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The geometry of minimal surfaces of finite genus I; curvature estimates and quasiperiodicity

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Abstract. Let $\mathcal{M}$ be the space of properly embedded minimal surfaces in $\mathbb{R}^3$ with genus zero and two limit ends, normalized so that every surface $M \in \mathcal{M}$ has horizontal limit tangent plane at infinity and vertical component of its flux equals one. We prove that if a sequence $\{M(i)\}_i \subset \mathcal{M}$ has horizontal part of the flux bounded from above, then the Gaussian curvature of the sequence is uniformly bounded. This curvature estimate yields compactness results and the techniques in its proof lead to a number of consequences, concerning the geometry of any properly embedded minimal surface in $\mathbb{R}^3$ with finite genus, and the possible limits through a blowing-up process on the scale of curvature of a sequence of properly embedded minimal surfaces with locally bounded genus in a homogeneously regular Riemannian 3-manifold.

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1. Introduction

The classical theory of minimal surfaces began in the middle of the 18th century with work by Lagrange [**14**] who described the variational equations that a function defined on a compact domain in the plane must satisfy for its graph to have least area with respect to its boundary values. It follows from Lagrange's equations that the plane, helicoid and catenoid are properly embedded surfaces in $\mathbb{R}^3$ that locally satisfy such equations. Results of Collin [**7**], López-Ros [**15**], and Meeks-Rosenberg [**26**] imply that these three classical examples are the only finite topology properly embedded minimal surfaces in $\mathbb{R}^3$ with genus zero.

In the 19th century Riemann [**31**] studied the question of which surfaces, other than the plane, catenoid and helicoid, are foliated by circles and lines in horizontal planes. He found a 1-parameter family $\{\mathcal{R}_t\}_t$, called the Riemann minimal examples, each of which is periodic in $\mathbb{R}^3$ with horizontal lines at integer heights planes and circles at other heights (the same conclusion holds after removing the hypothesis of the planes that contain the foliation circles and lines to be

parallel, see Enneper [**10**]). In [**23**] we proved that the Riemann examples are the only properly embedded minimal surfaces of genus zero with infinite topology and infinite symmetry group. We believe that our theorem holds without the hypothesis of infinite symmetry group. More generally, we make the following conjecture (see Section 2 for definitions):

Conjecture 1. *Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ of finite genus which is not a plane. Then:*

(1) M has bounded Gaussian curvature;
(2) M is recurrent for Brownian motion.
(3) If M has one end, then it is asymptotic to a helicoid;
(4) If M has a finite number $r \geq 2$ of ends, then M has finite total curvature;
(5) If M has finite total curvature with genus g and r ends, then $g + 2 \geq r$.
*(6) If M has an infinite number of ends, then M has two limit ends, each of which is asymptotic as $x_3 \to \pm\infty$ to translated top and bottom ends of one of the classical Riemann minimal examples (see [**23**] for a description of these beautiful singly-periodic minimal surfaces);*
(7) If M has genus zero, then M is a helicoid, a catenoid, or a Riemann minimal example. In particular, the genus zero examples are foliated by lines and circles in parallel planes.

In the cases *3, 4, 5* above, M has finite topology. In this setting, results in [**7, 15, 26**] imply that points *3, 4* of Conjecture 1 hold and point *2* holds when M has finite topology. If M has an infinite number of ends, Collin, Kusner, Meeks and Rosenberg [**8**] proved that M has at most two limit ends and it is recurrent for Brownian motion if it has exactly two limit ends. In [**18**] we prove that M cannot have one limit end; this result depends on several of the theorems, curvature estimates and other tools we develop here. Since M does not have one limit end, if it has an infinite number of ends, then it has two limit ends and so, it is recurrent for Brownian motion [**8**]. Thus, point *2* in Conjecture 1 holds. By Theorem 1 below, we conclude that point *1* of Conjecture 1 also holds. In fact, as an application of the results in the present paper, we eventually hope to be able to complete the proof of the above conjecture.

An important theoretical tool which we develop here appears in the proof of Lemma 8. This is a blow-up argument which we refer to as "blowing-up a sequence of properly embedded minimal surfaces on the scale of topology". This blow-up argument produces a new sequence of properly embedded minimal surfaces in $\mathbb{R}^3$ which is uniformly locally simply connected. Such a sequence can be analyzed by means of the local and global compactness and regularity theorems by Colding and Minicozzi [**2, 3, 5**]. Our proofs depend heavily on their technical results. Another important tool in the Colding-Minicozzi theory that will be used here is the 1-sided curvature estimate [**5**]. In Section 2 we prove a related curvature estimate which we will need for our proofs (see also Section 2 for further discussion on the Colding-Minicozzi results).

Theorems 1, 7 and 6 of this paper and Theorem 1 in [**18**] are useful for describing possible limits of sequences of properly embedded minimal surfaces in $\mathbb{R}^3$ with bounded genus. In [**19**] we will apply these theorems to obtain bounds on the number of ends and index of complete embedded

minimal surfaces of finite total curvature only in terms of the genus, which gives a partial result on part *5* of the above conjecture.

In our earlier proof of the uniqueness of the Riemann minimal examples under the infinite symmetry group assumption, it was essential to obtain uniform curvature estimates for the moduli space of periodic examples, appropriately normalized by rotations and scalings by homotheties. We will obtain here similar curvature estimates as in the periodic setting, but with the weaker hypothesis that the surface has two limit ends.

The curvature estimates obtained in this paper have a number of important consequences which we develop in Section 4. The most important application of our curvature estimates is to obtain the following descriptive theorem of the geometry of properly embedded minimal surfaces with finite genus and two limit ends. This theorem will play an essential role in our program to solve Conjecture 1. We have already made substantial applications in this direction in our papers [**18, 19, 20**]; we hope to give a final solution to Conjecture 1 in our proposed papers [**21, 22**].

THEOREM 1. *Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus and two limit ends. Assume that M is normalized by a rotation and homothety so that it has horizontal limit tangent plane at infinity and the vertical component of its flux equals 1. Then:*

(1) The middle ends $\{e_n \mid n \in \mathbb{Z}\}$ of M are planar and have heights $\mathcal{H} = \{x_3(e_n) \mid n \in \mathbb{Z}\}$ such that $x_3(e_n) < x_3(e_{n+1})$ for all $n \in \mathbb{Z}$;

(2) $\lim_{n\to\infty} x_3(e_n) = \infty$ and $\lim_{n\to-\infty} x_3(e_n) = -\infty$;

(3) Every horizontal plane sufficiently high or low intersects M in a simple closed curve when its height is not in $\mathcal{H}$ and in a single properly embedded arc when its height is in $\mathcal{H}$;

(4) M has bounded Gaussian curvature. If M has genus zero, then the bound of its curvature depends only on an upper bound of the horizontal component of the flux of M. If M has positive genus, then $\lim_{r\to\infty} \sup |K_{M-B(r)}|$ satisfies the same curvature estimates as in the genus zero case, in terms of the associated horizontal component of its flux (here $B(r)$ denotes the ball of radius r centered at the origin, and $K_{M-B(r)}$ is the Gauss curvature function of $M - B(r)$).

(5) If the Gaussian curvature of M is bounded from below in absolute value by ε^2, then M has a regular tubular neighborhood of radius $1/\varepsilon$ and so, the spacings $S(n) = x_3(e_{n+1}) - x_3(e_n)$ between consecutive ends are bounded from below by $2/\varepsilon$. Furthermore, these spacings are also bounded by above.

(6) M is quasiperiodic in the following sense. There exists a divergent sequence $V(n) \in \mathbb{R}^3$ such that the translated surfaces $M + V(n)$ converge to a properly embedded minimal surface of genus zero, two limit ends, horizontal limit tangent plane at infinity and with the same flux as M.

Finally, we remark that Conjecture 1, by means of normalized blow-ups (see Section 2), gives a precise geometric description of an embedded minimal surface in an homogeneously regular Riemannian 3-manifold, in a neighborhood of a point of sufficiently large curvature, with bounds on the genus of the surface in this neighborhood. In this way, our theorems explain the local

structure of minimal surfaces in 3-manifolds and should lead to important compactness/regularity theorems for limits of sequences of embedded minimal surfaces of bounded genus. In fact, one of our original motivations for proving Conjecture 1 was to use it as a variational tool to classify compact 3-manifolds with finite fundamental group, in part through the proposed Pitts-Rubenstein method [**30**] to resolve the spherical space form problem.

The authors would like to thank Bill Minicozzi and Toby Colding for explanations of several of their key results and for pointing our how these results relate to earlier proofs of theorems in our papers.

2. Preliminaries and perspectives.

In this section we recall some of the basic definitions and theorems for properly embedded minimal surfaces M in $\mathbb{R}^3$ that will be essential in the proof of our theorems. In the sequel, $B(x,r)$ will denote the ball centered at $x \in \mathbb{R}^3$ with radius $r > 0$, and for a surface Σ, K_Σ will stand for its Gaussian curvature.

Recall now the definition of the limit tangent plane at infinity for M. From the Weierstrass representation for minimal surfaces one knows that the finite collection of ends of a complete embedded noncompact minimal surface $\Sigma \subset \mathbb{R}^3$ of finite total curvature and compact boundary is asymptotic to a finite collection of pairwise disjoint ends of planes and catenoids, each of which has a well-defined unit normal at infinity. It follows that the limiting normals to the ends of Σ are parallel and one defines the limit tangent plane of Σ to be the plane passing through the origin and orthogonal to the normals of Σ at infinity. Suppose that Σ is contained in $\mathbb{R}^3 - M$. One defines a limit tangent plane for M to be the limit tangent plane of Σ. In [**1**] it is shown that if M has at least two ends, then M has a unique limit tangent plane which we call *the limit tangent plane at infinity* for M. We say that the limit tangent plane at infinity is *horizontal* if it is the (x_1, x_2)-plane.

The main result in [**12**] is that if M has more than one end and horizontal limit tangent plane at infinity, then the ends of M can be ordered by their "relative heights" over the (x_1, x_2)-plane and this ordering of the ends of M is a topological property of M, in the sense that if M is properly isotopic to another minimal surface M' with horizontal limit tangent plane at infinity, then the associated ordering of the ends of M' either agrees with or is opposite to the ordering coming from M.

Unless otherwise stated, we will assume that the limit tangent plane at infinity of M is horizontal, and so M is equipped with a particular ordering on its set of ends $\mathcal{E}(M)$, which has a natural topology of a totally disconnected compact Hausdorff space. The limit points of $\mathcal{E}(M)$ are called *limit ends* of M. Since $\mathcal{E}(M)$ is compact and embeds topologically in an ordering preserving manner in the closed unit interval $[0,1]$, there exist unique maximal and minimal elements of $\mathcal{E}(M)$ for this ordering. The maximal element is called the *top* end of M. The minimal element is called the *bottom* end of M. Otherwise the end is called a *middle* end of M.

The main theorem in [**8**] is that a limit end of M must be a top or bottom end, hence M has at most two limit ends. The proof of the following theorem can be found in Section 4. We will use

this result as a first step in proving the nonexistence of the surface appearing in the statement of the next theorem, which is the main result in [**18**].

THEOREM 2. *Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus and horizontal limit tangent plane at infinity. If M has just one limit end e_∞, which is its top end, then each middle end $e_n \in \mathcal{E}(M) = \{e_1, e_2, \ldots, e_\infty\}$ is asymptotic to a graphical annular end E_n of a vertical catenoid with logarithmic growth λ_n satisfying $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots < 0$. Here $\mathcal{E}(M)$ is ordered by the Ordering Theorem* [**12**].

We have normalized each properly embedded minimal surface M in $\mathbb{R}^3$ with more than one to have horizontal limit tangent plane at infinity. When M has finite genus and two limit ends, there is a second natural normalization which is to change the surface by a fixed homothety so that it has vertical component of its flux equal to one; in order to explain this second normalization, we need the partial description of the surface given in the next lemma.

LEMMA 1. *Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with finite genus, two limit ends and horizontal limit tangent plane at infinity. Then, statements 1, 2, 3 in Theorem 1 hold for M.*

PROOF. The main theorem in [**8**] implies that the middle ends of M are not limit ends and so, since M has finite genus, they can be represented by annuli. By Collin's theorem [**7**], an annular end of M is asymptotic to the end of a plane or of a catenoid. By another theorem in [**8**], the middle ends have representatives contained in slabs and so they are asymptotic to ends of horizontal planes rather than to ends of catenoids. In particular, there exists a plane P that intersects M transversely in a compact set.

Let $M(+)$ denote the portion of M above P. By the Halfspace Theorem [**13**], $x_3|_{M(+)}$ is not bounded from above and extends conformally across the middle ends. Conformal results in [**8**] state that $M(+)$ is parabolic. It follows that $M(+)$ is conformally diffeomorphic to a compact Riemann surface Σ with boundary, punctured in a countable number of points with one limit point p_∞ corresponding to the limit end of $M(+)$. Let $\bar{\Sigma}$ be the compactification of $M(+)$ by adding its middle ends and p_∞. It follows that there is a punctured disk coordinate neighborhood of p_∞, $D(*) = \{z \in \mathbb{C} \mid 0 < |z| < 1\} \subset \bar{\Sigma}$ so that $x_3|_{D(*)} = -\lambda \ln |z| + c$ where $\lambda, c \in \mathbb{R}, \lambda > 0$. Thus, M intersects planes above height c in simple closed curves at heights not corresponding to ends. Since different planar ends cannot be at the same height by the maximum principle at infinity [**27**], then all planes above height c and at the heights of ends intersect M in connected proper arcs. This proves the lemma. □

We are now ready to define the second natural normalization for a properly embedded minimal surface $M \subset \mathbb{R}^3$ with finite genus, two limit ends and horizontal tangent plane at infinity. By Lemma 1, all the middle ends of M are planar and there are horizontal planes intersecting M in a compact set. We may assume that $\{x_3 = 0\}$ intersects M transversely in a finite set of Jordan curves. Hence the integral of the upwards pointing conormal to $M(-) = \{p \in M \mid x_3(p) \leq 0\}$ along its boundary is a finite vector $F(M) \in \mathbb{R}^3$ which is independent of the vertical translation

of M by the Divergence Theorem. Here, $F(M) = V(M) + H(M)$ where $V(M)$ is a vertical vector and $H(M)$ is a horizontal vector. Since M intersects the plane $\{x_3 = 0\}$ transversely, $V(M)$ must be nonzero (in fact, we will prove in Section 4 that $H(M)$ is also nonzero). In the sequel, we will refer to $V(M)$ as the *vertical flux* and $H(M)$ as the *horizontal flux of* M.

After a homothety, we assume that $V(M) = (0, 0, 1)$, which is the natural normalization we take for M. In Sections 3 and 4, we prove that when M has vertical flux one, finite genus and two limit ends, then outside large compact subsets of M there is an estimate for the absolute curvature of M that only depends on any upper bound for the horizontal flux $\|H(M)\|$. This uniform estimate can be used to understand moduli space problems related to proving Conjecture 1. For example, in [**20**] we apply this estimate to prove that if M has genus zero, two limit ends and sufficiently small horizontal flux, then M is one of the Riemann minimal examples.

The curvature estimates we obtain in Sections 3 and 4 (or rather the proof of these curvature estimates) also play a fundamental role in the proof by Meeks and Rosenberg [**25**] that if M is a properly embedded minimal surface of finite genus in $\Sigma \times \mathbb{R}$, where Σ is a compact Riemannian surface, then M has bounded curvature, finite topology and linear area growth.

Let $O \subset \mathbb{R}^3$ be an open set. A sequence $\{M(n) \subset O\}_{n\in\mathbb{N}}$ of minimal surfaces is called *locally simply connected* if for each point in O, there exists a ball $B \subset O$ centered at that point such that for n large, every component of $M(n) \cap B$ is a disk with boundary in the boundary of B. In the case $O = \mathbb{R}^3$, we say that a sequence $\{M(n)\}_n$ is *uniformly locally simply connected* (ULSC) if there exists $r > 0$ such that for each $p \in \mathbb{R}^3$ and for n large, the ball centered at p with radius r intersects $M(n)$ in components which are disks with boundary lying in the boundary of that ball.

In a recent series of papers, Colding and Minicozzi [**2, 3, 4, 5, 6**] have attempted to describe the basic structure of compact embedded minimal surfaces M of fixed genus which are contained in the unit ball $B = B(\vec{0}, 1)$ and which have their boundary on the boundary of B. The most important case of their structure theorem is when M is a disk which passes through the origin $\vec{0}$ where its Gaussian curvature is large. In this case, Colding and Minicozzi prove that M has the appearance, in a smaller ball $B(\vec{0}, \varepsilon)$ and outside a cone centered at $\vec{0}$, of a highly sheeted double multigraph, similar to a homothetically shrunk helicoid. They then use this local picture as a tool for proving the next result, which we will need for our proofs in the next two sections.

Theorem 3 (Colding, Minicozzi [**5, 6**]). *Let $M_n \subset B(\vec{0}, R_n)$ be a* ULSC *sequence of embedded minimal planar domains with $\partial M_n \subset \partial B(\vec{0}, R_n)$, $R_n \to \infty$. If $\sup |K_{M_n \cap B(\vec{0},1)}| \to \infty$, then there exists a subsequence of the M_n (denoted in the same way) and a set $S(\mathcal{L})$ consisting of one or two Lipschitz curves such that after a rotation of $\mathbb{R}^3$:*

(1) Each Lipschitz curve S_k in $S(\mathcal{L})$ can be parametrized in $\mathbb{R}$ by its height.
(2) Each M_n is horizontally locally graphical away from $S(\mathcal{L})$.
(3) For each $\alpha \in (0, 1)$, $M_n - S(\mathcal{L})$ converges in the C^α-topology to the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes.
(4) $\sup |K_{M_n \cap B(S_k(t),r)}| \to \infty$ as $n \to \infty$, for all $t \in \mathbb{R}$ and $r > 0$.
(5) If additionally $M_n \cap B(\vec{0}, 2)$ contains a component which is not a disk for each n, then $S(\mathcal{L})$ consists of two Lipschitz curves.

Meeks and Rosenberg [**26**] applied this structure theorem for the case of disks in their proof that the plane and the helicoid are the only properly embedded simply connected minimal surfaces in $\mathbb{R}^3$. Meeks' regularity theorem [**17**] implies that the singular set $S(\mathcal{L})$ in Theorem 3 consists of vertical straight lines.

A sequence of possibly disconnected disjoint minimal graphs over the unit disk with bounded gradient and with nonempty limit set has a subsequence that converges to a minimal lamination of the open cylinder over that disk, whose leaves are graphical with the same gradient estimate. This fact, together with a standard diagonal argument has as a consequence the following lemma; see Theorem 4.39 in [**29**] or the proof of Theorem 1.6 in [**26**] for a similar analysis.

LEMMA 2. *Let $O \subset \mathbb{R}^3$ be an open set and take a sequence $\{M(n) \subset O\}_n$ of embedded minimal surfaces (possibly disconnected) with locally bounded Gaussian curvature in the sense that for any ball $B \subset O$, the $M(n) \cap B$ have uniformly bounded Gaussian curvature, with the bound of the curvature depending only on B. If for every n, any divergent curve of finite length on $M(n)$ has limit point in the boundary of O, then a subsequence of the $M(n)$ converges to a $C^{1,\alpha}$ minimal lamination of O with the same local bounds for its Gaussian curvature as the ones of $M(n)$.*

The following lemma is closely related to the 1-sided curvature estimate by Colding and Minicozzi in [**5**].

LEMMA 3. *Denote by $S = \{x = (x_1, x_2, x_3) \mid 0 \leq x_3 < 1\}$, $C(r) = \{x \in S \mid x_1^2 + x_2^2 \leq r^2\}$ for $r > 0$ and let $R\colon [0,1) \to (0,1)$ be a continuous function with $R(t) \leq 1-t$ for any $t \in [0,1)$. Given a divergent sequence $\{r_n\}_n \subset \mathbb{R}^+$, suppose that $\{\Sigma_n\}_n$ is a sequence of compact embedded minimal surfaces in $C(r_n)$ with $\partial\Sigma_n \subset \partial C(r_n) - \{x_3 = 0\}$, satisfying the following uniform local simply connected property: for any point $x = (x_1, x_2, x_3) \in S$ and for n sufficiently large, $\Sigma_n \cap B(x, R(x_3))$ consists of simply connected components. Then, a subsequence of the Σ_n converges smoothly on compact subsets of S to a lamination of S by horizontal planes.*

PROOF. Since $0 < R(0) < 1$, given a point $x \in \{x_3 = 0\}$, we have that for n large enough, $\Sigma_n \cap B(x, R(0))$ consists only of disks with boundaries on $\partial B(x, R(0)) \cap \{x_3 > 0\}$. By the 1-sided curvature estimate of Colding and Minicozzi [**5**] applied to $\Sigma_n \cap B(x, R(0))$, there exists an $\varepsilon > 0$ such that for n large the sequence $\{\Sigma_n\}_n$ has bounded absolute curvature at most ε in a fixed size small neighborhood of any point of the (x_1, x_2)-plane.

We claim that for any t with $0 < t < 1$ and for any point $x = (x_1, x_2, x_3)$ with $x_3 = t$, the curvature of the sequence $\{\Sigma_n \cap C\left(\frac{r_n}{2}\right)\}_n$ is also eventually bounded in a small ball centered at x whose radius depends only on t. If this property were to fail for some t, then it fails for a smallest such positive t, say t_0. After translating horizontally Σ_n at a rate slower than $\frac{r_n}{2}$ and taking a subsequence, we can obtain a new sequence of surfaces $\{\widetilde{\Sigma}_n\}_n$ satisfying the conditions of this lemma (with the radii $\widetilde{r}_n$ of the corresponding cylinders $C(\widetilde{r}_n)$ being $\widetilde{r}_n \geq \frac{r_n}{2} \to \infty$ as $n \to \infty$), and such that for any positive t, $0 < t < t_0$ and any $n \in \mathbb{N}$, the absolute curvature of $\widetilde{\Sigma}_n \cap \{x_3 \leq t\}$ is bounded from above by a fixed constant that only depends on t, and the point $p_0 = (0, 0, t_0) \in \widetilde{\Sigma}_n$ has absolute curvature at least n. It follows from Lemma 2 that a subsequence of these surfaces converges smoothly on compact subsets of $\{0 \leq x_3 < t\}$ to a minimal lamination

$\mathcal{L}_t$ with bounded curvature. Next we show that $\mathcal{L}_t$ consists only of horizontal planes, with the plane $\{x_3 = t_0\}$ inside the limit set of the $\mathcal{L}_t$ as $t \to t_0$. Arguing by contradiction, suppose that L is a nonplanar leaf of $\mathcal{L}_t$ and let $P = \{x_3 = t_1\}$ be the highest horizontal plane lying below L (here $0 \leq t_1 < t_0$). Using the bounded curvature of L, one can find a small ε such that the projection $\pi(x_1, x_2, x_3) = (x_1, x_2, 0)$ restricts to $L \cap \{t_1 < x_3 \leq t_1 + \varepsilon\}$ as a local diffeomorphism. Let $\mathcal{C}_L$ be a component of $L \cap \{t_1 < x_3 \leq t_1 + \varepsilon\}$. Then $\mathcal{C}_L$ is proper in the slab $\{t_1 < x_3 \leq t_1 + \varepsilon\}$ (otherwise the minimal lamination of $\{t_1 < x_3 \leq t_1 + \varepsilon\}$ given by the closure of $\mathcal{C}_L$ would have a limit leaf which can be proved to be a horizontal plane higher than P, which contradicts that P is the highest horizontal plane below L). Since $\mathcal{C}_L$ is properly embedded in the slab $\{t_1 < x_3 \leq t_1 + \varepsilon\}$ with $\partial\mathcal{C}_L \subset \{x_3 = t_1 + \varepsilon\}$, $\mathcal{C}_L$ must separate $\{t_1 < x_3 \leq t_1 + \varepsilon\}$ in two components. Since $\mathcal{C}_L$ is connected and submerses into the (x_1, x_2)-plane, it has a unique orientation induced by the vector $(0, 0, 1)$. By the separation property of $\mathcal{C}_L$, it follows that $\pi^{-1}(\pi(x)) = \{x\}$ for each $x \in \mathcal{C}_L$ (otherwise $\pi^{-1}(\pi(x))$ contains two consecutive points where $\mathcal{C}_L$ has opposite orientations with respect to $(0, 0, 1)$, a contradiction). Therefore $\mathcal{C}_L$ is a graph over its projection in the (x_1, x_2)-plane, which in turn implies that $\mathcal{C}_L$ is proper in the slab $\{t_1 \leq x_3 \leq t_1 + \varepsilon\}$. Now the proof of the Halfspace Theorem [**13**] applies to $\mathcal{C}_L$ giving a contradiction. Hence, $\mathcal{L}_t$ consists of horizontal planes. By the uniform local simply connected property of $\widetilde{\Sigma}_n$, we may assume that each $\widetilde{\Sigma}_n$ intersects the ball $B(p_0, R(t_0))$ in components which are disks. Let $\Delta(n)$ be the disk component of $\widetilde{\Sigma}(n) \cap B(p_0, R(t_0))$ that contains p_0. Choose δ_0 with $0 < \delta_0 < R(t_0)/2$. By the 1-sided curvature estimates of Colding-Minicozzi and after passing to a subsequence, we deduce that for δ_0 sufficiently small and for all positive $\delta < \delta_0$, $\Delta(n)$ intersects the closure of the horizontal disk $B(p_0, R(t_0)) \cap \{x_3 = t_0 - \delta\}$ for all n large. Now choose some t with $t_0 - \frac{\delta}{2} < t < t_0$. Note that the plane $\{x_3 = t_0 - \frac{\delta}{2}\}$ belongs to the lamination $\mathcal{L}_t$ (because $\widetilde{\Sigma}_n \cap \{0 \leq x_3 < t\}$ converges smoothly to $\mathcal{L}_t$ as $n \to \infty$ and $\Delta(n)$ intersects $\{x_3 = t_0 - \frac{\delta}{2}\}$ for all n large). Again the smooth convergence of $\widetilde{\Sigma}_n$ to $\mathcal{L}_t$ implies that $\widetilde{\Sigma}_n$ contains an almost flat almost horizontal disk $\Delta_1(n)$ arbitrarily close to $B(p_0, 2R(t_0)) \cap \{x_3 = t_0 - \frac{\delta}{2}\} \subset \mathcal{L}_t$. This is impossible, since $\Delta_1(n)$ separates p_0 from the closure of the disk $\{x_3 = t_0 - \delta\} \cap B(p_0, R(t_0))$, which also contains points of $\Delta(n)$. This contradiction proves our claim.

We have demonstrated that the sequence $\{\Sigma_n \cap C\left(\frac{r_n}{2}\right)\}_n$ has locally bounded curvature in the open slab $\{-\varepsilon < x_3 < 1\}$. By Lemma 2, a subsequence of the $\Sigma_n \cap C\left(\frac{r_n}{2}\right)$ converges to a $C^{1,\alpha}$ minimal lamination $\mathcal{L}$ of this slab. Since $x_3 > 0$ on $\Sigma_n \cap C\left(\frac{r_n}{2}\right)$ for each n, we have that $\mathcal{L} \subset \{0 \leq x_3 < 1\}$. A minor modification of the arguments in the last paragraph show that $\mathcal{L}$ consists only of horizontal planes. Finally, note that the conclusions of the lemma remain invariant if we replace the original sequence $\{\Sigma_n\}_n$ by $\{\Sigma_n \cap C\left(\frac{r_n}{2}\right)\}_n$, which finishes the proof. □

We now explain a useful technique for studying the local geometry of a sequence of embedded minimal surfaces $M(n)$ at points of large normalized curvature (Colding and Minicozzi refer to such points as *points of almost maximal curvature*), a concept that we now define. A *sequence of points of large normalized curvature* is a sequence $p(n) \in M(n)$ such that:

(1) $\lambda(n) := \sqrt{|K_{M(n)}|(p(n))}$ tends to ∞ as $n \to \infty$;
(2) $B(p(n), 1)$ does not contain points of the boundary of $M(n)$ for any n;

(3) There exists a sequence of positive numbers $\tau(n)$ diverging to infinity, such that $\tau(n) \leq \lambda(n)$ for each n;
(4) There exists $c > 0$ such that $|K_{M(n)}| \leq c\lambda(n)^2$ in $M(n) \cap B(p(n), \frac{\tau(n)}{\lambda(n)})$.

Whenever we have a sequence $p(n)$ of points of large normalized curvature, the rescaled surfaces $\widetilde{M}(n) = \lambda(n)[M(n) - p(n)]$ all pass through the origin with Gaussian curvature $K_{\widetilde{M}(n)}$ equal to -1 at $\vec{0}$, have their (possibly empty) boundaries outside the ball of radius $\tau(n)$ and have uniformly bounded curvature on $B(\vec{0}, \tau(n))$ since

$$K_{\widetilde{M}(n)\cap B(\vec{0},\tau(n))} = \lambda(n)^{-2} K_{M(n)\cap B\left(p(n),\frac{\tau(n)}{\lambda(n)}\right)} \geq -c.$$

(We have abused slightly notation in the equality above; the rigorous meaning of this equality relates the Gauss curvature function of $\widetilde{M}(n)$ at points inside $B(\vec{0}, \tau(n))$ with the Gauss curvature of $M(n)$ at the corresponding points under rescaling in $B\left(p(n), \frac{\tau(n)}{\lambda(n)}\right)$). It follows from the results in [**26**] that a subsequence of the surfaces $\widetilde{M}(n)$ converges with multiplicity 1 to a connected nonplanar properly embedded minimal surface $\widetilde{M}(\infty)$ of bounded Gaussian curvature, called a *normalized blow-up of the sequence* $\{M(n)\}_n$. By the convex hull property for compact minimal surfaces, the limit $\widetilde{M}(\infty)$ has genus less than or equal to the genus of the $M(n)$ and $\widetilde{M}(\infty)$ has at most as many generators in its fundamental group as the $M(n)$.

It follows that if $\{M(n)\}_n$ is a uniformly locally simply connected sequence of properly embedded minimal surfaces in $\mathbb{R}^3$ and $p(n) \in M(n)$ is a sequence of points of large normalized curvature, then the blow-up $\widetilde{M}(\infty)$ produced in the last paragraph must be simply connected. By [**26**], $\widetilde{M}(\infty)$ is a helicoid. Thus, in a small neighborhood of a point of $M(n)$ of very large normalized curvature, $M(n)$ has the appearance of a homothetically shrunk helicoid with a large number of sheets.

The curvature estimates we obtain in the next section have proven useful in describing the local geometry of a sequence $\{M(n)\}_n$ of properly embedded minimal surfaces of fixed finite genus, but not necessary with a bound on the number of generators of their fundamental groups, near a sequence of points of large normalized curvature. The desired description can be obtained from the special case where the genus of $M(n)$ is zero but $M(n)$ is not simply connected. If a sequence $\{M(n)\}_n$ is locally simply connected in an open set $O \subset \mathbb{R}^3$ (this can be easily extended to a locally simply connected sequence of properly embedded minimal surfaces in any homogeneously regular Riemannian 3-manifold), then the previous analysis implies that near points $p(n) \in M(n)$ of large normalized curvature, $M(n)$ has the appearance of a homothetically shrunk helicoid with many sheets. If $\{M(n)\}_n$ is not locally simply connected in O but it has *locally bounded genus* (i.e., for every point $p \in O$ there exists a geodesic ball $B(p, r) \subset O$ such that the genus of every component of $M(n) \cap B(p, r)$ is not greater than some uniform constant), then there exists a point $x \in O$, a subsequence of $\{M(n)\}_n$ (denoted in the same way) and a sequence of points $p(n) \in M(n)$ of large normalized curvature that converges to x. In particular, such a sequence of minimal surfaces has a normalized blow-up of bounded genus. Conjecture 1 in the Introduction describes what should be the possible normalized blow-ups of such a sequence. The next theorem is a partial result on the local description of these surfaces and its proof appears in Section 4. In [**18**] we will prove that item *3* below never holds.

THEOREM 4. *Suppose $\widetilde{M} \subset \mathbb{R}^3$ is the normalized blow-up of a sequence of properly embedded minimal surfaces in a homogeneously regular Riemannian 3-manifold where the surfaces have locally bounded genus. Then, there is another normalized blow-up $M \subset \mathbb{R}^3$ of the same sequence such that one of the following holds:*

(1) M is a helicoid;
(2) M has a finite number of ends greater than one and it has finite total curvature;
(3) M has finite genus, bounded Gaussian curvature and one limit end;
(4) M has genus zero and two limit ends.

Furthermore, if $\widetilde{M}$ has finite topology and genus zero, then $\widetilde{M}$ is a helicoid or a catenoid.

We finish this preliminaries section with some notation, to be used later on. If $\Sigma \subset \mathbb{R}^3$ is a minimal surface and $\delta \subset \Sigma$ is an embedded closed curve, $\mathrm{Flux}(\Sigma, \delta)$ will denote the flux of Σ along δ, i.e., the integral along δ of the unit tangent vector to Σ and orthogonal to δ, which is defined up to orientation on δ. Given $a, b \in [-\infty, \infty]$, $a < b$, we let $S(a, b) = \{x \in \mathbb{R}^3 \mid a < x_3 < b\}$ mean the open horizontal slab between heights a and b. Given $p, q \in \mathbb{R}^3$, we let $\overline{p, q} = \{tp + (1-t)q \mid t \in [0, 1]\}$ denote the closed segment joining p and q. Given $\theta \in [0, 2\pi)$, we denote by Rot_θ the rotation by angle θ around the x_3-axis.

3. Curvature estimates for the genus zero case.

Let $\mathcal{M}$ be the space of properly embedded minimal surfaces in $\mathbb{R}^3$ with genus zero, two limit ends and horizontal tangent plane at infinity. By Lemma 1, all middle ends of any surface $M \in \mathcal{M}$ are planar, and we can assume that such surfaces are normalized so that the flux $F(M)$ along the boundary of $M \cap \{x_3 \leq 0\}$ is $F(M) = (0, 0, 1) + H(M)$, where $H(M)$ is a horizontal vector. Note also that the arguments in the proof of Lemma 1 can be adapted to this setting and give that any horizontal section of M is a simple closed curve or a proper embedded arc (this last possibility occuring exactly at the heights of the planar ends), and the third coordinate function x_3 has no critical points on M, or equivalently the Gauss map of M omits the vertical directions.

Recall that the 1-parameter family $\{\mathcal{R}_t\}_t$ of Riemann minimal examples can be considered to be a curve in $\mathcal{M}$ parametrized by $t \in \mathbb{R}^+$ where $\|H(\mathcal{R}_t)\| = t$. It can be shown that $\lim_{t\to\infty} \max |K_{\mathcal{R}_t}| = +\infty$ and that for any $t_0 > 0$, the family $\{\mathcal{R}_t \mid t \leq t_0\}$ has uniformly bounded Gaussian curvature (see for instance Meeks, Pérez and Ros [**23**]). The main goal of this section is to prove a related basic curvature estimate.

THEOREM 5. *If $\{M(i)\}_i \subset \mathcal{M}$ is a sequence of surfaces with $\{H(M(i))\}_i$ bounded, then the Gaussian curvature of the $M(i)$ is uniformly bounded.*

In the proof of Theorem 5, we will need several lemmas. The next one follows directly from the curvature estimates for stable minimal surfaces by Schoen [**32**].

LEMMA 4. *Let $S \subset \mathbb{R}^3$ be a horizontal slab of width not greater than 1. Then, there exists $\tau_0 > 1$ such that for any properly embedded noncompact orientable stable minimal surface $\Delta \subset S$ with boundary inside a vertical cylinder of radius 1, the portion of Δ at distance greater than τ_0*

from the axis of the cylinder consists of a finite number of graphs over the outside of a disk of radius τ_0 in the (x_1, x_2)-plane.

In the next lemma we show that a normalized blow-up of a sequence $\{M(i)\}_i$ as in Theorem 5 is necessarily simply connected; in fact, such a blow-up must be a vertical helicoid.

LEMMA 5. *Let $\{M(i)\}_i \subset \mathcal{M}$ be a sequence in the hypotheses of Theorem 5. Assume that $\{|K_{M(i)}|\}_i$ is not uniformly bounded. Then, after possibly passing to a subsequence, there exist points $p(i) \in M(i)$ and positive numbers $\lambda(i) \to +\infty$ so that the surfaces $M'(i) = \lambda(i)(M(i) - p(i))$ converge uniformly on compact subsets of $\mathbb{R}^3$ with multiplicity 1 to a vertical helicoid H passing through the origin $\vec{0}$, with $|K_H| \leq 1$ and $|K_H(\vec{0})| = 1$.*

PROOF. As $\{|K_{M(i)}|\}_i$ is not uniformly bounded, after extracting a subsequence we find points $\widehat{p}(i) \in M(i)$ where $|K_{M(i)}(\widehat{p}(i))| \to \infty$ as i goes to ∞. On a ball of radius 1 centered at $\widehat{p}(i)$, the function $|K_{M(i)}|(1 - d(\widehat{p}(i), \cdot))^2$ attains a maximum value at a point $p(i) \in M(i)$. Define $\lambda(i) = \sqrt{|K_{M(i)}(p(i))|} > 0$ and consider the surfaces $M'(i) = \lambda(i)(M(i) - p(i))$, $i \in \mathbb{N}$. Up to a subsequence, $\{M'(i)\}_i$ converges smoothly on compact subsets of $\mathbb{R}^3$ with multiplicity 1 to a connected properly embedded minimal surface $H \subset \mathbb{R}^3$ with $\vec{0} \in H$, $|K_H(\vec{0})| = 1$ and $|K_H| \leq 1$ in H, see Meeks and Rosenberg [**26**] for a similar compactness result. It remains to show that H is a vertical helicoid.

Suppose that H is not simply connected. We claim that in this case there exists an embedded nontrivial cycle $\gamma \subset H$ such that $\mathrm{Flux}(H, \gamma) \neq 0$. To see this, we will discuss the possible cases for H. By standard arguments, H can be shown to have genus zero. If the number of ends of H is finite and greater than one, then it is a catenoid (Collin [**7**] and López, Ros [**15**]), and we choose γ as the waist circle of H. If H has infinitely many ends, then it must have one or two limit ends (Collin, Kusner, Meeks and Rosenberg [**8**]). In the one limit end case, the results in [**8**] imply, possibly after a rotation, that the limit end of H is the top end in the natural linear ordering given by the Ordering Theorem (Frohman and Meeks [**12**]). In this case, the bottom end E of H is an annulus and so, by Collin's Theorem [**7**] E is asymptotic to a plane or to a catenoid. If E is asymptotic to a plane, then H is contained is a halfspace, contradicting the Halfspace Theorem (Hoffman and Meeks [**13**]). Thus, E is of catenoidal type and a cycle generating the homology of E is a good choice for γ in this case. Finally in the two limit end case, we choose γ to be the generator of the homology of the cylinder obtained after attaching the planar ends to H. Thus the claim holds.

As the curve $\gamma \subset H$ is compact, it must be a smooth limit of embedded cycles $\gamma'(i) \subset M'(i)$, thus $\mathrm{Flux}(H, \gamma) = \lim_i \mathrm{Flux}(M'(i), \gamma'(i)) = \lim_i \lambda(i)\mathrm{Flux}(M(i), \gamma(i))$, where $\gamma(i) \subset M(i)$ is the embedded closed curve such that $\gamma'(i) = \lambda(i)(\gamma(i) - p(i))$. But the third component of $\mathrm{Flux}(H, \gamma)$ is bounded and the third component of $\mathrm{Flux}(M(i), \gamma(i))$ is equal to 1 for all i. This contradicts that $\lambda(i) \to \infty$, and proves that H is simply connected.

Since H is simply connected and nonflat, a theorem of Meeks and Rosenberg [**26**] implies that H is a helicoid. Finally, the helicoid H must be vertical because it is a smooth limit of the $M'(i)$, whose Gauss maps omit the vertical directions, and the Gauss map of H is an open map. □

The proof of Theorem 5 is rather long and delicate. For the sake of clarity, we will now give a sketch of this proof, skipping the details that will be developed later on. *We assume that the theorem fails.* Using Lemma 5 and passing to a subsequence, we have points $p(i) \in M(i)$ and positive numbers $\lambda(i) \to +\infty$ so that $\lambda(i)(M(i) - p(i))$ converges uniformly on compact subsets of $\mathbb{R}^3$ to a vertical helicoid H with $\vec{0} \in H$, $|K_H| \leq 1$ and $|K_H(\vec{0})| = 1$. We will show that there exist angles $\theta(i) \in [0, 2\pi)$ such that for any $\tau > \tau_0$ (this is the positive number given in Lemma 4), we can find positive numbers $\mu(\tau, i) > 0$ and embedded closed curves $\delta(\tau, i) \subset \mu(\tau, i)\mathrm{Rot}_{\theta(i)}(M(i) - p(i))$ such that

$$\mathrm{Flux}\left(\mu(\tau, i)\mathrm{Rot}_{\theta(i)}(M(i) - p(i)), \delta(\tau, i)\right) = V(\tau, i) + W(\tau, i) \tag{1}$$

where $V(\tau, i), W(\tau, i) \in \mathbb{R}^3$ are vectors such that $\lim_{i\to\infty} V(\tau, i) = (12\tau, 0, 0)$ and $\{\|W(r, i)\|\}_i$ is bounded by a constant independent of τ. Assuming these facts, the proof of Theorem 5 finishes as follows. First note that for any surface $M \in \mathcal{M}$, the angle between the flux vector $F(M)$ and its horizontal projection $H(M)$ is invariant under translations, homotheties and rotations around the x_3-axis. By (1), the corresponding angles for the flux vectors of the surfaces $\mu(\tau, i)\mathrm{Rot}_{\theta(i)}(M(i) - p(i))$ tend to zero as $i \to \infty$ and $\tau \to \infty$. But those angles are nothing but the angles for $M(i)$, which are bounded away from zero because of the hypothesis of $\{H(M(i))\}_i$ being bounded. This contradiction proves the theorem.

Now we will go into the details of what is assumed in the last paragraph. Consider the surfaces $M'(i)$ defined in Lemma 5. We may assume without loss of generality, after a possible small translation of the $M'(i)$, that the following three properties hold for all $i \in \mathbb{N}$:

(1) $\vec{0} \in M'(i)$;
(2) The horizontal section $\Gamma'(i) = M'(i) \cap \{x_3 = 0\}$ is a simple closed curve;
(3) $\{M'(i)\}_i$ converges on compact subsets of $\mathbb{R}^3$ to a vertical helicoid whose axis passes through $\vec{0}$.

For i large enough, consider the open arc $\alpha'(i) \subset \Gamma'(i)$ centered at $\vec{0}$ and with intrinsic length 1. Note the Gauss map N_i' of $M'(i)$ at the end points of $\alpha'(i)$ takes values in different hemispheres determined by the horizontal equator. Since N_i' restricted to $\Gamma'(i) - \alpha'(i)$ is continuous, there exists a point $q'(i) \in \Gamma'(i) - \alpha'(i)$ such that $N_i'(q'(i))$ is horizontal, $q'(i)$ being a point of $\Gamma'(i) - \alpha'(i)$ closest extrinsically to the origin with this property.

Fix $\tau > \tau_0$, where τ_0 is the positive number given by Lemma 4. We rescale and rotate $M'(i)$ around the x_3-axis, defining a new surface

$$\widetilde{M}(i) = \frac{6\tau}{\|q'(i)\|}\mathrm{Rot}_{\theta(i)}M'(i),$$

where the rotation angle $\theta(i)$ is chosen so that the point $\widetilde{q} = \frac{6\tau}{\|q'(i)\|}\mathrm{Rot}_{\theta(i)}q'(i)$ is exactly $(0, 6\tau, 0)$.

Remark 1. *With the notation used in* (1), $\mu(\tau, i)$ *is nothing but* $\frac{6\tau\lambda(i)}{\|q'(i)\|}$. *Also note that the dependence of* $\widetilde{M}(i)$ *in terms of* τ *is by a homothety, which makes the angle* $\theta(i)$ *be independent of* τ.

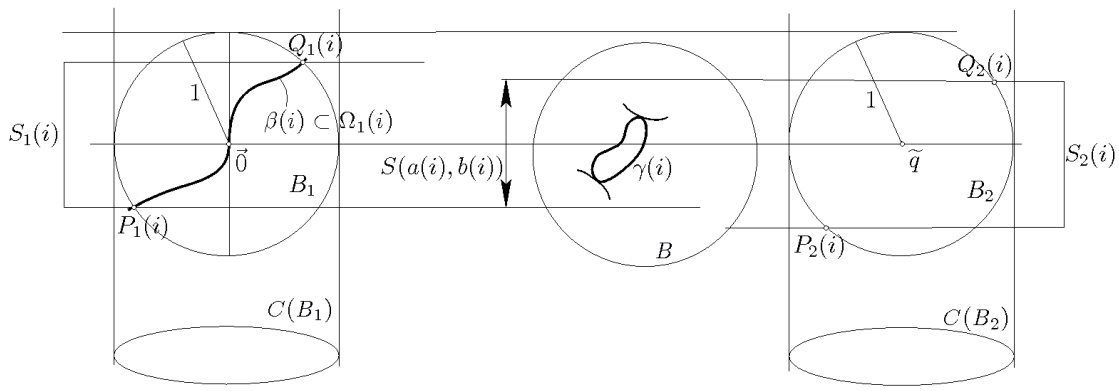

FIGURE 1. A nontrivial curve $\gamma(i) \subset B$ will produce a noncompact stable minimal surface crossing the curve $\beta(i)$.

It remains to find embedded cycles $\delta(r, i) \subset \widetilde{M}(i)$ such that $\mathrm{Flux}(\widetilde{M}(i), \delta(\tau, i))$ decomposes as in (1).

Let B_1 be the open ball of radius one centered at the origin and $\Omega_1(i)$ the component of $\widetilde{M}(i) \cap B_1$ passing through $\vec{0}$. $\Omega_1(i)$ has compact closure, so there exist points $P_1(i), Q_1(i) \in \partial\Omega_1(i)$ such that $S_1(i) = S(x_3(P_1(i)), x_3(Q_1(i)))$ is the smallest open slab containing $\Omega_1(i)$. Exchanging the origin by $\widetilde{q} = (0, 6\tau, 0)$, we similarly define B_2 as the ball of radius one centered at $\widetilde{q}$, $\Omega_2(i)$ to be the component of $\widetilde{M}(i) \cap B_2$ containing $\widetilde{q}$, $P_2(i)$, $Q_2(i)$ points in $\partial\Omega_2(i)$ at lowest and highest heights, and $S_2(i) = S(x_3(P_2(i)), x_3(Q_2(i)))$. The intersection of $S_1(i)$ and $S_2(i)$ is another open slab $S(a(i), b(i))$ with $a(i) < 0 < b(i)$.

LEMMA 6. *For any ball B of radius one, any component of $\widetilde{M}(i) \cap S(a(i), b(i)) \cap B$ is simply connected.*

PROOF. Suppose to the contrary, that there exists a ball B of radius one and a component of $\widetilde{M}(i) \cap S(a(i), b(i)) \cap B$ containing a homotopically nontrivial simple closed curve $\gamma(i)$. Let $C(B_1), C(B_2)$ be the vertical cylinders over the balls B_1, B_2.

First note that as the radius of B is $1 < \tau_0$ and the distance between $\vec{0}$ and $\widetilde{q}$ is $6\tau > 6\tau_0$, the distance from B to at least one of the cylinders $C(B_1)$, $C(B_2)$ must be greater than τ_0. Suppose that $\mathrm{dist}(B, C(B_1)) > \tau_0$ (the case $\mathrm{dist}(B, C(B_2)) > \tau_0$ can be solved similarly). Let $\beta(i) \subset \Omega_1(i)$ be an embedded arc joining $P_1(i)$ and $Q_1(i)$, see Figure 1.

Clearly $(\widetilde{M}(i) \cap S_1(i)) - \beta(i)$ is a connected planar domain whose fundamental group is generated by loops around its finite number of ends, and $\gamma(i) \subset (\widetilde{M}(i) \cap S_1(i)) - \beta(i)$ is a nontrivial embedded cycle. Therefore, $\gamma(i)$ bounds inside $(\widetilde{M}(i) \cap S_1(i)) - \beta(i)$ a compact disk $D(i)$ minus a finite number of points $e(i, 1), \ldots, e(i, k_i)$ corresponding to ends of $\widetilde{M}(i)$ (note that $k_i > 0$ because of the fact that $\gamma(i)$ is homotopically nontrivial and by the convex hull property). Consider a compact exhaustion $\{D(i, j) \mid j \in \mathbb{N}\}$ of $D(i) - \{e(i, 1), \ldots, e(i, k_i)\}$ obtained by removing neighborhoods of the punctures, so $\gamma(i) \subset \partial D(i, j)$, $D(i, j) \subset D(i, j + 1)$ for each j and $\cup_j D(i, j) = D(i) - \{e(i, 1), \ldots, e(i, k_i)\}$. Let $W(i)$ denote a component of $\mathbb{R}^3 - \widetilde{M}(i)$ in which

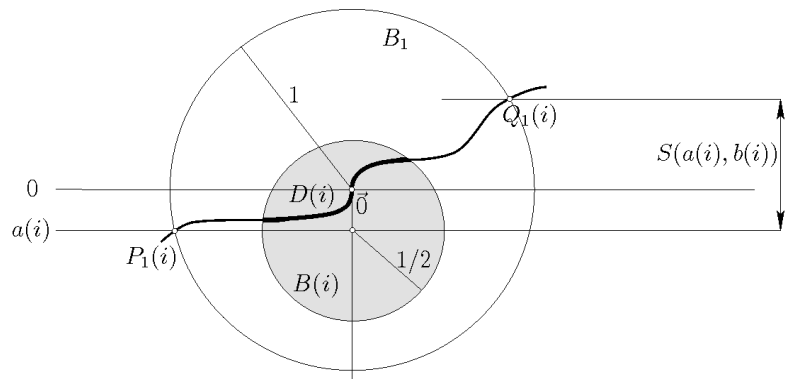

FIGURE 2. The tangent space of $D(i)$ at the origin is vertical, a contradiction.

$\gamma(i)$ is homologically nontrivial, which exists by elementary separation properties. Note that $\partial W(i) = \widetilde{M}(i)$ is a good barrier for solving Plateau problems. Consider the surface of least area $\Delta(i,j)$ in $W(i)$ with boundary $\partial\Delta(i,j) = \partial D(i,j)$. By standard compactness and regularity theorems (see Simon [**33**]), a subsequence of $\{\Delta(i,j)\}_j$ converges uniformly on compact subsets of $W(i)$ to a properly embedded noncompact orientable stable minimal surface $\Delta(i) \subset W(i)$ with $\partial\Delta(i) = \gamma(i)$. By the convex hull property, all of the $\Delta(i,j)$ are contained in the open slab $S_1(i)$ and thus, their limit $\Delta(i)$ lies inside the closure $\overline{S_1(i)}$. By the maximum principle at infinity, $\Delta(i)$ is contained in the open slab $S(a(i), b(i))$ since $\partial\Delta(i)$ is. Moreover, as $\partial\Delta(i)$ is contained in B, it follows that $\operatorname{dist}(\partial\Delta(i), C(B_1)) > \tau_0$. By Lemma 4, $\Delta(i) \cap B_1$ is a union of horizontal graphs, all contained in $S(a(i), b(i)) \cap B_1$. In particular, these graphs must cross the curve $\beta(i)$, a contradiction. This finishes the proof of the lemma. □

LEMMA 7. *There exists $\varepsilon > 0$ such that $\min(|a(i)|, b(i)) \geq \varepsilon$ for all i sufficiently large.*

PROOF. Arguing by contradiction, suppose (possibly after extracting a subsequence) that $\min(|a(i)|, b(i)) \to 0$ as $i \to \infty$. The proof of this lemma follows from the consideration of the following three cases:

CASE 1: *There exists a subsequence, also indexed by i, such that $S_1(i) \subseteq S_2(i)$ for all i (the case $S_2(i) \subseteq S_1(i)$ is analogous to this one).*
Note that in this case, $S_1(i) = S(a(i), b(i))$. Without loss of generality, we can assume that $|a(i)| \leq b(i)$ for all i. Let $B(i)$ be the ball of radius $1/2$ centered at the point $(0, 0, a(i))$, which is contained in B_1 for all i large enough. Using Lemma 6 we deduce that the component $D(i)$ of $\Omega_1(i) \cap S(a(i), b(i)) \cap B(i)$ that contains the origin is a disk for such i. Since the boundary of $D(i)$ is contained in the upper halfsphere $\partial B(i) \cap \{x_3 > a(i)\}$ and the center of $B(i)$ tends to $\vec{0}$ as $i \to \infty$, the 1-sided curvature estimates of Colding and Minicozzi [**5**] insure that the tangent space to $D(i)$ at $\vec{0}$ converges to the horizontal as $i \to \infty$, which contradicts that the origin is a point in $D(i)$ where the tangent space converges to vertical (this holds because suitable rescalings of the $D(i)$ converge smoothly around $\vec{0}$ to a vertical helicoid whose axis passes through $\vec{0}$), see Figure 2.

It is clear that if $b(i) \leq |a(i)|$ after passing to a subsequence, we can argue in a similar way exchanging $(0,0,a(i))$ by $(0,0,b(i))$.

Case 2: *Suppose that neither of the slabs $S_1(i), S_2(i)$ is contained in the other one, and that $\{b(i)/|a(i)|\}_i \to \infty$ as i goes to ∞ (the case $\{b(i)/|a(i)|\}_i \to 0$ can be solved similarly exchanging $a(i)$ by $b(i)$).*
Clearly, now $|a(i)| \leq b(i)$ for all i large and $|a(i)| \to 0$. This second case divides into two possibilities:

2.I: $\{x_3 = a(i)\} \subset \partial S_1(i)$ and $\{x_3 = b(i)\} \subset \partial S_2(i)$.
We rescale to have $b(i) = 1$, defining $\widetilde{M}'(i) = b(i)^{-1}\widetilde{M}(i)$. Consider the ball $B(i)$ of radius $1/2$ centered at the point $(0,0,a(i)/b(i))$, which tends to $\vec{0}$ as $i \to \infty$. Now the argument in Case 1 applies to $B(i)$ without changes, giving the desired contradiction.

2.II: $\{x_3 = a(i)\} \subset \partial S_2(i)$ and $\{x_3 = b(i)\} \subset \partial S_1(i)$.
Now define $\widetilde{M}'(i) = b(i)^{-1}\left(\widetilde{M}(i) - \widetilde{q}\right)$. Let $B(i)$ the ball of radius $1/2$ centered at the point $(0,0,a(i)/b(i))$. The argument of Case 1 applies to $\widetilde{M}'(i) \cap B(i)$ and finishes Case 2 (nevertheless, note that the situation at the origin may not be symmetric to the one at $\widetilde{q}$: nearby $\vec{0} \in \widetilde{M}(i)$ we know that a vertical helicoid is forming at a certain scale, while at $(0,6\tau,0)$ we only have a horizontal normal vector to $\widetilde{M}(i)$, but this is enough to follow the argument in Case 1).

Case 3: *Assume that neither of $S_1(i), S_2(i)$ is contained in the other one, and that $\{b(i)/|a(i)|\}_i$ is bounded.*
First note that we can assume that $\{b(i)/|a(i)|\}_i$ is bounded away from zero (see Case 2), thus $\varepsilon_1 := \frac{1}{2}\inf\{b(i)/|a(i)| \mid i \in \mathbb{N}\} > 0$. Without loss of generality, we can suppose that the plane $\{x_3 = a(i)\}$ is a boundary plane of $S_2(i)$ and $\{x_3 = b(i)\} \subset \partial S_1(i)$ (note that $|a(i)|$ may not be less than $b(i)$, and that both numbers tend to zero at the same rate as $i \to \infty$). The appropriate scale to deal with now will be having $a(i) = -1$, so let $D(i)$ be the component of $\left(|a(i)|^{-1}[\widetilde{M}(i) - \widetilde{q}]\right) \cap S(-1,\varepsilon_1) \cap \left\{x_1^2 + x_2^2 \leq \frac{|a(i)|^{-1}}{2}\right\}$ that passes through $\vec{0}$. Note that the surfaces $D(i)$ essentially satisfy the hypotheses in Lemma 3 after translating them by $e_3 = (0,0,1)$, see Figure 3. The fact that the tangent space of $D(i)$ at the origin is vertical contradicts the conclusion of Lemma 3. Now the lemma is proved. □

Proposition 1. *There exists $\varepsilon > 0$ such that the sequence $\{\widetilde{M}(i) \cap S(-\varepsilon,\varepsilon)\}_i$ is locally simply connected in $S(-\varepsilon,\varepsilon)$. In fact, given any ball B of radius one in $\mathbb{R}^3$, every component of $\widetilde{M}(i) \cap B \cap S(-\varepsilon,\varepsilon)$ is a disk for i sufficiently large.*

Proof. The second statement in the proposition is a direct consequence of Lemmas 6 and 7. Finally, the first statement in the proposition follows from the second one together with the convex hull property. □

Lemma 8. *The sequence of surfaces $\{\widetilde{M}(i)\}_i$ is* ULSC *in $\mathbb{R}^3$ and, after extracting a subsequence, it converges to the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes, with singular set of convergence $S(\mathcal{L}) = \Gamma \cup \Gamma'$ where Γ is the x_3-axis and Γ' is the vertical line passing through $\widetilde{q}$.*

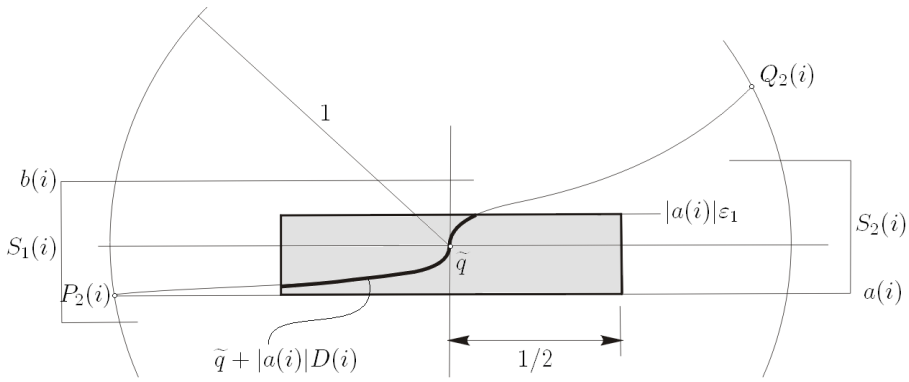

FIGURE 3. The component $D(i)$ represented in the scale of $\widetilde{M}(i)$.

We will postpone the proof of Lemma 8 and show how Theorem 5 follows from it. Recall that given $\tau > \tau_0$, we had translated and rescaled the original surfaces $M(i)$ to get a new sequence $\{\widetilde{M}(i)\}_i$, such that each surface $\widetilde{M}(i)$ passes through $\vec{0}$ and through the point $\widetilde{q} = (0, 6\tau, 0)$. The tangent space of $\widetilde{M}(i)$ at $\widetilde{q}$ is always vertical. After blowing-up, these surfaces converge (up to a subsequence) to a vertical helicoid with axis passing through $\vec{0}$. In the present scale, $\{\widetilde{M}(i)\}_i$ converges (again up a to subsequence) to the foliation $\{x_3 = t\}_{t\in\mathbb{R}}$ outside of two vertical straight lines, namely the x_3-axis and the line passing through $\widetilde{q}$.

REMARK 2. *To get the above description of $\widetilde{M}(i)$, we have fixed $\tau > \tau_0$. Nevertheless, the end of the proof of Theorem 5 is based on the behavior of certain fluxes as $i, \tau \to \infty$.*

The next step in the proof of Theorem 5 consists of finding embedded closed curves $\delta(\tau, i) \subset \widetilde{M}(i)$ where (3) holds. Roughly speaking, we will use the structure of *parking garage* for the surface $\widetilde{M}(i)$ in order to define $\delta(\tau, i)$ by joining an approximation $L(\tau, i)$ of a horizontal segment from $\vec{0}$ to $\widetilde{q}$ with a "short" arc $\alpha_2(\tau, i)$ close to $\widetilde{q}$ that goes up exactly one level in the parking garage structure, then traveling from $\widetilde{q}$ to $\vec{0}$ by an approximation $\widetilde{L}(\tau, i)$ of a horizontal segment directly above $L(\tau, i)$ and finally coming down exactly one level nearby the origin along another "short" arc $\alpha_1(\tau, i)$ to close the curve $\delta(\tau, i)$ at $\vec{0}$. Here "short" means that the corresponding flux contribution is bounded independently of τ ("short" also means that the lengths of the arcs $\alpha_1(\tau, i)$ can be made arbitrarily small for i large; for an alternative approach to proving the existence of the short arc $\alpha_1(\tau, i)$, see the statement of the Limit Lamination Metric Theorem in [**16**]). In order to construct $\delta(\tau, i)$ one could appeal to the local structure of double multigraph by Colding-Minicozzi (Theorem 3) around points in the singular set of convergence $S(\mathcal{L})$ that appears in Lemma 8, although this approach has the difficulty of getting the "short " arc $\alpha_2(\tau, i)$ nearby $\widetilde{q}$ (because Colding-Minicozzi theory describes the surfaces $\widetilde{M}(i)$ outside a vertical double cone centered at $\widetilde{q}$ and the arc $\alpha_2(\tau, i)$ should lie inside that cone; note that the arc $\alpha_1(\tau, i)$ is easier to construct since we know that suitable blow-ups of the $\widetilde{M}(i)$ at the origin converge to a vertical helicoid). Instead, we will analyze directly our situation and produce explicitly the needed arcs.

We now choose a small positive number $\rho \in (0,1)$ and consider the solid cylinder $C(\Gamma,\rho)$ of radius ρ with axis Γ. As $\tau_0 > 1$, $\widetilde{q}$ cannot lie in $C(\Gamma,\rho)$.

REMARK 3. *It is important to notice that since $\widetilde{M}(i)$ depends on τ by an homothety and τ is always greater that one, the radius ρ of the cylinder $C(\Gamma,\rho)$ can be chosen independent of τ.*

We claim that the Gaussian curvature of $\widetilde{M}(i)$ at $\widetilde{q}$ blows up as $i \to \infty$: this follows because otherwise, Lemmas 2 and 8 would imply that for certain $\varepsilon > 0$, a subsequence of $\{\widetilde{M}(i) \cap B(\widetilde{q},\varepsilon)\}_i$ would converge smoothly to the foliation of $B(\widetilde{q},\varepsilon)$ by horizontal disks. As the tangent space to $\widetilde{M}(i)$ at $\widetilde{q}$ is vertical for every i, this smooth convergence to horizontal disks is impossible.

Let $C(\Gamma',\rho)$ be the solid cylinder of radius ρ and axis given by the vertical straight line passing through $\widetilde{q}$. Of course, $C(\Gamma,\rho) \cap C(\Gamma',\rho) = \emptyset$.

REMARK 4. *As in Remark 3, ρ can be chosen independently of $\tau > \tau_0$.*

Let $A_1 = \overline{\vec{0},\widetilde{q}} \cap \partial C(\Gamma,\rho/2)$ and $A_2 = \overline{\vec{0},\widetilde{q}} \cap \partial C(\Gamma',\rho/2)$, both points in the leaf $L_0 = \{x_3 = 0\}$ of $\mathcal{L}$. Take $\varepsilon > 0$ such that Proposition 1 holds. Given $\delta \in (0, \min\{\rho/2,\varepsilon\})$, denote by $R(\delta) = \{(x_1,x_2,x_3) \mid \mathrm{dist}((x_1,x_2),\overline{\vec{0},\widetilde{q}}) \leq \delta,\ |x_3| \leq \delta\}$. Hence, $R(\delta)$ is a closed tubular neighborhood of $\overline{\vec{0},\widetilde{q}}$ in $\mathbb{R}^3$. Let F be the topological closed disk

$$F = \left\{ p = (x_1,x_2,0) \in R(\delta) \mid \mathrm{dist}(p,\vec{0}) \geq \frac{\rho}{2}, \mathrm{dist}(p,\widetilde{q}) \geq \frac{\rho}{2} \right\}, \tag{2}$$

where both distances are measured in the plane L_0, see Figure 4 below. Thus, $A_1, A_2 \in \partial F$. Let π be the projection $\pi(x_1,x_2,x_3) = (x_1,x_2)$, $(x_1,x_2,x_3) \in \mathbb{R}^3$.

Note that the compact set $R = R(\delta) \cap \pi^{-1}(F)$ is disjoint from $S(\mathcal{L})$. In particular, a subsequence of $\{\widetilde{M}(i) \cap R\}_i$ (denoted in the same way) converges smoothly to F as $i \to \infty$. Without loss of generality, we can assume that for i large, any component $D(i)$ of $\widetilde{M}(i) \cap R$ sufficiently close to F is a closed disk and π restricts to $D(i)$ as a diffeomorphism onto F. Restricting to the union of such components, we may view π as being a disconnected covering whose number of sheets increases to ∞ as $i \to \infty$. As $\widetilde{M}(i)$ is embedded, the $D(i)$-type components of $\widetilde{M}(i) \cap R$ are naturally ordered by heights and the Gauss map of $\widetilde{M}(i)$ takes almost vertical opposite values on consecutive sheets. Let $D(i) \subset \widetilde{M}(i) \cap R$ be one of the sheets of this covering, chosen so that the unique point $A_1(i) \in \partial D(i)$ that projects on A_1 through π is the lowest point of $\widetilde{M}(i) \cap R \cap \pi^{-1}(A_1) \cap \{x_3 \geq 0\}$. Let $L(\tau,i)$ denote a lift of the segment $\overline{A_1,A_2} \subset F$ to the sheet $D(i)$ through π (we write explicitly the dependence on τ for this lift since it will be part of the embedded closed curve $\delta(\tau,i)$ we are looking for, see formula (1)). The ends of $L(\tau,i)$ are $A_1(i)$ together with another point $A_2(i) \in \partial D(i) \cap \pi^{-1}(A_2)$. Since the lifts of $\overline{A_1,A_2}$ to $\widetilde{M}(i) \cap R$ are also ordered by heights, we can choose another lift $\widetilde{L}(\tau,i)$ of $\overline{A_1,A_2}$, lying inside the component $\widetilde{D}(i)$ of $\widetilde{M}(i) \cap R$ directly above $D(i)$. Denote by $\widetilde{A}_1(i), \widetilde{A}_2(i)$ the end points of $\widetilde{L}(\tau,i)$ so that $\pi(\widetilde{A}_j(i)) = A_j$ for $j = 1,2$, see Figure 4. We now join $A_1(i)$ to $\widetilde{A}_1(i)$ by an embedded curve $\alpha_1(\tau,i)$ contained in $\widetilde{M}(i) \cap C(\Gamma,\rho)$ (recall that $\widetilde{M}(i)$ depends on $\tau > \tau_0$, which explains the notation $\alpha_1(\tau,i)$). Since suitable expansions of the $\widetilde{M}(i) \cap C(\Gamma,\rho)$ converge smoothly to a vertical helicoid as $i \to \infty$, we can choose $\alpha_1(\tau,i)$ so that its intrinsic length in $\widetilde{M}(i)$ is bounded above by 2ρ for all i. By Remark 3, this length property of $\alpha_1(\tau,i)$ can be obtained independently of $\tau > \tau_0$.

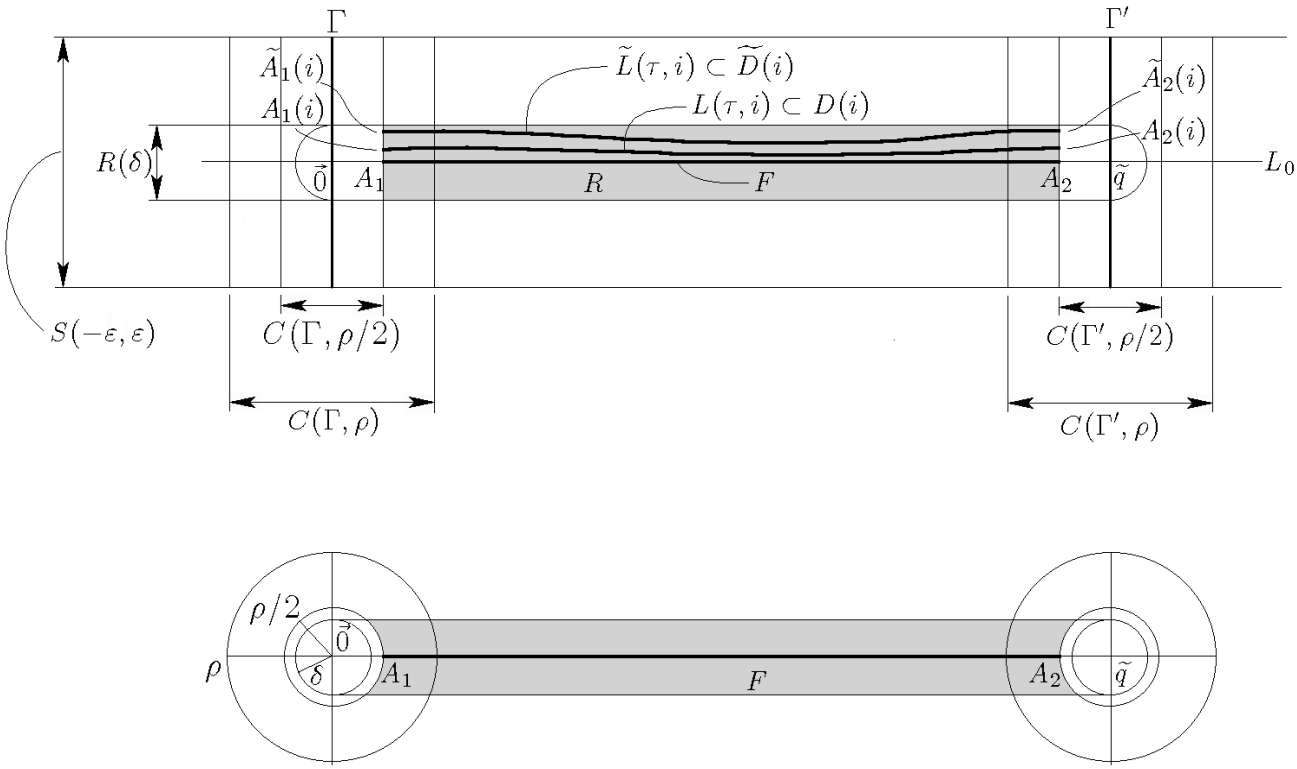

FIGURE 4. Above, a horizontal view (the shaded region is R). Below, a vertical view of the plane L_0 (the shaded region is F).

LEMMA 9. *For all i large, the points $A_2(i)$ and $\widetilde{A}_2(i)$ can be joined by an embedded arc $\alpha_2(\tau,i)\subset\widetilde{M}(i)\cap C(\Gamma',\rho)\cap S(-\varepsilon,\varepsilon)$ with intrinsic length bounded above by a constant that does not depend neither on i nor on τ.*

PROOF. Given $0\le r\le r'$, we let $A(\widetilde{q},r,r')=\{p\in L_0\mid r\le \mathrm{dist}(p,\widetilde{q})\le r'\}$. Thus, $A(\widetilde{q},0,r')$ is nothing but the horizontal closed disk of radius r' centered at $\widetilde{q}$, while for $r=r'>0$, $A(\widetilde{q},r,r)$ is the horizontal circle of radius r and center $\widetilde{q}$. By our choice of ε and Proposition 1, the component $\Theta(i)$ of $\widetilde{M}(i)\cap\pi^{-1}(A(\widetilde{q},0,\rho))\cap S(-\varepsilon,\varepsilon)$ that contains $\widetilde{q}$ is a disk, for all i sufficiently large. Note that the singular straight line Γ' coincides with the axis of the solid cylinder $\pi^{-1}(A(\widetilde{q},0,\rho))$. For i large, $\partial\Theta(i)$ consists of two disjoint helicoidal-type curves $c(i,\rho),\widetilde{c}(i,\rho)$ spinning along the vertical cylinder $\pi^{-1}(A(\widetilde{q},\rho,\rho))$ from height $-\varepsilon$ to ε, together with two embedded planar curves $h_\varepsilon(i)\subset\{x_3=\varepsilon\},h_{-\varepsilon}(i)\subset\{x_3=-\varepsilon\}$. Moreover, the spinning numbers of $c(i,\rho),\widetilde{c}(i,\rho)$ increase without bound as i goes to infinity. $\Theta(i)\cap\pi^{-1}(A(\widetilde{q},\rho/2,\rho))$ consists of two multivalued almost horizontal sublinear graphs $G(i),\widetilde{G}(i)$ over $A(\widetilde{q},\rho/2,\rho)$. $G(i)$ is topologically a disk whose boundary consists of $c(i,\rho)$ together with arcs in $h_\varepsilon(i),h_{-\varepsilon}(i)$, and another helicoidal curve $c(i,\rho/2)$ spinning along the vertical cylinder $\pi^{-1}(A(\widetilde{q},\rho/2,\rho/2))$ from height $-\varepsilon$ to ε. A similar description holds for $\widetilde{G}(i)$ and its boundary, giving rise to a new helicoidal curve $\widetilde{c}(i,\rho/2)$ spinning along the vertical

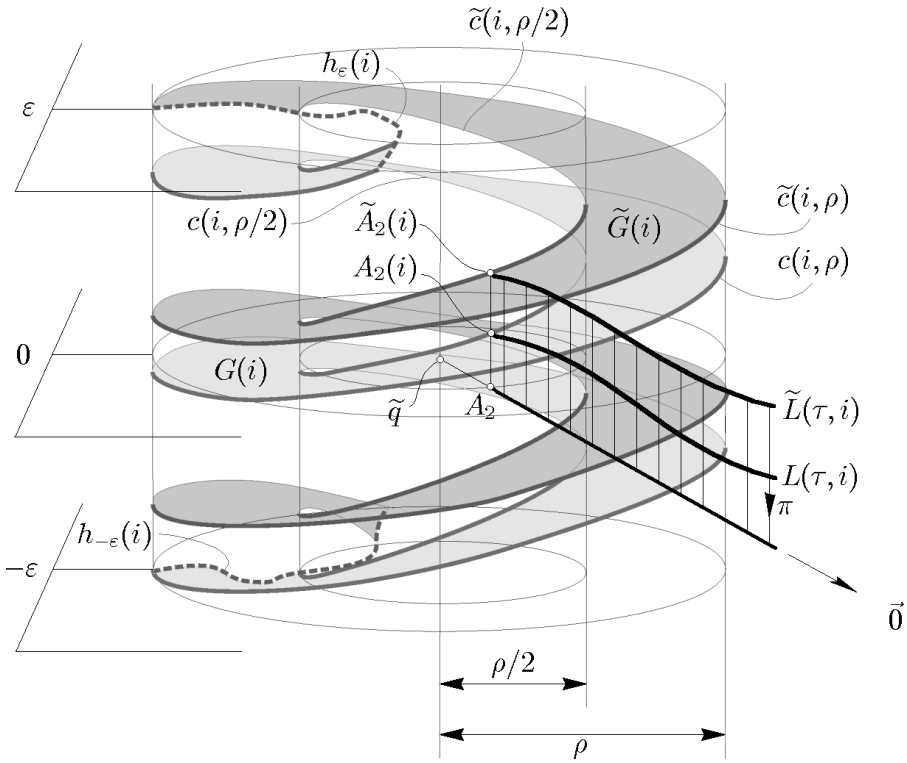

FIGURE 5. The multivalued sublinear graphs $G(i), \widetilde{G}(i) \subset \Theta(i)$.

cylinder $\pi^{-1}(A(\widetilde{q}, \rho/2, \rho/2))$. Without loss of generality, we can assume that $A_2(i) \in c(i, \rho/2)$ and $\widetilde{A}_2(i) \in \widetilde{c}(i, \rho/2)$. Finally, let $\Theta'(i) = \Theta(i) - [G(i) \cup \widetilde{G}(i)]$, see Figure 5.

In [**26**] Meeks and Rosenberg proved that for i large there exist minimal disks $D(i,t)$ with $|t| \leq \varepsilon$, which are graphs over $A(\widetilde{q}, 0, \rho)$ of approximate height t, and such that $D(i,t) \cap \widetilde{M}(i) \subset \Theta(i)$ consists of a single compact arc. In this case, $D(i,t)$ converges to the flat horizontal disk at height t as $i \to \infty$. Since for i large $\widetilde{\Theta}(i) = \cup_{|t| \leq \varepsilon}[D(i,t) \cap \widetilde{M}(i)]$ is essentially all of $\Theta(i)$, we do not lose generality in our later arguments by assuming that $\Theta(i) = \widetilde{\Theta}(i)$. Under this assumption, $D(i, -\varepsilon) \cap \Theta(i)$ and $D(i, \varepsilon) \cap \Theta(i)$ are both compact arcs.

It will be useful for our purposes to work with the following differentiable representation of $\Theta(i)$: We will identify $\Theta(i)$ with a planar rectangle $[-1,1] \times [-\varepsilon, \varepsilon]$ in the (u,v)-plane, so that

- The boundary curves $c(i,\rho)$, $c(i,\rho/2)$, $\widetilde{c}(i,\rho/2)$, $\widetilde{c}(i,\rho)$ correspond respectively to the vertical segments $\{-1\} \times [-\varepsilon,\varepsilon], \{-1/2\} \times [-\varepsilon,\varepsilon], \{1/2\} \times [-\varepsilon,\varepsilon], \{1\} \times [-\varepsilon,\varepsilon]$;
- $D(i,\varepsilon) \cap \widetilde{M}(i), D(i,-\varepsilon) \cap \widetilde{M}(i)$ correspond respectively to the horizontal segments $[-1,1] \times \{\varepsilon\}, [-1,1] \times \{-\varepsilon\}$;
- The vertical coordinate $v \in [-\varepsilon, \varepsilon]$ is defined by $v(p) = t$ if $p \in D(i,t) \cap \widetilde{M}(i)$.

Thus, $\Theta'(i)$ is represented in this model by the rectangle $(-1/2, 1/2) \times [-\varepsilon, \varepsilon]$, see Figure 6(a).

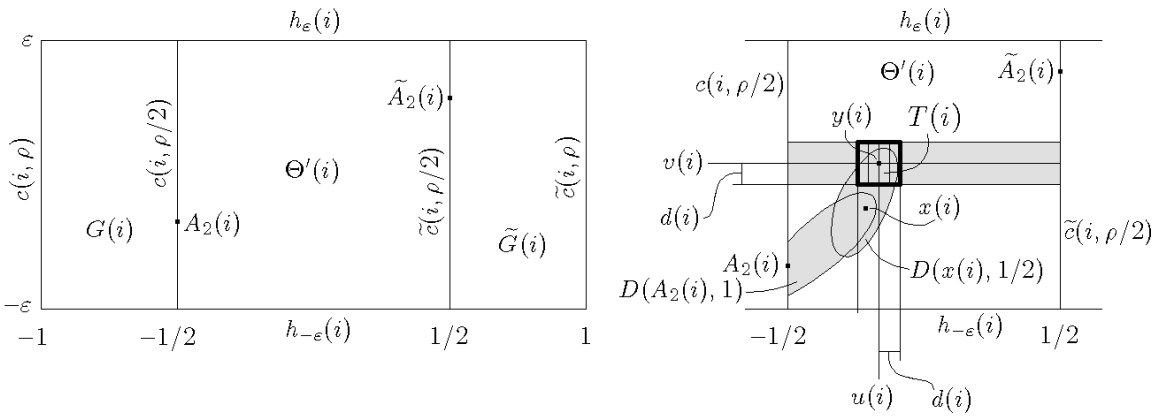

FIGURE 6. (a) The disk $\Theta(i)$. (b) The domain $H(i)$ is the shaded area.

Let $ds^2(i)$ be the Riemannian metric on $\Theta(i) = [-1,1] \times [-\varepsilon,\varepsilon]$ induced by the natural inner product on $\widetilde{M}(i)$. Clearly, Lemma 9 will be proved if we check the validity of the following:

ASSERTION 1. *For all i large, $A_2(i)$ and $\widetilde{A}_2(i)$ can be joined by a curve in $\Theta'(i)$ of $ds^2(i)$-length less than a constant that does not depend either on i or on τ.*

To prove Assertion 1, let $D(A_2(i),1)$ denote the intersection of the closure of $\Theta'(i)$ with the $ds^2(i)$-metric ball of radius 1 in $\widetilde{M}(i)$ centered at $A_2(i)$. If for all i large $\widetilde{A}_2(i)$ lies in $D(A_2(i),1)$, then Assertion 1 follows. Therefore, we can assume that, after passing to a subsequence, $\widetilde{A}_2(i) \notin D(A_2(i),1)$ for all i.

Next we check that there exist points $x(i) \in D(A_2(i),1)$ such that the Gaussian curvature of $\widetilde{M}(i)$ at $x(i)$ blows up as i goes to ∞: this occurs because otherwise, there would exist a radius $c \in (0,1)$ independent of i such that a subdisk inside $D(A_2(i),1)$ centered at $A_2(i)$ can be written as a graph $\Delta(i)$ over a disk inside the tangent space at $A_2(i)$ of radius c. As such tangent space is arbitrarily close to horizontal when i is large enough, we conclude that $\Delta(i)$ would divide the cylinder $C(\Gamma',\rho/2)$ in two components, provided that ρ is chosen small enough. In particular, $\Delta(i)$ would cross the singular line Γ', which contradicts the curvature estimates in [**5**] for the surfaces $\widetilde{M}(i) \cap C(\Gamma',\rho/2)$. Hence, we find points $x(i) \in D(A_2(i),1)$ such that $|K_{\widetilde{M}(i)}(x(i))| \to \infty$ as $i \to \infty$.

Using similar arguments as in the proof of Lemma 5, we can deduce that there exist points $y(i)$ in the intersection $D(x(i),1/2)$ of $\Theta'(i)$ with the $ds^2(i)$-disk of radius $1/2$ centered at $x(i)$, such that $|K_{\widetilde{M}(i)}(y(i))| \to \infty$ as $i \to \infty$ and $\widetilde{M}'(i) = \sqrt{K_{\widetilde{M}(i)}(y(i))}\left(\widetilde{M}(i) - y(i)\right)$ converges on compact subsets of $\mathbb{R}^3$ to a vertical helicoid H that passes through $\vec{0}$. As $\widetilde{M}(i)$ is a multivalued sublinear graph along a neighborhood of $c(i,\rho/2) \cup \widetilde{c}(i,\rho/2)$, the property that the Gauss curvature of $\widetilde{M}(i)$ blows up at $y(i)$ implies that $y(i)$ can be supposed to lie in the interior of $\Theta'(i)$ for all i. On the other hand, the smooth convergence of $\widetilde{M}'(i)$ to H implies that for i large, the intersection of $\widetilde{M}'(i)$ with the annular region $\{(x_1,x_2,x_3) \mid k \leq x_1^2 + x_2^2 \leq k+1,\ |x_3| \leq 1\}$ ($k > 0$ large) consists of two multivalued sublinear graphs over their common vertical projection, with almost vertical opposite Gauss maps. Coming back to $\Theta'(i)$, this implies that there exists

$d(i) > 0$ small such that if we write $y(i) = (u(i), v(i))$ (here $|u(i)| < 1/2$, $|v(i)| < \varepsilon$), then the image set in $\mathbb{R}^3$ of the square $T(i) = [u(i) - d(i), u(i) + d(i)] \times [v(i) - d(i), v(i) + d(i)] \subset \Theta'(i)$ is a minimal disk inside the closed horizontal slab $\overline{S(v(i) - 2d(i), v(i) + 2d(i))}$ and both strips $[u(i) - d(i), u(i) - \frac{d(i)}{2}] \times [v(i) - d(i), v(i) + d(i)]$, $[u(i) + \frac{d(i)}{2}, u(i) + d(i)] \times [v(i) - d(i), v(i) + d(i)]$ consist of almost horizontal multivalued sublinear graphs. Colding and Minicozzi's paper [**2**] insures that for i sufficiently large, these multivalued graphs can be extended until hitting the boundary curves $c(i, \rho/2), \widetilde{c}(i, \rho/2)$ of $\Theta'(i)$, preserving the property of being almost horizontal multivalued graphs. In summary, we have proven that all points in the domain (see Figure 6(b))

$$H(i) = D(A_2(i), 1) \cup D(x(i), 1/2) \cup ([-1/2, 1/2] \times [v(i) - d(i), v(i) + d(i)]) \subset \Theta'(i)$$

are at $ds^2(i)$-distance from $A_2(i)$ less than some fixed positive number that only depends on ρ (recall that ρ is independent of i and of τ, see Remarks 3 and 4).

Note that $\partial H(i)$ has points on both boundary curves $c(i, \rho/2), \widetilde{c}(i, \rho/2) \subset \partial\Theta'(i)$. If $\widetilde{A}_2(i)$ lies in $\partial H(i)$ for all i large, then Assertion 1 holds. Thus, we can suppose from now on that, after extracting a subsequence, $\widetilde{A}_2(i) \notin \partial H_1(i)$ for all i. Exchanging $A_2(i)$ by $\widetilde{A}_2(i)$ and reasoning as before, we produce a second domain $\widetilde{H}(i) \subset \Theta'(i)$ all whose points are at $ds^2(i)$-distance from $\widetilde{A}_2(i)$ less than some fixed positive number that only depends on ρ, and such that $\partial\widetilde{H}(i)$ contains points on both $c(i, \rho/2), \widetilde{c}(i, \rho/2)$. By the triangle inequality, if $H(i) \cap \widetilde{H}(i) \neq \emptyset$ for all i large, then Assertion 1 holds. Therefore, we can assume, again passing to a subsequence, that $H(i) \cap \widetilde{H}(i) = \emptyset$ for all i large.

Let $z(i) \in c(i, \rho/2)$ be a point $ds^2(i)$-equidistant from both $H(i)$ and $\widetilde{H}(i)$. Reasoning as above, we find a third domain $H'(i) \subset \Theta'(i)$ at $ds^2(i)$-distance from $z(i)$ less than some constant that only depends on ρ, such that $\partial H'(i)$ contains points of $c(i, \rho/2)$ and of $\widetilde{c}(i, \rho/2)$. If $H'(i)$ has nonvoid intersection with $H(i)$ for all i large, then $z(i)$ is at uniform $ds^2(i)$-distance from $H(i)$ (i.e., this distance is less that some fixed number independent of i, τ). By its $ds^2(i)$-equidistance property, $z(i)$ must also be at uniform $ds^2(i)$-distance from $\widetilde{H}(i)$, thus by the triangle inequality $H(i)$ and $\widetilde{H}(i)$ are also at uniform $ds^2(i)$-distance, and Assertion 1 is also true. The same argument works if $H'(i) \cap \widetilde{H}(i) \neq \emptyset$ for all i large. Thus we can assume, again after extracting a subsequence, that $H'(i) \cap H(i) = \emptyset$ and $H'(i) \cap \widetilde{H}(i) = \emptyset$ for all i large. We will finish the proof of Assertion 1 by showing that this last case leads to a contradiction.

As both $H'(i)$ and $\Theta'(i)$ are connected domains and $\partial H'(i) \cap c(i, \rho/2) \neq \emptyset$, $\partial H'(i) \cap \widetilde{c}(i, \rho/2) \neq \emptyset$, we deduce that $H'(i)$ divides $\Theta'(i)$ into several components, one of which contains $H(i)$ and another one which contains $\widetilde{H}(i)$. Since $H(i), \widetilde{H}(i), H'(i)$ contain horizontal strips joining $c(i, \rho/2)$ with $\widetilde{c}(i, \rho/2)$ (for instance, $[-1/2, 1/2] \times [v(i) - d(i), v(i) + d(i))] \subset H(i)$), it follows that $[-1/2, 1/2] \times \{\varepsilon\} \subset h_\varepsilon(i)$ is contained in the boundary of exactly one of the components of $\Theta'(i) - H'(i)$ (which we will refer as the *upper component*) and $[-1/2, 1/2] \times \{-\varepsilon\} \subset h_{-\varepsilon}(i)$ is contained in the remaining component of $\Theta'(i) - H'(i)$ (this one will be the *lower component*). Since the point $A_2(i) \in H(i)$ is strictly below $\widetilde{A}_2(i) \in \widetilde{H}(i)$ in $\mathbb{R}^3$, we have that $\widetilde{H}(i)$ must be contained in the upper component of $\Theta'(i) - H'(i)$ while $H(i)$ lies inside the lower component of $\Theta'(i) - H'(i)$.

Consider the vertical segment $\overline{A_2, \widetilde{A}_2(i)} \subset \mathbb{R}^3$. The intersections

$$\Xi(i) := \overline{A_2, \widetilde{A}_2(i)} \cap c(i, \rho/2) \quad \text{and} \quad \widetilde{\Xi}(i) := \overline{A_2, \widetilde{A}_2(i)} \cap \widetilde{c}(i, \rho/2)$$

consist of two finite subsets whose points are naturally ordered by heights. With this order, the natural increasing parametrization in $\mathbb{R}^3$ of $\overline{A_2, \widetilde{A}_2(i)}$ passes alternatively through points in $\Xi(i)$ and in $\widetilde{\Xi}(i)$. In the planar rectangle model of $\Theta'(i)$ as $[-1/2, 1/2] \times [-\varepsilon, \varepsilon]$, both $\Xi(i)$ and $\widetilde{\Xi}(i)$ are finite subsets each one inside one of the vertical boundary segments $\{-1/2\} \times [-\varepsilon, \varepsilon]$, $\{1/2\} \times [-\varepsilon, \varepsilon]$, respectively. As $H'(i)$ contains a horizontal strip joining $c(i, \rho/2)$ with $\widetilde{c}(i, \rho/2)$, it follows that $\Xi(i) \cap H'(i)$ is nonvoid for i large (in fact, the number of points of $\Xi(i) \cap H'(i)$ increases without bound as $i \to \infty$ thanks to the multivalued graph inside $H'(i)$). Clearly, all points in $\Xi(i) \cap H'(i)$ are strictly above $A_2(i)$ and all points in $\widetilde{\Xi}(i) \cap H'(i)$ are strictly below $\widetilde{A}_2(i)$. Thanks to the almost horizontal multivalued graph inside $H'(i)$, we can insure that for any two points $a, b \in \Xi(i) \cap H'(i)$ with $x_3(a) < x_3(b)$ there exists at least one $c \in \widetilde{\Xi}(i) \cap H'(i)$ with $x_3(a) < x_3(c) < x_3(b)$. In particular, such c is strictly below $\widetilde{A}_2(i)$. Since the number of points in $\Xi(i) \cap H'(i)$ goes to ∞ as $i \to \infty$, we contradict that $\widetilde{A}_2(i)$ was chosen as the point in $\Theta'(i)$ directly above $A_2(i)$ in the fiber of A_2 by the vertical projection π. This contradiction finishes both the proof of Assertion 1 and of Lemma 9. □

Finally, we define the desired embedded closed curve $\delta(\tau, i) \subset \widetilde{M}(i)$ by joining the arcs $\alpha_1(\tau, i), L(\tau, i), \alpha_2(\tau, i), \widetilde{L}(\tau, i)$ by their common ends. We also consider a coherent choice of orientations on each of these arcs, so $\delta(\tau, i)$ is globally oriented. To compute the flux of $\widetilde{M}(i)$ along $\delta(\tau, i)$, we just note that

$$\mathrm{Flux}(\widetilde{M}(i), \delta(\tau, i)) = \int_{L(\tau,i)} \eta(i)\, ds + \int_{\widetilde{L}(\tau,i)} \eta(i)\, ds + \int_{\alpha_1(\tau,i)} \eta(i)\, ds + \int_{\alpha_2(\tau,i)} \eta(i)\, ds,$$

where $\eta(i)$ is the unit conormal to $\widetilde{M}(i)$ along the corresponding arc, and ds stands for the length element. Each of the first two integrals converges to $(6\tau - \rho, 0, 0)$ as $i \to \infty$ (use that the disks $D(i), \widetilde{D}(i)$ converge smoothly as $i \to \infty$ to the flat disk F defined in equation (2)), while the third and fourth integrals can be bounded in length by the lengths of $\alpha_1(\tau, i)$ and $\alpha_2(\tau, i)$, which are bounded independently of i and τ. Now the decomposition (1) holds and the proof of Theorem 5 is complete provided that Lemma 8 is true. □

It remains to prove Lemma 8 to complete this section.

Proof of Lemma 8.

Recall that the surface $\widetilde{M}(i)$ is obtained from $M(i)$ by a translation of vector $p(i) \in M(i)$ followed by a rotation around the x_3-axis by angle $\theta(i) \in [0, 2\pi)$ and a homothety by a scalar factor $\mu(\tau, i) \in \mathbb{R}^+$, $\widetilde{M}(i) = \mu(\tau, i)\mathrm{Rot}_{\theta(i)}(M(i) - p(i))$. Furthermore, the surfaces $M'(i) = \lambda(i)(M(i) - p(i))$ converge smoothly as $i \to \infty$ to a vertical helicoid, where $\lambda(i) = \sqrt{|K_{M(i)}(p(i))|} = \frac{\mu(\tau,i)}{6\tau}\|q'(i)\| \to \infty$ and $q'(i) \in M'(i) \cap \{x_3 = 0\}$ is a point different from the origin $\vec{0}$ such that the tangent plane to $M'(i)$ at $q'(i)$ is vertical, see the arguments just before Remark 1. This property and the convergence of the $M'(i)$ to a vertical helicoid imply that $\|q'(i)\| \to \infty$ and since $\widetilde{M}(i) = \frac{6\tau}{\|q'(i)\|}\mathrm{Rot}_{\theta(i)}M'(i)$, we deduce that in any small neighborhood of $\vec{0}$, the surface $\widetilde{M}(i)$

has the appearance of a highly sheeted homothetically shrunk vertical helicoid. Also recall from Proposition 1 that there exists $\varepsilon > 0$ such that every component of the intersection of $\widetilde{M}(i)$ with the ball $B(\vec{0}, \varepsilon)$ of radius ε centered at the origin is a disk for i large. In this situation Colding and Minicozzi prove that, in a ball $B = B(\vec{0}, \varepsilon')$ of smaller radius $\varepsilon' \in (0, \varepsilon)$, a subsequence of the disks in $\widetilde{M}(i) \cap B$ converges to a minimal lamination $\widetilde{\mathcal{L}}_B$ with singular set of convergence $S(\widetilde{\mathcal{L}}_B)$ that contains $\vec{0}$, such that $\widetilde{\mathcal{L}}_B$ contains a disk leaf D with $\vec{0} \in D$ and with horizontal tangent plane at $\vec{0}$. Furthermore, ε' can be chosen small enough so that $D \cap S(\widetilde{\mathcal{L}}_B) = \{\vec{0}\}$. The 1-sided curvature estimate by Colding-Minicozzi implies that in a small neighborhood of $D - \{\vec{0}\}$, these disks converge to a sublamination of $\widetilde{\mathcal{L}}'_B \subset \widetilde{\mathcal{L}}_B$ with empty singular set of convergence, such that $D - \{\vec{0}\}$ is a limit leaf of $\widetilde{\mathcal{L}}'_B$.

We now check that there exists $\delta = \delta(\tau) > 0$ such that $\mu(\tau, i) > \delta$ for all $i \in \mathbb{N}$. Reasoning by contradiction, suppose that $\lim_{i\to\infty} \mu(\tau, i) = 0$. Since the vertical fluxes of the surfaces $M(i)$ are normalized to $(0, 0, 1)$, we conclude that the vertical fluxes of the $\widetilde{M}(i)$ converge to zero as $i \to \infty$. This fact and the convergence of portions of $\widetilde{M}(i) \cap B$ to $D - \{\vec{0}\} \in \widetilde{\mathcal{L}}'_B$ imply that $D - \{\vec{0}\}$ cannot contain points with nonhorizontal tangent plane. Hence, D is the horizontal disk $B \cap \{x_3 = 0\}$ and similarly all the leaves in $\widetilde{\mathcal{L}}'_B$ are pieces of horizontal planes as well. With this result in mind and using that $\{\widetilde{M}(i) \cap S(-\varepsilon, \varepsilon)\}_i$ is simply connected in balls of radius 1 by Proposition 1, the proof of the Colding-Minicozzi limit foliation theorem for planar domains (Theorem 3) can be adapted to show that $\{\widetilde{M}(i) \cap S(-\varepsilon, \varepsilon)\}_i$ converges to the foliation $\mathcal{L}_\varepsilon$ of $S(-\varepsilon, \varepsilon)$ by horizontal planes, with singular set of convergence $S(\mathcal{L}_\varepsilon)$ consisting of Lipschitz curves transverse to the planes, which by Meeks' regularity theorem [**17**] consist of vertical line segments from height $-\varepsilon$ to ε. Note that one of these line segments passes through $\vec{0}$ and another one passes through the point $\widetilde{q} = (0, 6\tau, 0)$ where $\widetilde{M}(i)$ has a vertical tangent plane. In fact, the singular set of convergence of $\mathcal{L}_\varepsilon$ reduces to these two line segments, because otherwise one could construct two simple closed curves on $\widetilde{M}(i)$ which intersect transversely at a single point, which would contradict that $\widetilde{M}(i)$ is a planar domain (see the proof of Lemma 1 in [**19**] or the proof of Theorem 3 by Colding and Minicozzi). Now one can argue as in the proof of Theorem 5 to produce an embedded closed curve $\delta(\tau, i) \subset \widetilde{M}(i)$ where the flux diverges horizontally to infinity, which is impossible. This contradiction proves that there exists $\delta > 0$ such that $\mu(\tau, i) > \delta$ for all $i \in \mathbb{N}$.

Assertion 2. *The sequence $\{\widetilde{M}(i)\}_i$ is ULSC.*

We will prove Assertion 2 by contradiction. Suppose that $\{\widetilde{M}(i)\}_i$ is not ULSC, and we will carry out a modification of this sequence which will produce a ULSC sequence of minimal surfaces $\overline{M}(i) \subset \mathbb{R}^3$. For each i define the function $r_i \colon \mathbb{R}^3 \to \mathbb{R}^+$ that assigns to each $p \in \mathbb{R}^3$ the radius of the largest open ball $B(p, r_i(p))$ such that every component of $\widetilde{M}(i) \cap B(p, r_i(p))$ is simply connected but $\widetilde{M}(i) \cap \overline{B}(p, r_i(p))$ contains a nonsimply connected component; here $\overline{B}(p, r_i(p))$ is the corresponding closed ball. It follows from the convex hull property for compact minimal surfaces that there exists a homotopically nontrivial simple closed curve $\gamma_i(p)$ in $\widetilde{M}(i)$ whose image lies in $\overline{B}(p, r_i(p))$.

Since the sequence $\widetilde{M}(i)$ is not ULSC, after choosing a subsequence, there exist points $q(i) \in \mathbb{R}^3$ such that $r_i(q(i)) \to 0$ as $i \to \infty$. Let $\widetilde{p}(i)$ be a point in $B(q(i), 1)$ where the function $p \mapsto$

$\frac{1-\|p-q(i)\|}{r_i(p)} = \frac{d(p)}{r_i(p)}$, d being the extrinsic distance to $\partial B(q(i),1)$, has a maximum value $\mu(i) > 0$ (in particular, $\lim_{i\to\infty}\mu(i) = \infty$ since $\frac{d(q(i))}{r_i(q(i))} = \frac{1}{r_i(q(i))} \to \infty$ as $i \to \infty$). Thus, we obtain a new normalized sequence of properly embedded minimal surfaces in $\mathbb{R}^3$,

$$\overline{M}(i) = \frac{1}{r_i(\widetilde{p}(i))}(\widetilde{M}(i) - \widetilde{p}(i)).$$

Next we show that this new sequence $\{\overline{M}(i)\}_i$ is ULSC. Given $\overline{x} \in \mathbb{R}^3$ and $R > 0$, the intersection of $\overline{M}(i)$ with the ball $B(\overline{x}, R)$ centered at $\overline{x}$ with radius R is the rescaled image of the intersection of $\widetilde{M}(i)$ with the ball $B(x_i, R\cdot r_i(\widetilde{p}(i)))$, where $x_i = \widetilde{p}(i)+r_i(\widetilde{p}(i))\overline{x}$. Hence, to prove that $\{\overline{M}(i)\}_i$ is ULSC it suffices to show that there exists $R > 0$ depending only on $\overline{x}$ such that all the components of $\widetilde{M}(i) \cap B(x_i, R\cdot r_i(\widetilde{p}(i)))$ are simply connected for all i large. To see this last property, one first checks that $x_i \in B(q(i),1)$ for all i large enough since

$$\|x_i - q(i)\| \le r_i(\widetilde{p}(i))\|\overline{x}\| + \|\widetilde{p}(i) - q(i)\| < 1 + r_i(\widetilde{p}(i))\left(\|\overline{x}\| - \mu(i)\right) < 1,$$

where the last inequality uses that $\mu(i) \to \infty$ as $i \to \infty$. Secondly, the definition of x_i and the triangle inequality imply that

$$\frac{1-\|x_i - q(i)\|}{1-\|\widetilde{p}(i) - q(i)\|} \ge \frac{1-\|\widetilde{p}(i) - q(i)\| - r_i(\widetilde{p}(i))\|\overline{x}\|}{1-\|\widetilde{p}(i) - q(i)\|} = 1 - \frac{\|\overline{x}\|}{\mu(i)}, \tag{3}$$

which is greater than or equal to $\frac{1}{2}$ for all i large enough, and finally comparing the value of $\frac{d}{r_i}$ at $x_i \in B(q(i),1)$ with its maximum attained at $\widetilde{p}(i)$ it follows that

$$r_i(\widetilde{p}(i)) \le \frac{1-\|\widetilde{p}(i) - q(i)\|}{1-\|x_i - q(i)\|} r_i(x_i) \le 2r_i(x_i),$$

where we have used (3) in the last inequality. Therefore for any $0 < R < \frac{1}{2}$, the strict inclusion $B(x_i, R\cdot r_i(\widetilde{p}(i))) \subset B(x_i, r_i(x_i))$ holds. By definition of r_i and by the convex hull property for compact minimal surfaces, it follows that all the components of $\widetilde{M}(i) \cap B(x_i, R\cdot r_i(\widetilde{p}(i)))$ are simply connected for all i large. This proves $\{\overline{M}(i)\}_i$ is ULSC.

On the other hand, by construction $\overline{M}(i)$ contains a homotopically nontrivial simple closed curve $\overline{\gamma}_i = \frac{1}{r_i(\widetilde{p}(i))}(\gamma_i(\widetilde{p}(i)) - \widetilde{p}(i))$ with $\overline{\gamma}_i \subset \overline{B}(\vec{0},1)$. The existence of $\overline{\gamma}_i$ implies that there is some $\varepsilon > 0$ and a point $\overline{p}(i) \in \overline{M}(i) \cap \overline{B}(\vec{0},2)$ with $|K_{\overline{M}(i)}|(\overline{p}(i)) \ge \varepsilon$.

Next suppose that $\{\overline{M}(i)\}_i$ has locally bounded Gaussian curvature in $\mathbb{R}^3$ and we will find a contradiction. Lemma 2 implies that, after choosing a subsequence, we have that $\{\overline{M}(i)\}_i$ converges to a $C^{1,\alpha}$ minimal lamination $\overline{\mathcal{L}}$ of $\mathbb{R}^3$ (here $0 < \alpha < 1$). By the proved inequality $|K_{\overline{M}(i)}|(\overline{p}(i)) \ge \varepsilon > 0$, $\overline{\mathcal{L}}$ must contain at least one leaf L which is not flat. Now Theorem 1.6 in [**26**] gives the following description of this limit leaf L:

- L is properly embedded in $\mathbb{R}^3$, in a halfspace of $\mathbb{R}^3$ or in a slab in $\mathbb{R}^3$.
- if L has finite topology, then L is properly embedded in $\mathbb{R}^3$.
- If L is the only leaf in $\overline{\mathcal{L}}$, then L is properly embedded in $\mathbb{R}^3$.

Since L is orientable (because L separates the simply connected region $W \subset \mathbb{R}^3$ in which L is properly embedded) and it is not a plane, it follows that L is not stable (do Carmo and Peng [**9**], Fischer-Colbrie and Schoen [**11**]), which implies that the convergence of the $\overline{M}(i)$ to L in W is of multiplicity 1. A standard curve lifting argument then implies that L is a planar domain. Suppose

that L is properly embedded in $\mathbb{R}^3$. Since $\{\overline{M}(i)\}_i$ converges smoothly to L with multiplicity 1 in $\mathbb{R}^3$ and $\overline{M}(i) \cap \overline{B}(\vec{0}, 2)$ is not simply connected, we deduce that L is not simply connected. Since L is a planar domain, L must have more than one end, which implies that L contains a closed curve with nonzero finite flux. This is impossible, since L is obtained as a limit of a sequence of rescalings of the original surfaces $M(i)$ by unbounded scaling factors $\frac{\mu(\tau,i)}{r_i(\widetilde{p}(i))} > \frac{\delta}{r_i(\widetilde{p}(i))}$ (see our earlier proof of Lemma 5 for a similar argument). Hence we deduce that L is not properly embedded in $\mathbb{R}^3$. By the above description, L is not the only leaf in $\overline{\mathcal{L}}$, it is properly embedded in a region W which is a halfspace or a slab, and L has infinite topology (thus it has an infinite number of ends). The proof of the Ordering Theorem in [**12**] (or see the proof of Theorem 7 in Section 4) implies the existence of an end Σ of a plane or catenoid in the complement of L respect to W, such that Σ "separates" two ends of L. By the maximum principle at infinity, Σ stays at positive distance from L. It follows that there exists a plane $\Pi \subset W$ which intersects L transversely and $\Pi \cap L$ contains a simple closed curve γ_L along which L has nonzero flux, which we have already seen that is impossible. This contradiction proves that the sequence $\{\overline{M}(i)\}_i$ does not have locally bounded Gaussian curvature in $\mathbb{R}^3$.

Using that $\{\overline{M}(i)\}_i$ is ULSC without locally bounded curvature in $\mathbb{R}^3$ and that $\overline{M}(i) \cap B(\vec{0}, 2)$ contains a homotopically nontrivial simple closed curve, Theorem 3 gives that up to a subsequence, $\{\overline{M}(i)\}_i$ converges to a foliation $\overline{\mathcal{L}}$ of $\mathbb{R}^3$ by parallel planes, with singular set of convergence $S(\overline{\mathcal{L}})$ consisting of two Lipschitz curves. By Meeks' regularity theorem [**17**], these singular curves are straight lines orthogonal to the planes in $\mathcal{L}$. By Lemma 5, at points near the straight lines in $S(\overline{\mathcal{L}})$ the approximating surfaces $\overline{M}(i)$ contain small disks which closely approximate highly sheeted homothetically shrunk vertical helicoids, which implies by the unique extension of multigraphs result of [**5**] that the planes in $\overline{\mathcal{L}}$ are horizontal. As we have already seen in our earlier arguments following the statement of Lemma 8, this situation allows to define a simple closed curve in $\overline{M}(i)$ where the horizontal part of the flux divided by the vertical component of the flux tends to infinity as $i \to \infty$, which contradicts that the original surfaces $M(i)$ have vertical flux 1 and bounded horizontal flux. This contradiction finishes the proof of Assertion 2.

We now complete the proof of the lemma. Note that the Gaussian curvature of the surfaces $\widetilde{M}(i)$ blows up at the origin (otherwise a subsequence of the $\widetilde{M}(i)$ would converge to a vertical plane or to a vertical helicoid, contradicting the horizontality of the Gauss map of $\widetilde{M}(i)$ at $\widetilde{q}$). By Theorem 3, a subsequence of the $\widetilde{M}(i)$ converges to a minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by planes with singular set of convergence $S(\mathcal{L})$ consisting of one or two Lipschitz curves. Since the helicoid which is forming at the origin at a smaller scale is vertical, it follows that $\mathcal{L}$ is the foliation of $\mathbb{R}^3$ by horizontal planes. Since the tangent space to $\widetilde{M}(i)$ at $\widetilde{q}$ is vertical for all i, we deduce that $\widetilde{q} \in S(\mathcal{L})$. Meeks' regularity theorem [**17**] implies that $S(\mathcal{L}) = \Gamma \cup \Gamma'$ with Γ being the x_3-axis and Γ' the vertical line passing through $\widetilde{q}$. This completes the proof of the lemma.

□

4. Applications

In this section we will make a number of applications of the curvature estimates, Theorem 5, that we obtained for the two limit end minimal surfaces of genus zero in $\mathcal{M}$. We first prove Theorem 1 stated in the Introduction.

Proof. Statements *1, 2, 3* of Theorem 1 hold by Lemma 1. It follows from statements *1, 2, 3* that for some $a, b \in \mathbb{R}$ the horizontal slab $S(a,b)$ intersects M in a connected domain bounded by two simple closed curves and all horizontal planes outside $S(a,b)$ intersect M transversely in connected level sets. By the proof of curvature estimates in Section 3, each of the components of $M - S(a,b)$ have bounded curvature and the asymptotic value of the maximum of this curvature is given by the same curvature estimates that one obtains in Theorem 5. Since the curvature of $M \cap S(a,b)$ is asymptotic to zero, the curvature of this portion of M is also bounded. This proves statement *4* holds.

The direct argument of the existence of a regular neighborhood given in [**23**] proves that the spacing between consecutive ends of M is bounded from below by a constant that only depends on the curvature estimates. More generally, the results in [**24**] imply that the spacing $S(n)$ is at least equal to $2\sqrt{C}$ where C is the supremum of the Gaussian curvature on M.

We now prove that M is quasiperiodic. We first check that for n large and for any two consecutive ends e_n, e_{n+1}, asymptotic to planes P_n and P_{n+1}, there is a point on M between P_n and P_{n+1} where the tangent plane is vertical. By statement *3*, the intersection of the slab $S(x_3(P_n), x_3(P_{n+1}))$ with M is an annulus $A(n)$ with two proper arcs on its boundary. Since M separates $\mathbb{R}^3$, the asymptotic values of the Gauss map on the boundary component of $A(n)$ are the north and south poles of $\mathbb{S}^2$. By continuity, there is a point $p(n) \in A(n)$ where the tangent plane is vertical.

Next note that for ε fixed and sufficiently small, for all n large there must exist a point $q(n)$ in the extrinsic ball $B(p(n), 2)$ with $|K_M(q(n))| > \varepsilon$. If not, then we could find a subsequence of the $p(n)$ such that would have $\sup |K_{M \cap B(p(n),2)}| < \frac{1}{n}$ and so under translation of $M \cap B(p(n), 2)$ by $-p(n)$ we obtain a limit component which is a flat vertical disk of radius 2. This implies that the length of the vertical flux of M is at least 2 but by assumption it is 1.

Now consider the translated surfaces $M(n) = M - q(n)$. By [**23**] the $M(n)$ have local area estimates (coming from the existence of a regular neighborhood of M with fixed radius, see the proof of the lower bound for the spacing between the ends) and bounded absolute curvature which is at least ε at $\vec{0}$. It follows from the arguments in [**26**] that a subsequence of the $M(n)$ converges with multiplicity 1 to a connected nonflat properly embedded minimal surface $M(\infty)$ of genus zero, with length of its vertical flux at most 1. Moreover, the Gauss map of $M(\infty)$ omits the vertical directions (by statement *3* and by the open mapping property for the Gauss map of minimal surfaces). Since the flux of $M(\infty)$ is finite, $M(\infty)$ is not a helicoid and so by [**26**] it is not simply connected. Since $M(\infty)$ is not simply connected and has genus zero, it has at least two ends. In particular, it must have a planar or a catenoid type end and since the Gauss map of $M(\infty)$ misses the vertical directions, this planar or catenoid type end must be horizontal. Thus, $M(\infty)$ has a horizontal limit tangent plane at infinity.

If $M(\infty)$ has a finite number of ends or one limit end, then it has a top or bottom end which is asymptotic to a catenoid. Hence, there is a horizontal plane that intersects $M(\infty)$ transversely with a component which is a strictly convex Jordan curve. But since the $M(n)$ converge smoothly to $M(\infty)$, some horizontal plane $Q(n)$ intersects M in a component which is also a compact convex curve. As each closed curve in $M(\infty)$ is homologous to a finite sum of cycles around finite total curvature ends, we deduce that the flux of $M(\infty)$ along any closed curve is vertical, and therefore the same holds to the portion of M between planes $Q(n), Q(m)$ with n, m large. Since for n large $Q(n) \cap M$ is connected, one can apply a variant of the original López-Ros argument [**15, 28**] (see also the proof of Theorem 6 below) to obtain a contradiction. Hence, $M(\infty)$ has two limit ends, genus zero and horizontal limit tangent plane at infinity. If the spacing between the ends of M is unbounded, then a variation of the quasiperiodic proof of M would yield a catenoid limit rather than a limit surface with two limit ends, where a similar flux argument leads to a contradiction. This completes the proof of Theorem 1. □

Next we generalize a result in [**28**] for periodic minimal surfaces of genus zero and two limit ends to the quasiperiodic setting given by Theorem 1.

THEOREM 6. *If M is a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus, two limit ends and horizontal limit tangent plane, then M has nonzero horizontal flux on any compact horizontal section.*

PROOF. By the proof of Theorem 1, after taking limits by sequence of translations of M we obtain a limit which has genus zero, two limit ends, horizontal limit tangent plane at infinity and the same flux as the original surface. So, assume that M has genus zero.

Suppose to the contrary that M has vertical flux and we will derive a contradiction. The main tool used here is a variant of the López-Ros deformation for minimal surfaces with vertical flux. Basically, one takes the Weierstrass data g, η where g is the meromorphic Gauss map of M and η is the holomorphic form $\frac{1}{g}(dx_3 + idx_3^*)$. Define new Weierstrass data $g_t = t \cdot g$ and $\eta_t = \frac{1}{t} \cdot \eta$ where $t > 0$. The vertical flux assumption shows that the immersion $f_t \colon M \to \mathbb{R}^3$ associated to these Weierstrass data exists on M. Clearly $f_1(M) = M$ (up to a translation). It is not difficult to prove, using that the middle ends of M are planar, that for t large f_t is not an embedding, see [**28**]. Let $t_0 \in [1, \infty)$ be the largest $t, 1 \leq t \leq t_0$ such that f_t is injective. Since limits of embedded minimal surfaces are embedded, it is straightforward to prove t_0 exists. Thus, the goal is to prove that if f_t is injective, then for nearby parameter values t', $f_{t'}$ is also injective which contradicts the existence of t_0.

Since during the López-Ros deformation, the heights of the planar ends are constant, the openness property of the injective parameter values holds easily when M satisfies our hypotheses and is periodic. It is clear that in the quasiperiodic setting of Theorem 1, it suffices to show that we can preserve the embeddedness property of $f_t(M)$ for t close to 1 (we could take M in this case to be $f_{t_0}(M)$, which also satisfies the description in Theorem 1).

As M is quasiperiodic, there exists a sequence of points $p_n \in M$ with $x_3(p_n) \to \infty$ and a number $\varepsilon > 0$ such that if we denote $a_n = x_3(p_n)$, then $M(a_n) = \overline{S}(a_n - \varepsilon, a_n + \varepsilon) \cap M$

is a compact annulus (here $\overline{S}(a,b)$ is the closure of the open slab $S(a,b)$), and the translated surfaces $\widetilde{M}(a_n) = M(a_n) - p_n$ converge smoothly to a compact embedded annulus A passing through the origin. Similarly, we obtain points $q_n \in M$ with $b_n = x_3(q_n) \to -\infty$, compact annuli $M(b_n) = \overline{S}(b_n - \varepsilon, b_n + \varepsilon) \cap M$ and a limit of $\widetilde{M}(b_n) = M(b_n) - q_n$ to a compact embedded minimal annulus B. Since A and B are compact and embedded, $f_t|_A$ and $f_t|_B$ are also embedded for t close to 1. Since the $\widetilde{M}(a_n)$ converge smoothly to A and the $\widetilde{M}(b_n)$ converge smoothly to B as $n \to \infty$, it follows that $f_t|_{\widetilde{M}(a_n)}$ and $f_t|_{\widetilde{M}(b_n)}$ are embeddings for t very close to 1 and $n \geq N_0$ large. Hence there exists $\delta > 0$ such that $f_t|_{M(a_n) \cup M(b_n)}$ is injective, for all $t \in [1, 1+\delta]$ and $n \geq N_0$.

Now suppose that $f_{t'}$ is not injective for some $t' \in [1, 1+\delta]$ and we will derive the desired contradiction. Let $x, y \in M$ be distinct points with $p = f_{t'}(x) = f_{t'}(y)$. Then $p \in S(b_n, a_n) \cap M = M(*)$ for some fixed large n, and so $f_{t'}|_{M(*)}$ is not injective. Note that for arbitrary $t \in [1, 1+\delta]$, f_t is injective on $\partial M(*)$ and injective outside some large compact set. Since $f_t|_{M(*)}$ is continuous in the t parameter, then there is a first $t'' \in [1, t']$ such that $f_{t''}|_{M(*)}$ is not an embedding. But since $f_{t''}|_{\partial M(*)}$ is injective, the usual maximum principle for minimal surfaces gives a contradiction. This proves the theorem. □

It follows from the results in [**26**] that a complete connected embedded minimal surface M in $\mathbb{R}^3$ which has bounded curvature in balls, called a surface of *locally bounded Gaussian curvature*, is either proper in $\mathbb{R}^3$, proper in an open halfspace H (with limit set the boundary plane of H) or proper in an open slab S (with limit set the boundary planes of S). Moreover if $M \subset H$, P is a plane in H and $S(P, H)$ is the slab of $\mathbb{R}^3$ bounded by $P \cup \partial H$, then $S(P, H) \cap M$ is connected. If $M \subset S$, then either halfspace determined by a plane $P \subset S$ intersects M in a connected set. Also Meeks and Rosenberg [**26**], using the 1-sided curvature estimate of Colding-Minicozzi, proved that any complete connected embedded minimal surface $M \subset \mathbb{R}^3$ with finite topology and locally bounded Gaussian curvature must be proper. We now generalize this result.

THEOREM 7. *If M is a complete connected embedded minimal surface in $\mathbb{R}^3$ with finite genus and locally bounded Gaussian curvature, then M is properly embedded in $\mathbb{R}^3$.*

PROOF. Since the theorem is true for M with finite topology (Lemma 1.5 in [**26**]), we will assume that M has an infinite number of ends. Since M is properly embedded in a region W, where we may assume that $W = \{x_3 > 0\}$ or $W = \{0 < x_3 < 1\}$, the standard minimization procedure of using M as barrier against itself implies that there exist stable complete minimal surfaces in W with compact boundary which "separate" any two ends of M. Following the work in [**12**], one can show that there is a linear ordering on the ends of M by their relative heights over the (x_1, x_2)-plane. The results in [**8**] imply that the middle ends of M are simple ends contained between minimal graphs over annular domains in $\mathbb{R}^2$ and so, such middle ends are planar or catenoidal type.

If there is a catenoidal middle end of M, then the maximum principle at infinity implies $W - M$ contains an end C of a real catenoid. In this case W must be a halfspace. Next we check that there can only be a finite number of annular ends of M above C. Note that each middle end of M above C has logarithmic growth at least as great as C. Since the planar disk D bounding the circle

boundary of C intersects M in a compact set, it follows that the number of middle ends above C times the flux of C cannot be greater than the finite length of $D \cap M$; here we are using the fact that M is proper in the domain above $C \cup D$ and that the harmonic function x_3 on this part of M is proper. It follows in this case that M has exactly one limit end. Using the locally bounded curvature property and Lemma 1.1 in [**26**], this limit end has limit set the (x_1, x_2)-plane.

Assume that some middle end of M is planar (say at height $t_0 > 0$), and we will derive a contradiction. Note that in this case, all of the middle ends of M below height t_0 are also planar. If there are only a finite number of middle ends below height t_0, then the proof of the finite topology case considered in [**26**] applies to give a contradiction. So, in this case the bottom end of M is a limit end. However, the property that the portion of M below any horizontal plane is connected (Lemma 1.3 in [**26**]) implies in the case of finite genus that for t_0 sufficiently small, every plane $x_3^{-1}(t), 0 < t < t_0$, intersects M transversely in a connected component. The proof of the curvature estimates given in Theorem 5 now applies to prove that $M \cap \{0 < x_3 < t_0\}$ has bounded curvature from which we obtain, as in [**26**], a contradiction.

So far we have shown that if M is not properly embedded in $\mathbb{R}^3$, then it is properly embedded in $W = \{x_3 > 0\}$, it has one limit end which is its bottom end and all of the other ends are of catenoid type. Since M has finite genus, there exists an $\varepsilon > 0$ such that every simple closed curve in $M_{2\varepsilon} = M \cap (\mathbb{R}^2 \times (0, 2\varepsilon])$ separates M. As discussed before the statement of the theorem, $M_{2\varepsilon}$ is connected and, for a generic choice of ε, it has smooth boundary. If $M_{2\varepsilon}$ has bounded curvature, then each tangent plane to $M_{2\varepsilon}$ is arbitrarily close to horizontal, and so each component Ω of $M_{2\varepsilon}$ submerses to $\mathbb{R}^2 \times \{0\}$ under the orthogonal projection. But Ω is properly embedded in $\mathbb{R}^2 \times (0, 2\varepsilon]$ so it separates this region. Hence, the orthogonal projection from Ω to $\mathbb{R}^2 \times \{0\}$ is one-to-one, thus Ω is a graph. This implies Ω is properly embedded in $\mathbb{R}^2 \times [0, 2\varepsilon]$ and the proof of the Halfspace Theorem gives a contradiction. This shows that $M_{2\varepsilon}$ does not have bounded curvature for any $\varepsilon > 0$. Hence, there exists points $p(n) \in M_{1/n} = M \cap (\mathbb{R}^2 \times (0, \frac{1}{n}])$ such that the absolute Gaussian curvature $|K|$ of M satisfies $|K(p(n))| \geq n$.

Consider the sequence of horizontally translated surfaces $M(n) = M - \overline{p}(n)$, where $\overline{p}(n)$ is the orthogonal projection of $p(n)$ to the (x_1, x_2)-plane. We claim that the sequence $\{M(n)\}_n$ is not locally simply connected in any neighborhood of the origin. Reasoning by contradiction, suppose that for some ball B centered at the origin, a subsequence $M(n_i) \cap B$ consists only of simply connected components. Then, the locally bounded curvature assumption for M implies that every component of $M(n_i) \cap B$ is compact. In this case, let $C(n_i)$ be the disk component containing the point $q(n_i) = p(n_i) - \overline{p}(n_i)$. Since the $q(n_i)$ converge to $\vec{0}$ and the $\partial C(n_i)$ lie in the upper half sphere of ∂B, we obtain a contradiction to the 1-sided curvature estimate in [**5**]. Thus, the sequence $M(n)$ is not locally simply connected in any neighborhood of the origin.

Now apply the blow-up of $M(n) \cap B(\varepsilon)$ described in the proof of Lemma 8. Here, $B(\varepsilon)$ is the ball of radius ε centered at the origin. In other words, consider the new normalized sequence $\overline{M}(n) = \frac{1}{r_n(q(n))}(M(n) - q(n))$, where $q(n)$ is a point of $B(\varepsilon)$ where the function $p \mapsto \frac{\varepsilon - \|p\|}{r_n(p)}$ has its maximum value, $r_n(p)$ being the radius of the largest open ball centered at p which intersects $M(n)$ in simply connected components. It follows that $\overline{M}(n)$ is a ULSC sequence of minimal surfaces

in $\mathbb{R}^3$. Now we have two possibilities, depending on whether the sequence $\{\overline{M}(n)\}_n$ has locally bounded Gaussian curvature in $\mathbb{R}^3$ or not.

First assume that the Gaussian curvature of $\{\overline{M}(n)\}_n$ is not locally bounded in $\mathbb{R}^3$. Note that for n large, any simple closed curve in $\overline{M}(n)$ comes from rescaling of a simple closed curve in $M(n) \cap B(\varepsilon)$, hence it must separate this surface. By Theorem 3, a subsequence of the $\overline{M}(n)$ (denoted in the same way) converges to a foliation $\mathcal{L}$ of $\mathbb{R}^3$ by parallel planes with singular set of convergence $S(\mathcal{L})$ being exactly two Lipschitz curves. By Meeks' regularity theorem, such Lipschitz curves are in fact straight lines orthogonal to the planes in $\mathcal{L}$. From the proof of Theorem 5, we can construct a simple closed curve $\delta(n) \subset \overline{M}(n)$ which separates $\overline{M}(n)$ into two noncompact components, such that $\delta(n)$ is arbitrarily close to a doubly covered straight line segment l contained in one of the planes of $\mathcal{L}$, l joining orthogonally the two straight lines in $S(\mathcal{L})$. Furthermore, the flux of $\overline{M}(n)$ along $\delta(n)$ converges to a vector of length $2l$ contained in one of the planes in $\mathcal{L}$. On the other hand, since $\overline{M}(n)$ has just one limit end and its limit tangent plane is horizontal, $\delta(n)$ is the boundary of a subdomain $\Sigma(n) \subset \overline{M}(n)$ with a finite number of vertical catenoidal ends with positive logarithmic growth. Hence, $\delta(n)$ has vertical flux. By the maximum principle, $x_3|_{\Sigma(n)}$ has its minimum value at some point of $\delta(n)$. Since $\Sigma(n)$ has points at any fixed positive distance from the convex hull of $\delta(n)$ and at such points it has uniformly bounded curvature, the limit set of $\Sigma(n)$ as $n \to \infty$ must contain the end of one of the planes in $\mathcal{L}$. If this plane were not horizontal, then we would contradict that $x_3|_{\Sigma(n)}$ is bounded from below by $\min\left(x_3|_{\delta(n)}\right)$. This shows that the planes in $\mathcal{L}$ are horizontal, which in turn implies that the flux of $\overline{M}(n)$ along $\delta(n)$ converges to a nonzero finite horizontal vector, which contradicts that such a flux is vertical.

Secondly assume that the Gaussian curvature of $\{\overline{M}(n)\}_n$ is locally bounded in $\mathbb{R}^3$. Then Lemma 2 implies that after passing to a subsequence, the $\overline{M}(n)$ converge to a $C^{1,\alpha}$ minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$. As in the proof of Lemma 8, there is some leaf L of $\mathcal{L}$ which is not flat and not simply connected. Further, L is a planar domain and the portions of $\overline{M}(n)$ which converge to L, converge with multiplicity 1. Since L is a smooth limit of surfaces with vertical flux, then L has vertical flux. We will finish the proof of the theorem by discarding all possible limit surfaces L as above.

Suppose that L is properly embedded in $\mathbb{R}^3$. Since L is a nonsimply connected planar domain, it must have more than one end. Hence, L has a well-defined limit tangent plane at infinity which we denote by Π. If L has two limit ends, then Lemma 1 insures that there exists a plane parallel to Π which intersects L in a simple closed curve δ. As above, there exist simple closed curves $\delta(n) \subset \overline{M}(n)$ arbitrarily close to δ, and each $\delta(n)$ bounds a subdomain $\Omega(n) \subset \overline{M}(n)$ with a finite number of vertical catenoidal ends. Arguing as before, one concludes that Π is horizontal, which is a contradiction with Theorem 6 since L has vertical flux. Therefore, L cannot have two limit ends and so, it must have some vertical top or bottom catenoid type end. Thus, there exists a horizontal plane P whose intersection with L contains a convex curve γ and the open planar disk in P bounded by γ is disjoint from L. It follows that there exists a sequence of horizontal planes $P(n) \subset \mathbb{R}^2 \times (0, \varepsilon)$ whose heights converge to zero as $n \to \infty$, such that $P(n) \cap M$ contains a component $\gamma(n)$ which is a simple closed convex curve and the open planar disk $D(n)$ that $\gamma(n)$

bounds is disjoint from M. Further, the lengths of $\gamma(n)$ tend to zero as $n \to \infty$. Let $\Omega(n) \subset M$ be the component of $M - \gamma(n)$ with a finite number of ends. If $\Omega(n)$ is an annulus, then $\Omega(n)$ must be a representative of the top end of M (this is true because $M - \Omega(n)$ clearly lies below the topological plane $\Omega(n) \cup D(n)$). Since the lengths of $\gamma(n)$ converge to zero as $n \to \infty$ and the flux of the top end of M is fixed, we may assume that $\Omega(n)$ is never an annulus. Notice also that, after choosing a subsequence, the flux of $\gamma(n)$ is strictly monotonically decreasing to zero, as well as its height. Thus, for all n large enough the domains $\Omega(n)$ are pairwise disjoint. Since the genus of M is finite, there exists a $\Omega(n)$ which is a planar domain and has more than one end. Then, the López-Ros deformation argument applied to this $\Omega(n)$ gives a contradiction (see Theorem 2 in [**28**]). This contradiction shows that L cannot be properly embedded in $\mathbb{R}^3$.

Since L is not properly embedded in $\mathbb{R}^3$ and it is a leaf of a minimal lamination of $\mathbb{R}^3$, L must be properly embedded in a halfspace of in a slab. In fact, our arguments at the beginning of the proof imply that L is a connected minimal surface of genus zero and one limit end that embeds properly in a halfspace, which is horizontal since L has vertical flux. It follows that L has a top or bottom catenoid type end and so, there exists a horizontal plane that intersects L in a convex curve which bounds an open convex planar disk which is disjoint from L. Arguing now as in the previous paragraph where L was assumed to be proper in $\mathbb{R}^3$, we obtain a contradiction. The theorem now follows. □

Since a leaf of a minimal lamination of $\mathbb{R}^3$ has locally bounded curvature, we obtain the following Corollary.

Corollary 1. *If $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3$ with more than one leaf, then every non-planar leaf of $\mathcal{L}$ has infinite genus.*

We now prove Theorem 2 in Section 2.

Proof of Theorem 2. Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus and horizontal limit tangent plane at infinity. Assume also that M has exactly one limit end e_∞, which is its top end. Then the bottom end is annular and so, by Collin [**7**] it is asymptotic to an end of a horizontal plane or to an end of a catenoid. If the bottom end were planar or catenoidal type with positive logarithmic growth, then M would lie in a halfspace, which is impossible by the Halfspace Theorem [**13**]. So the bottom end of M is a catenoid end of negative logarithmic growth λ_1. It is clear from the Ordering Theorem [**12**] that $\lambda_n \leq \lambda_{n+1}$ for all n. From flux considerations (see for example the proof of Theorem 7), every λ_n is nonpositive. Therefore, if some $\lambda_n = 0$ then e_n would be planar as would all ends above it. A slight variant of the proof of statement *3* in Theorem 1 shows that there is a halfspace of the form $\{x_3 \geq c\}$, $c \in \mathbb{R}$, such that every plane in this halfspace intersects M in a connected component which is a proper arc at the heights of ends and otherwise is a simple closed curve. Using the fact that M has finite vertical flux, then the proof of Theorem 1 implies that under some divergent sequence of translations by $-q(n)$, the surfaces $M(n) = M - q(n)$ converge with multiplicity 1 to a properly embedded minimal surface $M(\infty)$ with genus zero, two limit ends and horizontal limit tangent plane at infinity. Furthermore,

the flux vector of $M(\infty)$ is vertical since this is the case for the $M(n)$. But by Theorem 6, $M(\infty)$ cannot have vertical flux, which proves the theorem. □

Finally, we prove Theorem 4 stated in Section 2.

Proof of Theorem 4. It is clear from the proof of curvature estimates in Section 3 and other theorems in this section that the normalized blow-up $\widetilde{M}$ is a properly embedded minimal surface in $\mathbb{R}^3$ of bounded curvature and finite genus. If $\widetilde{M}$ has one end, then, by the results in [**26**], $\widetilde{M}$ has a helicoidal end. In this case one can clearly take a new normalized blow-up sequence which has the helicoid as a limit, and statement *1* holds.

If $\widetilde{M}$ has a finite number of ends greater than one, then $\widetilde{M}$ has finite total curvature by Collin's Theorem [**7**]. If the number of ends of $\widetilde{M}$ is infinite, then it has one or two limit ends [**8**]. The one limit end case is stated as statement *3* in our Theorem 4. Finally if $\widetilde{M}$ has two limit ends, then the proof of the quasiperiodic property in Theorem 1 makes it clear that there is a new normalized blow-up with genus zero and two limit ends. This completes the proof of Theorem 4. □

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