suppose that the surfaces M_n converge as $n \to \infty$ to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$, with singular set of convergence $S(\mathcal{L})$ being a non-empty, locally finite set of Lipschitz curves transverse to the leaves of $\mathcal{L}$. Then,*

1. *$\mathcal{L}$ is a foliation of $\mathbb{R}^3$ by planes, and $S(\mathcal{L})$ is a locally finite set of lines orthogonal to the planes in $\mathcal{L}$.*

2. *The convergence of the M_n to $\mathcal{L}$ gives rise to a limiting parking garage structure on $\mathbb{R}^3$.*

Given $p, q \in \mathbb{R}^3$, let $\mathcal{A}(p,q)$ denote the set of all piecewise linear, unit speed arcs Γ joining p and q, such that each of the line segments of Γ is contained either in a plane of $\mathcal{L}$ or in a line in $S(\mathcal{L})$.

3. *If γ_n is a unit speed geodesic on M_n with fixed length $l > 0$ and γ_n intersects a given compact set of $\mathbb{R}^3$ for all $n \in \mathbb{N}$, then a subsequence of the γ_n converges to an element γ_∞ of $\mathcal{A}(p,q)$ of length l, and the convergence is C^1 except at the corners of γ_∞.*

4. *Given $p, q \in \mathbb{R}^3$, we define the lamination distance $d(p,q)$ to be the infimum of the lengths of all elements of $\mathcal{A}(p,q)$, which is always attained at some $\Gamma(p,q) \in \mathcal{A}(p,q)$. Then, d endows $\mathbb{R}^3$ with a path connected metric space structure which does not admit a countable basis. Furthermore, there exist sequences of points $p_n, q_n \in M_n$ and unit speed length-minimizing geodesics $\gamma_n \subset M_n$ joining p_n and q_n such that the γ_n converge C^1 except at a finite number of points to $\Gamma(p,q)$ and the lengths of the γ_n converge to $d(p,q)$ as $n \to \infty$.*

11 Local pictures, local removable singularities and dynamics.

As we have explained in previous chapters, results by Colding and Minicozzi on removable singularities for certain limit minimal laminations of $\mathbb{R}^3$, and their subsequent applications by Meeks and Rosenberg to the uniqueness of the helicoid or to the Limit Lamination Closure Theorem explained above, demonstrate the importance of removable singularities results for obtaining a deep understanding of the geometry of complete, embedded minimal surfaces in three-manifolds.

Removable singularities theorems for limit minimal laminations also play a central role in a series of papers by Meeks, Pérez and Ros [130, 131, 133, 134, 137, 139, 140], where these authors obtain topological bounds and descriptive results for complete, embedded minimal surfaces of finite genus in $\mathbb{R}^3$. In this chapter, we will explain some of these results by Meeks, Pérez and Ros contained in [133, 134], including a crucial local removable singularity theorem for certain minimal laminations with isolated singularities in a Riemannian three-manifold (Theorem 11.0.6 below). We view this local result as a tool in developing a general removable singularities theory for possibly singular minimal laminations of $\mathbb{R}^3$. We also describe a number of applications of this result to the classical theory of minimal surfaces.

Another important building block of this emerging theory is a basic compactness result (Theorem 11.0.9) which has as limit objects properly embedded minimal surfaces in $\mathbb{R}^3$ and minimal parking garage structures on $\mathbb{R}^3$. This result gives us for the first time a glimpse at the extrinsic geometric structure of an arbitrary embedded minimal surface in a three-manifold, near points of the surface with small injectivity radius.

An important application of the Local Removable Singularity Theorem 11.0.6 is a characterization of all complete, embedded minimal surfaces of quadratic decay of curvature (Theorem 11.0.11 below). This characterization result leads naturally to a dynamical theory for the space $D(M)$ consisting of all properly embedded, non-flat minimal surfaces which are obtained as smooth limits of a given properly embedded minimal surface $M \subset \mathbb{R}^3$, under sequences of homotheties and translations with translational part diverging to infinity, see Theorem 11.0.13. This dynamical theory can be used as a tool to obtain insight and simplification strategies for solving several important outstanding problems in the classical theory of minimal surfaces (see [134]), of which we emphasize the following conjecture.

Conjecture 11.0.5 (Fundamental Singularity Conjecture) *Suppose $\mathcal{A} \subset \mathbb{R}^3$ is a closed set whose 1-dimensional Hausdorff measure is zero. If $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3 - \mathcal{A}$, then the closure $\overline{\mathcal{L}}$ in $\mathbb{R}^3$ has the structure of a minimal lamination of $\mathbb{R}^3$.*

Since the union of catenoid with a plane passing through its waist circle is a singular minimal lamination of $\mathbb{R}^3$ whose singular set is the intersecting circle, the above conjecture represents the best possible extension result in terms of the size of the singular set of the lamination to be removed.

Conjecture 11.0.5 has a global nature, because there exist interesting minimal laminations of the open unit ball in $\mathbb{R}^3$ punctured at the origin which do not extend across the origin, see Examples I and II in Chapter 4.2. In Example III of the same section, we saw that in hyperbolic three-space $\mathbb{H}^3$, there are rotationally invariant, global minimal laminations which have a similar unique isolated singularity. The existence of these global, singular minimal laminations of $\mathbb{H}^3$ demonstrate that the validity of Conjecture 11.0.5 depends on the metric properties of $\mathbb{R}^3$. However, Meeks, Pérez and Ros obtained the following remarkable local removable singularity result in any Riemannian three-manifold N for certain possibly singular minimal laminations.

Given a three-manifold N and a point $p \in N$, we will denote by d the distance function in N to p and $\mathbb{B}_N(p,r)$ the metric ball of center p and radius $r > 0$. For a lamination $\mathcal{L}$ of N, we will denote by $|K_{\mathcal{L}}|$ the absolute curvature function on the leaves of $\mathcal{L}$.

Theorem 11.0.6 (Local Removable Singularity Theorem) *A minimal lamination $\mathcal{L}$ of a punctured ball $\mathbb{B}_N(p,r) - \{p\}$ in a Riemannian three-manifold N extends to a minimal lamination of $\mathbb{B}_N(p,r)$ if and only if there exists a positive constant c such that $|K_{\mathcal{L}}|d^2 < c$ in some subball*[1].

Since stable minimal surfaces have local curvature estimates which satisfy the hypothesis of Theorem 11.0.6 and every limit leaf of a minimal lamination is stable (Theorem 4.1.5), we obtain the next extension result for

[1] Equivalently by the Gauss theorem, for some positive constant c', $|A_{\mathcal{L}}|d < c'$, where $|A_{\mathcal{L}}|$ is the norm of the second fundamental form of the leaves of $\mathcal{L}$.

the sublamination of limit leaves of any minimal lamination in a punctured three-manifold.

Corollary 11.0.7 *Suppose that N is a Riemannian three-manifold, which is not necessarily complete. If $W \subset N$ is a closed countable subset*[2] *and $\mathcal{L}$ is a minimal lamination of $N - W$, then:*

1. *The sublamination of $\mathcal{L}$ consisting of the closure of any collection of its stable leaves extends across W to a minimal lamination $\mathcal{L}_1$ of N. Furthermore, each leaf of $\mathcal{L}_1$ is stable.*

2. *The sublamination of $\mathcal{L}$ consisting of its limit leaves extends across W to a minimal lamination of N.*

3. *If $\mathcal{L}$ is a minimal foliation of $N - W$, then $\mathcal{L}$ extends across W to a minimal foliation of N.*

In particular, the above theorem implies that the absolute curvature function $|K_{\mathcal{L}}|$ for the laminations $\mathcal{L}$ given in Examples I, II and III in Chapter 4.2, blows up near the origin strictly faster that any positive constant over the distance squared to $\vec{0}$.

The natural generalizations of the above Local Removable Singularity Theorem and of Conjecture 11.0.5 fail badly for codimension-one minimal laminations of $\mathbb{R}^n$, for $n = 2$ and for $n > 3$. In the case $n = 2$, consider the cone C over any two non-antipodal points on the unit circle; C consists of two infinite rays making an acute angle at the origin. The punctured cone $C - \{\vec{0}\}$ is totally geodesic and so, the norm of the second fundamental form is zero but C is not a smooth lamination at the origin. For $n > 3$, one can consider cones over any embedded, compact minimal hypersurface in $\mathbb{S}^{n-1}$ which is not an equator. These examples demonstrate that Theorem 11.0.6 is precisely a two-dimensional result.

Recall that Corollary 5.0.4 insured that every complete, embedded minimal surface in $\mathbb{R}^3$ with bounded curvature is properly embedded. The next theorem shows that any complete minimal surface in $\mathbb{R}^3$ which is not properly embedded, has natural limits under dilations, which are properly embedded minimal surfaces. By *dilation,* we mean the composition of a homothety and a translation.

Theorem 11.0.8 (Local Picture on the Scale of Curvature, Meeks, Pérez, Ros [134])
Suppose M is a complete, embedded minimal surface with unbounded curvature in a homogeneously regular three-manifold N. Then, there exists a sequence of points $p_n \in M$ and positive numbers $\varepsilon_n \to 0$, such that the following statements hold.

[2] An argument based on the classical Baire's theorem allows us to pass from the isolated singularity setting of Theorem 11.0.6 to a closed countable set of singularities, see [134].

1. *For all n, the component M_n of $\mathbb{B}_N(p_n, \varepsilon_n) \cap M$ that contains p_n is compact, with boundary $\partial M_n \subset \partial \mathbb{B}_N(p_n, \varepsilon_n)$.*

2. *Let $\lambda_n = \sqrt{|K_{M_n}|(p_n)}$. Then, the absolute curvature function $|K_{M_n}|$ satisfies $\frac{\sqrt{|K_{M_n}|}}{\lambda_n} \leq 1 + \frac{1}{n}$ on M_n, with $\lim_{n \to \infty} \varepsilon_n \lambda_n = \infty$.*

3. *The metric balls $\lambda_n \mathbb{B}_N(p_n, \varepsilon_n)$ of radius $\lambda_n \varepsilon_n$ converge uniformly to $\mathbb{R}^3$ with its usual metric (so that we identify p_n with $\vec{0}$ for all n), and there exists a connected, properly embedded minimal surface M_∞ in $\mathbb{R}^3$ with $\vec{0} \in M_\infty$, $|K_{M_\infty}| \leq 1$ on M_∞ and $|K_{M_\infty}|(\vec{0}) = 1$ (here K_{M_∞} stands for the Gaussian curvature function of M_∞), such that for any $k \in \mathbb{N}$, the surfaces $\lambda_n M_n$ converge C^k on compact subsets of $\mathbb{R}^3$ to M_∞ with multiplicity one as $n \to \infty$.*

The above theorem gives a *local picture* or description of the local geometry of an embedded minimal surface M in an extrinsic neighborhood of a point $p_n \in M$ of concentrated curvature. The points $p_n \in M$ appearing in Theorem 11.0.8 are called *blow-up points on the scale of curvature.*

Now assume that for any positive ε, the intrinsic ε-balls of such a minimal surface M are not always disks. Then, the curvature of M certainly blows up at some points in these non-simply-connected intrinsic ε-balls as $\varepsilon \to 0$. Thus, one could blow-up M on the scale of curvature as in Theorem 11.0.8 but this process could create a simply-connected limit, hence a helicoid. Now imagine that we want to avoid this helicoidal blow-up limit, and note that the injectivity radius of M is zero, i.e. there exists an intrinsically divergent sequence of points $p_n \in M$ where the injectivity radius function of M tends to zero; such points are called *points of concentrated topology.* If we blow-up M around these points p_n in a similar way as we did in the above theorem, but exchanging the former ratio of rescaling (which was the square root of the absolute Gaussian curvature at p_n) by the inverse of the injectivity radius at these points, then we will obtain a sequence of minimal surfaces which are non-simply-connected in a fixed ball of space (after identifying again p_n with $\vec{0} \in \mathbb{R}^n$), and it is natural to ask about possible limits of such a blow-up sequence. This is the purpose of the next result.

Theorem 11.0.9 (Local Picture on the Scale of Topology, Meeks, Pérez, Ros [133]) *Suppose M is a complete, embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold N. Then, there exists a sequence of points $p_n \in M$ and positive numbers $\varepsilon_n \to 0$ such that the following statements hold.*

1. *For all n, the component M_n of $\mathbb{B}_N(p_n, \varepsilon_n) \cap M$ that contains p_n is compact, with boundary $\partial M_n \subset \partial \mathbb{B}_N(p_n, \varepsilon_n)$.*

2. *Let* $\lambda_n = 1/I_{M_n}(p_n)$, *where* I_{M_n} *denotes the injectivity radius function of* M *restricted to* M_n. *Then,* $\lambda_n I_{M_n} \geq 1 - \frac{1}{n+1}$ *on* M_n, *and* $\lim_{n\to\infty} \varepsilon_n \lambda_n = \infty$.

3. *The metric balls* $\lambda_n \mathbb{B}_N(p_n, \varepsilon_n)$ *of radius* $\lambda_n \varepsilon_n$ *converge uniformly to* $\mathbb{R}^3$ *with its usual metric (so that we identify* p_n *with* $\vec{0}$ *for all* n*).*

Furthermore, one of the following three possibilities occurs.

4. *The surfaces* $\lambda_n M_n$ *have uniformly bounded curvature on compact subsets of* $\mathbb{R}^3$ *and there exists a connected, properly embedded minimal surface* $M_\infty \subset \mathbb{R}^3$ *with* $\vec{0} \in M_\infty$, $I_{M_\infty} \geq 1$ *and* $I_{M_\infty}(\vec{0}) = 1$ *(here* I_{M_∞} *denotes the injectivity radius function of* M_∞*), such that for any* $k \in \mathbb{N}$, *the surfaces* $\lambda_n M_n$ *converge* C^k *on compact subsets of* $\mathbb{R}^3$ *to* M_∞ *with multiplicity one as* $n \to \infty$.

5. *The surfaces* $\lambda_n M_n$ *converge*[3] *to a limiting minimal parking garage structure on* $\mathbb{R}^3$, *consisting of a foliation* $\mathcal{L}$ *by planes with columns based on a locally finite set* $S(\mathcal{L})$ *of lines orthogonal to the planes in* $\mathcal{L}$ *(which is the singular set of convergence of* $\lambda_n M_n$ *to* $\mathcal{L}$*), and:*

   5.1 $S(\mathcal{L})$ *contains a line* L_1 *which passes through the origin and another line* L_2 *at distance* 1 *from* L_1.

   5.2 *All of the lines in* $S(\mathcal{L})$ *have distance at least* 1 *from each other.*

   5.3 *If there exists a bound on the genus of the surfaces* $\lambda_n M_n$, *then* $S(\mathcal{L})$ *consists of just two components* L_1, L_2 *with associated limiting double multigraphs being oppositely handed.*

6. *There exists a non-empty, closed set* $\mathcal{S} \subset \mathbb{R}^3$ *and a lamination* $\mathcal{L}$ *of* $\mathbb{R}^3 - \mathcal{S}$ *such that the surfaces* $(\lambda_n M_n) - \mathcal{S}$ *converge to* $\mathcal{L}$ *outside some singular set of convergence* $S(\mathcal{L}) \subset \mathbb{R}^3 - \mathcal{S}$. *Furthermore:*

   6.1 *There exists* $R_0 > 0$ *such that sequence of surfaces* $\{(\lambda_n M_n) \cap \mathbb{B}(\vec{0}, R_0)\}_n$ *does not have bounded genus.*

   6.2 *The sublamination* $\mathcal{P}$ *of flat leaves in* $\mathcal{L}$ *is non-empty.*

   6.3 *The set* $\mathcal{S} \cup S(\mathcal{L})$ *is a closed set of* $\mathbb{R}^3$ *which is contained in the union of planes* $\bigcup_{P\in\mathcal{P}} \overline{P}$. *Furthermore, there are no planes in* $\mathbb{R}^3 - \mathcal{L}$.

   6.4 *If* $P \in \mathcal{P}$, *then the plane* $\overline{P}$ *intersects* $\mathcal{S} \cup S(\mathcal{L})$ *in an infinite set of points, which are at least distance 1 from each other in* $\overline{P}$, *and either* $\overline{P} \cap \mathcal{S} = \emptyset$ *or* $\overline{P} \cap S(\mathcal{L}) = \emptyset$.

[3] This convergence must be understood similarly as those in Theorems 4.1.2 and 4.1.7, outside the singular set of convergence of $\lambda_n M_n$ to $\mathcal{L}$.

The next corollary follows immediately from the above theorem, the Local Picture Theorem on the Scale of Curvature (Theorem 11.0.8), and the fact that the set of properly embedded minimal surfaces in $\mathbb{R}^3$ passing through the origin with uniformly bounded Gaussian curvature and absolute Gaussian curvature 1 at the origin, forms a compact metric space with the uniform topology on compact subsets of $\mathbb{R}^3$. By the Regular Neighborhood Theorem (Corollary 2.6.6), the surfaces in this compact metric space all have cubical area growth at most CR^3 in extrinsic balls of any radius $R > 0$, where $C > 0$ depends on the uniform bound of the curvature.

Corollary 11.0.10 *Suppose M is a complete, embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold N, and suppose M does not have a local picture on the scale of curvature which is a helicoid*[4]*. Let $T(M)$ be the set of all local pictures for M on the scale of topology. Then, the following assertions hold.*

1. *Every element in $T(M)$ is a properly embedded, non-simply-connected minimal surface M_∞ satisfying item 4 of Theorem 11.0.9*[5]*. Furthermore, M_∞ has Gaussian curvature bounded by a uniform constant.*
2. *$T(M)$ is compact in the uniform topology on compact sets of $\mathbb{R}^3$, and there is a positive constant C such that every surface in $T(M)$ has area growth at most CR^3.*

The local picture theorems on the scales of curvature and topology deal with limits of a sequence of blow-up rescalings for a complete, embedded minimal surface. Next we will study an interesting function which is invariant by rescalings, namely the Gaussian curvature of a surface in $\mathbb{R}^3$ times the squared distance to a given point. A complete Riemannian surface M is said to have *intrinsic quadratic curvature decay constant $C > 0$ with respect to a point $p \in M$*, if the absolute curvature function $|K_M|$ of M satisfies

$$|K_M(q)| \leq \frac{C}{d_M(p,q)^2} \qquad \text{for all } q \in M,$$

where d_M denotes the Riemannian distance function. Note that if such a Riemannian surface M is a complete surface in $\mathbb{R}^3$ with $p = \vec{0} \in M$, then it also has extrinsic quadratic decay constant C with respect to the radial distance R to $\vec{0}$, i.e. $|K_M|R^2 \leq C$ on M. For this reason, when we say that a minimal surface in $\mathbb{R}^3$ has *quadratic decay of curvature*, we will always refer to curvature decay with respect to the extrinsic distance R to $\vec{0}$, independently of whether or not M passes through or limits to $\vec{0}$.

[4] This means that the helicoid cannot be found as a limit surface M_∞ of a sequence $\lambda_n M_n$ as in item *4* of Theorem 11.0.9.

[5] In other words, items *5* and *6* of Theorem 11.0.9 do not occur and thus, only item *4* occurs, with M_∞ being different from a helicoid.

Theorem 11.0.11 (Quadratic Curvature Decay Theorem, Meeks, Pérez, Ros [134]) *Let $M \subset \mathbb{R}^3 - \{\vec{0}\}$ be an embedded minimal surface with compact boundary (possibly empty), which is complete outside the origin $\vec{0}$; i.e. all divergent paths of finite length on M limit to $\vec{0}$. Then, M has quadratic decay of curvature if and only if its closure in $\mathbb{R}^3$ has finite total curvature. In particular, every connected, complete, embedded minimal surface $M \subset \mathbb{R}^3$ with compact boundary and quadratic decay of curvature is properly embedded in $\mathbb{R}^3$. Furthermore, if C is the maximum of the logarithmic growths of the ends of M, then*

$$\lim_{R\to\infty} \sup_{M-\mathbb{B}(R)} |K_M|R^4 = C^2,$$

where $\mathbb{B}(R)$ denotes the extrinsic ball of radius R centered at $\vec{0}$.

Theorem 11.0.11 has been used in two different ways by Meeks, Pérez and Ros. Their first application is the Dynamics Theorem 11.0.13 that we explain below. A second consequence of Theorem 11.0.11, which appears in [142], deals with the space $\mathcal{F}_C$ of all connected, complete, embedded minimal surfaces $M \subset \mathbb{R}^3$ with prescribed quadratic curvature decay constant $C > 0$, normalized so that the maximum of the function $|K_M|R^2$ occurs at a point of $M \cap \mathbb{S}^2$. A result in [142] asserts that for $C > 0$ fixed, $\mathcal{F}_C$ is naturally a compact metric space, there is a bound on the topology of surfaces in $\mathcal{F}_C$ and that the subsets of $\mathcal{F}_C$ whose surfaces have fixed topology are compact.

Next we will explain the Dynamics Theorem mentioned above. To clarify its statement, we need some definitions.

Definition 11.0.12 Let $M \subset \mathbb{R}^3$ be a properly embedded, non-flat minimal surface. Then:

1. M is called *periodic,* if it is invariant under a non-trivial translation or a screw motion symmetry.

2. M is said to be *quasi-translation-periodic,* if there exists a divergent sequence $\{p_n\}_n \subset \mathbb{R}^3$ such that $\{M - p_n\}_n$ converges in a C^2-manner on compact subsets of $\mathbb{R}^3$ to M (note that every periodic surface is also quasi-translation-periodic, even in the case the surface is invariant under a screw motion symmetry).

3. M is called *quasi-dilation-periodic,* if there exists a sequence of homotheties[6] $\{h_n\}_n$ and a divergent sequence $\{p_n\}_n \subset \mathbb{R}^3$ such that $\{h_n(M - p_n)\}_n$ converges in a C^2-manner on compact subsets of $\mathbb{R}^3$ to M. Since M is not flat, it is not stable and, thus, the convergence of $\{h_n(M-p_n)\}_n$ to M has multiplicity one by Proposition 4.2.1 in [186]. Clearly, every quasi-translation-periodic surface is quasi-dilation-periodic.

[6] We do not assume any property on the ratios of these homotheties.

4. Let $D(M)$ be the set of properly embedded, non-flat minimal surfaces in $\mathbb{R}^3$ which are obtained as C^2-limits (these are again limits with multiplicity one since they are not stable) of a divergent sequence of dilations of M (i.e. the translational part of the dilations diverges). A subset $\Delta \subset D(M)$ is called *D-invariant* if it satisfies:

$$D(\Sigma) \subset \Delta, \text{ for every } \Sigma \in \Delta.$$

A D-invariant subset $\Delta \subset D(M)$ is called a *minimal D-invariant set* if it contains no proper non-empty D-invariant subsets. We say that $\Sigma \in D(M)$ is a *minimal element* if Σ is an element of a minimal D-invariant subset of $D(M)$. Note that $D(M)$ is D-invariant, and so, by item *4* in Theorem 11.0.13 below, $D(M)$ always contains minimal elements provided that $D(M) \neq \emptyset$.

We next illustrate the above concepts via some examples. We first consider the case where $M \subset \mathbb{R}^3$ is a properly embedded, non-flat minimal surface of finite topology. If M has finite genus and one end, then Theorem 1.0.4 implies that M is asymptotic to a helicoid $\mathcal{H}$, and so, $D(M)$ is path connected and consists of all translated images and dilations of the helicoid $\mathcal{H}$. If M has finite genus and a finite number $k \geq 2$ of ends, then M has finite total curvature. Hence, the normal lines to M are arbitrarily close to a given direction outside of a ball $\mathbb{B}(\vec{0}, R)$ for $R > 0$ sufficiently large. Thus, the only dilation limits of M with divergent translational parts are collections of planes and so, $D(M) = \emptyset$.

When M is properly embedded with infinite topology, then the set $D(M)$ is always non-empty by the next theorem, and in general it has a more interesting structure. For example, consider a planar, unbounded, convex unitary[7] polygon $\mathcal{P} \subset \mathbb{R}^2$ with infinitely many edges lying on two half-lines $l_1, l_2 \subset \mathbb{R}^2$ as in Figure 11.1. Mazet, Rodriguez and Traizet [112] have proven that there exists a minimal graph over the interior of $\mathcal{P}$, which takes alternatively the values $+\infty, -\infty$ on consecutive edges of its boundary. The closure $\mathcal{G}$ of this graph has vertical lines over the vertices of $\mathcal{P}$. By a theorem of Krust (unpublished, see Theorem 2.4.1 in Karcher [95]), the interior of the conjugate surface $\mathcal{G}^*$ of $\mathcal{G}$ is a graph (hence embedded). The boundary of $\mathcal{G}^*$ consists of infinitely many geodesics contained in two different horizontal planes. After reflection in one of these planes (see Footnote 15), one obtains a fundamental region of a properly embedded minimal surface $M \subset \mathbb{R}^3$ which is invariant under a vertical translation T (furthermore, $M/T \subset \mathbb{R}^3/T$ has genus zero and exactly one limit end), see Figure 11.2. Since M is periodic, then $M \in D(M)$. Furthermore, $D(M)$ has the following description depending on the case in Figure 11.1:

[7] *Unitary* means that the edges of $\mathcal{P}$ have length one.

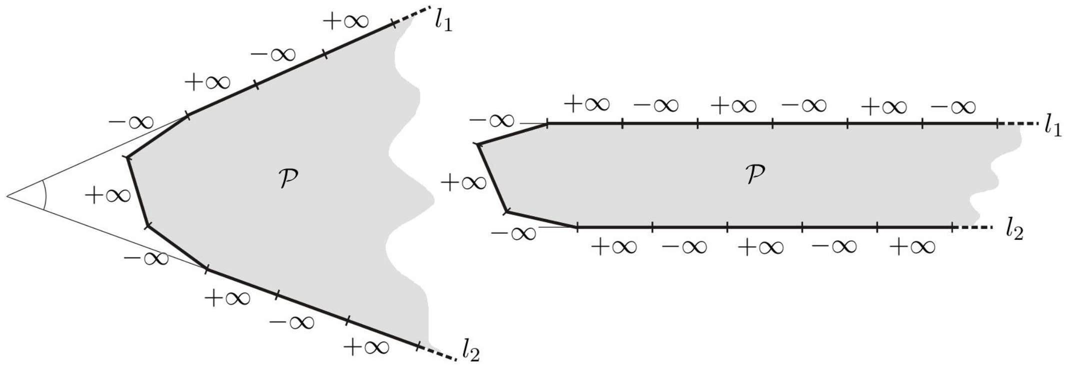

Figure 11.1: Two possibilities for the planar, unbounded, convex unitary polygon $\mathcal{P}$, and the boundary values for the minimal graph $\mathcal{G}$ over the interior of $\mathcal{P}$.

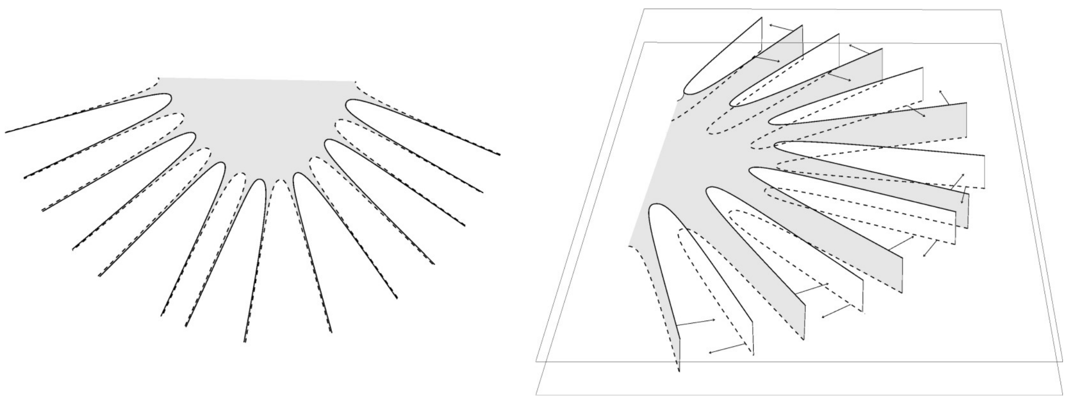

Figure 11.2: Left: The planar domain on which $\mathcal{G}^*$ is a graph. Right: The boundary of $\mathcal{G}^*$ consists of geodesics in two horizontal planes. Images courtesy of M. M. Rodríguez.

- Figure 11.1 left. $D(M)$ consists of three path components, each generated under dilations and translations either by M, by the doubly-periodic Scherk minimal surface $M(\theta)$ of angle $\theta = \pi/2$ and period vectors v_1, v_3 of the same length, with v_1 in the direction of l_1 and $v_3 = (0,0,2)$, or by the rotated image $M'(\theta)$ of $M(\theta)$ around the x_3-axis so that the lattice of periods of $M'(\theta)$ is generated by v_2, v_3, with v_2 in the direction of l_2. In this case, $M(\theta)$ and $M'(\theta)$ are minimal elements in $D(M)$, but M is not.

- Figure 11.1 right. $D(M)$ consists of two path components, each generated under dilations and translations either by M or by a certain KMR doubly-periodic torus M_∞ (see Chapter 2.5 for a description of these surfaces). Now, M_∞ is a minimal element in $D(M)$ and M is not. For details, see [112] and Rodríguez [192].

We now state the Dynamics Theorem for the space $D(M)$ of dilation limits of a properly embedded minimal surface $M \subset \mathbb{R}^3$.

Theorem 11.0.13 (Dynamics Theorem, Meeks, Pérez, Ros [134]) *Let $M \subset \mathbb{R}^3$ be a properly embedded, non-flat minimal surface. Then, $D(M) = \emptyset$ if and only if M has finite total curvature. Now assume that M has infinite total curvature, and consider $D(M)$ endowed with the topology of C^k-convergence on compact sets of $\mathbb{R}^3$ for all k. Then:*

1. *$D_1(M) = \{\Sigma \in D(M) \mid \vec{0} \in \Sigma,\ |K_\Sigma| \leq 1,\ |K_\Sigma|(\vec{0}) = 1\}$ is a non-empty compact subspace of $D(M)$.*

2. *For any $\Sigma \in D(M)$, $D(\Sigma)$ is a closed D-invariant set of $D(M)$. If $\Delta \subset D(M)$ is a D-invariant set, then its closure $\overline{\Delta}$ in $D(M)$ is also D-invariant.*

3. *Suppose that $\Delta \subset D(M)$ is a minimal D-invariant set which does not consist of exactly one surface of finite total curvature. If $\Sigma \in \Delta$, then $D(\Sigma) = \Delta$ and the closure in $D(M)$ of the path connected subspace of all dilations of Σ equals Δ. In particular, any minimal D-invariant set is connected and closed in $D(M)$.*

4. *Any non-empty D-invariant subset of $D(M)$ contains minimal elements.*

5. *Let $\Delta \subset D(M)$ be a D-invariant subset. If no $\Sigma \in \Delta$ has finite total curvature, then $\Delta_1 = \{\Sigma \in \Delta \mid \vec{0} \in \Sigma,\ |K_\Sigma| \leq 1,\ |K_\Sigma|(\vec{0}) = 1\}$ contains a minimal element Σ' with $\Sigma' \in D(\Sigma')$ (which in particular, is a quasi-dilation-periodic surface of bounded curvature).*

6. *If a minimal element Σ of $D(M)$ has finite genus, then either Σ has finite total curvature, or Σ is a helicoid, or Σ is a Riemann minimal example.*

We note that the last item in Theorem 11.0.13 requires Theorem 1.0.2 for its proof. To finish this chapter, we point out a consequence of Theorem 11.0.13: every properly embedded minimal surface $M \subset \mathbb{R}^3$ with infinite total curvature admits a limit surface $\Sigma \in \Delta_1$ which is a minimal element and is a limit of itself under a sequence of homotheties with divergent translational part. These properties allow to prove the following periodicity property for Σ (see [134] for details):

> *Each compact subdomain of Σ can be approximated with arbitrarily high precision (under dilation) by an infinite collection of pairwise disjoint compact subdomains of Σ.*

Embedded minimal surfaces of finite genus. 12

12.1 The Hoffman-Meeks conjecture.

By Theorem 1.0.5, every complete, embedded minimal surface in $\mathbb{R}^3$ with finite topology is proper, and hence, orientable. Until the early eighties of the last century, no complete embedded minimal surfaces of finite topology other than the plane, the helicoid (both with genus zero, one end) and the catenoid (genus zero, two ends) were known. For a long time, some geometers supported the conjecture that no other examples of finite topology would exist. The discovery in 1982 of a new genus-one, three-ended embedded example (Costa [41], Hoffman and Meeks [79]) not only disproved this conjecture, but also revitalized enormously the interest of geometers in classical minimal surface theory. Since then, a number of new embedded examples have appeared, sometimes even coming in multiparameter families [73, 75, 80, 92, 215, 222].

For complete embedded minimal surfaces with finite topology, there is an interesting dichotomy between the one-end case and those surfaces with more that one end: surfaces in this last case always have finite total curvature (Collin's Theorem 1.0.3). Only the simplest finite topologies with more than one end have been characterized: the unique examples with genus zero and finite topology are the plane and the catenoid (López and Ros's Theorem 3.1.2), the catenoid is the unique example with finite genus and two ends (Schoen's Theorem 3.1.1), and Costa [42, 43] showed that the examples with genus one and three ends lie inside the one-parameter family of surfaces $\{M_{1,a} \mid 0 < a < \infty\}$ that appears in Chapter 2.5. Today we know many more examples of more complicated finite topologies and more than one end, and up to this date all known examples support the following conjecture.

Conjecture 12.1.1 (Finite Topology Conjecture, Hoffman, Meeks) *A connected surface of finite topology, genus g and r ends, $r > 2$, can be properly minimally embedded in $\mathbb{R}^3$ if and only if $r \leq g + 2$.*

Meeks, Pérez and Ros [130] proved the existence of an upper bound for the number of ends of a minimal surface as in the above conjecture solely in terms of the genus, see Theorem 1.0.6 stated in the Introduction. We next give a sketch of the proof of this result.

Theorem 12.1.2 (Meeks, Pérez, Ros [130]) *For every $g \in \mathbb{N} \cup \{0\}$, there exists a positive integer $e(g)$ such that if $M \subset \mathbb{R}^3$ is a complete, embedded minimal surface of finite topology with genus g, then the number of ends of M is at most $e(g)$.*

Sketch of the proof. The argument is by contradiction. The failure of Theorem 1.0.6 to hold would produce an infinite sequence $\{M_n\}_n$ of complete, embedded minimal surfaces in $\mathbb{R}^3$ with fixed finite genus g and a strictly increasing number of ends, and which are properly embedded in $\mathbb{R}^3$ since they have finite total curvature. Then one analyzes the non-simply-connected limits of subsequences of $\{M_n\}_n$. The key idea used to achieve these limits is to normalize M_n by a translation followed by a homothety on the scale of topology. What we mean here is that we assume each renormalized surface intersects the closed unit ball of $\mathbb{R}^3$ centered at the origin in some non-simply-connected component, but that every open ball of radius 1 intersects the surface in simply-connected components. Note that we are using here the expression *on the scale of topology* for a sequence of minimal surfaces, rather than for only one minimal surface as we did in Theorem 11.0.9.

Using the uniformly locally simply-connected property of the renormalized sequence $\{M_n\}_n$ and the fact that the M_n have genus g, we analyze what are the possible limits of subsequences of $\{M_n\}_n$ through application of Theorem 11.0.9; these limits are either properly embedded, non-simply-connected minimal surfaces with genus at most g and possibly infinitely many ends, or parking garage structures on $\mathbb{R}^3$ with exactly two columns. The limits which are surfaces with infinitely many ends are then discarded by an application of either a descriptive theorem for the geometry of any properly embedded minimal surface in $\mathbb{R}^3$ with finite genus and two limit ends (Meeks, Pérez and Ros [136]), or a non-existence theorem for properly embedded minimal surfaces in $\mathbb{R}^3$ with finite genus and one limit end (Meeks, Pérez and Ros [140] or Theorem 12.2.1 below). The parking garage structure limits are also discarded, by a modification of the argument that eliminates the two-limit-ended limits. Hence, any possible limit M_∞ of a subsequence of $\{M_n\}_n$ must be a non-simply-connected minimal surface, which either has finite total curvature or is a helicoid with positive genus at most g. Then, a surgery argument allows one to modify the surfaces M_n by replacing compact pieces of M_n close to the limit M_∞ by a finite number of disks, obtaining a new surface $\widetilde{M}_n$ with strictly simpler topology than M_n and which is not minimal in the replaced part. A careful study of the replaced parts during the sequence allows one to iterate the process of finding non-simply-connected minimal limits of the modified surfaces $\widetilde{M}_n$. The fact that all the M_n have the same finite genus, allows one to arrive to a stage in the process of producing limits from which all subsequent limits have genus zero, and so they are catenoids. From this point in the proof, one shows that there exists a large integer n such that M_n contains a non-compact planar domain $\Omega \subset M_n$ whose boundary consists of two convex planar curves Γ_1, Γ_2 in parallel planes, such that each Γ_i separates M_n and has flux orthogonal to the plane that contains Γ_i. In this setting, the López-Ros deformation defined in Chapter 2.2 (see [107, 186]) applies to Ω giving the desired contradiction. This finishes the sketch of the proof of Theorem 1.0.6. □

Recall that the Jorge-Meeks formula (equation (2.9) in Chapter 2.3) relates the degree of the Gauss map of a complete, embedded minimal surface with finite total curvature in $\mathbb{R}^3$, with its genus and number of ends. Since the finite index of stability of a complete minimal surface of finite total curvature can be estimated from above in terms of the degree of its Gauss map (Tysk [220]), Theorem 1.0.6 has the following important theoretical consequence.

Corollary 12.1.3 (Meeks, Pérez, Ros [130]) *For every $g \in \mathbb{N} \cup \{0\}$, there exists an integer $i(g)$ such that if $M \subset \mathbb{R}^3$ is a complete, embedded minimal surface with genus g and a finite number of ends greater than 1, then the index of stability of M is at most $i(g)$.*

The above paragraphs deal with complete, embedded minimal surfaces of finite topology and more than one end. The case of one-ended surfaces of finite genus was covered by Theorems 1.0.1 and 1.0.4, see also the discussion about genus-k helicoids in Chapters 2.5 and 9.2, together with Conjecture 17.0.20 below.

12.2 Nonexistence of one-limit-ended examples.

Next we consider properly embedded minimal surfaces with finite genus and infinite topology. Note that we have included the hypothesis "proper" instead of "complete" since we can no longer apply Theorem 1.0.5 in this setting[1]. Since the number of ends of such a surface $M \subset \mathbb{R}^3$ is infinite and

[1]However, a result of Meeks, Pérez and Ros [131] states that a connected, complete, embedded minimal surface of finite genus in $\mathbb{R}^3$ which has at most a countable number of limit ends is always properly embedded.

the set of ends $\mathcal{E}(M)$ of M is compact (see Chapter 2.7), then M must have at least one limit end. Up to a rotation, we can assume that the limit tangent plane at infinity of M (Chapter 6) is horizontal. Theorem 7.3.1 insures that M has no middle limit ends, hence either it has one limit end (this one being the top or the bottom limit end) or both top and bottom ends are the limit ends of M, like in a Riemann minimal example. Meeks, Pérez and Ros [140] have discarded the one limit end case through the following result.

Theorem 12.2.1 (Meeks, Pérez, Ros [140]) *If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with finite genus, then M cannot have exactly one limit end. Furthermore, M is recurrent.*

Sketch of the proof. Assume M is a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus and exactly one limit end. After a rotation, we can suppose that M has horizontal limit tangent plane at infinity and its set of ends, linearly ordered by increasing heights (see the Ordering Theorem 6.0.11), is $\mathcal{E}(M) = \{e_1, e_2, \ldots, e_\infty\}$ with the limit end of M being its top end e_∞. One first needs a previous description of the asymptotic behavior of M by Meeks, Pérez and Ros (Theorem 2 in [139]): each non-limit end $e_n \in \mathcal{E}(M)$ is asymptotic to a graphical annular end E_n of a vertical catenoid with negative logarithmic growth a_n satisfying $a_1 \leq \ldots \leq a_n \leq \ldots < 0$. The next step consists of a detailed analysis of the limits (after passing to a subsequence) of homothetic shrinkings $\{\lambda_n M\}_n$, where $\{\lambda_n\}_n \subset \mathbb{R}^+$ is any sequence of numbers decaying to zero; one first shows that $\{\lambda_n M\}_n$ is locally simply-connected in $\mathbb{R}^3 - \{\vec{0}\}$ (see Definition 4.1.6). This is a difficult technical part of the proof, where the results of Colding and Minicozzi again play a crucial role. Then one proves that the limits of subsequences of $\{\lambda_n M\}_n$ consist of (possibly singular) minimal laminations $\mathcal{L}$ of $H(*) = \{x_3 \geq 0\} - \{\vec{0}\} \subset \mathbb{R}^3$ containing $\partial H(*)$ as a leaf. Subsequently, one checks that every such limit lamination $\mathcal{L}$ is smooth and that the singular set of convergence $S(\mathcal{L})$ of $\lambda_n M$ to $\mathcal{L}$ is empty (we will discuss this step a bit more after this sketched proof). In particular, taking $\lambda_n = \|p_n\|^{-1}$ where p_n is any divergent sequence on M, the fact that $S(\mathcal{L}) = \emptyset$ for the corresponding limit minimal lamination $\mathcal{L}$ insures that the Gaussian curvature of M decays at least quadratically in terms of the distance function to the origin. By the Quadratic Curvature Decay Theorem (Theorem 11.0.11), this implies that M has finite total curvature. This is impossible in our situation with infinitely many ends, and finishes the outline of the proof of the first statement of Theorem 12.2.1.

In order to finish the sketch of proof, it only remains to check that M is recurrent. If M has exactly one end, then M is conformally a compact Riemann surface minus one point (Theorem 1.0.4) and so, M is recurrent. If M has a finite number of ends greater than one, then M has finite total curvature (Theorem 1.0.3). By the Huber-Osserman Theorem 2.3.1, M is conformally a compact Riemann surface minus a finite number of points, thus it is again recurrent. Finally, if M has infinitely many ends, then M has exactly two limit ends, see the paragraph just before the statement of Theorem 12.2.1. In this situation, Theorem 7.3.3 asserts that M is recurrent, thereby finishing our sketch of proof. □

There is a crucial point in the above sketch of the proof of Theorem 12.2.1, that should be emphasized. We mentioned above that any limit lamination $\mathcal{L}$ of $H(*)$ obtained as a limit of (a subsequence of) homothetic shrinkings $\{\lambda_n M\}_n$ with $\lambda_n \searrow 0$, has no singularities and empty singular set of convergence $S(\mathcal{L})$. These properties are consequences of what we call the *Stability Lemma* (see Lemma 12.2.2), which we want to explain now. If such a lamination $\mathcal{L}$ had singularities or if $S(\mathcal{L})$ were non-empty for a given sequence of shrinkings of M, then we would find some smooth limit leaf L of $\mathcal{L}$, and hence L would be stable. The difficulty in discarding this possibility lies in the fact that the stable leaves of $\mathcal{L}$, while perhaps proper in $\{x_3 > 0\}$, may not be complete and so, we do not know they must be planes. It is not difficult to prove that the smooth stable leaves in $\mathcal{L}$ in fact satisfy the hypotheses of the Stability Lemma 12.2.2 below, and so they are planes. Once one has that the smooth stable leaves in $\mathcal{L}$ are planes, then the proof of Theorem 4.1.2 leads to a contradiction, thereby showing that $\mathcal{L}$ has no singularities and $S(\mathcal{L})$ is empty.

We include below the proof of the Stability Lemma for three reasons: firstly, to illustrate how one can obtain information on the conformal structure of possibly incomplete minimal surfaces by studying conformally related metrics, and then how to apply such information to constrain their geometry. Secondly, because Theorem 2.8.4 is a direct consequence of Lemma 12.2.2. And thirdly, because the short proof of Lemma 12.2.2 below indicates some new techniques and insights for possibly solving the Isolated Singularities Conjecture stated in Conjecture 17.0.16 below. The next lemma by Meeks, Pérez, Ros [134] was also found independently by Colding and Minicozzi [24], and it is motivated by a similar result of Meeks, Pérez and Ros in [140].

Lemma 12.2.2 (Stability Lemma) *Let $L \subset \mathbb{R}^3 - \{\vec{0}\}$ be a stable, orientable, minimal surface which is complete outside the origin; i.e. all divergent paths of finite length on L limit to the origin $\vec{0}$. Then, its closure $\overline{L}$ is a plane.*

Proof. Consider the metric $\widetilde{g} = \frac{1}{R^2} g$ on L, where g is the metric induced by the usual inner product $\langle , \rangle$ of $\mathbb{R}^3$. Note that if L were a plane through $\vec{0}$, then $\widetilde{g}$ would be the metric on L of an infinite cylinder of radius 1 with ends at $\vec{0}$ and at infinity. We will show that in general, this metric is complete on L and that the assumption of stability can be used to show that (L, g) is flat. Since $(\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$ with $\widehat{g} = \frac{1}{R^2}\langle , \rangle$, is isometric to $\mathbb{S}^2 \times \mathbb{R}$ and L is complete outside $\vec{0}$, then $(L, \widetilde{g}) \subset (\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$ is complete.

We now prove that (L, g) is flat. The laplacians and Gauss curvatures of $g, \widetilde{g}$ are related by the equations $\widetilde{\Delta} = R^2\Delta$ and $\widetilde{K} = R^2(K_L + \Delta \log R)$. Since $\Delta \log R = \frac{2(1-\|\nabla R\|^2)}{R^2} \geq 0$, then

$$-\widetilde{\Delta} + \widetilde{K} = R^2(-\Delta + K_L + \Delta \log R) \geq R^2(-\Delta + K_L).$$

Since $K_L \leq 0$ and (L, g) is stable, $-\Delta + K_L \geq -\Delta + 2K_L \geq 0$, and so, $-\widetilde{\Delta} + \widetilde{K} \geq 0$ on $(L, \widetilde{g})$. As $\widetilde{g}$ is complete, the universal covering of L is conformally $\mathbb{C}$ (Fischer-Colbrie and Schoen [58]). Since (L, g) is stable, there exists a positive Jacobi function u on L (Fischer-Colbrie [57]). Passing to the universal covering $\widehat{L}$, $\Delta\widehat{u} = 2K_{\widehat{L}}\widehat{u} \leq 0$, and so, $\widehat{u}$ is a positive superharmonic on $\mathbb{C}$, and hence constant. Thus, $0 = \Delta u - 2K_L u = -2K_L u$ on L, which means $K_L = 0$. □

12.3 Uniqueness of the Riemann minimal examples.

If a properly embedded minimal surface $M \subset \mathbb{R}^3$ has finite genus and infinite topology, then Theorems 7.3.1 and 12.2.1 imply M has two limit ends which are its top and bottom ends (after a rotation so that the limit tangent plane at infinity of M is horizontal). The classical model in this setting is any of the surfaces in the one-parameter family of Riemann minimal examples, see Chapter 2.5. Meeks, Pérez and Ros [138] have characterized these last surfaces as the unique properly embedded minimal surfaces in $\mathbb{R}^3$ with genus zero and infinite topology (Theorem 12.3.1 below), a uniqueness result that, together with Theorems 1.0.3, 1.0.4 and 3.1.2, finishes the classification of all properly embedded minimal planar domains in $\mathbb{R}^3$ stated in Theorem 1.0.2. We will explain the main steps in the proof of this characterization below, see also [128] for further details.

Theorem 12.3.1 *Every properly embedded, minimal planar domain $M \subset \mathbb{R}^3$ with infinite topology is a Riemann minimal example.*

Sketch of the proof. We first rotate M in $\mathbb{R}^3$ to have horizontal limit tangent plane at infinity. Next, we rescale M by a suitable homothety so that the vertical component of the flux vector of M along a compact horizontal section $M \cap \{x_3 = \text{constant}\}$ is equal to 1. At this point we need a previous analytic and geometric description of M (Theorem 1 in Meeks, Pérez and Ros [139]):

(A) M can be conformally parameterized by the cylinder $\mathbb{C}/\langle i \rangle$ (here $i = \sqrt{-1}$) punctured in an infinite discrete set of points $\{p_j, q_j\}_{j \in \mathbb{Z}}$ which correspond to the planar ends of M.

(B) The stereographically projected Gauss map $g\colon (\mathbb{C}/\langle i\rangle) - \{p_j, q_j\}_{j\in\mathbb{Z}} \to \mathbb{C}\cup\{\infty\}$ extends through the planar ends of M to a meromorphic function g on $\mathbb{C}/\langle i\rangle$ which has double zeros at the points p_j and double poles at the q_j.

(C) The height differential of M is $dh = dz$ with z being the usual conformal coordinate on $\mathbb{C}$, hence the third coordinate function of M is $x_3(z) = \Re(z)$.

(D) The planar ends of M are ordered by their heights so that $\Re(p_j) < \Re(q_j) < \Re(p_{j+1})$ for all j with $\Re(p_j) \to \infty$ (resp. $\Re(p_j) \to -\infty$) when $j \to \infty$ (resp. $j \to -\infty$).

(E) The flux vector F of M along a compact horizontal section has non-zero horizontal component; hence after a suitable rotation around the x_3-axis, $F = (h, 0, 1)$ for some $h > 0$.

(F) With the normalizations above, the Gaussian curvature of M is bounded and the vertical spacings between consecutive planar ends are bounded from above and below by positive constants, with all these constants depending only on h. This is a non-trivial application of Colding-Minicozzi theory [139].

(G) For every divergent sequence $\{z_k\}_k \subset \mathbb{C}/\langle i\rangle$, there exists a subsequence of the meromorphic functions $g_k(z) = g(z + z_k)$ which converges uniformly on compact subsets of $\mathbb{C}/\langle i\rangle$ to a non-constant meromorphic function $g_\infty\colon \mathbb{C}/\langle i\rangle \to \mathbb{C}\cup\{\infty\}$ (we will refer to this property saying that g is *quasiperiodic*). In fact, g_∞ corresponds to the Gauss map of a minimal surface M_∞ satisfying the same properties as M, which is the limit of a related subsequence of translations of M by vectors whose x_3-components are $\Re(z_k)$. Such a limit exists by an application of item **(F)** above.

Let $\mathcal{M}$ be the space of properly embedded, minimal planar domains $M \subset \mathbb{R}^3$ with two limit ends and flux $F = (h, 0, 1)$ for some $h = h(M) > 0$, identified up to translations. Every minimal surface $M \in \mathcal{M}$ admits a classical Jacobi function called its *Shiffman function* S_M, which measures the curvature variation of the parallel sections of M. Analytically, S_M is given by

$$S_M = \Lambda \frac{\partial \kappa_c}{\partial y} = \Im\left[\frac{3}{2}\left(\frac{g'}{g}\right)^2 - \frac{g''}{g} - \frac{1}{1+|g|^2}\left(\frac{g'}{g}\right)^2\right], \tag{12.1}$$

where Λ is the conformal factor between the induced metric on M by the inner product of $\mathbb{R}^3$ and the flat metric in the conformal coordinate $z = x + iy$, i.e. $ds^2 = \Lambda^2|dz|^2$, and $\kappa_c(y)$ is the planar curvature of the level curve $M \cap \{x_3 = c\}$ ($c \in \mathbb{R}$), which is parameterized by $z_c(y) = c + iy$ since $x_3(z) = \Re(z) = x$. The second equality in (12.1) expresses S_M solely as a rational expression of g and its derivatives with respect to z up to order 2.

Since Λ is a positive function, the zeros of S_M coincide with the critical points of $\kappa_c(y)$. Thus, S_M vanishes identically if and only if M is foliated by circles and straight lines in horizontal planes. In a posthumously published paper, B. Riemann [190, 191] classified all minimal surfaces with such a foliation property: they reduce to the plane, catenoid, helicoid and the 1-parameter family of Riemann minimal examples (which took this name from this discovery of Riemann). Therefore, a possible approach to proving Theorem 12.3.1 is to verify that $S_M = 0$, but instead of proving this fact directly, Meeks, Pérez and Ros demonstrated that S_M can be integrated in the following sense:

> *Given a minimal surface $M \in \mathcal{M}$, there exists a one-parameter family $\{M_t\}_t \subset \mathcal{M}$ such that $M_0 = M$ and the normal variational vector field for this variation, when restricted to each M_t, is the Shiffman Jacobi function S_{M_t} multiplied by the unit normal vector field to M_t.*

In fact, the parameter t of this deformation can be extended to be a complex number, and $t \mapsto M_t$ can be viewed as the real part of a complex valued *holomorphic* curve in a certain complex variety. This is a very special integration property for S_M, which we refer to by saying that the Shiffman function can be *holomorphically integrated* for every surface $M \in \mathcal{M}$.

The method for proving that the Shiffman function S_M can be holomorphically integrated for every such M is through the *Korteweg-de Vries equation* (KdV) and its *hierarchy,* two fundamental tools in integrable systems theory. We recommend the interested reader to consult the excellent survey by Gesztesy and Weikard [66] for an overview of the notions and properties which we will use in the sequel. Briefly, the way to obtain the KdV equation from the Shiffman function is as follows.

It turns out that S_M admits a globally defined conjugate Jacobi function[2] S_M^* on $(\mathbb{C}/\langle i\rangle) - B(N)$, where $B(N)$ denotes the set of branch points of the Gauss map N of M. Thus, $S_M + iS_M^*$ can be viewed as the support function of a holomorphic map from $(\mathbb{C}/\langle i\rangle) - B(N)$ into $\mathbb{C}^3$, namely $S_M + iS_M^* = \langle X_u + i(X_u)^*, N\rangle$. The holomorphicity of $X_u + i(X_u)^*$ allows us to identify $S_M + iS_M^*$ with an infinitesimal deformation in the space $\mathcal{W}$ of quasiperiodic meromorphic functions on $\mathbb{C}/\langle i\rangle$ with double zeros and double poles, i.e. with the derivative $\dot{g}_S = \frac{d}{dt}\big|_{t=0} g_t$ of a holomorphic curve $t \in \mathbb{D}(\varepsilon) = \{t \in$

[2]The *conjugate Jacobi function* of a Jacobi function u over a minimal surface M is the (locally defined) support function u^* of the conjugate surface $(X_u)^*$ of the branched minimal surface X_u associated to u by the Montiel-Ros correspondence, see Theorem 2.8.8. In particular, both X_u and $(X_u)^*$ have the same Gauss map as M, and u^* also satisfies the Jacobi equation.

$\mathbb{C} \mid |t| < \varepsilon\} \mapsto g_t \in \mathcal{W}$ with $g_0 = g$, which can be computed from (12.1) as

$$\dot{g}_S = \frac{i}{2}\left(g''' - 3\frac{g'g''}{g} + \frac{3}{2}\frac{(g')^3}{g^2}\right). \tag{12.2}$$

Therefore, to integrate S_M holomorphically one needs to find a holomorphic curve $t \in \mathbb{D}(\varepsilon) \mapsto g_t \in \mathcal{W}$ with $g_0 = g$, such that for all t the pair (g_t, dz) is the Weierstrass data of a minimal surface $M_t \in \mathcal{M}$ and $\frac{d}{dt}\big|_t g_t = \frac{i}{2}\left(g_t''' - 3\frac{g_t' g_t''}{g_t} + \frac{3}{2}\frac{(g_t')^3}{g_t^2}\right)$.

Viewing (12.2) as an evolution equation (in complex time t), one could apply general techniques to find solutions $g_t = g_t(z)$ defined locally around a point $z_0 \in (\mathbb{C}/\langle i\rangle) - g^{-1}(\{0,\infty\})$ with the initial condition $g_0 = g$, but such solutions are not necessarily defined on the whole cylinder, can develop essential singularities, and even if they were meromorphic on $\mathbb{C}/\langle i\rangle$, it is not clear *a priori* that they would have only double zeros and poles and other properties necessary to give rise to minimal surfaces M_t in $\mathcal{M}$. All these problems are solved by arguments related to the theory of the (meromorphic) KdV equation, as we will next explain.

The change of variables

$$u = -\frac{3(g')^2}{4g^2} + \frac{g''}{2g}. \tag{12.3}$$

transforms (12.2) into the evolution equation

$$\frac{\partial u}{\partial t} = -u''' - 6uu', \tag{12.4}$$

which is the KdV equation[3]. The reader may wonder why the KdV equation appears in connection to the Shiffman function. The change of variables $x = g'/g$ transforms the expression (12.2) for $\dot{g}_S$ into the evolution equation $\dot{x} = \frac{i}{2}(x''' - \frac{3}{2}x^2x')$, called a *modified KdV equation* (mKdV). It is well-known that mKdV equations in x can be transformed into KdV equations in u through the so called *Miura transformations,* $x \mapsto u = ax' + bx^2$ with a, b suitable constants, see for example [66] page 273. Equation (12.3) is nothing but the composition of $g \mapsto x$ and a Miura transformation. The holomorphic integration of the Shiffman function S_M could be performed just in terms of the theory of the mKdV equation, but we will instead use the more standard KdV theory.

Coming back to the holomorphic integration of S_M, this integration amounts to solving globally in $\mathbb{C}/\langle i\rangle$ the Cauchy problem for equation (12.4), i.e. finding a meromorphic solution $u(z,t)$ of (12.4) with $z \in \mathbb{C}/\langle i\rangle$ and

[3]In the literature one can find different normalizations of the KdV equation, given by different coefficients for u''', uu' in equation (12.4); all of them are equivalent up to a change of variables.

$t \in \mathbb{D}(\varepsilon)$, whose initial condition is $u(z,0) = u(z)$ given by (12.3). It is a well-known fact in KdV theory (see for instance [66] and also see Segal and Wilson [207]) that such a Cauchy problem can be solved globally producing a holomorphic curve $t \mapsto u_t$ of meromorphic functions $u(z,t) = u_t(z)$ on $\mathbb{C}/\langle i\rangle$ (with controlled Laurent expansions in poles of u_t) provided that the initial condition $u(z)$ is an *algebro-geometric potential* for the KdV equation. To understand this last notion, one must view (12.4) as the case $n = 1$ of a sequence of evolution equations in u called the *KdV hierarchy,*

$$\left\{ \frac{\partial u}{\partial t_n} = -\partial_z \mathcal{P}_{n+1}(u) \right\}_{n\geq 0}, \tag{12.5}$$

where $\mathcal{P}_{n+1}(u)$ is a differential operator given by a polynomial expression of u and its derivatives with respect to z up to order $2n$. These operators, which are closely related to Lax Pairs (see Section 2.3 in [66]) are defined by the recurrence law

$$\begin{cases} \partial_z \mathcal{P}_{n+1}(u) = (\partial_{zzz} + 4u\,\partial_z + 2u')\mathcal{P}_n(u), \\ \mathcal{P}_0(u) = \frac{1}{2}. \end{cases} \tag{12.6}$$

In particular, $\mathcal{P}_1(u) = u$ and $\mathcal{P}_2(u) = u'' + 3u^2$ (plugging $\mathcal{P}_2(u)$ in (12.5) one obtains the KdV equation). Hence, for each $n \in \mathbb{N} \cup \{0\}$ one must consider the right-hand-side of the n-th equation in (12.5) as a polynomial expression of $u = u(z)$ and its derivatives with respect to z up to order $2n+1$. We will call this expression a *flow*, denoted by $\frac{\partial u}{\partial t_n}$. A function $u(z)$ is said to be an *algebro-geometric potential* of the KdV equation if there exists a flow $\frac{\partial u}{\partial t_n}$ which is a linear combination of the lower order flows in the KdV hierarchy.

With all these ingredients, one needs to check that for every minimal surface $M \in \mathcal{M}$, the function $u = u(z)$ defined by equation (12.3) in terms of the Gauss map $g(z)$ of M, is an algebro-geometric potential of the KdV equation. This property follows from a combination of the following two facts:

1. Each flow $\frac{\partial u}{\partial t_n}$ in the KdV hierarchy produces a *bounded,* complex valued Jacobi function v_n on $\mathbb{C}/\langle i\rangle$ in a similar manner to the way that $\frac{\partial u}{\partial t_1}$ produces the complex Shiffman function $S_M + iS_M^*$.

2. Since the Jacobi functions v_n produced in item 1 are bounded on $\mathbb{C}/\langle i\rangle$, they can be considered to lie in the kernel of a Schrödinger operator L_M on $\mathbb{C}/\langle i\rangle$ with bounded potential; namely, $L_M = (\Delta_{\mathbb{S}^1} + \partial_t^2) + V_M$ where $\mathbb{C}/\langle i\rangle$ has been isometrically identified with $\mathbb{S}^1 \times \mathbb{R}$ endowed with the usual product metric $d\theta^2 \times dt^2$, and the potential V_M is the square of the norm of the differential of the Gauss map of M with respect to $d\theta^2 \times dt^2$ (V_M is bounded since M has bounded Gaussian curvature, see property **(F)** above). Finally, the kernel of L_M restricted to bounded functions is

finite dimensional; this finite dimensionality was proved by Meeks, Pérez and Ros[4] in [136] and also follows from a more general result by Colding, de Lellis and Minicozzi [23]).

The aforementioned control on the Laurent expansions in poles of u_t, coming from the integration of the Cauchy problem for the KdV equation, is enough to prove that the corresponding meromorphic function g_t associated to u_t by (12.3) has the correct behavior in poles and zeros; this property together with the fact that both S_M, S_M^* preserve infinitesimally the complex periods along any closed curve in $\mathbb{C}/\langle i \rangle$, suffice to show that the Weierstrass data (g_t, dz) solves the period problem and defines $M_t \in \mathcal{M}$ with the desired properties.

Finally, once the holomorphic integrability of S_M for any $M \in \mathcal{M}$ by a curve $t \mapsto M_t \in \mathcal{W}$ with $M_0 = M$ is proved, a compactness argument based on the fact that $t \mapsto M_t$ preserves the flux along every compact horizontal section gives that $\dot{g}_S$ given by (12.2) vanishes identically, which in turn implies that $S_M + iS_M^*$ is a complex valued linear function of the Gauss map, $S_M + iS_M^* = \langle a, N \rangle$ for some $a \in \mathbb{C}^3$. This property can be used to show that M is invariant by a translation T, inducing a properly embedded minimal torus in some quotient $\mathbb{R}^3/T$ with total curvature -8π. Under these conditions, a previous classification theorem by Meeks, Pérez and Ros [138] leads to the desired property that M is a Riemann minimal example, which finishes the sketch of the proof. $\square$

The classification Theorem 12.3.1 can be used to analyze the asymptotic behavior of properly embedded minimal surfaces with finite genus and infinitely many ends. Roughly speaking, it asserts that if M is a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus g and an infinite number of ends, then M has two limit ends and each of its limit ends is asymptotic as $x_3 \to \pm\infty$ to one of the limit ends of the Riemann minimal example with horizontal tangent plane at infinity and the same flux vector as M (see Theorem 9.1 in [136] for details).

12.4 Colding-Minicozzi theory (fixed genus).

Colding and Minicozzi have given a general description for the structure of limits of sequences of properly embedded minimal surfaces of finite genus. In [24], they consider a sequence of compact, embedded minimal surfaces $M_n \subset \mathbb{B}(R_n) = \mathbb{B}(\vec{0}, R_n)$ with finite genus at most k, $\partial M_n \subset \partial \mathbb{B}(R_n)$ and $R_n \to \infty$ as $n \to \infty$, and prove that after choosing a subsequence, there exists a sequence of intrinsic disks $D_n \subset M_n \cap \mathbb{B}(1)$ which converges *smoothly*

[4]Following arguments by Pacard (personal communication), which in turn are inspired in a paper by Lockhart and McOwen [105].

to a minimal disk $D_\infty \subset \mathbb{B}(1)$. Depending on whether D_∞ is flat or not, the description of the surfaces M_n for n large is as follows.

1. If D_∞ is not flat, then results by Colding and Minicozzi [24] and results by Meeks, Pérez and Ros explained in this chapter imply that the M_n converge smoothly on compact subsets of $\mathbb{R}^3$ with multiplicity one to a properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ in one of the following three cases.

   1.A. M_∞ is a helicoid with at most k handles.

   1.B. M_∞ is a surface of finite total curvature with genus at most k.

   1.C. M_∞ is a surface with two limit ends and genus at most k.

   In particular, in this case M_∞ has bounded Gaussian curvature, and so, the collection $\{M_n\}_n$ has uniformly bounded curvature on compact sets of $\mathbb{R}^3$.

2. If D_∞ is flat, then $\{M_n\}_n$ converge to a lamination $\mathcal{L}$ of $\mathbb{R}^3$ by parallel planes, which we may assume to be horizontal. After choosing a subsequence and composing with a fixed rotation of $\mathbb{R}^3$, the nature of this convergence can be of two types:

   2.A. The convergence is of class C^1 on all compact subsets of $\mathbb{R}^3$.

   2.B. The convergence is of class C^1 on compact subsets away from a non-empty, closed set $S(\mathcal{L})$ (in fact, $S(\mathcal{L})$ might be all of $\mathbb{R}^3$). In this case, either Theorem 4.1.7 holds for a subsequence of the M_n, or small necks form on M_n around *every* point $p \in S(\mathcal{L})$. In the case that small necks form on the M_n near a $p \in S(\mathcal{L})$, then one of the following two possibilities holds:

      1. A pair of almost-horizontal graphs G_n, G'_n form on M_n nearby p. In this case, both G_n, G'_n are graphs over annuli in the horizontal plane $\{x_3 = x_3(p)\}$ and converge smoothly away from p to this plane as $n \to \infty$.
      2. A "pair of paints decomposition" occurs on M_n nearby $p \in S(\mathcal{L})$. This means that after choosing a subsequence, there exist graphs $G_n \subset M_n$ converging to the plane $\{x_3 = x_3(p)\}$ as $n \to \infty$. This convergence is smooth away from p and possibly also another point $q \in S(\mathcal{L}) \cap \{x_3 = x_3(p)\}$; if the convergence is away from p, q, then the G_n are graphs defined over disks in $\{x_3 = x_3(p)\}$ with two subdisks removed (this is a *pair of paints* surface), see Figure 12.1.

   Note that each of these cases of forming necks can occur by taking either homothetic shrinkings of a fixed catenoid, or a sequence of homothetic shrinkings of appropriately chosen Riemann minimal

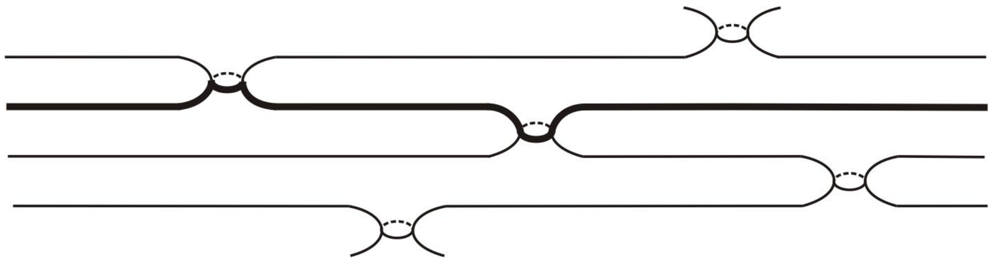

Figure 12.1: Small necks forming. In bold, “a pair of pants” piece.

examples, see [24] for more details. In the case of small necks forming, there is a conjecture for the structure of the singular set $S(\mathcal{L})$, see Conjecture 17.0.38 below.

We finish this chapter with a brief comment about properly embedded minimal surfaces of infinite genus. The collection of such surfaces with one end is extremely rich. One reason for this is that there are many doubly-periodic examples[5], and as we mentioned in Chapter 2.7 (see also Chapter 13 below), every doubly-periodic, properly embedded minimal surface in $\mathbb{R}^3$ has infinite genus and one end (Callahan, Hoffman and Meeks [16]). Besides Theorem 7.3.1 on the non-existence of middle limit ends, very little is known about properly embedded minimal surfaces with infinite genus and infinitely many ends. The first known examples arise from singly-periodic surfaces with planar ends and positive genus (Callahan, Hoffman and Meeks [15]), but these examples are better studied as minimal surfaces with finite total curvature in the corresponding quotient space. A tentative example of (truly) infinite genus and one limit end might be constructed as follows. As explained in Chapter 3.2, Weber and Wolf [222] proved the existence of a complete, immersed minimal surface $M(g)$ with finite total curvature, arbitrary odd genus g, horizontal ends (two catenoidal and g planar), which is embedded outside a ball in $\mathbb{R}^3$. Furthermore, computer graphics pictures indicate that all these surfaces $M(g)$ are embedded. Assuming this embeddedness property holds, a suitable normalization of the $M(g)$ should give as a limit when $g \to \infty$, a properly embedded minimal surface with a bottom catenoid end, infinitely many middle planar ends and a top limit end. By Theorem 12.2.1, this limit surface could not have finite genus.

[5] Note that any triply-periodic example can be viewed as a doubly-periodic one.

Topological aspects of minimal surfaces. 13

Two of the main challenges in the classical theory of minimal surfaces are to decide which non-compact topological types are admissible as properly embedded minimal surfaces in space, and given an admissible topological type in the previous sense, to show that there exists a unique way (up to ambient isotopy) of properly embedding this topological type as a minimal surface in $\mathbb{R}^3$. We discussed some aspects of the first of these problems in Chapter 12, and we will devote this chapter to explain the amazing advances achieved in the last decade for the second problem, including its final solution.

Definition 13.0.1 Two properly embedded surfaces in a three-manifold N are said to be *ambiently isotopic* if one can be deformed to the other by a continuous 1-parameter family $\{f_t \mid t \in [0,1]\}$ of diffeomorphisms of N. This notion gives rise to an equivalence relation on the set of all properly embedded surfaces in N, whose equivalence classes are called *isotopy classes.*

The problem of the topological uniqueness (up to ambient isotopy) of properly embedded minimal surfaces in three-manifolds has been classically tackled by different authors. In 1970, Lawson [101] showed that two embedded, compact, diffeomorphic minimal surfaces in $\mathbb{S}^3$ are ambiently isotopic. Meeks [116] generalized a key result of Lawson to the case of orientable, compact minimal surfaces in a compact three-manifold with non-negative Ricci curvature and also proved that any two compact, diffeomorphic minimal surfaces embedded in a convex body B in $\mathbb{R}^3$, each with boundary a simple closed curve on the boundary of B, are ambiently isotopic in B (this result fails for more than one boundary curve, as demonstrated by a counterexample of Hall [69]). Later Meeks, Simon and Yau [151] generalized Lawson's

Theorem to the ambient case of $\mathbb{S}^3$ with a metric of non-negative scalar curvature. Frohman [60] proved that two triply-periodic minimal surfaces in $\mathbb{R}^3$ are always ambiently isotopic. Although published two years later than Frohman's result, Meeks and Yau [158] had shown a decade earlier that if M_1, M_2 are properly embedded minimal surfaces in $\mathbb{R}^3$ with the same finite topological type, then they are ambiently isotopic.

An essential point in the proofs of these topological uniqueness results is to obtain a good understanding of the closed complements of the minimal surface in the ambient space. This problem of the topological classification of the closed complements and the related uniqueness of the surfaces up to isotopy depend on the concept of a Heegaard surface in a general three-manifold, although here we will only deal with the case of the ambient space being $\mathbb{R}^3$.

Definition 13.0.2 A three-manifold with boundary is a *handlebody* if it is homeomorphic to a closed regular neighborhood of a properly embedded, one-dimensional CW-complex in $\mathbb{R}^3$. A properly embedded surface $M \subset \mathbb{R}^3$ is called a *Heegaard surface* if each of the closed complements of M in $\mathbb{R}^3$ are handlebodies.

In 1997, Frohman and Meeks [62] proved that every properly embedded, one-ended minimal surface in $\mathbb{R}^3$ is a Heegaard surface. Additionally, they obtained a topological uniqueness result for Heegaard surfaces in $\mathbb{R}^3$: two such surfaces of the same genus (possibly infinite) are ambiently isotopic. Joining these two results they obtained the following statement.

Theorem 13.0.3 (Frohman, Meeks [62]) *Two properly embedded, one-ended, minimal surfaces in $\mathbb{R}^3$ with the same genus (possibly infinite) are ambiently isotopic. Furthermore, given every such minimal surface M, there exists a diffeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(M)$ equals the Heegaard surface $\Sigma \subset \mathbb{R}^3$ constructed as follows.*

- *If M has finite genus g, then Σ is the surface obtained after attaching g trivial unknotted handles to the (x_1, x_2)-plane P, each one in a small neighborhood of each of the points in the set $\{(1,0,0), (2,0,0), ..., (g,0,0)\} \subset P$, see Figure 13.1.*
- *If M has infinite genus, then Σ is the singly-periodic surface obtained after attaching infinitely (countably) many unknotted handles to the (x_1, x_2)-plane P with the attaching performed in neighborhoods of the points in $(n, 0, 0)$, $n \in \mathbb{Z}$.*

Furthermore, let W be the handlebody of $\mathbb{R}^3$ with boundary Σ, such that the closed lower half-space is contained in W. Then, one can choose the diffeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that a prescribed closed complement of M in $\mathbb{R}^3$ maps through h to the handlebody W.

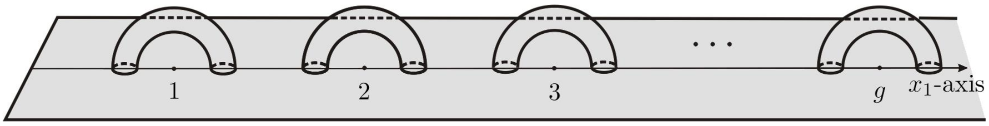

Figure 13.1: Topological model for a properly embedded minimal surface with one end and finite genus g.

To appreciate the power of Theorem 13.0.3, consider the singly and doubly-periodic Scherk minimal surfaces defined in Chapter 2.5. By Theorem 13.0.3, there exists a diffeomorphism of $\mathbb{R}^3$ that takes one surface to the other, although they look very different in space. We should recall here a result by Callahan, Hoffman and Meeks [16] that insures that every connected, doubly-periodic minimal surface in $\mathbb{R}^3$ has infinite genus and one end. Joining this result with Theorem 13.0.3, it follows that any two doubly-periodic minimal surfaces in $\mathbb{R}^3$ are ambiently isotopic (for example, this applies to the classical Schwarz Primitive minimal surface and the doubly-periodic Scherk minimal surface!).

All of these topological uniqueness results are special cases of a theorem by Frohman and Meeks, which represents the final solution to the topological classification problem. It is also based on a deep topological analysis of the complements of a properly embedded minimal surface in space, using previous work by Freedman [59], and shows what are the topological roles of the ordering on the set of ends given by Theorem 6.0.11 and the parity of each middle end defined in Theorem 7.3.1.

Theorem 13.0.4 (Topological Classification Theorem, Frohman, Meeks [63]) *Two properly embedded minimal surfaces in $\mathbb{R}^3$ are ambiently isotopic if and only if there exists a homeomorphism between the surfaces that preserves both the ordering of their ends and the parity of their middle ends.*

We now give an explicit cookbook-type recipe for representing the isotopy class of a properly embedded minimal surface $M \subset \mathbb{R}^3$ by a smooth surface $\widehat{M}$ constructed from basic building blocks (actually $\widehat{M}$ is only piecewise smooth, but easily smoothable). In the case that M has one end, then $\widehat{M}$ is the Heegaard surface Σ constructed just after the statement of Theorem 13.0.3. Thus, in this case the isotopy class of M is determined only by its topology. Suppose now that M has more than one end. After a rotation, assume that M has horizontal limit tangent plane at infinity, and the space of ordered ends of M (in the sense of Theorem 6.0.11) is $\mathcal{E}(M) = \{e_i \mid i \in I\}$, where

$$I = \begin{cases} \{1, 2, ..., k\} & \text{if } M \text{ has } k \text{ ends,} \\ \mathbb{N} \cup \{\infty\} & \text{if } M \text{ has one limit end,} \\ \mathbb{Z} \cup \{-\infty, \infty\} & \text{if } M \text{ has two limit ends.} \end{cases}$$

Let P denote the (x_1, x_2)-plane, $P_1 = \{x_3 = 1\}$ denote the horizontal plane at height 1, and for $m, n \in \mathbb{Z}$ define:

- $D(m, n) = \{(x_1, x_2, n) \mid (x_1 - m)^2 + x_2^2 \leq \frac{1}{3}\}$.
- $C(m, n) = \{(x_1, x_2, t) \mid (x_1, x_2, 0) \in \partial D(m, 0),\ n \leq t \leq n + 1\}$.

Using these planes, disks and cylinders, we obtain eight basic ends which will be the building blocks for constructing $\widehat{M}$ (see Figure 13.2).

1. $P_T = (P - D(0, 0)) \cup C(0, -1)$. This surface has one boundary circle, genus 0, and one annular end.

2. $P_B = (P - D(0, 0)) \cup C(0, 0)$. This surface is the reflected image of P_T upside down. P_B (resp. P_T) is used to represent a bottom (resp. top) simple end of genus zero.

3. For any positive integer g, $P_T(g)$ is P_T with a trivial handle attached in a small neighborhood of each of the points in $\{(1, 0, 0), (2, 0, 0), ..., (g, 0, 0)\}$. For $g = \infty$, $P_T(\infty)$ is P_T with a trivial handle attached in a small neighborhood of each of the points $\{(n, 0, 0) \mid n \in \mathbb{N}\}$. $P_T(\infty)$ is used to represent simple top ends of infinite genus.

4. $P_B(g)$ is obtained from P_B in a manner similar to how $P_T(g)$ was obtained from P_T (actually, we will only need $P_B(\infty)$ in our constructions).

5. $MO = (P_T - D(-1, 0)) \cup C(-1, 0)$. This surface has genus 0, two boundary circles and one annular end. MO is used to represent middle ends of $\widehat{M}$ of genus 0, which must always have odd parity.

6. $MO(1) = (P_T(1) - D(-1, 0)) \cup C(-1, 0)$. This surface has genus 1, two boundary circles and one annular end. $MO(1)$ is used to represent middle ends of $\widehat{M}$ of genus 0 (hence with odd parity), but only ones which occur either on certain infinite genus limit ends of $\widehat{M}$, or when $\widehat{M}$ has two limit ends and finite positive genus.

7. $ME = [(P \cup P_1) - \bigcup_{k=0}^{\infty} (D(k, 0) \cup D(k, 1))] \cup [\bigcup_{k=1}^{\infty} C(k, 0)] \cup [C(0, -1) \cup C(0, 1)]$. This surface has infinite genus, two boundary circles and one end. It is used to represent even middle ends.

8. $MO(\infty) = (P_T(\infty) - D(-1, 0)) \cup C(-1, 0)$. This surface has infinite genus and two boundary circles. It is used to represent odd middle ends of infinite genus.

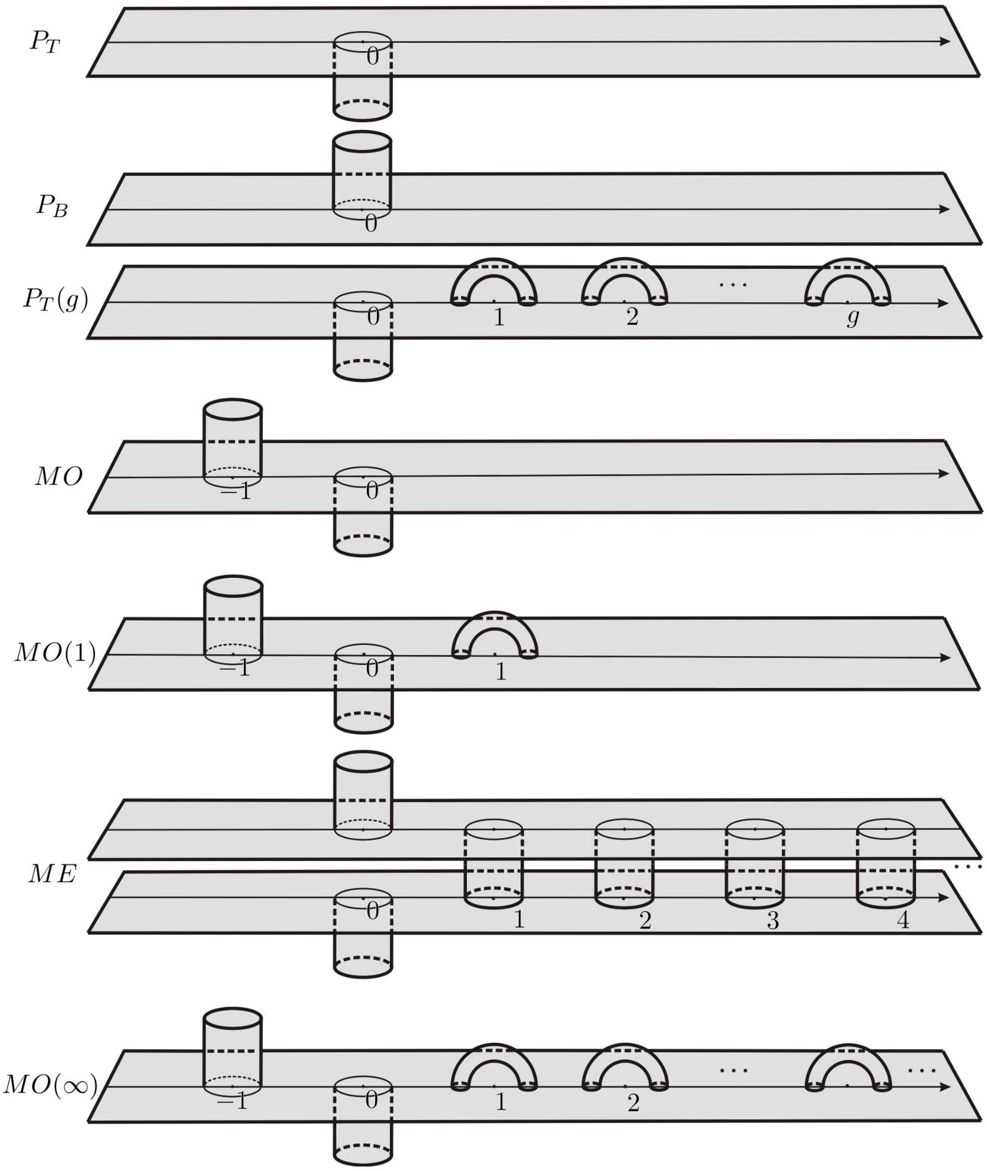

Figure 13.2: Building blocks for the basic ends.

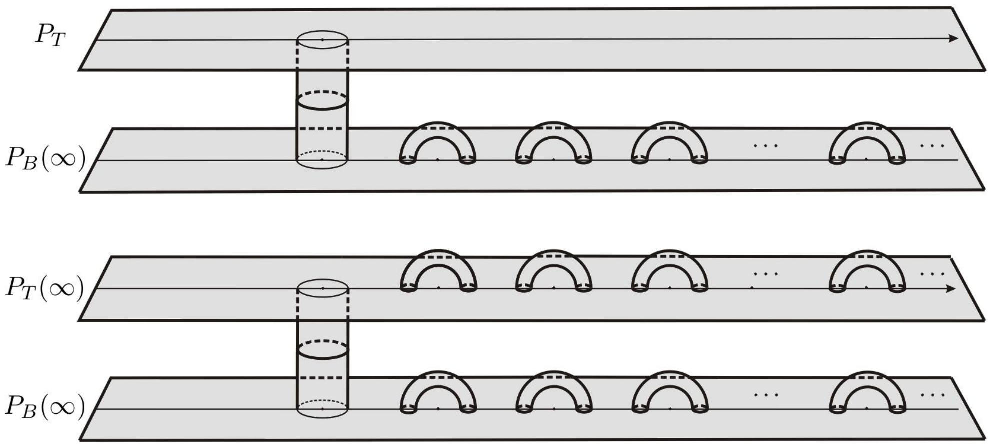

Figure 13.3: Topological model for the cases 1.II (up) and 1.III (down) of a properly embedded minimal surface with two ends.

We now construct a representative surface $\widehat{M}$ of the proper isotopy class of M in terms of topological invariants of M.

1. M **has two ends.**

   1.I. M has finite genus. Then, by Collin's Theorem 1.0.3, M has finite total curvature, and by Schoen's Theorem 3.1.1, M is a catenoid. Hence, $\widehat{M}$ can be made by gluing together P_T with a translated copy of P_B (namely $P_B - (0, 0, 2)$).

   1.II. M has one end of genus zero and one end of infinite genus. Then $\widehat{M}$ can be made by gluing together P_T with a translated copy of $P_B(\infty)$, see Figure 13.3 up.

   1.III. Each end of M has infinite genus. Then $\widehat{M}$ can be made by gluing together $P_T(\infty)$ with $P_B(\infty)$, see Figure 13.3 down.

2. M **has a finite number of ends** $\{e_1, e_2, ..., e_n\}$ **with** $n > 2$**.**

   2.I. M has finite genus g. Then by Collin's Theorem and the López-Ros Theorem 3.2, g is positive. In this case, $\widehat{M}$ can be made stacking $n-2$ translated copies of MO to P_B and then attaching a translated copy of $P_T(g)$ to the top of the stack, see Figure 13.4.

   2.II. M has infinite genus. In this case, one uses a combination of the basic ends $P_B(\infty)$, $P_T(\infty)$, $MO(\infty)$, ME, MO, P_B, P_T and stacks them according to their ordering (each of these basic ends is used according to whether an end of M has genus zero or infinite genus, and whether a middle end has even or odd parity). Note that even middle ends must have infinite genus, see Figure 13.5.

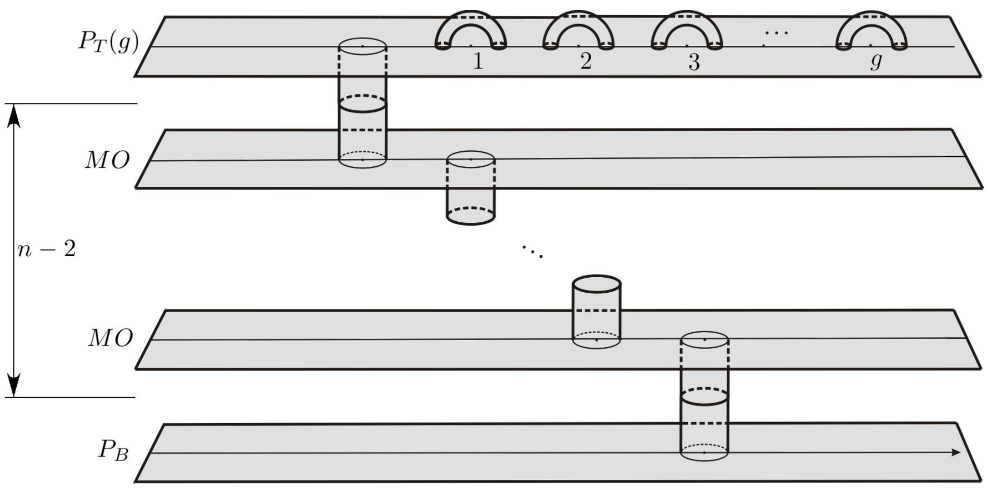

Figure 13.4: Topological model for the case 2.I.

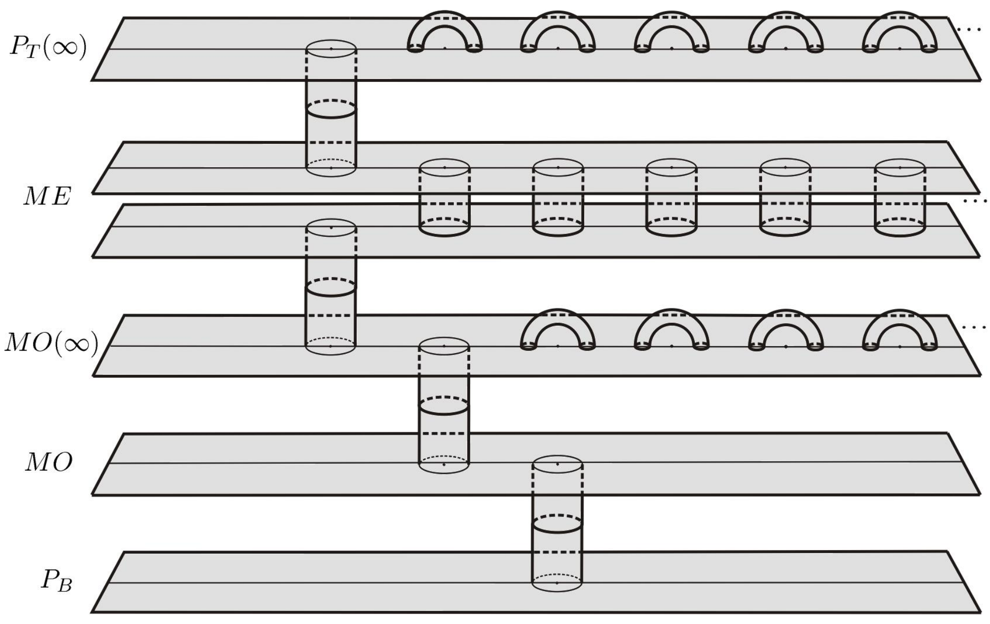

Figure 13.5: One example corresponding to the case 2.II; in this example, $n = 5$.

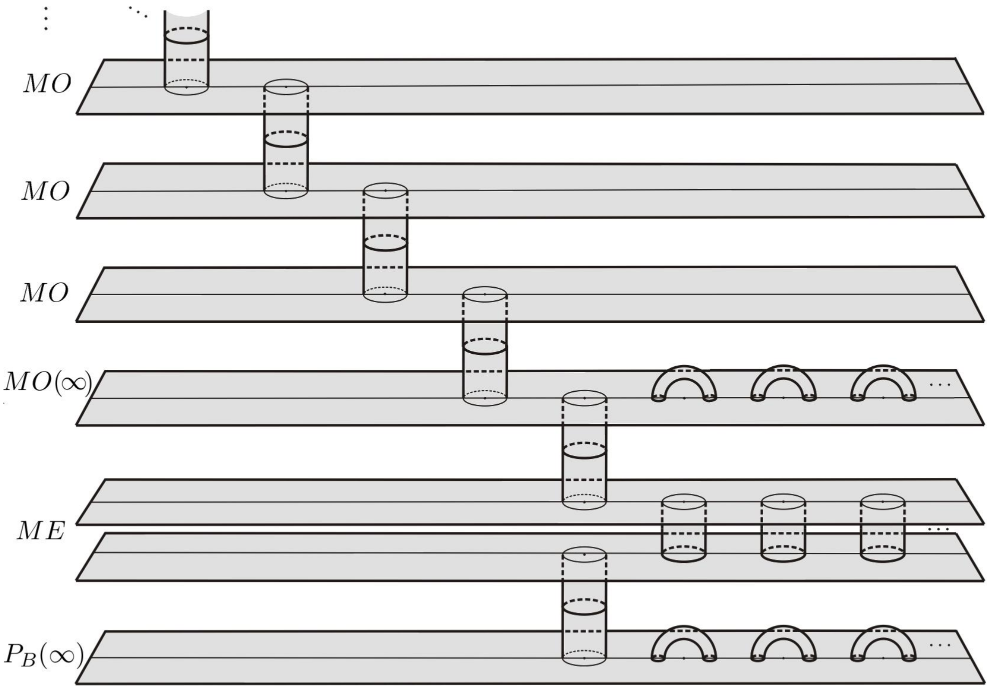

Figure 13.6: One example of the surface $\widehat{M}$ in the case 3.I.

3. M **has one limit end which we may assume is the top end** e_∞**.** Note that by Theorem 12.2.1, the genus of M must be infinite.

   3.I. e_∞ has genus zero. In this case, we construct the limit end of $\widehat{M}$ by stacking consecutively and upward, infinitely many basic ends of the type MO; then one glues from below this infinite stack, a configuration of finitely many ends, at least one of which has infinite genus, using the basic ends $P_B(\infty), P_B, MO, ME$ and $MO(\infty)$, chosen according to the ordering, genus and parity of the ends of M, see Figure 13.6.

   3.II. e_∞ has infinite genus. In other words, every representative of e_∞ has infinite genus. In this case, one uses the basic ends $P_B(\infty)$, P_B, $MO(1)$, ME, $MO(\infty)$ and stacks them according to their ordering and type, using $MO(1)$ for middle ends of genus zero (recall that although $MO(1)$ has genus 1, its end is an annulus and so, it represents an end of M with genus 0!).

4. M **has two limit ends.** Recall that by Theorem 7.3.1, the limit ends of M are its top end e_∞ and its bottom end $e_{-\infty}$.

   4.I. M has genus zero. In this case, M is a Riemann minimal example by Theorem 12.3.1. Therefore, one only uses the basic ends MO to stack up and down to make $\widehat{M}$.

4.II. <u>M has finite positive genus g.</u> In this case, one first stacks g consecutive copies of $MO(1)$ and then stacks up and down from this middle region with infinitely many copies of the basic end MO to make $\widehat{M}$.

4.III. <u>e_∞ has genus zero and $e_{-\infty}$ has infinite genus</u> (or vice versa). In this case, the top end $\widehat{e}_\infty$ of $\widehat{M}$ is constructed by stacking consecutively and upward infinitely many basic ends of the type MO, as we did with the limit end of the case 3.I above; the bottom end $\widehat{e}_{-\infty}$ of $\widehat{M}$ is constructed with a similar process as in case 3.II, and in between $\widehat{e}_\infty, \widehat{e}_{-\infty}$ we will place a finite stack of basic ends chosen among MO, ME and $MO(\infty)$.

4.IV. <u>Both limit ends of M have infinite genus.</u> In this case, both limit ends of $\widehat{M}$ follow the same rules described in case 3.II; in between $\widehat{e}_\infty, \widehat{e}_{-\infty}$ we will place a finite stack of basic ends chosen from among $MO(1), ME$ and $MO(\infty)$.

14 Partial results on the Liouville Conjecture.

The classical Liouville theorem for the plane asserts that every positive harmonic function on the complex plane is constant. The open unit disk $\mathbb{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ does not satisfy the same property, since the function $x+1 \colon \mathbb{D} \to (0,2)$ is a non-constant, bounded, positive harmonic function.

The existence or non-existence of non-constant, positive harmonic functions can be viewed as a tool for understanding the so-called *type problem* of classifying open Riemann surfaces. There are related properties useful for tackling this problem on a non-compact Riemann surface M without boundary, among which we emphasize the following ones:

1. M admits a non-constant, positive superharmonic function (equivalently, M is *transient*).
2. M admits a non-constant, positive harmonic function.
3. M admits a non-constant, bounded harmonic function.

We defined recurrency and transience in Chapter 7.1 for an n-dimensional Riemannian manifold without boundary; see e.g. Theorem 5.1 in Grigor'yan [67] for the equivalence in item 1 above and for general properties of the Brownian motion on manifolds. Clearly (3) $\Rightarrow$ (2) $\Rightarrow$ (1). These three conditions are known to be equivalent if M has finite genus, or more generally, if M has *almost-finite genus* (see [202] pages 193, 194 for this definition and result).

Recall that Theorem 7.2.4 insures that properly embedded minimal surfaces with bounded Gaussian curvature do not admit non-constant bounded harmonic functions. Closely related to this result is the Liouville Conjecture for properly embedded minimal surfaces, which states that *positive* *harmonic functions on properly embedded minimal surfaces in* $\mathbb{R}^3$ *are constant* (see Meeks [122], also see Conjecture 17.0.22 below). In relation with the above properties and the Liouville Conjecture, we note that by Theorem 12.2.1, every properly embedded minimal surface with finite genus in $\mathbb{R}^3$ is recurrent, and so, the Liouville Conjecture holds for these surfaces. Also note that embeddedness in this conjecture is essential since there exists a proper minimal immersion of the open unit disk into $\mathbb{R}^3$ (Morales [164]). Since every complete Riemannian manifold with quadratic volume growth is recurrent (see Corollary 7.4 in [67]), each of the classical singly-periodic Scherk minimal surfaces in $\mathbb{R}^3$ is recurrent. Another class of properly embedded minimal surfaces in $\mathbb{R}^3$ which are known to be recurrent are those with two limit ends by Theorem 7.3.3.

Meeks, Pérez and Ros [141] have determined which of the above three properties are satisfied by certain complete, embedded minimal surfaces $M \subset \mathbb{R}^3$, in terms of their topology and geometry. Let G be a group of orientation preserving isometries that acts properly and discontinuously on $\mathbb{R}^3$, which leaves a properly embedded minimal surface $M \subset \mathbb{R}^3$ invariant. Note that the properties we are interested in —recurrence, transience, existence of unbounded positive harmonic functions— hold simultaneously for the base and the total space of a finitely sheeted covering space, and bounded harmonic functions lift to arbitrary covers.

By classification, G lies in one of the following cases:

- G is a finite group. In this case, we will say that M is 0-*periodic.*
- G contains a finite index subgroup generated by a translation or screw motion symmetry. In this case, M is called 1-*periodic.*
- G contains a finite index subgroup generated by two (resp. three) independent translations, in which case M is called 2-*periodic* (resp. 3-*periodic*).

Next we summarize four main results of Meeks, Pérez and Ros appearing in [141].

Theorem 14.0.5 *A complete, embedded,* 1*-periodic minimal surface with bounded curvature in* $\mathbb{R}^3$ *does not admit non-constant, bounded harmonic functions and its quotient surface is recurrent.*

Theorem 14.0.6 *A properly immersed,* 2*-periodic minimal surface in* $\mathbb{R}^3$ *does not admit non-constant, bounded harmonic functions and its quotient surface is recurrent.*

Theorem 14.0.7 *Let* $M \subset \mathbb{R}^3$ *be a non-flat,* k*-periodic, complete embedded minimal surface, such that* $M/G \subset \mathbb{R}^3/G$ *has finite topology. Then,* M *is recurrent if and only if* $k \in \{0, 1\}$*.*

Theorem 14.0.8 *Let $M \subset \mathbb{R}^3$ be a complete, embedded minimal surface such that M/G has finite topology. Then, every positive harmonic function on M is constant.*

A long standing question in the classical theory of minimal surfaces asked whether or not the Liouville Conjecture held for the doubly-periodic Scherk minimal surfaces. We now see, by Theorem 14.0.8, that each of these surfaces satisfies the Liouville Conjecture. In contrast to this restrictive result on the conformal structure of a doubly-periodic Scherk minimal surface, Theorem 14.0.7 demonstrates that each doubly-periodic Scherk minimal surface is transient.

15 The Scherk Uniqueness Theorem.

In Chapter 3.2, we described the method of Kapouleas [92] for producing complete, embedded minimal surfaces of finite total curvature with an arbitrarily prescribed number of ends $k \geq 3$ and sufficiently large genus, starting with a configuration $\mathcal{F}$ of horizontal planes and catenoids with axis the x_3-axis, so that the total number of ends of the collection $\mathcal{F}$ is k. Recall that Kapouleas' method has two stages, the first one being the replacement of each circle C of intersection in $\mathcal{F}$ by a suitable bent, slightly deformed and scaled down singly-periodic Scherk minimal surface, glued along its wings to slight perturbations of the remaining bounded pieces and annular ends in $\mathcal{F} - \bigcup C$. The second stage in the method was to find a small minimal graph over the non-minimal surface constructed in the previous part, which is the desired minimal surface.

Kapouleas' method can be also applied to two non-degenerate (i.e. non-zero Jacobi functions with zero boundary values do not exist), compact, embedded minimal surfaces X_1, X_2 with boundary, such that X_1, X_2 intersect transversely in their interiors (see Kapouleas [93] for a detailed discussion). Again, the self-intersecting union can be replaced by an almost minimal, smooth surface X made by sewing in singly-periodic Scherk necklaces with small perturbations of $(X_1 \cup X_2) - (X_1 \cap X_2)$, and then one finds a small graph u over this auxiliary surface X, such that $X + uN_X$ is truly minimal, where N_X denotes the unit normal to X.

On the other hand, if a sequence $\{Y(n)\}_n$ of embedded minimal surfaces converges to the union of two embedded, transversely intersecting, compact minimal surfaces X_1, X_2, then it is conjectured that $Y(n)$ has the appearance of singly-periodic Scherk necklaces near $X_1 \cap X_2$. This conjecture means that for any point $p \in X_1 \cap X_2$, appropriate dilation scalings $\lambda_n(Y(n)-p)$ of $Y(n)$

with $\lambda_n \nearrow \infty$, should converge to one of the singly-periodic Scherk minimal surfaces, namely the one which desingularizes the planes passing through the origin and parallel to the tangent planes T_pX_1, T_pX_2. If one could prove this property for a sequence $\{Y(n)\}_n \to X_1 \cup X_2$ at all points $p \in X_1 \cup X_2$, then one would know that the desingularization method of Kapouleas is the only possibility. We believe that the proof of this desingularization conjecture should follow from the same methods that could be used to prove the following *Scherk Uniqueness Conjecture* (see also Conjecture 17.0.27 below):

> *The catenoid and the singly-periodic Scherk minimal surfaces are the only connected, properly immersed minimal surfaces in $\mathbb{R}^3$ with area growth constant equal to 2π, which is the area growth constant of two planes.*

Recall that the Monotonicity Formula (Theorem 2.6.2) implies that for any properly immersed minimal surface $M \subset \mathbb{R}^3$ with area $A(R)$ in Euclidean balls $\mathbb{B}(R)$ centered at the origin of radius $R > 0$, the function $A(R)R^{-2}$ is monotonically non-decreasing in R. In particular, $A(R)R^{-2}$ has a limit value $A_M \in [\pi, \infty]$ as $R \to \infty$. As defined in Footnote 6, the minimal surface M is said to have *quadratic area growth* A_M if this limit value A_M is finite. It follows easily from the Monotonicity Formula that when M is connected and $A_M < 3\pi$, then M is embedded and either $A_M = \pi$ or $A_M = 2\pi$ (see [155]). Recall from Theorem 2.6.2 that if $A_M = \pi$, then M is a plane. Note that $A_M = 2\pi$ just means that M has the area growth constant of two planes.

Under the assumption that M has infinite symmetry group and $A_M = 2\pi$, Meeks and Wolf proved that the Scherk Uniqueness Conjecture holds, which is Theorem 1.0.9 stated in the Introduction. We now outline the proof of Theorem 1.0.9. First we recall that the plane and the catenoid are the only complete minimal surfaces of revolution. Suppose $M \subset \mathbb{R}^3$ is a connected minimal surface satisfying the hypothesis in Theorem 1.0.9, and M is not a plane or a catenoid. Since M has quadratic area growth, the Monotonicity Formula and basic compactness results from geometric measure theory, imply that for any sequence of positive numbers λ_n with $\lambda_n \searrow 0$, a subsequence of $\{\lambda_n M\}_n$ converges to a cone $\Delta(M)$ over a finite collection of geodesic arcs in $\mathbb{S}^2$ with positive integer multiplicities, joined by their common end points. This set $\Delta(M)$ is called a *limit tangent cone at infinity for M* (it is unknown if $\Delta(M)$ is independent of the sequence $\{\lambda_n\}_n$, see Conjecture 17.0.28 below). In our case, the fact that the area growth constant of M equals 2π implies that $\Delta(M)$ consists of the union of two planes passing through the origin or of a single plane passing through the origin with multiplicity two.

Since M has infinite symmetry group but it is not a surface of revolution, then M must be invariant under a screw motion symmetry which leaves

$\Delta(M)$ invariant. It follows that both $M, \Delta(M)$ are invariant under the screw motion symmetry raised to the fourth power, which must correspond to a pure translation $\tau\colon \mathbb{R}^3 \to \mathbb{R}^3$. Using the translational symmetry τ, it is not difficult to prove that $\Delta(M)$ is unique and, from that property, it follows that $M/\langle\tau\rangle \subset \mathbb{R}^3/\langle\tau\rangle$ is a complete, embedded minimal surface of finite genus g and four annular ends, which are Scherk-type ends parallel to the four flat annular ends of $\Delta(M)/\langle\tau\rangle \subset \mathbb{R}^3/\langle\tau\rangle$. A fairly simple application of the maximum principle shows that $\Delta(M)$ must correspond to two distinct planes meeting at an angle $\Theta(M) \in (0, \pi/2]$.

Now normalize M by a homothety and rotation, so that $\tau(x) = x + (0,0,1)$ and the angle that the planes in $\Delta(M)$ make with the half-plane $\{(x_1, 0, x_3) \mid x_1 > 0, x_3 \in \mathbb{R}\}$ is $\Theta(M)/2$. A straightforward application of the Alexandrov reflection method (see Alexandrov [2] and Schoen [206]) shows that, after a translation, M is invariant under reflection in the (x_1, x_3)-plane and in the (x_2, x_3)-plane, and each of these planes separates M into connected graphs over these planes, each graph $\mathcal{G}$ being orthogonal to the corresponding plane along the intersection of the boundary $\partial\mathcal{G}$ with that plane. The union of these symmetry planes decomposes M into 4 simply-connected pieces M_1, M_2, M_3, M_4 contained in the four sectors of $\mathbb{R}^3$ determined by these planes.

Let $G = \{(0,0,n) \mid n \in \mathbb{Z}\}$ and $\mathcal{M}_g$ denote the moduli space of all complete, embedded minimal surfaces in $\mathbb{R}^3/G$, with finite genus g, four Scherk-type ends, normalized as in the previous paragraph and whose liftings to $\mathbb{R}^3$ pass through the origin. Inside $\mathcal{M}_g$ we have the curve $\mathcal{S}$ of singly-periodic Scherk minimal surfaces (quotiented appropriately to have genus g in the quotient). The goal is then to prove that $\mathcal{M}_g$ reduces to $\mathcal{S}$. By elementary topology, it suffices to demonstrate the following key properties for the *angle map* $\Theta\colon \mathcal{M}_g \to (0, \frac{\pi}{2}]$, where $\mathcal{M}_g$ is considered to be a metric space with distance induced by the Hausdorff distance on compact subsets of $\mathbb{R}^3$:

1. Each connected component C of $\mathcal{M}_g$ is a curve and $\Theta|_C\colon C \to (0, \frac{\pi}{2}]$ is a local diffeomorphism.

2. $\Theta\colon \mathcal{M}_g \to (0, \frac{\pi}{2}]$ is a proper map.

3. There exists an $\varepsilon > 0$ such that if $M \in \mathcal{M}_g$ satisfies $\Theta(M) < \varepsilon$, then $M \in \mathcal{S}$.

Proving these key properties for $\Theta\colon \mathcal{M}_g \to (0, \frac{\pi}{2}]$ constitutes the three main steps in the proof of Theorem 1.0.9. We now explain briefly how these steps are performed.

The argument to prove item 1 relies on some aspects of the method of Weber and Wolf explained in Chapter 3.2. One considers the planar domains $\Omega_{g\,dh}, \Omega_{g^{-1}\,dh} \subset \mathbb{C}$ obtained by developing the flat structures $|g\,dh|, |g^{-1}\,dh|$

associated to the Weierstrass data (g, dh) of a surface $M \in \mathcal{M}_g$. The domains $\Omega_{g\,dh}, \Omega_{g^{-1}\,dh}$ are conformally diffeomorphic to each of the fundamental domains M_1, M_2, M_3, M_4 obtained above, embed in $\mathbb{C} = \mathbb{R}^2$ in a periodic way and have boundaries which are straight line segments of slope ± 1. We call these two flat domains $\Omega_{g\,dh}, \Omega_{g^{-1}\,dh}$ *orthodisks* and call their boundaries *zig-zags*. As explained in Chapter 3.2, the complex period condition in equation (2.5)-left is equivalent to the fact that the orthodisks $\Omega_{g\,dh}, \Omega_{g^{-1}\,dh}$ are conjugate. The real period condition in (2.5)-right can be also checked here, since it reduces to an algebraic condition on the sum of the cone angles at the vertices of the zig-zags, which is explicit in our setting. Then it only remains to find what pairs of orthodisks $\Omega_{g\,dh}, \Omega_{g^{-1}\,dh}$ match up to give a well-defined Riemann surface on which $g\,dh$, $g^{-1}dh$ both exist (equivalently, the complex conjugate region $\overline{\Omega_{g\,dh}}$ of $\Omega_{g\,dh} \subset \mathbb{C}$ is conformally diffeomorphic to $\Omega_{g^{-1}\,dh}$ by a conformal map that preserves the vertices of their zig-zag boundaries). By studying the appropriate Teichmüller space of pairs of conjugate periodic orthodisks and applying the Implicit Function Theorem, one shows that the map $\Theta \colon \Omega_g \to (0, \frac{\pi}{2}]$ is a local diffeomorphism, which proves that point 1 holds.

Proving that point 2 holds is essentially a compactness argument. In other words, given a sequence $\{M_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_g$ with $\Theta(M_n) > \varepsilon > 0$, one checks that a subsequence of these surfaces converges to another example in $\mathcal{M}_g$. Since the surfaces in $\mathcal{M}_g$ have local area estimates coming from the Monotonicity Formula, compactness follows by proving that the second fundamental forms, or Gaussian curvatures, of the M_n are uniformly bounded. The failure to have such a uniform curvature estimate leads to a contradiction through a fairly standard blow-up argument on the scale of curvature.

Finally to check point 3, one argues by contradiction. Suppose $\{M_n\}_n$ is a sequence in $\mathcal{M}_g$ such that $\Theta(M_n) \to 0$ as $n \to \infty$ and $M_n \notin \mathcal{S}$ for all n. By explicitly parameterizing the related orthodisks $\Omega_{g(n)\,dh(n)}$, $\Omega_{g(n)^{-1}dh(n)}$ by *Schwarz-Christoffel maps* (see e.g. [170] page 189) defined on the closed upper half-space $\{(x, y) \mid y \geq 0\}$, one shows that the asymptotics of these mappings approach the asymptotics of the corresponding Schwarz-Christoffel map of the singly-periodic Scherk minimal surface with the same value of Θ as M_n. A careful application of the Implicit Function Theorem then proves that for n large, the M_n lie on the curve $\mathcal{S}$, which is limiting to a Riemann surface with nodes that corresponds to two copies of the (x_1, x_3)-plane glued together along a periodic sequence of points on the x_3-axis. This contradiction proves that point 3 holds, which completes our outline of the proof of Theorem 1.0.9.

Calabi-Yau problems. 16

The Calabi-Yau problems or conjectures refer to a series of questions concerning the non-existence of a complete, minimal immersion $f\colon M \to \mathbb{R}^3$ whose image $f(M)$ is constrained to lie in a particular region of $\mathbb{R}^3$ (see [13], page 212 in [21], problem 91 in [229] and page 360 in [230]). Calabi's original conjecture states that a complete non-flat minimal surface cannot be contained either in the unit ball $\mathbb{B}(1)$ or in a slab. The first important negative result on the Calabi-Yau problem was given by Jorge and Xavier [91], who proved the existence of a complete, embedded minimal surface contained in an open slab of $\mathbb{R}^3$. This result was improved by Rosenberg and Toubiana [200], who showed that there exists a complete, properly immersed minimal annulus in the slab $\mathbb{R}^2 \times (0, 1)$ which intersects each plane $\mathbb{R}^2 \times \{t\}$, $0 < t < 1$, in an immersed closed curve. In 1996, Nadirashvili [166] constructed a complete minimal disk in $\mathbb{B}(1)$; this minimal disk cannot be embedded by the Colding-Minicozzi Theorem 1.0.5. A clever refinement of the ideas used by Nadirashvili, allowed Morales [164] to construct a conformal minimal immersion of the open unit disk that is proper in $\mathbb{R}^3$. These same techniques were then applied by Martín and Morales [110] to prove that if $\mathcal{D} \subset \mathbb{R}^3$ is either a smooth open bounded domain or a possibly non-smooth open convex domain, then there exists a complete, properly immersed minimal disk in $\mathcal{D}$. This disk cannot be embedded by Theorem 1.0.5. In fact, embeddedness creates a dichotomy in results concerning the Calabi-Yau questions, as we have already seen in Theorems 1.0.5 and 10.1.2.

In contrast to the existence results described in the previous paragraph, Martín, Meeks and Nadirashvili have shown that there exist many bounded non-smooth domains in $\mathbb{R}^3$ which do not admit any complete, properly immersed minimal surfaces with at least one annular end.

Theorem 16.0.9 (Martín, Meeks, Nadirashvili [109]) *Given any bounded domain $\mathcal{D}' \subset \mathbb{R}^3$, there exists a proper family $\mathcal{F}$ of horizontal simple closed curves in $\mathcal{D}'$ such that the bounded domain $\mathcal{D} = \mathcal{D}' - \bigcup \mathcal{F}$ does not admit any complete, properly immersed minimal surfaces with an annular end. More generally, the same result holds for non-compact minimal surfaces with compact boundary.*

The next theorem generalizes the previous theorem to the case of surfaces of bounded mean curvature.

Theorem 16.0.10 (Martín, Meeks [108]) *Given any smooth bounded domain $\mathcal{D}' \subset \mathbb{R}^3$, there exists a proper family $\mathcal{F}$ of simple closed curves in $\mathcal{D}'$ such that the bounded domain $\mathcal{D} = \mathcal{D}' - \bigcup \mathcal{F}$ does not admit any complete, properly immersed surfaces with an annular end and bounded mean curvature. More generally, the same result holds for non-compact surfaces with compact boundary and bounded mean curvature.*

Ferrer, Martín and Meeks have given the following general and essentially final result on the classical Calabi-Yau problem.

Theorem 16.0.11 (Ferrer, Martín and Meeks [56]) *Let M be an open, connected orientable surface and let $\mathcal{D}$ be a domain in $\mathbb{R}^3$ which is either convex or bounded and smooth. Then, there exists a complete, proper minimal immersion $f\colon M \to \mathcal{D}$.*

In a somewhat different direction, Martín and Nadirashvili have proven the following surprising theorem related to the classical Calabi-Yau problem.

Theorem 16.0.12 (Martín, Nadirashvili [111]) *There exists a simple closed curve $\Gamma \subset \mathbb{R}^3$ and a continuous map $F\colon \overline{\mathbb{D}} \to \mathbb{R}^3$ from the closed unit disk such that $F|_{\partial\overline{\mathbb{D}}}$ parameterizes Γ and $F(\mathbb{D})$ is a complete, immersed minimal disk*[1]*. Furthermore, such simple closed curves Γ are dense in the metric space of all simple closed curves in $\mathbb{R}^3$ with respect to the Hausdorff distance.*

If a complete, minimal immersion $f\colon M \to \mathbb{R}^3$ has image contained in a half-space, then M certainly admits a non-constant positive harmonic function, and so, it is transient (equivalently, its conformal type is *hyperbolic*). In particular, the examples discussed in this chapter have this property. Until recently, complete minimal surfaces of hyperbolic type played a marginal role in the global theory of minimal surfaces, partly because there are fairly

[1]By the maximum principle for harmonic functions, $F(\mathbb{D})$ lies in the convex hull of its boundary, and so, it lies in a bounded region.

natural conditions on a complete, *embedded*, minimal surface that imply recurrence, see Theorems 7.2.3, 7.3.3, 12.2.1, 14.0.5, 14.0.6, 14.0.7, and see also Conjecture 17.0.23 below. However, the following result suggests that complete hyperbolic minimal surfaces in $\mathbb{R}^3$ are present in some of the most interesting aspects of minimal surface theory.

Theorem 16.0.13 (Density Theorem, Alarcón, Ferrer and Martín [1]) *Properly immersed, hyperbolic minimal surfaces of finite topology are dense in the space of all properly immersed minimal surfaces in $\mathbb{R}^3$, endowed with the topology of smooth convergence on compact sets.*

Certainly one of the important outstanding remaining problems in this subject is how embeddedness relates to the Calabi-Yau conjectures. As we have already discussed (see Theorems 1.0.5 and 10.1.2, Chapter 5 and Chapter 10.1), certain complete, embedded minimal surfaces with restricted topology or geometry in $\mathbb{R}^3$ must be proper. On the other hand, Martín, Meeks and Nadirashvili have conjectured that given a smooth, bounded domain $\mathcal{D} \subset \mathbb{R}^3$, a necessary and sufficient condition that a connected, open topological surface M properly minimally embeds in $\mathcal{D}$ as a complete surface is that M is orientable and every end of M has infinite genus. We refer the reader to Conjecture 17.0.35 for further open problems on this fascinating research topic.

17 Outstanding problems and conjectures.

In this last chapter, we present many of the fundamental conjectures in classical minimal surface theory. Hopefully, our presentation and discussion of these problems will speed up their solution and stimulate further interest in this beautiful subject. We have listed in the statement of each conjecture the principal researchers to whom the conjecture might be attributed. We consider all of these problems to be of fundamental importance and we note that they are not listed in order of significance, difficulty or presumed depth.

Some of these problems and others appear in [119] or in [122], along with further discussions. Also see the first author's 1978 book [113] for a long list of conjectures in the subject, some of whose solutions we have discussed in our survey presented here.

Conjecture 17.0.14 (Convex Curve Conjecture, Meeks) *Two convex Jordan curves in parallel planes cannot bound a compact minimal surface of positive genus.*

There are some partial results on the Convex Curve Conjecture, under the assumption of some symmetry on the curves (see Meeks and White [153], Ros [194] and Schoen [206]). Also, the results of [153, 154] indicate that the Convex Curve Conjecture probably holds in the more general case where the two convex planar curves do not necessarily lie in parallel planes, but rather lie on the boundary of their convex hull; in this case, the planar Jordan curves are called *extremal.* Results by Ekholm, White and Wienholtz [50] imply that every compact, orientable minimal surface that arises as a counterexample to the Convex Curve Conjecture is embedded. Based on work in [50], Tinaglia [212] proved that for a fixed pair of extremal, convex planar curves, there is a bound on the genus of such a minimal surface. More generally, Meeks [113] has conjectured that if $\Gamma = \{\alpha, \beta_1, \beta_2, .., \beta_n\} \subset \mathbb{R}^3$ is a finite collection of planar, convex, simple closed curves with α in one plane and such that $\beta_1, \beta_2, ..., \beta_n$ bound a pairwise disjoint collection of disks in a parallel plane, then any compact minimal surface with boundary Γ must have genus zero.

Conjecture 17.0.15 (4π-Conjecture, Meeks, Yau, Nitsche) *If Γ is a simple closed curve in $\mathbb{R}^3$ with total curvature at most 4π, then Γ bounds a unique compact, orientable, branched minimal surface and this unique minimal surface is an embedded disk.*

As partial results to this conjecture, it is worth mentioning that Nitsche [174] proved that a regular analytic Jordan curve in $\mathbb{R}^3$ whose total curvature is at most 4π bounds a unique minimal disk; recall also (Theorem 2.9.2 above) that Meeks and Yau [157] demonstrated the conjecture if Γ is a C^2-extremal curve (they even allowed the minimal surface spanned by Γ to be non-orientable). Concerning this weakening of Conjecture 17.0.15 by removing the orientability assumption on the minimal surface spanning Γ, we mention the following generalized conjecture due to Ekholm, White and Wienholtz [50]:

> *Besides the unique minimal disk given by Nitsche's Theorem [174], only one or two Möbius strips can occur; and if the total curvature of Γ is at most 3π, then there are no such Möbius strip examples.*

Passing to a different conjecture, Gulliver and Lawson [68] proved that if Σ is an orientable, stable minimal surface with compact boundary that is properly embedded in the punctured unit ball $\mathbb{B} - \{\vec{0}\}$ of $\mathbb{R}^3$, then its closure is a compact, embedded minimal surface. If Σ is not stable, then the corresponding result is not known. Nevertheless, Meeks, Pérez and Ros [134, 140] proved that every properly embedded minimal surface M in $\mathbb{B} - \{\vec{0}\}$ with $\partial M \subset \mathbb{S}^2$ extends across the origin provided that $K|R|^2$ is bounded on M, where K is the Gaussian curvature function of M and $R^2 = x_1^2 + x_2^2 + x_3^2$ (Theorem 2.8.6 implies that this inequality holds if M is stable). In fact, the boundedness of $|K|R^2$ is equivalent to the removability of the singularity of M at the origin, and this removable singularity result holds true if we replace $\mathbb{R}^3$ by an arbitrary Riemannian three-manifold (Theorem 11.0.6). Meeks, Pérez and Ros have conjectured that this removable singularity result holds true if we replace the origin by any closed set in $\mathbb{R}^3$ with zero 1-dimensional Hausdorff measure and the surface is assumed to be properly embedded in the complement of this set, see Conjecture 17.0.17 below. It is easy to prove that the following conjecture holds for any minimal surface of finite topology (in fact, with finite genus).

Conjecture 17.0.16 (Isolated Singularities Conjecture, Gulliver, Lawson)
The closure of a properly embedded minimal surface with compact boundary in the punctured ball $\mathbb{B} - \{\vec{0}\}$ is a compact, embedded minimal surface.

In Chapter 11, we saw how the Local Removable Singularity Theorem 11.0.6 is a cornerstone for the proof of the Quadratic Curvature Decay Theorem 11.0.11 and the Dynamics Theorem 11.0.13, which illustrates the usefulness of removable singularities results.

The most ambitious conjecture about removable singularities for minimal surfaces is the following one, which deals with lamination instead of with surfaces. Recall that Examples I, II and III in Chapter 4.2 indicate that one cannot expect the next conjecture to be true if we replace $\mathbb{R}^3$ by $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$ or by a ball in $\mathbb{R}^3$.

Conjecture 17.0.17 (Fundamental Singularity Conjecture, Meeks, Pérez, Ros) *If $A \subset \mathbb{R}^3$ is a closed set with zero 1-dimensional Hausdorff measure and $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3 - A$, then $\mathcal{L}$ extends to a minimal lamination of $\mathbb{R}^3$.*

In Chapter 8, we mentioned that a key part of the proof of the uniqueness of the helicoid by Meeks and Rosenberg relies on a finiteness result for the number of components of a minimal graph over a proper domain of $\mathbb{R}^2$ with zero boundary values. More precisely, they proved that if $\mathcal{D}$ is a proper, possibly disconnected domain of $\mathbb{R}^2$ and $u\colon \mathcal{D} \to \mathbb{R}$ is a solution of the minimal surface equation (2.1) in $\mathcal{D}$ with zero boundary values and bounded gradient, then $\mathcal{D}$ has at most a finite number of components where u is non-zero. This technical property can be viewed as an important partial result in the direction of the solution of the following conjecture, made by Meeks a number of years earlier.

Conjecture 17.0.18 (Connected Graph Conjecture, Meeks) *A minimal graph in $\mathbb{R}^3$ with zero boundary values over a proper, possibly disconnected domain in $\mathbb{R}^2$ can have at most two non-planar components. Furthermore, if the graph also has sublinear growth, then such a graph with no planar components is connected.*

Consider a proper, possibly disconnected domain $\mathcal{D}$ in $\mathbb{R}^2$ and a solution $u\colon \mathcal{D} \to \mathbb{R}$ of the minimal surface equation with zero boundary values, such that u is non-zero on each component of $\mathcal{D}$. There are several partial results related to Conjecture 17.0.18. In 1981, by Mikljukov [160] proved that if each component of $\mathcal{D}$ is simply-connected with a finite number of boundary components, then $\mathcal{D}$ has at most three components (in fact, with current tools, it can be shown that his method applies to the case that $\mathcal{D}$ has finitely generated first homology group [161]). Earlier, Nitsche observed [173] that no component of $\mathcal{D}$ can be contained in a proper wedge (angle less than π). Collin [38] proved that at most one component of $\mathcal{D}$ can lie in any given half-plane. Spruck [211] demonstrated that under the assumption of sublinear growth in a suitably strong sense, $\mathcal{D}$ has at most two components. Without any assumption of the growth of the minimal graph, Li and Wang [104] proved that the number of disjointly supported minimal graphs with zero boundary values over an open subset of $\mathbb{R}^n$ is at most $(n+1)2^{n+1}$. Later, Tkachev [213] improved this exponential bound by a polynomial one, and in the case $n = 3$, he obtained that the number of disjointly supported minimal graphs is at most three. Also, Weitsman [224] has some related results that suggest that if $\mathcal{D}$ has finitely generated first homology group and u has sublinear growth, then the number of components of $\mathcal{D}$ should be at most one. We refer the reader to the end of his paper [223], where he discusses several interesting unsolved problems concerning the growth of u defined on a proper domain contained in a half-plane.

In the discussion of the conjectures that follow, it is helpful to fix some notation for certain classes of complete embedded minimal surfaces in $\mathbb{R}^3$.

- Let $\mathcal{C}$ be the space of connected, $\mathcal{C}$omplete, embedded minimal surfaces.
- Let $\mathcal{P} \subset \mathcal{C}$ be the subspace of $\mathcal{P}$roperly embedded surfaces.
- Let $\mathcal{M} \subset \mathcal{P}$ be the subspace of surfaces with $\mathcal{M}$ore than one end.

Conjecture 17.0.19 (Finite Topology Conjecture I, Hoffman, Meeks) *An orientable surface M of finite topology with genus g and r ends, $r \neq 0, 2$, occurs as a topological type of a surface in $\mathcal{C}$ if and only if $r \leq g + 2$.*

See [75, 80, 215, 222], the discussion in Chapter 3.2 (the method of Weber and Wolf) and Chapter 12 for partial existence results which seem to indicate that the existence implication in the Finite Topology Conjecture holds when $r > 2$. Recall that Theorem 1.0.6 insures that for each positive genus g, there exists an upper bound $e(g)$ on the number of ends of an $M \in \mathcal{M}$ with finite topology and genus g. Hence, the non-existence implication in Conjecture 17.0.19 will be proved if one can show that $e(g)$ can be taken as $g + 2$. Concerning the case $r = 2$, Theorems 1.0.3 and 3.1.1 imply that the only examples in $\mathcal{M}$ with finite topology and two ends are catenoids. Also, by Theorems 1.0.3 and 3.1.2, if M has finite topology, genus zero and at least two ends, then M is a catenoid.

On the other hand, one of the central results in this monograph (Theorem 1.0.1) characterizes the helicoid among complete, embedded, non-flat minimal surfaces in $\mathbb{R}^3$ with genus zero and one end. Concerning one-ended surfaces in $\mathcal{C}$ with finite positive genus, first note that all these surfaces are proper by Theorem 1.0.5. Furthermore, every example $M \in \mathcal{P}$ of finite positive genus and one end has a special analytic representation on a once punctured compact Riemann surface, as follows from the works of Bernstein and Breiner [4] and Meeks and Pérez [126], see Theorem 1.0.4. In fact, these authors showed that any such minimal surface has *finite type*[1] and is asymptotic to a helicoid. This finite type condition could be used to search computationally for possible examples of genus-g helicoids, $g \in \mathbb{N}$. Along these lines, we have already mentioned the rigorous proofs by Hoffman, Weber and Wolf [84] and by Hoffman and White [86] of existence of a genus-one helicoid, a mathematical fact that was earlier computationally indicated by Hoffman, Karcher and Wei [77]. For genera $g = 2, 3, 4, 5, 6$, there are numerical existence results [7, 8, 204, 214] as mentioned in Chapter 2.5, see also the last paragraph in Chapter 8. All these facts motivate the next conjecture, which appeared in print for the first time in the paper [148] by Meeks and Rosenberg, although several versions of it as questions were around a long time before appearing in [148].

Conjecture 17.0.20 (Finite Topology Conjecture II, Meeks, Rosenberg) *For every non-negative integer g, there exists a unique non-planar $M \in \mathcal{C}$ with genus g and one end.*

The Finite Topology Conjectures I and II together propose the precise topological conditions under which a non-compact orientable surface of finite topology can be properly minimally embedded in $\mathbb{R}^3$. What about the case where the non-compact orientable surface M has infinite topology? In this case, either M has infinite genus or M has an infinite number of ends. By Theorem 7.3.1, such an M must have at most two limit ends. Theorem 12.2.1 states that such an M cannot have one limit end and finite genus. We claim that these restrictions are the only ones.

Conjecture 17.0.21 (Infinite Topology Conjecture, Meeks) *A non-compact, orientable surface of infinite topology occurs as a topological type of a surface in $\mathcal{P}$ if and only if it has at most one or two limit ends, and when it has one limit end, then its limit end has infinite genus.*

We now discuss two conjectures related to the underlying conformal structure of a minimal surface.

Conjecture 17.0.22 (Liouville Conjecture, Meeks) *If $M \in \mathcal{P}$ and $h\colon M \to \mathbb{R}$ is a positive harmonic function, then h is constant.*

The above conjecture is closely related to work in [39, 141, 150]. For example, from the discussion in Chapters 7 and 12, we know that if $M \in \mathcal{P}$ has finite genus or two limit ends, then M is recurrent, which implies M satisfies the Liouville Conjecture. From results in [141], Chapter 14, we know

[1] See Definition 9.2.2 for the concept of minimal surface of finite type.

the conjecture holds for all of the classical examples listed in Chapter 2.5. We also remark that Neel [169] proved that if a surface $M \in \mathcal{P}$ has bounded Gaussian curvature, then M does not admit non-constant bounded harmonic functions. A related conjecture is the following one:

Conjecture 17.0.23 (Multiple-End Recurrency Conjecture, Meeks) *If $M \in \mathcal{M}$, then M is recurrent.*

Assuming that one can prove the last conjecture, the proof of the Liouville Conjecture would reduce to the case where $M \in \mathcal{P}$ has infinite genus and one end. Note that in this setting, a surface could satisfy Conjecture 17.0.22 while at the same time being transient. For example, every doubly or triply-periodic minimal surface with finite topology quotient satisfies the Liouville Conjecture, and these minimal surfaces are never recurrent (both properties follow from [141]). On the other hand, every doubly or triply-periodic minimal surface has exactly one end (Callahan, Hoffman and Meeks [16]), which implies that the assumption in Conjecture 17.0.23 that $M \in \mathcal{M}$, not merely $M \in \mathcal{P}$, is a necessary one. It should be also noted that the previous two conjectures need the hypothesis of global embeddedness, since there exist properly immersed minimal surfaces with two embedded ends and which admit bounded non-constant harmonic functions [39].

An end $e \in \mathcal{E}(M')$ of a non-compact Riemannian manifold M' is called *massive* or *non-parabolic* if every open, proper subdomain $\Omega \subset M'$ with compact boundary that represents e is massive (i.e. there exists a bounded subharmonic function $v\colon M' \to [0,\infty)$ such that $v = 0$ in $M' - \Omega$ and $\sup_\Omega v > 0$, see Grigor'yan [67]). Theorem 5.1 in [67] implies that M' has a massive end if and only if M' is transient. If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface, then M can have at most one massive end (if M has two massive ends e_1, e_2, then arguments in [61] imply that there exist end representatives E_1, E_2 with compact boundary of e_1, e_2, and an end of a plane or of a catenoid that separates $E_1 - \mathbb{B}(R)$ from $E_2 - \mathbb{B}(R)$, where $\mathbb{B}(R) = \mathbb{B}(\vec{0}, R)$ and $R > 0$ is sufficiently large; in this situation, the work by Collin, Kusner, Meeks and Rosenberg [39] insures that at least one of E_1, E_2 is a parabolic surface with boundary, which easily contradicts massiveness). We now state a basic conjecture, due to Meeks, Pérez and Ros [131], which concerns the relationship between the Liouville Conjecture and transience for a properly embedded minimal surface in $\mathbb{R}^3$ with more than one end, as well as two unanswered structure conjectures for the ends of properly embedded minimal surfaces which arise from [39].

Conjecture 17.0.24 *Let $M \in \mathcal{M}$ with horizontal limit tangent plane at infinity. Then:*

1. *M has a massive end if and only if it admits a non-constant, positive harmonic function* **(Massive End Conjecture)***.*

2. *Any proper, one-ended representative E with compact boundary for a middle end of M has vertical flux* **(Middle End Flux Conjecture)**.

3. *Suppose that there exists a half-catenoid $\mathcal{C}$ with negative logarithmic growth in $\mathbb{R}^3 - M$. Then, any proper subdomain of M that only represents ends that lie below $\mathcal{C}$ (in the sense of the Ordering Theorem [61]) has quadratic area growth* **(Quadratic Area Growth Conjecture)**.

Conjecture 17.0.25 (Isometry Conjecture, Choi, Meeks, White) *If $M \in \mathcal{C}$, then every intrinsic isometry of M extends to an ambient isometry of $\mathbb{R}^3$. More generally, if M is not a helicoid, then it is minimally rigid, in the sense that any isometric minimal immersion of M into $\mathbb{R}^3$ is congruent to M.*

The Isometry Conjecture is known to hold if $M \in \mathcal{P}$ and either $M \in \mathcal{M}$ (Choi, Meeks and White [22]), M is doubly-periodic (Meeks and Rosenberg [143]), M is periodic with finite topology quotient (Meeks [117] and Pérez [180]) or M has finite genus (this follows from Theorem 1.0.4).

It can be shown that one can reduce the validity of the Isometry Conjecture to checking that whenever $M \in \mathcal{P}$ has one end and infinite genus, then there exists a plane in $\mathbb{R}^3$ that intersects M in a set that contains a simple closed curve. If $M \in \mathcal{P}$ and there exists such a simple closed intersecting curve γ of M with a plane, then the flux of M along γ is not zero, and hence, none of the associate surfaces to M are well-defined (see Footnote 16 for the definition of associate surface). But Calabi [14] proved that the associate surfaces are the only isometric minimal immersions from M into $\mathbb{R}^3$, up to congruence.

A consequence of the Dynamics Theorem 11.0.13 and the Local Picture Theorem on the Scale of Topology (Theorem 11.0.9) is that the first statement in the Isometry Conjecture holds for surfaces in $\mathcal{C}$ if and only if it holds for those surfaces in $\mathcal{P}$ which have bounded curvature and are quasi-dilation-periodic. Meeks (unpublished) has shown that any example in $\mathcal{P}$ with bounded curvature and invariant under a translation satisfies the conjecture. Since non-zero flux ($\mathcal{F} \neq \{0\}$ with the notation of the next conjecture) implies uniqueness of an isometric minimal immersion, the One-Flux Conjecture below implies the Isometry Conjecture.

Conjecture 17.0.26 (One-Flux Conjecture, Meeks, Pérez, Ros) *Let $M \in \mathcal{C}$ and let $\mathcal{F} = \{F(\gamma) = \int_\gamma \mathrm{Rot}_{90^\circ}(\gamma') \mid \gamma \in H_1(M, \mathbb{Z})\}$ be the abelian group of flux vectors of M. If $\mathcal{F}$ has rank at most 1, then M is a plane, a helicoid, a catenoid, a Riemann minimal example or a doubly-periodic Scherk minimal surface.*

Conjecture 17.0.27 (Scherk Uniqueness Conjecture, Meeks, Wolf) *If M is a connected, properly immersed minimal surface in $\mathbb{R}^3$ and* $\mathrm{Area}(M \cap$

$\mathbb{B}(R)) \leq 2\pi R^2$ holds in balls $\mathbb{B}(R)$ of radius R, then M is a plane, a catenoid or one of the singly-periodic Scherk minimal surfaces.

By the Monotonicity Formula, any connected, properly immersed minimal surface in $\mathbb{R}^3$ with

$$\lim_{R\to\infty} R^{-2}\mathrm{Area}(M \cap \mathbb{B}(R)) \leq 2\pi,$$

is actually embedded. A related conjecture on the uniqueness of the doubly-periodic Scherk minimal surfaces was solved by Lazard-Holly and Meeks [103]; they proved that if $M \in \mathcal{P}$ is doubly-periodic and its quotient surface has genus zero, then M is one of the doubly-periodic Scherk minimal surfaces. The basic approach used in [103] was adapted later on by Meeks and Wolf [155] to prove that Conjecture 17.0.27 holds under the assumption that the surface is singly-periodic; see this precise statement in Theorem 1.0.9 and a sketch of its proof in Chapter 15. We recall that Meeks and Wolf's proof uses that the Unique Limit Tangent Cone Conjecture below holds in their periodic setting; this approach suggests that a good way to solve the general Conjecture 17.0.27 is first to prove Conjecture 17.0.28 on the uniqueness of the limit tangent cone of M, from which it follows (unpublished work of Meeks and Ros) that M has two Alexandrov-type planes of symmetry. Once M is known to have these planes of symmetry, one can describe the Weierstrass representation of M, which Meeks and Wolf (unpublished) claim would be sufficient to complete the proof of the conjecture. As explained in Chapter 15, much of the interest in Conjecture 17.0.27 arises from the role that the singly-periodic Scherk minimal surfaces play in desingularizing two transversally intersecting minimal surfaces by means of the Kapouleas' method.

Conjecture 17.0.28 (Unique Limit Tangent Cone at Infinity Conjecture, Meeks) *If $M \in \mathcal{P}$ is not a plane and has quadratic area growth, then $\lim_{t\to\infty} \frac{1}{t}M$ exists and is a minimal, possibly non-smooth cone over a finite balanced configuration of geodesic arcs in the unit sphere, with common ends points and integer multiplicities. Furthermore, if M has area not greater than $2\pi R^2$ in balls of radius R, then the limit tangent cone of M is either the union of two planes or consists of a single plane with multiplicity two passing through the origin.*

By unpublished work of Meeks and Wolf, the above conjecture is closely tied to the validity of the next classical one.

Conjecture 17.0.29 *Let $f\colon M \to \mathbb{B} - \{\vec{0}\}$ be a proper immersion of a surface with compact boundary in the punctured unit ball, such that $f(\partial M) \subset \partial\mathbb{B}$ and whose mean curvature function is bounded. Then, $f(M)$ has a unique limit tangent cone at the origin under homothetic expansions.*

If $M \in \mathcal{C}$ has finite topology, then M has finite total curvature or is asymptotic to a helicoid by Theorems 1.0.3, 1.0.4 and 1.0.5. It follows that for any such surface M, there exists a constant $C_M > 0$ such that the injectivity radius function $I_M\colon M \to (0, \infty]$ satisfies

$$I_M(p) \geq C_M \|p\|, \quad p \in M.$$

Work of Meeks, Pérez and Ros in [133, 134] indicates that this linear growth property of the injectivity radius function should characterize the examples in $\mathcal{C}$ with finite topology, in a similar manner that the inequality $K(p)\|p\|^2 \leq C_M$ characterizes finite total curvature for a surface $M \in \mathcal{C}$ (Theorem 11.0.11, here K denotes the Gaussian curvature function of M).

Conjecture 17.0.30 (Injectivity Radius Growth Conjecture, Meeks, Pérez, Ros)
A surface $M \in \mathcal{C}$ has finite topology if and only if its injectivity radius function grows at least linearly with respect to the extrinsic distance from the origin.

A positive solution of Conjecture 17.0.30 would give rise to an interesting dynamics theorem on the scale of topology, similar to Theorem 11.0.13. The results in [133, 134] and the earlier described Theorems 4.1.2 and 4.1.7 also motivated several conjectures concerning the limits of locally simply-connected sequences of minimal surfaces in $\mathbb{R}^3$, like the following one.

Conjecture 17.0.31 (Parking Garage Structure Conjecture, Meeks, Pérez, Ros)
Suppose $M_n \subset \mathbb{B}(R_n)$ is a locally simply-connected sequence of embedded minimal surfaces with $\partial M_n \subset \partial \mathbb{B}(R_n)$ and $R_n \to \infty$ as $n \to \infty$. Assume also that the sequence M_n does not have uniformly bounded curvature in $\mathbb{B}(1)$. Then:

1. *After a rotation and choosing a subsequence, the M_n converge to a minimal parking garage structure on $\mathbb{R}^3$ consisting of the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes, with singular set of convergence being a locally finite collection $\mathcal{S}(\mathcal{L})$ of vertical lines which are the columns of the parking garage structure.*

2. *For any two points $p, q \in \mathbb{R}^3 - \mathcal{S}(\mathcal{L})$, the ratio of the vertical spacing between consecutive sheets of the double multigraphs defined by M_n near p and q, converges to one as $n \to \infty$. Equivalently, the Gaussian curvature of the sequence M_n blows up at the same rate along all the columns as $n \to \infty$.*

We next deal with the question of when a surface $M \in \mathcal{C}$ has strictly negative Gaussian curvature. Suppose again that a surface $M \in \mathcal{C}$ has finite topology, and so, M either has finite total curvature or is a helicoid with handles. It is straightforward to check that such a surface has negative curvature if and only if it is a catenoid or a helicoid (note that if $g\colon M \to \mathbb{C} \cup \{\infty\}$ is the stereographically projected Gauss map of M, then after a suitable rotation of M in $\mathbb{R}^3$, the meromorphic differential $\frac{dg}{g}$ vanishes exactly at the zeros of the Gaussian curvature of M; from here one deduces easily that if M has finite topology and strictly negative Gaussian curvature, then the genus of M is zero). More generally, if we allow a surface $M \in \mathcal{C}$ to be invariant under a proper discontinuous group G of isometries of $\mathbb{R}^3$, with M/G having finite topology, then M/G is properly embedded in $\mathbb{R}^3/G$ by an elementary application of the Minimal Lamination Closure Theorem (see Proposition 1.3 in [187]). Hence, in this case M/G has finite total curvature by a result of Meeks and Rosenberg [143, 146]. Suppose additionally that M/G has negative curvature, and we will discuss which surfaces are possible. If the ends of M/G are helicoidal or planar, then a similar argument using $\frac{dg}{g}$ gives that M has genus zero, and so, it is a helicoid. If M/G is doubly-periodic, then M is a Scherk minimal surface, see [135]. In the case M/G is singly-periodic, then M must have Scherk-type ends but we still do not know if the surface must be a Scherk singly-periodic minimal surface. These considerations motivate the following conjecture.

Conjecture 17.0.32 (Negative Curvature Conjecture, Meeks, Pérez, Ros)
If $M \in \mathcal{C}$ has negative curvature, then M is a catenoid, a helicoid or one of the singly or doubly-periodic Scherk minimal surfaces.

Passing to a different question, some of the techniques developed by Meeks, Pérez and Ros in [134] and discussed in Chapter 11 provide a beginning theory for analyzing and possibly characterizing examples in $\mathcal{C}$ whose Gauss maps exclude two or more points on $\mathbb{S}^2$. A classical result of Fujimoto [64] establishes that the Gauss map of any orientable, complete, non-flat, minimally immersed surface in $\mathbb{R}^3$ cannot exclude more than 4 points, which improved the earlier result of Xavier [227] that the Gauss map of such a surface cannot miss more than 6 points. If one assumes that a surface $M \in \mathcal{C}$ is periodic with finite topology quotient, then Meeks, Pérez and Ros solved the first item in the next conjecture [135]. Also see Kawakami, Kobayashi and Miyaoka [97] for related results on this problem, including some partial results on the conjecture of Osserman that states that the Gauss map of an orientable, complete, non-flat, immersed minimal surface with finite total curvature in $\mathbb{R}^3$ cannot miss 3 points of $\mathbb{S}^2$.

Conjecture 17.0.33 (Four Point Conjecture, Meeks, Pérez, Ros)

Suppose $M \in \mathcal{C}$. Then:

1. *If the Gauss map of M omits 4 points on $\mathbb{S}^2$, then M is a singly or doubly-periodic Scherk minimal surface.*

2. *If the Gauss map of M omits exactly 3 points on $\mathbb{S}^2$, then M is a singly-periodic Karcher saddle tower whose flux polygon is a convex unitary hexagon*[2] *(note that any three points in a great circle are omitted by one of these examples, see [187] for a description of these surfaces).*

3. *If the Gauss map of M omits exactly 2 points, then M is a catenoid, a helicoid, one of the Riemann minimal examples or one of the KMR doubly-periodic minimal tori described in Chapter 2.5. In particular, the pair of points missed by the Gauss map of M must be antipodal.*

The following three conjectures are related to the embedded Calabi-Yau problem.

Conjecture 17.0.34 (Finite Genus Properness Conjecture, Meeks, Pérez, Ros)
If $M \in \mathcal{C}$ and M has finite genus, then $M \in \mathcal{P}$.

In [131], Meeks, Pérez and Ros proved Conjecture 17.0.34 under the additional hypothesis that M has a countable number of ends (recall that this assumption is necessary for M to be proper in $\mathbb{R}^3$ by Theorem 7.3.1). A stronger conjecture by Meeks, Pérez and Ros [139] states that if $M \in \mathcal{C}$ has finite genus, then M has bounded Gaussian curvature; note that Theorem 5.0.8 implies that if $M \in \mathcal{C}$ has locally bounded Gaussian curvature in $\mathbb{R}^3$ and finite genus, then M is properly embedded.

Conjecture 17.0.35 (Embedded Calabi-Yau Conjectures, Martín, Meeks, Nadirashvili, Pérez, Ros)

1. *There exists an $M \in \mathcal{C}$ contained in a bounded domain in $\mathbb{R}^3$.*

2. *There exists an $M \in \mathcal{C}$ whose closure $\overline{M}$ has the structure of a minimal lamination of a slab, with M as a leaf and with two planes as limit leaves. In particular, $\mathcal{P} \neq \mathcal{C}$.*

3. *A necessary and sufficient condition for a connected, open topological surface M to admit a complete bounded minimal embedding in $\mathbb{R}^3$ is that every end of M has infinite genus.*

4. *A necessary and sufficient condition for a connected, open topological surface M to admit a proper minimal embedding in every smooth bounded domain $\mathcal{D} \subset \mathbb{R}^3$ as a complete surface is that M is orientable and every end of M has infinite genus.*

[2] Not necessarily regular.

5. *A necessary and sufficient condition for a connected, non-orientable open topological surface M to admit a proper minimal embedding in some bounded domain $\mathcal{D} \subset \mathbb{R}^3$ as a complete surface is that every end of M has infinite genus.*

The next conjecture deals with the embedded Calabi-Yau problem in a Riemannian three-manifold N. We remark that Conjecture 17.0.34 can be shown to follow from the next conjecture.

Conjecture 17.0.36 (Finite Genus Conjecture in three-manifolds, Meeks, Pérez, Ros)
Suppose M is a connected, complete, embedded minimal surface with empty boundary and finite genus in a Riemannian three-manifold N. Let $\overline{M} = M \cup L(M)$, where $L(M)$ is the set of limit points[3] *of M. Then, one of the following possibilities holds.*

1. *$\overline{M}$ has the structure of a minimal lamination of N.*

2. *$\overline{M}$ fails to have a minimal lamination structure, $L(M)$ is a non-empty minimal lamination of N consisting of stable leaves and M is properly embedded in $N - L(M)$.*

Using arguments contained in the proofs of Theorems 1.0.5 and 10.1.2, we believe that Conjecture 17.0.36 can be reduced to be a consequence of the next conjecture, which is motivated by partial results in [37, 131]. Recall that given $\delta > 0$, we let $\mathbb{B}(\delta) = \{x \in \mathbb{R}^3 \mid |x| < \delta\}$.

Conjecture 17.0.37 (Chord-Arc Conjecture, Meeks, Pérez, Ros)
For any $k \geq 0$, there exists a positive number $\delta(k) < \frac{1}{2}$ such that if Σ is a compact, embedded minimal surface with genus k in the closed unit ball $\overline{\mathbb{B}} \subset \mathbb{R}^3$ with $\partial\Sigma \subset \partial\mathbb{B}$ and $\vec{0} \in \Sigma$, then the component $\Sigma(\vec{0}, \delta(k))$ of $\Sigma \cap \mathbb{B}(\delta(k))$ containing $\vec{0}$ satisfies

$$\Sigma(\vec{0}, \delta(k)) \subset B_\Sigma(\vec{0}, \tfrac{1}{2}),$$

where $B_\Sigma(\vec{0}, \frac{1}{2})$ is the intrinsic ball in Σ of radius $\frac{1}{2}$ centered at the point $\vec{0} \in \Sigma$.

The next two conjectures are closely related to the previous two conjectures. They are motivated by results of Colding and Minicozzi discussed in Chapter 12.4, as well as by the work by Meeks, Pérez and Ros in [131, 137].

Conjecture 17.0.38 (Small Necks Conjecture, Meeks, Pérez, Ros)
Let $M_n \subset \mathbb{B}(R_n)$ be a sequence of embedded minimal surfaces with uniformly

[3] See the paragraph just before Theorem 10.1.2 for the definition of $L(M)$.

bounded genus, $\partial M_n \subset \partial\mathbb{B}(R_n)$ and $R_n \to \infty$. If $\vec{0} \in M_n$ for all n and the injectivity radius function I_{M_n} of M_n satisfies $I_{M_n}(\vec{0}) \to 0$ as $n \to \infty$, then there exists a subsequence of the M_n (denoted in the same way) and a rotation of $\mathbb{R}^3$ such that one of the following possibilities holds.

i) $\{M_n\}_n$ admits a local picture on the scale of topology which is a properly embedded minimal surface M_∞ with two limit ends and horizontal limit tangent plane at infinity. In this case, the M_n converge to the foliation of $\mathbb{R}^3$ by horizontal planes, with singular set of C^1-convergence being a non-horizontal straight line S which passes through the origin and points to the same direction as the period of the Riemann minimal example with horizontal ends and the same flux vector as M_∞.

ii) $\{M_n\}_n$ admits a local picture on the scale of topology which is a complete, embedded minimal surface with finite total curvature and horizontal ends. Moreover, the components $\widehat{M}_n$ of $M_n \cap \mathbb{B}(\sqrt{R_n})$ with $\vec{0} \in \widehat{M}_n$, converge to the (x_1, x_2)-plane away from the singular set of convergence, which is either a horizontal line passing through the origin, or the discrete set $\mathbb{Z}v = \{nv \mid n \in \mathbb{Z}\}$ for some horizontal vector v.

The $C^{1,1}$-regularity Theorem 10.2.1 and results in [131, 136] motivate the next conjecture.

Conjecture 17.0.39 ($C^{1,1}$-regularity Conjecture, Meeks, Pérez, Ros) *Let $M_n \subset N$ be a sequence of embedded minimal surfaces of fixed genus in a three-manifold N. Suppose that $\{M_n\}_n$ converges to a minimal foliation $\mathcal{L}$ of N with singular set of C^1-convergence being a Lipschitz curve $\mathcal{S} \subset N$ which is transverse to $\mathcal{L}$. Also assume that the M_n do not admit a local picture on the scale of curvature near $\mathcal{S}$ which is a catenoid. Then, $\mathcal{S}$ is of class $C^{1,1}$.*

Our next problem is related to integral curves of harmonic functions. Theorem 7.2.3 implies that for any properly immersed minimal surface M in $\mathbb{R}^3$ and for any $t \in \mathbb{R}$, the surface $M(t) = M \cap \{x_3 \leq t\}$ is parabolic. In [122], Meeks used the parabolicity of $M(t)$ to show that the scalar flux of ∇x_3 across $\partial M(t)$ does not depend on t (this result is called the *Algebraic Flux Lemma*). If M were recurrent, then it is known (Tsuji [219]) that the following stronger property holds: almost all integral curves of ∇x_3 begin at $x_3 = -\infty$ and end at $x_3 = \infty$. The next conjecture is a strengthening of this geometric flux property to arbitrary properly embedded minimal surfaces in $\mathbb{R}^3$.

Conjecture 17.0.40 (Geometric Flux Conjecture, Meeks, Rosenberg) *Let $M \in \mathcal{P}$ and $h\colon M \to \mathbb{R}$ be a non-constant coordinate function on*

M. Consider the set I of integral curves of ∇h. Then, there exists a countable set $C \subset I$ such that for any integral curve $\alpha \in I - C$, the composition $h \circ \alpha \colon \mathbb{R} \to \mathbb{R}$ is a diffeomorphism[4].

One could weaken the hypothesis in the above conjecture that "except for a countable subset of I, h restricted to an element $\alpha \in I$ is a diffeomorphism with $\mathbb{R}$" to the hypothesis that "for almost-all elements of I, h restricted to an an element $\alpha \in I$ is a diffeomorphism with $\mathbb{R}$". We feel that the proof of the Geometric Flux Conjecture will have important theoretical consequences.

We have seen examples of how one can produce stable minimal surfaces by using barrier constructions (Chapter 2.9), and how these stable minimal surfaces act as guide posts which are useful for deciphering the structure of complete, embedded minimal surfaces (e.g. in the proofs of Theorems 2.9.1 and 6.0.11). Below, we have collected some outstanding problems that concern stable minimal surfaces. Regarding item *1* in the next conjecture, Ros (unpublished) proved that a complete, non-orientable minimal surface without boundary which is stable outside a compact set[5] must have finite total curvature. The validity of item *2* implies that the sublamination of limit leaves of the lamination $\mathcal{L}$ in Conjecture 17.0.17 extends to a lamination of $\mathbb{R}^3$ by planes. In reference to item *3,* we remark that complete, stable minimal surfaces with boundary are not in general parabolic (see pages 22 and 23 in our survey [129]). Concerning item *4,* Pérez [183] proved this conjecture under the additional assumptions that the surface is proper and has quadratic area growth.

Conjecture 17.0.41 (Stable Minimal Surface Conjectures)

1. *A complete, non-orientable, stable minimal surface in $\mathbb{R}^3$ with compact boundary has finite total curvature* **(Ros)***.*

2. *If $A \subset \mathbb{R}^3$ is a closed set with zero* 1*-dimensional Hausdorff measure and $M \subset \mathbb{R}^3 - A$ is a connected, stable, minimally immersed surface which is complete outside of A, then the closure of M is a plane* **(Meeks)***.*

3. *If $M \subset \mathbb{R}^3$ is a complete, stable minimal surface with boundary, then M is δ-parabolic, i.e. given $\delta > 0$, the set $M(\delta) = \{p \in M \mid \mathrm{dist}_M(p, \partial M) \geq \delta\}$ is parabolic* **(Meeks, Rosenberg)***.*

4. *A complete, embedded, stable minimal surface in $\mathbb{R}^3$ with boundary a straight line is a half-plane, a half of the Enneper minimal surface or a half of the helicoid* **(Pérez, Ros, White)***.*

[4] After a choice of $p \in \alpha$, we are identifying α with the parameterized curve $\alpha \colon \mathbb{R} \to M$ such that $\alpha(0) = p$ and $\alpha'(t) = \nabla h(\alpha(t))$, $t \in \mathbb{R}$.

[5] See Footnote 18 for the definition of stability in the non-orientable case.

Any end of a surface $M \in \mathcal{C}$ with finite total curvature is C^2-asymptotic to the end of a plane or catenoid (equation (2.10)). Our last conjecture can be viewed as a potential generalization of this result.

Conjecture 17.0.42 (Standard Middle End Conjecture, Meeks) *If $M \in \mathcal{M}$ and $E \subset M$ is a one-ended representative for a middle end of M, then E is C^0-asymptotic to the end of a plane or catenoid. In particular, if M has two limit ends, then each middle end is C^0-asymptotic to a plane.*

For a non-annular, one-ended middle end representative E (i.e. E has infinite genus) in the above conjecture, $\lim_{t\to\infty} \frac{1}{t}E$ is a plane P passing through the origin with positive integer multiplicity at least two by Theorem 7.3.1. Also, if M has two limit ends and horizontal limit tangent plane at infinity, then for such a middle end representative E, every divergent sequence of horizontal translates of E has a subsequence which converges to a finite collection of horizontal planes. This limit collection might depend on the sequence; in this case it remains to prove there is only one plane in limit collections of this type.

Bibliography

[1] A. Alarcón, L. Ferrer, and F. Martín. Density theorems for complete minimal surfaces in $\mathbb{R}^3$. *Geom. Funct. Anal.*, 18(1):1–49, 2008. MR2399094, Zbl 1144.53013.

[2] A. D. Alexandrov. Uniqueness theorems for surfaces in the large I. *Vestnik Leningrad Univ. Math.*, 11(19):5–17, 1956. MR0150706.

[3] D. Anderson, C. Henke, D. Hoffman, and E. L. Thomas. Periodic area-minimizing surfaces in block copolymers. *Nature*, 334(6184):598–601, 1988. August 18 issue.

[4] J. Bernstein and C. Breiner. Conformal structure of minimal surfaces with finite topology. Preprint available at http://arxiv.org/abs/0810.4478v1.

[5] J. Bernstein and C. Breiner. Helicoid-like minimal disks and uniqueness. Preprint available at http://arxiv.org/abs/0802.1497.

[6] S. Bernstein. Uber ein geometrisches theorem und seine anwendung auf die partiellen differentialglechungen vom auf die partiellen differentialglechungen vom elliptischen typus. *Math. Z.*, 26:551–558, 1927. MR1544873, Zbl JFM 53.0670.01.

[7] A. I. Bobenko. Helicoids with handles and Baker-Akhiezer spinors. *Math. Z.*, 229(1):9–29, 1998. MR1649381, Zbl 0970.53010.

[8] A. I. Bobenko and M. Schmies. Computer graphics experiments for helicoids with handles. Personal communication.

[9] F. Bonahon. Geometric structures on 3-manifolds. In *Handbook of geometric topology*, pages 93–164. North-Holland, Amsterdam, 2002.

[10] O. Bonnet. Mémoire sur l'emploi d'un nouveau systeme de variables dans l'etude des surfaces courbes. *J. Mathemém. p. appl.*, 2:153–266, 1860.

[11] J. C. Borda. Eclaircissement sur les méthodes de trouver ler courbes qui jouissent de quelque propiété du maximum ou du minimum. *Mém. Acad. Roy. Sci. Paris*, pages 551–565, 1770. Presented in 1767.

[12] C. V. Boys. *Soap bubbles: Their colours and the forces which mold them.* Dover Publications, New York, 1959.

[13] E. Calabi. Problems in differential geometry. In *Proceedings of the United States-Japan Seminar in Differential Geometry, Kyoto, Japan 1965*, page 170. Nippon Hyoronsha Co. Ltd., Tokyo, 1966.

[14] E. Calabi. Quelques applications de l'Analyse complexe aux surfaces d'Aire minima. In *Topics in Complex Manifolds*, pages 59–81. Les Presses de l'Université de Montréal, 1967. H. Rossi, editor.

[15] M. Callahan, D. Hoffman, and W. H. Meeks III. Embedded minimal surfaces with an infinite number of ends. *Invent. Math.*, 96:459–505, 1989. MR0996552, Zbl 0676.53004.

[16] M. Callahan, D. Hoffman, and W. H. Meeks III. The structure of singly-periodic minimal surfaces. *Invent. Math.*, 99:455–481, 1990. MR1032877, Zbl 695.53005.

[17] E. Catalan. Sur les surfaces réglées dont l'aire est un minimum. *J. Mathem. p. appl.*, 7:203–211, 1842.

[18] I. Chavel. *Riemannian Geometry: a modern introduction.* Cambridge University Press, 1993. MR1271141, Zbl 0810.53001.

[19] C. C. Chen and F. Gackstatter. Elliptic and hyperelliptic functions and complete minimal surfaces with handles. *Instituto de Matemática e Estatistica-Universidade de São Paulo*, 27, 1981.

[20] C. C. Chen and F. Gackstatter. Elliptische und Hyperelliptische Functionen und vollständige Minimalflächen von Enneperschen Typ. *Math. Ann.*, 259:359–369, 1982. MR0661204, Zbl 0468.53008.

[21] S. S. Chern. The geometry of G-structures. *Bull. Amer. Math. Soc.*, 72:167–219, 1966. MR0192436, Zbl 0136.17804.

[22] T. Choi, W. H. Meeks III, and B. White. A rigidity theorem for properly embedded minimal surfaces in $\mathbb{R}^3$. *J. Differential Geom.*, 32:65–76, 1990. MR1064865, Zbl 0704.53008.

[23] T. H. Colding, C. de Lellis, and W. P. Minicozzi II. Three circles theorems for Schrödinger operators on cylindrical ends and geometric applications. *Comm. Pure Appl. Math.*, 61(11):1540–1602, 2008. MR2444375, Zbl pre05358518.

[24] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold V; Fixed genus. Preprint math.DG/0509647 (2005).

[25] T. H. Colding and W. P. Minicozzi II. *Minimal surfaces*, volume 4 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 1999. MR1683966,Zbl 0987.49025.

[26] T. H. Colding and W. P. Minicozzi II. Complete properly embedded minimal surfaces in $\mathbb{R}^3$. *Duke Math. J.*, 107:421–426, 2001. MR1823052, Zbl 1010.49025.

[27] T. H. Colding and W. P. Minicozzi II. Multi-valued minimal graphs and properness of disks. *International Mathematical Research Notices*, 21:1111–1127, 2002. MR1904463, Zbl 1008.58012.

[28] T. H. Colding and W. P. Minicozzi II. Disks that are double spiral staircases. *Notices of the AMS*, 50(3):327–339, 2003. MR1954009, Zbl pre02115046.

[29] T. H. Colding and W. P. Minicozzi II. Embedded minimal disks: proper versus nonproper - global versus local. *Transactions of the AMS*, 356(1):283–289, 2003. MR2020033, Zbl 1046.53001.

[30] T. H. Colding and W. P. Minicozzi II. An excursion into geometric analysis. In *Surveys of Differential Geometry IX - Eigenvalues of Laplacian and other geometric operators*, pages 83–146. International Press, edited by Alexander Grigor'yan and Shing Tung Yau, 2004. MR2195407, Zbl 1076.53001.

[31] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold I; Estimates off the axis for disks. *Ann. of Math.*, 160:27–68, 2004. MR2119717, Zbl 1070.53031.

[32] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold II; Multi-valued graphs in disks. *Ann. of Math.*, 160:69–92, 2004. MR2119718, Zbl 1070.53032.

[33] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold III; Planar domains. *Ann. of Math.*, 160:523–572, 2004. MR2123932, Zbl 1076.53068.

[34] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold IV; Locally simply-connected. *Ann. of Math.*, 160:573–615, 2004. MR2123933, Zbl 1076.53069.

[35] T. H. Colding and W. P. Minicozzi II. Embedded minimal disks. In *Global theory of minimal surfaces*, pages 405–438. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167253, Zbl 1078.53002.

[36] T. H. Colding and W. P. Minicozzi II. Shapes of embedded minimal surfaces. *Proc. National Academy of Sciences*, 103:11106–11111, 2006. MR2242650, Zbl pre05272457.

[37] T. H. Colding and W. P. Minicozzi II. The Calabi-Yau conjectures for embedded surfaces. *Ann. of Math.*, 167:211–243, 2008. MR2373154, Zbl 1142.53012.

[38] P. Collin. Topologie et courbure des surfaces minimales de $\mathbb{R}^3$. *Ann. of Math. (2)*, 145–1:1–31, 1997. MR1432035, Zbl 886.53008.

[39] P. Collin, R. Kusner, W. H. Meeks III, and H. Rosenberg. The geometry, conformal structure and topology of minimal surfaces with infinite topology. *J. Differential Geom.*, 67:377–393, 2004. MR2153082.

[40] C. Costa. *Imersöes minimas en $\mathbb{R}^3$ de gênero un e curvatura total finita.* PhD thesis, IMPA, Rio de Janeiro, Brasil, 1982.

[41] C. Costa. Example of a complete minimal immersion in $\mathbb{R}^3$ of genus one and three embedded ends. *Bull. Soc. Bras. Mat.*, 15:47–54, 1984. MR0794728, Zbl 0613.53002.

[42] C. Costa. Uniqueness of minimal surfaces embedded in $\mathbb{R}^3$ with total curvature 12π. *J. Differential Geom.*, 30(3):597–618, 1989. MR1021368, Zbl 0696.53001.

[43] C. Costa. Classification of complete minimal surfaces in $\mathbb{R}^3$ with total curvature 12π. *Invent. Math.*, 105(2):273–303, 1991. MR1115544, Zbl 0741.53009.

[44] R. Courant. Soap film experiments with minimal surfaces. *Amer. Math. Monthly*, 47:167–174, 1940. MR0001622, Zbl 0024.41704.

[45] Dr. Crypton. Shapes that eluded discovery. *Science Digest*, pages 50–55, 78, April 1986.

[46] U. Dierkes, S. Hildebrandt, A. Küster, and O. Wohlrab. *Minimal Surfaces I.* Grundlehren der mathematischen Wissenschaften 295. Springer-Verlag, 1992. MR1215267, Zbl 0777.53012.

[47] U. Dierkes, S. Hildebrandt, A. Küster, and O. Wohlrab. *Minimal Surfaces II.* Grundlehren der mathematischen Wissenschaften 296. Springer-Verlag, 1992. MR1215268, Zbl 0777.53013.

[48] M. do Carmo and C. K. Peng. Stable complete minimal murfaces in $\mathbb{R}^3$ are planes. *Bulletin of the AMS*, 1:903–906, 1979. MR0546314, Zbl 442.53013.

[49] G. Donnay and D. L. Pawson. X-ray diffraction studies of echinoderm plates. *Science*, 166:1147–1150, 1969.

[50] T. Ekholm, B. White, and D. Wienholtz. Embeddedness of minimal surfaces with total curvature at most 4π. *Ann. of Math.*, 155:209–234, 2002. MR1888799, Zbl 1017.53013.

[51] A. Enneper. Analytisch-geometrische Untersuchungen. *Z. Math. und Phys.*, 9:96–125, 1864.

[52] L. Euler. *Methodus inveniendi lineas curvas maximi minimive propietate gaudeates sive solutio problematis isoperimetrici latissimo sensu accepti.* Harvard Univ. Press, Cambridge, MA, 1969. *Opera omnia(1)*, **24**, Fussli, Turici, 1952. A source book in mathematics, partially translated by D. J. Struik, see pages 399–406.

[53] Y. Fang. On minimal annuli in a slab. *Comment. Math. Helv.*, 69(3):417–430, 1994. MR1289335, Zbl 0819.53006.

[54] Y. Fang. Minimal annuli in $\mathbb{R}^3$ bounded by non-compact complete convex curves in parallel planes. *J. Austral. Math. Soc. Ser. A*, 60(3):369–388, 1996. MR1385149, Zbl 0867.53005.

[55] H. Federer. *Geometric measure theory.* Springer-Verlag, Berlin-Heidelberg, New York, 1969. MR0257325, Zbl 0176.00801.

[56] L. Ferrer, F. Martín, and W. H. Meeks III. The existence of proper minimal surfaces of arbitrary topological type. Preprint.

[57] D. Fischer-Colbrie. On complete minimal surfaces with finite Morse index in 3-manifolds. *Invent. Math.*, 82:121–132, 1985. MR0808112, Zbl 0573.53038.

[58] D. Fischer-Colbrie and R. Schoen. The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature. *Comm. on Pure and Appl. Math.*, 33:199–211, 1980. MR0562550, Zbl 439.53060.

[59] M. Freedman. An unknotting result for complete minimal surfaces in $\mathbb{R}^3$. *Invent. Math*, 109(1):41–46, 1992. MR1168364, Zbl 0773.53003.

[60] C. Frohman. The topological uniqueness of triply-periodic minimal surfaces in $\mathbb{R}^3$. *J. Differential Geom.*, 31:277–283, 1990. MR1030674, Zbl 0689.53002.

[61] C. Frohman and W. H. Meeks III. The ordering theorem for the ends of properly embedded minimal surfaces. *Topology*, 36(3):605–617, 1997. MR1422427, Zbl 878.53008.

[62] C. Frohman and W. H. Meeks III. The topological uniqueness of complete one-ended minimal surfaces and Heegaard surfaces in $\mathbb{R}^3$. *J. Amer. Math. Soc.*, 10(3):495–512, 1997. MR1443545, Zbl 0886.57015.

[63] C. Frohman and W. H. Meeks III. The topological classification of minimal surfaces in $\mathbb{R}^3$. MR2415385, 2008.

[64] H. Fujimoto. On the number of exceptional values of the Gauss maps of minimal surfaces. *J. of the Math. Society of Japan*, 40(2):235–247, 1988. MR0930599, Zbl 0629.53011.

[65] F. Gackstatter. Über die Dimension einer Minimalfläche und zur Ungleichung von St. Cohn-Vossen. *Arch. Rational Mech. Anal.*, 61(2):141–152, 1976. MR0420447, Zbl 0328.53002.

[66] F. Gesztesy and R. Weikard. Elliptic algebro-geometric solutions of the KdV and AKNS hierarchies—an analytic approach. *Bull. Amer. Math. Soc. (N.S.)*, 35(4):271–317, 1998. MR1638298 (99i:58075).

[67] A. Grigor'yan. Analytic and geometric background of recurrence and non-explosion of Brownian motion on Riemannian manifolds. *Bull. of A.M.S*, 36(2):135–249, 1999. MR1659871, Zbl 0927.58019.

[68] R. Gulliver and H. B. Lawson. The structure of minimal hypersurfaces near a singularity. *Proc. Symp. Pure Math.*, 44:213–237, 1986. MR0840275, Zbl 0592.53005.

[69] P. Hall. Two topological examples in minimal surface theory. *J. Differential Geom.*, 19:475–481, 1984. MR0755235, Zbl 0523.53004.

[70] L. Hauswirth, J. Pérez, and P. Romon. Embedded minimal ends of finite type. *Transactions of the AMS*, 353:1335–1370, 2001. MR1806738, Zbl 0986.53005.

[71] S. Hildebrandt. Boundary behavior of minimal surfaces. *Archive Rational Mech. Anal.*, 35:47–81, 1969. MR0248650, Zbl 0183.39402.

[72] S. Hildebrandt. The calculus of variations today. *Mathematical Intelligencer*, 11(4):50–60, 1989. MR1016107, Zbl 0689.49003.

[73] D. Hoffman. The computer-aided discovery of new embedded minimal surfaces. *Mathematical Intelligencer*, 9(3):8–21, 1987. MR0895770, Zbl 0616.53007.

[74] D. Hoffman. Computing minimal surfaces. In *Global theory of minimal surfaces*, pages 259–282. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167253, Zbl 1078.53002.

[75] D. Hoffman and H. Karcher. Complete embedded minimal surfaces of finite total curvature. In R. Osserman, editor, *Encyclopedia of Mathematics, Vol. 90, Geometry V*, pages 5–93. Springer-Verlag, 1997. MR1490038, Zbl 0890.53001.

[76] D. Hoffman, H. Karcher, and F. Wei. Adding handles to the helicoid. *Bulletin of the AMS, New Series*, 29(1):77–84, 1993. MR1193537, Zbl 0787.53003.

[77] D. Hoffman, H. Karcher, and F. Wei. The genus one helicoid and the minimal surfaces that led to its discovery. In *Global Analysis and Modern Mathematics*. Publish or Perish Press, 1993. K. Uhlenbeck, editor, p. 119–170. MR1278754, Zbl 990.54130.

[78] D. Hoffman, H. Karcher, and F. Wei. The singly periodic genus-one helicoid. *Comment. Math. Helv.*, 74:248–279, 1999. MR1691949, Zbl 0958.53006.

[79] D. Hoffman and W. H. Meeks III. A complete embedded minimal surface in $\mathbb{R}^3$ with genus one and three ends. *J. Differential Geom.*, 21:109–127, 1985. MR0806705, Zbl 0604.53002.

[80] D. Hoffman and W. H. Meeks III. Embedded minimal surfaces of finite topology. *Ann. of Math.*, 131:1–34, 1990. MR1038356, Zbl 0695.53004.

[81] D. Hoffman and W. H. Meeks III. Limits of minimal surfaces and Scherk's Fifth Surface. *Arch. Rat. Mech. Anal.*, 111(2):181–195, 1990. MR1057654, Zbl 0709.53006.

[82] D. Hoffman and W. H. Meeks III. Minimal surfaces based on the catenoid. *Amer. Math. Monthly, Special Geometry Issue*, 97(8):702–730, 1990. MR1072813, Zbl 0737.53006.

[83] D. Hoffman and W. H. Meeks III. The strong halfspace theorem for minimal surfaces. *Invent. Math.*, 101:373–377, 1990. MR1062966, Zbl 722.53054.

[84] D. Hoffman, M. Weber, and M. Wolf. An embedded genus-one helicoid. *Ann. of Math.*, 169(2):347–448, 2009. MR2480608.

[85] D. Hoffman and F. Wei. Deforming the singly periodic genus-one helicoid. *Experimental Mathematics*, 11(2):207–218, 2002. MR1959264, Zbl 1116.53301.

[86] D. Hoffman and B. White. Genus-one helicoids from a variational point of view. *Comm. Math. Helv.*, 83(4):767–813, 2008.

[87] H. Hopf. *Differential Geometry in the Large*, volume 1000 of *Lecture Notes in Math.* Springer-Verlag, 1989. MR1013786, Zbl 0669.53001.

[88] A. Huber. On subharmonic functions and differential geometry in the large. *Comment. Math. Helvetici*, 32:181–206, 1957. MR0094452, Zbl 0080.15001.

[89] Y. Imayoshi and M. Taniguchi, editors. *An introduction to Teichmüller spaces*. Springer-Verlag, 1992. MR1215481, Zbl 754.30001.

[90] L. Jorge and W. H. Meeks III. The topology of complete minimal surfaces of finite total Gaussian curvature. *Topology*, 22(2):203–221, 1983. MR0683761, Zbl 0517.53008.

[91] L. Jorge and F. Xavier. A complete minimal surface in $\mathbb{R}^3$ between two parallel planes. *Ann. of Math.*, 112:203–206, 1980. MR0584079, Zbl 0455.53004.

[92] N. Kapouleas. Complete embedded minimal surfaces of finite total curvature. *J. Differential Geom.*, 47(1):95–169, 1997. MR1601434, Zbl 0936.53006.

[93] N. Kapouleas. Constructions of minimal surfaces by gluing minimal immersions. In *Global theory of minimal surfaces*, pages 489–524. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167253, Zbl 1078.53002.

[94] H. Karcher. Embedded minimal surfaces derived from Scherk's examples. *Manuscripta Math.*, 62:83–114, 1988. MR0958255, Zbl 658.53006.

[95] H. Karcher. Construction of minimal surfaces. *Surveys in Geometry*, pages 1–96, 1989. University of Tokyo, 1989, and Lecture Notes No. 12, SFB256, Bonn, 1989.

[96] H. Karcher. The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions. *Manuscripta Math.*, 64:291–357, 1989. MR1003093, Zbl 0687.53010.

[97] Y. Kawakami, R. Kobayashi, and R. Miyaoka. The Gauss map of pseudo-algebraic minimal surfaces. *Mathematische Nachrichten*, 282(2):157–306, 2009.

[98] N. Korevaar, R. Kusner, and B. Solomon. The structure of complete embedded surfaces with constant mean curvature. *J. Differential Geom.*, 30:465–503, 1989. MR1010168, Zbl 0726.53007.

[99] J. L. Lagrange. Essai d'une nouvelle methode pour determiner les maxima et les minima des formules integrales indefinies. *Miscellanea Taurinensia 2*, 325(1):173–199, 1760.

[100] R. Langevin and H. Rosenberg. A maximum principle at infinity for minimal surfaces and applications. *Duke Math. J.*, 57(3):819–828, 1988. MR0975123, Zbl 0667.49024.

[101] H. B. Lawson. The unknottedness of minimal embeddings. *Invent. Math.*, 11:183–187, 1970. MR0287447, Zbl 0205.52002.

[102] H. B. Lawson, Jr. *Lectures on Minimal Submanifolds*. Publish or Perish Press, Berkeley, 1980. MR0576752, Zbl 0434.53006.

[103] H. Lazard-Holly and W. H. Meeks III. The classification of embedded doubly–periodic minimal surfaces of genus zero. *Invent. Math.*, 143:1–27, 2001. MR1802791, Zbl 992.18190.

[104] P. Li and J. Wang. Finiteness of disjoint minimal graphs. *Math. Research Letters*, 8(6):771–777, 2001. MR1879819, Zbl 0999.05055.

[105] R. B. Lockhart and R. C. McOwen. Elliptic differential operators on noncompact manifolds. *Ann. Scuola Norm. Sup. Pisa*, 12(3):409–447, 1985. MR0837256, Zbl 0615.58048.

[106] F. J. López and F. Martín. Complete minimal surfaces in $\mathbb{R}^3$. *Publ. Mat.*, 43(2):341–449, 1999. MR1744617, Zbl 0951.53001.

[107] F. J. López and A. Ros. On embedded complete minimal surfaces of genus zero. *J. Differential Geom.*, 33(1):293–300, 1991. MR1085145, Zbl 719.53004.

[108] F. Martin and W. H. Meeks III. Calabi-Yau domains in three-manifolds. Preprint.

[109] F. Martin, W. H. Meeks III, and N. Nadirashvili. Bounded domains which are universal for minimal surfaces. *American Journal of Math.*, 129(2):455–461, 2007. MR2306042, Zbl pre05152573.

[110] F. Martin and S. Morales. Complete proper minimal surfaces in convex bodies of $\mathbb{R}^3$. *Duke Math. J.*, 128(3):559–593, 2005. MR2145744, Zbl 1082.53009.

[111] F. Martin and N. Nadirashvili. A Jordan curve spanned by a complete minimal surface. *Arch. Ration. Mech. Anal.*, 184(2):285–301, 2007. MR2299764, Zbl 1114.49039.

[112] L. Mazet, M.M. Rodríguez, and M. Traizet. Saddle towers with infinitely many ends. *Indiana Univ. Math. J.*, 56(6):2821–2838, 2007. MR2375703.

[113] W. H. Meeks III. *Lectures on Plateau's Problem.* Instituto de Matematica Pura e Aplicada (IMPA), Rio de Janeiro, Brazil, 1978.

[114] W. H. Meeks III. The classification of complete minimal surfaces with total curvature greater than -8π. *Duke Math. J.*, 48:523–535, 1981. MR0630583, Zbl 0472.53010.

[115] W. H. Meeks III. A survey of the geometric results in the classical theory of minimal surfaces. *Bol. Soc. Brasil Mat.*, 12:29–86, 1981. MR0671473, Zbl 0577.53007.

[116] W. H. Meeks III. The topological uniqueness of minimal surfaces in three-dimensional Euclidean space. *Topology*, 20:389–410, 1981. MR0617373, Zbl 0458.53003.

[117] W. H. Meeks III. The theory of triply-periodic minimal surfaces. *Indiana Univ. Math. J.*, 39(3):877–936, 1990. MR1078743, Zbl 721.53057.

[118] W. H. Meeks III. The geometry, topology, and existence of periodic minimal surfaces. *Proceedings of Symposia in Pure Math.*, 54:333–374, 1993. Part I. MR1216594, Zbl 812.49030.

[119] W. H. Meeks III. Geometric results in classical minimal surface theory. In *Surveys in Differential Geometry*, volume 8, pages 269–306. International Press, edited by S.T. Yau, 2003. MR2039993, Zbl 1063.53008.

[120] W. H. Meeks III. The geometry and topology of singly-periodic minimal surfaces. *Asian J. of Math.*, 7(3):297–302, 2003. MR2129324, Zbl pre02148286.

[121] W. H. Meeks III. The regularity of the singular set in the Colding and Minicozzi lamination theorem. *Duke Math. J.*, 123(2):329–334, 2004. MR2066941, Zbl pre02127998.

[122] W. H. Meeks III. Global problems in classical minimal surface theory. In *Global theory of minimal surfaces*, pages 453–470. American

Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167253, Zbl 1078.53002.

[123] W. H. Meeks III. The lamination metric for a Colding-Minicozzi minimal lamination. *Illinois J. of Math.*, 49(2):645–658, 2005. MR2164355, Zbl pre02212258.

[124] W. H. Meeks III. Proofs of some classical results in minimal surface theory. *Indiana J. of Math.*, 54(4):1031–1045, 2005. MR2164416, Zbl 1085.53010.

[125] W. H. Meeks III and J. Pérez. The classical theory of minimal surfaces. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[126] W. H. Meeks III and J. Pérez. Embedded minimal surfaces of finite topology. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[127] W. H. Meeks III and J. Pérez. Finite type annular ends for harmonic functions. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[128] W. H. Meeks III and J. Pérez. Properly embedded minimal planar domains with infinite topology are Riemann minimal examples. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[129] W. H. Meeks III and J. Pérez. Conformal properties in classical minimal surface theory. In *Surveys of Differential Geometry IX - Eigenvalues of Laplacian and other geometric operators*, pages 275–336. International Press, edited by Alexander Grigor'yan and Shing Tung Yau, 2004. MR2195411, Zbl 1086.53007.

[130] W. H. Meeks III, J. Pérez, and A. Ros. Bounds on the topology and index of classical minimal surfaces. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[131] W. H. Meeks III, J. Pérez, and A. Ros. The embedded Calabi-Yau conjectures for finite genus. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[132] W. H. Meeks III, J. Pérez, and A. Ros. Limit leaves of a CMC lamination are stable. To appear in J. Differential Geometry, available at arXiv:0801.4345 and at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[133] W. H. Meeks III, J. Pérez, and A. Ros. The local picture theorem on the scale of topology. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[134] W. H. Meeks III, J. Pérez, and A. Ros. Local removable singularity theorems for minimal and H-laminations. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[135] W. H. Meeks III, J. Pérez, and A. Ros. Minimal surfaces whose Gauss map misses four points. Work in progress.

[136] W. H. Meeks III, J. Pérez, and A. Ros. Properly embedded minimal planar domains. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[137] W. H. Meeks III, J. Pérez, and A. Ros. Structure theorems for singular minimal laminations. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[138] W. H. Meeks III, J. Pérez, and A. Ros. Uniqueness of the Riemann minimal examples. *Invent. Math.*, 133:107–132, 1998. MR1626477, Zbl 916.53004.

[139] W. H. Meeks III, J. Pérez, and A. Ros. The geometry of minimal surfaces of finite genus I; curvature estimates and quasiperiodicity. *J. Differential Geom.*, 66:1–45, 2004. MR2128712, Zbl 1068.53012.

[140] W. H. Meeks III, J. Pérez, and A. Ros. The geometry of minimal surfaces of finite genus II; nonexistence of one limit end examples. *Invent. Math.*, 158:323–341, 2004. MR2096796, Zbl 1070.53003.

[141] W. H. Meeks III, J. Pérez, and A. Ros. Liouville-type properties for embedded minimal surfaces. *Communications in Analysis and Geometry*, 14(4):703–723, 2006. MR2273291, Zbl 1117.53009.

[142] W. H. Meeks III, J. Pérez, and A. Ros. Stable constant mean curvature surfaces. In *Handbook of Geometrical Analysis*, volume 1, pages 301–380. International Press, edited by Lizhen Ji, Peter Li, Richard Schoen and Leon Simon, ISBN: 978-1-57146-130-8, 2008. MR2483369, Zbl 1154.53009.

[143] W. H. Meeks III and H. Rosenberg. The global theory of doubly periodic minimal surfaces. *Invent. Math.*, 97:351–379, 1989. MR1001845, Zbl 676.53068.

[144] W. H. Meeks III and H. Rosenberg. The maximum principle at infinity for minimal surfaces in flat three-manifolds. *Comment. Math. Helvetici*, 65:255–270, 1990. MR1057243, Zbl 713.53008.

[145] W. H. Meeks III and H. Rosenberg. The geometry and conformal structure of properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$. *Invent. Math.*, 114:625–639, 1993. MR1244914, Zbl 0803.53007.

[146] W. H. Meeks III and H. Rosenberg. The geometry of periodic minimal surfaces. *Comment. Math. Helvetici*, 68:538–578, 1993. MR1241472, Zbl 807.53049.

[147] W. H. Meeks III and H. Rosenberg. The theory of minimal surfaces in $M \times \mathbb{R}$. *Comment. Math. Helv.*, 80:811–858, 2005. MR2182702, Zbl pre02242664.

[148] W. H. Meeks III and H. Rosenberg. The uniqueness of the helicoid. *Ann. of Math.*, 161:723–754, 2005. MR2153399, Zbl 1102.53005.

[149] W. H. Meeks III and H. Rosenberg. The minimal lamination closure theorem. *Duke Math. Journal*, 133(3):467–497, 2006. MR2228460, Zbl 1098.53007.

[150] W. H. Meeks III and H. Rosenberg. Maximum principles at infinity. *J. Differential Geometry*, 79(1):141–165, 2008. MR2401421, Zbl pre05285650.

[151] W. H. Meeks III, L. Simon, and S. T. Yau. The existence of embedded minimal surfaces, exotic spheres and positive Ricci curvature. *Ann. of Math.*, 116:221–259, 1982. MR0678484, Zbl 0521.53007.

[152] W. H. Meeks III and M. Weber. Bending the helicoid. *Mathematische Annalen*, 339(4):783–798, 2007. MR2341900, Zbl 1156.53011.

[153] W. H. Meeks III and B. White. Minimal surfaces bounded by convex curves in parallel planes. *Comment. Math. Helvetici*, 66:263–278, 1991. MR1107841, Zbl 0731.53004.

[154] W. H. Meeks III and B. White. The space of minimal annuli bounded by an extremal pair of planar curves. *Communications in Analysis and Geometry*, 1(3):415–437, 1993. MR1266474, Zbl 0854.53009.

[155] W. H. Meeks III and M. Wolf. Minimal surfaces with the area growth of two planes; the case of infinite symmetry. MR2276776, Zbl 1115.53008, 2007.

[156] W. H. Meeks III and S. T. Yau. The classical Plateau problem and the topology of three-dimensional manifolds. *Topology*, 21(4):409–442, 1982. MR0670745, Zbl 0489.57002.

[157] W. H. Meeks III and S. T. Yau. The existence of embedded minimal surfaces and the problem of uniqueness. *Math. Z.*, 179:151–168, 1982. MR0645492, Zbl 0479.49026.

[158] W. H. Meeks III and S. T. Yau. The topological uniqueness of complete minimal surfaces of finite topological type. *Topology*, 31(2):305–316, 1992. MR1167172, Zbl 0761.53006.

[159] J. B. Meusnier. Mémoire sur la courbure des surfaces. *Mém. Mathém. Phys. Acad. Sci. Paris, prés. par div. Savans*, 10:477–510, 1785. Presented in 1776.

[160] V. V. Mikljukov. Some singularities and the behavior of equations of minimal surface type in unbounded domains. *Math. Sbornik*, 116(1):72–86, 1981. English translation in Math. USSR Sbornik 44(1): 61–73, 1983. Zbl 0556.49022.

[161] V. V. Miklyukov and A. Weitsman. Carleman's method and solutions to the minimal surface equation. Preprint.

[162] P. Mira. Complete minimal Möbius strips in $\mathbb{R}^n$ and the Björling problem. *J. of Geometry and Physics*, 56:1506–1515, 2006.

[163] S. Montiel and A. Ros. Schrödinger operators associated to a holomorphic map. In *Global Differential Geometry and Global Analysis (Berlin, 1990)*, volume 1481 of *Lecture Notes in Mathematics*, pages 147–174. Springer-Verlag, 1991. MR1178529, Zbl 744.58007.

[164] S. Morales. On the existence of a proper minimal surface in $\mathbb{R}^3$ with a conformal type of disk. *GAFA*, 13(6):1281–1301, 2003. MR2033839, Zbl pre02067276.

[165] C. B. Morrey. The problem of Plateau on a Riemannian manifold. *Ann. of Math.*, 49:807–851, 1948. MR0027137, Zbl 0033.39601.

[166] N. Nadirashvili. Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces. *Invent. Math.*, 126(3):457–465, 1996. MR1419004, Zbl 0881.53053.

[167] S. Nayatani. On the Morse index of complete minimal surfaces in Euclidean space. *Osaka J. Math.*, 27:441–451, 1990. MR1066637, Zbl 0704.53007.

[168] R. Neel. Brownian motion and the parabolicity of minimal graphs. ArXiv:0810.0669v1 [math.DG].

[169] R. Neel. A martingale approach to minimal surfaces. *J. Funct. Anal.* doi:10.1016/j.jfa.2008.06.033 (in press).

[170] Z. Nehari. *Conformal mapping.* Dover Publications Inc., New York, 1975. MR0377031, Zbl 0048.31503.

[171] H. U. Nissen. Crystal orientation and plate structure in echinoid skeletal units. *Science*, 166:1150–1152, 1969.

[172] J. C. C. Nitsche. A characterization of the catenoid. *J. of Math. Mech.*, 11:293–302, 1962. MR0137043, Zbl 0106.14602.

[173] J. C. C. Nitsche. On new results in the theory of minimal surfaces. *Bull. Amer. Math. Soc.*, 71:195–270, 1965. MR0173993, Zbl 0135.21701.

[174] J. C. C. Nitsche. A new uniqueness theorem for minimal surfaces. *Arch. Rat. Mech. Anal.*, 52:319–329, 1973. MR0341258, Zbl 0285.49001.

[175] J. C. C. Nitsche. *Lectures on Minimal Surfaces*, volume 1. Cambridge University Press, 1989. MR1015936, Zbl 0688.53001.

[176] R. Osserman. Global properties of minimal surfaces in E^3 and E^n. *Ann. of Math.*, 80(2):340–364, 1964. MR0179701, Zbl 0134.38502.

[177] R. Osserman. The convex hull property of immersed manifolds. *J. Differential Geom.*, 6(2):267–270, 1971.

[178] R. Osserman. *A Survey of Minimal Surfaces*. Dover Publications, New York, 2nd edition, 1986. MR0852409, Zbl 0209.52901.

[179] J. Pérez. On singly-periodic minimal surfaces with planar ends. *Transactions of the AMS*, 6:2371–2389, 1997. MR1407709, Zbl 882.53007.

[180] J. Pérez. A rigidity theorem for periodic minimal surfaces. *Comm. in Analysis and Geom.*, 7(1):95–104, 1999. MR1674113, Zbl 0947.53006.

[181] J. Pérez. Parabolicity and minimal surfaces. In *Global theory of minimal surfaces*, pages 163–174. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167253, Zbl 1078.53002.

[182] J. Pérez. Uniqueness of the Riemann minimal surfaces. In *Global theory of minimal surfaces*, pages 597–610. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. Based on joint work with W.H. Meeks III and A. Ros. MR2167253, Zbl 1078.53002.

[183] J. Pérez. Stable embedded minimal surfaces bounded by a straight line. *Calculus of Variations and PDE*, 29(2):267–279, 2007. MR2307776, Zbl 1113.53006.

[184] J. Pérez, M. Rodríguez, and M. Traizet. The classification of doubly periodic minimal tori with parallel ends. *Journal of Diff. Geometry*, 69(3):523–577, 2005. MR2170278, Zbl pre05004289.

[185] J. Pérez and A. Ros. The space of properly embedded minimal surfaces with finite total curvature. *Indiana Univ. Math. J.*, 45(1):177–204, 1996. MR1406689, Zbl 864.53008.

[186] J. Pérez and A. Ros. Properly embedded minimal surfaces with finite total curvature. In *The Global Theory of Minimal Surfaces in Flat Spaces*, pages 15–66. Lecture Notes in Math 1775, Springer-Verlag, 2002. G. P. Pirola, editor. MR1901613, Zbl 1028.53005.

[187] J. Pérez and M. Traizet. The classification of singly periodic minimal surfaces with genus zero and Scherk type ends. *Transactions of the AMS*, 359(3):965–990, 2007. MR2262839,Zbl 1110.53008.

[188] I. Peterson. Spiral proof. *Science News*, 168(25):393–397, 2005.

[189] A. V. Pogorelov. On the stability of minimal surfaces. *Soviet Math. Dokl.*, 24:274–276, 1981. MR0630142, Zbl 0495.53005.

[190] B. Riemann. Über die Fläche vom kleinsten Inhalt bei gegebener Begrenzung. *Abh. Königl, d. Wiss. Göttingen, Mathem. Cl.*, 13:3–52, 1867. K. Hattendorf, editor. JFM 01.0218.01.

[191] B. Riemann. *Ouevres Mathématiques de Riemann.* Gauthiers-Villars, Paris, 1898.

[192] M. M. Rodríguez. A Jenkins-Serrin problem on the strip. *Mat. Contemp.*, 30:135–143, 2006. MR2289652.

[193] A. Ros. Compactness of spaces properly embedded minimal surfaces with finite total curvature. *Indiana Univ. Math. J.*, 44(1):139–152, 1995. MR1336435, Zbl 0826.53005.

[194] A. Ros. Embedded minimal surfaces: forces, topology and symmetries. *Calc. Var.*, 4:469–496, 1996. MR1402733, Zbl 861.53008.

[195] A. Ros. The isoperimetric problem. In *Global theory of minimal surfaces*, pages 175–209. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167260, Zbl 1125.49034.

[196] A. Ros. One-sided complete stable minimal surfaces. *Journal Differential Geometry*, 74:69–92, 2006. MR2260928, Zbl 1110.53009.

[197] H. Rosenberg. Minimal surfaces of finite type. *Bull. Soc. Math. France*, 123:351–354, 1995. MR1373739, Zbl 0848.53004.

[198] H. Rosenberg. Some recent developments in the theory of minimal surfaces in 3-manifolds. In *24th Brazilian Mathematics Colloquium*, Instituto de Matematica Pura e Aplicada (IMPA), Rio de Janeiro, 2003. IMPA Mathematical Publications. MR2028922 (2005b:53015), Zbl 1064.53007.

[199] H. Rosenberg, R. Souam, and E. Toubiana. General curvature estimates for stable H-surfaces in 3-manifolds and applications. Preprint.

[200] H. Rosenberg and E. Toubiana. Complete minimal surfaces and minimal herissons. *J. Differential Geom.*, 28:115–132, 1988. MR0950557, Zbl 0657.53004.

[201] M. Ross. Schwarz' P and D surfaces are stable. *Differential Geom. Appl.*, 2(2):179–195, 1992. MR1245555, Zbl 0747.53010.

[202] L. Sario and M. Nakai. *Classification theory of Riemann surfaces*. Grundlehren der mathematischen Wissenschaften 164. Springer-Verlag, 1970. MR0264064, Zbl 0199.40603.

[203] H. F. Scherk. Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen. *J. R. Angew. Math.*, 13:185–208, 1835. ERAM 013.0481cj.

[204] M. Schmies. *Computational methods for Riemann surfaces and helicoids with handles*. PhD thesis, Technical University of Berlin, Berlin, Germany, 2005.

[205] A. Schoen. Infinite periodic minimal surfaces without self-intersections. Technical Note D-5541, NASA, Cambridge, Mass., May 1970. Zbl 1071.53507.

[206] R. Schoen. Uniqueness, symmetry, and embeddedness of minimal surfaces. *J. Differential Geom.*, 18:791–809, 1983. MR0730928, Zbl 0575.53037.

[207] G. Segal and G. Wilson. Loop groups and equations of KdV type. *Pub. Math. de I.H.E.S.*, 61:5–65, 1985. MR0783348, Zbl 0592.35112.

[208] M. Shiffman. On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes. *Ann. of Math.*, 63:77–90, 1956. MR0074695, Zbl 0070.16803.

[209] B. Solomon. On foliations of $\mathbb{R}^{n+1}$ by minimal hypersurfaces. *Comm. Math. Helv.*, 61:67–83, 1986. MR0847521, Zbl 0601.53025.

[210] M. Soret. Maximum principle at infinity for complete minimal surfaces in flat 3-manifolds. *Annals of Global Analysis and Geometry*, 13:101–116, 1995. MR1336206, Zbl 0873.53039.

[211] J. Spruck. Two-dimensional minimal graphs over unbounded domains. *Journal of the Inst. of Math. Jussieu*, 1(4):631–640, 2002. MR1954438, Zbl 1038.53012.

[212] G. Tinaglia. Curvature bounds for minimal surfaces with total boundary curvature less than 4π. To appear in Proceedings of the American Mathematical Society, arXiv:0712.1500.

[213] V. G. Tkachev. Disjoint minimal graphs. *Annals of Global Analysis and Geometry*, 35(2):139–155, 2009.

[214] M. Traizet. The genus 2 helicoid. Personal Communication.

[215] M. Traizet. An embedded minimal surface with no symmetries. *J. Differential Geometry*, 60(1):103–153, 2002. MR1924593, Zbl 1054.53014.

[216] M. Traizet. A balancing condition for weak limits of minimal surfaces. *Comment. Math. Helvetici*, 79(4):798–825, 2004. MR2099123, Zbl 1059.53016.

[217] M. Traizet. On the genus of triply periodic minimal surfaces. *J. Differential Geom.*, 79:243–275, 2008. MR2420019, Zbl pre05294962.

[218] M. Traizet and M. Weber. Hermite polynomials and helicoidal minimal surfaces. *Invent. Math.*, 161(1):113–149, 2005. MR2178659, Zbl 1075.53010.

[219] M. Tsuji. *Potential theory in modern function theory, 2nd ed.* Chelsea Publishing Company, New York, N. Y., 1975. MR0414898, Zbl 0322.30001.

[220] J. Tysk. Eigenvalue estimates with applications to minimal surfaces. *Pacific J. of Math.*, 128:361–366, 1987. MR0888524, Zbl 0594.58018.

[221] M. Weber and M. Wolf. Minimal surfaces of least total curvature and moduli spaces of plane polygonal arcs. *Geom. Funct. Anal.*, 8:1129–1170, 1998. MR1664793, Zbl 0954.53007.

[222] M. Weber and M. Wolf. Teichmüller theory and handle addition for minimal surfaces. *Ann. of Math.*, 156:713–795, 2002. MR1954234, Zbl 1028.53009.

[223] A. Weitsman. Growth of solutions to the minimal surface equation over domains in a half plane. *Communications in Analysis and Geometry*, 13:1077–1087, 2005. MR2216153.

[224] A. Weitsman. On the growth of minimal graphs. *Indiana Univ. Math Journal*, 54(2):617–625, 2005. MR2136824, Zbl pre02182484.

[225] B. White. The bridge principle for stable minimal surfaces. *Calc. Var.*, 2:405–425, 1994. MR1383916, Zbl 0830.49025.

[226] B. White. The bridge principle for unstable and for singular minimal surfaces. *Comm. Anal. and Geom.*, 2(4):513–532, 1994. MR1336893, Zbl 0860.58009.

[227] F. Xavier. The Gauss map of a complete non-flat minimal surface cannot omit 7 points of the sphere. *Ann. of Math.*, 113:211–214, 1981. MR0604048, Zbl 0477.53007.

[228] F. Xavier. Convex hulls of complete minimal surfaces. *Math. Ann.*, 269:179–182, 1984. MR0759107, Zbl 0528.53009.

[229] S.T. Yau. Problem section, Seminar on Differential Geometry. In *Annals of Math Studies*, volume 102, pages 669–706, 1982. MR0645762, Zbl 0479.53001.

[230] S.T. Yau. Review of geometry and analysis. In *Mathematics: frontiers and prospectives*, pages 353–401. Amer. Math. Soc., Providence, RI, 2000. MR1754787, Zbl 0969.53001.