CMC surfaces in metric Lie groups

Joaquín Pérez
email: jperez@ugr.es
http://www.ugr.es/local/jperez/

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Definition

1. $(X, \langle , \rangle)$ homogeneous if $I(X, \langle , \rangle) = \{$ isometries of X $\}$ transitive.
2. $(X, \langle , \rangle)$ locally homogenous if $\forall p, q \in X$, $\exists \varepsilon = \varepsilon(p, q) > 0$ s.t. $B(p, \varepsilon)$, $B(q, \varepsilon) \subset X$ isometric.

$(X, \langle , \rangle)$ homogeneous $\Rightarrow$ complete, loc. homogeneous
Converse not true: (M_k, g_{-1}), $k > 1$.
$(X, \langle , \rangle)$ complete, loc. homogeneous $\Rightarrow$ univ cover is homogeneous.

Definition

$(G, *)$ Lie group $=$ mfd $+$ $[(x, y) \in G \times G \to G \mapsto x * y^{-1}$ smooth$]$.
G_1, G_2 Lie groups are *isomorphic* if $\exists \phi \colon G_1 \to G_2$ diffeo + group iso.

$a \in G$, l_a : left translation, r_a : right translation.
$\langle , \rangle$ Riemannian metric on G is left invariant if $l_a \in I(G, \langle , \rangle)$ $\forall a \in G$.

Definition

Metric Lie group (MLG): $(G, *)$ Lie group + $\langle , \rangle$ left invariant metric.

Theorem

$(X^3, \langle,\rangle)$ *connected homogeneous,* $\pi_1(X) = 0$
$\Rightarrow\ (M, \langle,\rangle) = (\mathbb{S}^2(k) \times \mathbb{R}, \mathit{standard})$ *or* $(X^3, \langle,\rangle)$ *is a MLG.*

Proof. $\lambda\colon I(X, \langle,\rangle) \times X \to X$, $\lambda(\phi, p) = \phi(p)$ transitive action
$p \in X \rightsquigarrow \mathrm{Stab}_p = \{\phi \in I(X) \mid \phi(p) = p\}$; Stab_p, Stab_q conjugate $\forall p, q$
$\mathrm{Stab}_p \to I(X) \to X$ principal bundle; classify?
$I_0(X)$: component of 1_X in $I(X)$, $\mathrm{Stab}_0(p) = \{\phi \in I_0(X) \mid \phi(p) = p\}$
$\mathrm{Stab}_0(p) \to I_0(X) \to X$ principal bundle; classify? $\mathfrak{h}$: Lie algebra $\mathrm{Stab}_0(p)$
$\mathrm{Stab}_0(p) \leq SO(3)$ closed conn subgr $\leftrightarrow \mathfrak{h} \leq \mathcal{A}_3(\mathbb{R}) \cong (\mathbb{R}^3, \times)$ subalgebra

- $\dim \mathfrak{h} = 3$ ($\dim I(X) = 6$) $\Rightarrow \mathrm{Stab}_0(p) = \mathrm{SO}(3) \Rightarrow (X, \langle,\rangle)$ CSC: $(\mathbb{R}^3, \mathbb{H}^3, \mathbb{S}^3;\text{standard})$
- $\dim \mathfrak{h} = 2$: impossible ($\nexists$ 2-dim subalgebras of $(\mathbb{R}^3, \times)$)
- $\dim \mathfrak{h} = 1$ ($\dim I(X) = 4$) $\Rightarrow \mathfrak{h} = \mathrm{Span}\{A\}$, $A \in \mathcal{A}_3(\mathbb{R})$ (^{1} up to isom) $\Rightarrow \mathrm{Stab}_0(p) \cong \mathrm{SO}(2) = \mathbb{S}^1 \Rightarrow \mathbb{E}(\kappa, \tau)$, all MLG except $\mathbb{S}^2(k) \times \mathbb{R}$.
- $\dim \mathfrak{h} = 0$ ($\dim I(X) = 3$) $\Rightarrow \mathrm{Stab}_0(p) \overset{\text{discrete}}{\leq} I_0(X)$, $I_0(X) \overset{\text{cover}}{\to} I_0(X)/\mathrm{Stab}_0(p_0) \overset{\text{cover}}{\to} X \Rightarrow \mathrm{Stab}_0(p_0) = \{1_X\}$, $X \cong I_0(X)$.

Generalities on MLG

G: Lie group. Given $a, p \in G$ and $v \in T_pG$,

$$av \equiv (l_a)_*(v) \in T_{ap}G, \quad va \equiv (r_a)_*(v) \in T_{pa}G).$$

Definition

$X \in \mathfrak{X}(G)$ left invariant if $X = aX$ $\forall a \in G$.
right invariant $X = Xa$
$\mathfrak{g} = (\{\text{left inv vector fields}\} \equiv T_eG, [\cdot,\cdot]) \quad \leftarrow$ Lie algebra of G.
$\mathfrak{g}^* = (\{\text{right inv vector fields}\} \equiv T_eG, [\cdot,\cdot]) \cong \mathfrak{g}$ (Lie algebra isom)

If G is simply-connected, then $\mathfrak{g}$ determines G up to isomorphism.

$X \in \mathfrak{g} \rightsquigarrow \gamma\colon \mathbb{R} \to G$ Lie group morphism (1-parameter subgroup of G)
$t \mapsto \exp(tX)$
$l_a \circ \gamma$: integral curve of X passing through $a \in G$ at $t = 0$,
$\{r_{\gamma(t)} \mid t \in \mathbb{R}\} \leq \mathrm{Diff}(G) \quad \leftarrow$ 1-parameter group of X.

$X \in \mathfrak{g}^* \rightsquigarrow \gamma\colon \mathbb{R} \to G$ integral curve of X passing through e at $t = 0$,
$\{l_{\gamma(t)} \mid t \in \mathbb{R}\} \leq I(G, \langle,\rangle) \quad \leftarrow$ 1-parameter group of isometries associated to X (Killing) $\forall$ left inv metric $\langle,\rangle$.

Examples of MLG and first properties

1. $((\mathbb{R}^n,+),\text{standard metric})$: commutative MLG ($[\cdot,\cdot]=0$).
   - $(\mathbb{R},+)$: unique simply-connected 1d Lie group.
   - If G commutative $\Rightarrow$ every left inv metric on G is flat (write $K(e_i\wedge e_j)$ in terms of coefs which only depend on $\langle [e_i,e_j],e_k\rangle$).
2. G^n Lie group, $n\geq 2$, s.t. $[X,Y]\in \text{Span}\{X,Y\}\quad \forall X,Y\in\mathfrak{g}$ $\Rightarrow \exists l\colon \mathfrak{g}\to\mathbb{R}$ linear s.t.

$$(\star)\qquad [X,Y]=l(X)Y-l(Y)X,\quad \forall X,Y\in\mathfrak{g}.$$

   Furthermore:
   - If $l=0\Rightarrow$ every left inv metric on G is flat,
   - If $l\neq 0\Rightarrow$ every left inv metric on G has $K\equiv -\|l\|<0$.
3. The only simply-connected 2d MLG are $\mathbb{R}^2,\mathbb{H}^2$.
4. $\mathbb{H}^n$ as the (non-commutative) Lie group of similarities of $\mathbb{R}^{n-1}$:

$$(\mathbf{a},a_n)\in\mathbb{H}^n\equiv(\mathbb{R}^n)^+\mapsto\phi_{(\mathbf{a},a_n)}\colon\quad \begin{array}{l}\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}\\ \mathbf{x}\mapsto a_n\mathbf{x}+\mathbf{a}\end{array}$$

   (Its Lie algebra satisfies $(\star)$).

5. $SO(3) = \{A \in Gl(3, \mathbb{R}) \mid A \cdot A^T = I_3,\ \det A = 1\}$
   - $SO(3)$ diffeo to $\mathbb{RP}^3$.
   - Universal cover is $\mathbb{S}^3 \subset \mathbb{R}^4 = \{a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \mid a, b, c, d \in \mathbb{R}\}$ (unit length quaternions).
   - Left multiplication by a unit length quaternion is isometry of $(\mathbb{R}^4, g_0) \ \Rightarrow\ \langle , \rangle = g_0|_{\mathbb{S}^3}$ (standard metric) is a left invariant metric.
   - $SO(3)$ is diffeomorphic to the unit tangent bundle of $\mathbb{S}^2$:
     $U\mathbb{S}^2 = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|x\| = \|y\| = 1, \langle x, y\rangle = 0\}$:
     $F\colon SO(3) \to U\mathbb{S}^2$, (diffeo)
     $F(\text{column}_1, \text{column}_2, \text{column}_3 = c_1 \times c_2) = (\text{column}_1, \text{column}_2)$.
   - $\mathbb{E}(\kappa = 1, \tau)$-metrics on $\mathbb{S}^3$ (Berger spheres):
     Lift $g_\lambda = \sum_{i=1}^3 dx_i^2 + \lambda \sum_{i=1}^3 dy_i^2$ on $U\mathbb{S}^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{S}^3$ ($\lambda > 0$)
   - 1-parameter subgroups of $SO(3)$ are the circle subgroups (rotations around every fixed axis).
   - $SO(3)$ has no 2-dimensional subgroups.

Semidirect products: definition and examples

$$\left.\begin{array}{l}(V,+),(H,\star) \text{ groups}\\ \varphi\colon V\to \mathrm{Aut}(H)\\ \text{group morphism}\end{array}\right\} \quad z\in V\mapsto \quad \varphi(z)=\varphi_z\colon \begin{array}{ccc} H & \to & H\\ \mathbf{p} & \mapsto & \varphi_z(\mathbf{p}),\end{array}$$

$H \rtimes_\varphi V = (V\times\mathbb{R},*)$ where

$$(\mathbf{p}_1,z_1)*(\mathbf{p}_2,z_2)=(\mathbf{p}_1\star\varphi_{z_1}(\mathbf{p}_2),z_1+z_2)$$

$\mathbb{R}^2\rtimes_A\mathbb{R}$: Take $H=\mathbb{R}^2$, $V=\mathbb{R}$ $\Rightarrow$ $\varphi_z(\mathbf{p})=e^{zA}\mathbf{p}$, $A\in\mathcal{M}_2(\mathbb{R})$.

- $A=0\in\mathcal{M}_2(\mathbb{R})$ $\rightsquigarrow$ $\mathbb{R}^2\rtimes_0\mathbb{R}=\mathbb{R}^2\times\mathbb{R}=\mathbb{R}^3$.
- If $V=\mathbb{R}$, $H=\mathbb{H}^2$ and $\varphi\equiv 1_H\colon\mathbb{R}\to\mathrm{Aut}(H)$ $\rightsquigarrow$ $H\rtimes_{1_H}\mathbb{R}=\mathbb{H}^2\times\mathbb{R}$.
- $A=I_2$ $\rightsquigarrow$ $\mathbb{R}^2\rtimes_{I_2}\mathbb{R}=\mathbb{H}^3$, with group isomorphism

$$(p,z)\in\mathbb{R}^2\rtimes_{I_2}\mathbb{R}\mapsto(p,e^z)\in(\mathbb{R}^3)^+$$

- $A=(1)$ $\rightsquigarrow$ $\mathbb{R}\rtimes_{I_1}\mathbb{R}=\mathbb{H}^2\Rightarrow(\mathbb{R}\rtimes_{I_1}\mathbb{R})\times\mathbb{R}=\mathbb{H}^2\times\mathbb{R}$, with

$$(x,y,z)*(x',y',z')=(x+e^y x',y+y',z+z')$$

- $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $\rightsquigarrow$ $\mathbb{R}^2\rtimes_A\mathbb{R}=\mathbb{H}^2\times\mathbb{R}$, with

$$(x,y,z)*(x',y',z')=(x+e^z x',y+y',z+z')$$

- $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow \mathbb{R}^2 \rtimes_A \mathbb{R} = \widetilde{E}(2)$, univ cover of the group of orientation preserving rigid motions of $\mathbb{R}^2$, with $(x,y,z)*(x',y',z') = (x+x'\cos z - y'\sin z, y+x'\sin z+y'\cos z, z+z')$, $(x,y,z) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mapsto \mathrm{Rot}_z(\cdot) + \begin{pmatrix} x \\ y \end{pmatrix} \in \widetilde{E}(2)$ group isomorphism
- $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow \mathbb{R}^2 \rtimes_A \mathbb{R} = \mathrm{Sol}_3 = E(1,1)$, group of orientation preserving rigid motions of $\mathbb{L}^2$, with $(x,y,z) * (x',y',z')$ $= (x + x'\cosh z + y'\sinh z, y + x'\sinh z + y'\cosh z, z + z')$ $(x,y,z) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mapsto \mathrm{Roth}_z(\cdot) + \begin{pmatrix} x \\ y \end{pmatrix} \in E(1,1)$ group isom, where $\mathrm{Roth}_z = \begin{pmatrix} \cosh z & \sinh z \\ \sinh z & \cosh z \end{pmatrix}$
- $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \mathbb{R}^2 \rtimes_A \mathbb{R} = \mathrm{Nil}_3$ with $(x,y,z) * (x',y',z') = (x + x' + zy', y + y', z + z')$. $(x,y,z) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mapsto \begin{pmatrix} 1 & z & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Nil}_3$ group isomorphism

Semidirect products: basic properties

$\mathbb{R}^2 \rtimes_A \mathbb{R}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Coordinates $(x, y) \in \mathbb{R}^2$, $z \in \mathbb{R} \rightsquigarrow \partial_x, \partial_y, \partial_z$ basis of tangent bundle.

$$(\mathbf{p}_1, z_1) * (\mathbf{p}_2, z_2) = (\mathbf{p}_1 + e^{z_1 A}\mathbf{p}_2, z_1 + z_2)$$

Taking derivatives w.r.t. $(\mathbf{p}_1, z_1)$ we obtain

$$F_1 = \partial_x, \quad F_2 = \partial_y, \quad F_3(x, y, z) = (ax + by)\partial_x + (cx + dy)\partial_y + \partial_z$$

basis of right invariant vector fields on $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Taking derivatives w.r.t. $(\mathbf{p}_2, z_2)$ we obtain

$$E_1(x, y, z) = a_{11}(z)\partial_x + a_{21}(z)\partial_y, \; E_2(x, y, z) = a_{12}(z)\partial_x + a_{22}(z)\partial_y, \; E_3 = \partial_z$$

basis of left invariant vector fields on $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where $e^{zA} = (a_{ij}(z))_{i,j}$.

Lie bracket: $[E_1, E_2] = 0, \quad [E_3, E_1] = aE_1 + cE_2, \quad [E_3, E_2] = bE_1 + dE_2$.

- A determines $(\mathfrak{g}, [\cdot, \cdot])$ completely.
- $\mathrm{ad}_{E_3} : \mathrm{Span}\{E_1, E_2\} \to \mathrm{Span}\{E_1, E_2\}$ (endomorphism) $Y \mapsto [E_3, Y]$ has $\mathrm{trace}(\mathrm{ad}_{E_3}) = \mathrm{trace}(A)$.
- $\mathrm{Span}\{E_1, E_2\}$ integrable distribution $\rightsquigarrow \mathcal{F} = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$

Unimodular versus non-unimodular

G Lie group, $G \to \mathrm{Aut}(G)$ inner automorphisms

$$g \mapsto a_g\colon G \to G, \quad a_g(h) = ghg^{-1}\ \forall h \in G$$

$a_g(e) = e \ \Rightarrow (da_g)_e\colon T_eG = \mathfrak{g} \to \mathfrak{g}$ automorphism $\rightsquigarrow \mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$

(Adjoint representation) $\quad \to \quad g \mapsto \mathrm{Ad}_g = d(a_g)_e$

$a_{gh} = a_g \circ a_h \ \Rightarrow$ Ad Lie group morphism $\rightsquigarrow ad := d(Ad)\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

Given $X \in \mathfrak{g}$, $\quad \mathrm{ad}_X = \mathrm{ad}(X)\colon \mathfrak{g} \to \mathfrak{g}$

$Y \mapsto \mathrm{ad}_X(Y) = [X, Y]$ (Warner)

$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y] \ \Rightarrow \mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ a Lie algebra morphism.

Definition

A Lie group G is unimodular $\Leftrightarrow$ $\det(\mathrm{Ad}_g) = \pm 1,\ \forall g \in G$.
$\Leftrightarrow$ $\mathrm{trace}(\mathrm{ad}_X) = 0,\ \forall X \in \mathfrak{g}$.

- $\mathbb{R}^2 \rtimes_A \mathbb{R}$ unimodular $\Leftrightarrow$ $\mathrm{trace}(A) = 0$.
- $\mathbb{R}^3, \mathrm{Nil}_3, \widetilde{E}(2), \mathrm{Sol}_3$ are unimodular.
- $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$ are non-unimodular. $\widetilde{\mathrm{SL}}(2, \mathbb{R})$: Warning!

Canonical left invariant metric on a semidirect product

$A \in \mathcal{M}_2(\mathbb{R}) \rightsquigarrow \mathbb{R}^2 \rtimes_A \mathbb{R}, \quad \mathfrak{g} = \mathrm{Span}\{E_1, E_2, E_3\}.$

Definition

Canonical left invariant metric $\langle , \rangle_A$ on $\mathbb{R}^2 \rtimes_A \mathbb{R}$: declare the left invariant basis $\{E_1, E_2, E_3\}$ to be orthonormal (equivalently, extend the usual inner product of $\mathbb{R}^3 = T_e(\mathbb{R}^2 \rtimes_A \mathbb{R})$ by left translations).

Koszul formula gives Levi-Civita connection ∇ for $\langle , \rangle_A$:

$$
\begin{array}{l|l|l}
\nabla_{E_1} E_1 = a\, E_3 & \nabla_{E_1} E_2 = \frac{b+c}{2}\, E_3 & \nabla_{E_1} E_3 = -a\, E_1 - \frac{b+c}{2}\, E_2 \\
\nabla_{E_2} E_1 = \frac{b+c}{2}\, E_3 & \nabla_{E_2} E_2 = d\, E_3 & \nabla_{E_2} E_3 = -\frac{b+c}{2}\, E_1 - d\, E_2 \\
\nabla_{E_3} E_1 = \frac{c-b}{2}\, E_2 & \nabla_{E_3} E_2 = \frac{b-c}{2}\, E_1 & \nabla_{E_3} E_3 = 0.
\end{array}
$$

- $z \overset{\Gamma}{\mapsto} (x_0, y_0, z)$ is a geodesic for every $(x_0, y_0) \in \mathbb{R}^2$ and $(x, y, z) \overset{\phi}{\mapsto} (-x + 2x_0, -y + 2y_0, z)$ is an isometry ($\mathrm{Fix}(\phi) = \Gamma$).
- Each leaf of $\mathcal{F} = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ is intrinsically flat, has unit normal vector field E_3 and CMC $H = \mathrm{trace}(A)/2$ w.r.t. E_3 (stable)

Canonical left invariant metric on a semidirect product

- Canonical metric in (x, y, z)-coordinates:
  $\langle,\rangle_A = \left[a_{11}(-z)^2 + a_{21}(-z)^2\right] dx^2 + \left[a_{12}(-z)^2 + a_{22}(-z)^2\right] dy^2 + dz^2$
  $+ \left[a_{11}(-z)a_{12}(-z) + a_{21}(-z)a_{22}(-z)\right](dx \otimes dy + dy \otimes dx)$
- $(x, y, z) \overset{\phi}{\mapsto} (-x + 2x_0, -y + 2y_0, z)$ isometry of $\langle,\rangle_A$ (rotation by angle π around geodesic $\Gamma = \{(x_0, y_0, z) \mid z \in \mathbb{R}\}$)
- $\mathbb{R}^2 \rtimes_A \mathbb{R} \leftrightarrow \mathbb{R}^2 \rtimes_B \mathbb{R}$, $\quad \langle,\rangle_A \leftrightarrow \langle,\rangle_B$ in terms of $A \leftrightarrow B$?
  - If $B = P^{-1}AP \;\Rightarrow\; \psi \colon \mathbb{R}^2 \rtimes_A \mathbb{R} \to \mathbb{R}^2 \rtimes_B \mathbb{R}$
    $(\mathbf{p}, t) \mapsto (P^{-1}\mathbf{p}, t)$ Lie group isomorphism
  - If $B = P^T AP$, $P \in O(2) \;\Rightarrow\; \langle,\rangle_A = \langle,\rangle_B$
  - $\lambda > 0 \;\Rightarrow \Psi \colon \mathbb{R}^2 \rtimes_A \mathbb{R} \to \mathbb{R}^2 \rtimes_{\lambda A} \mathbb{R}$
    $(\mathbf{p}, z) \mapsto (\mathbf{p}, z/\lambda)$ Lie group isomorphism
    $\Theta \colon \mathbb{R}^2 \rtimes_A \mathbb{R} \to \mathbb{R}^2 \rtimes_{\lambda A} \mathbb{R}$
    $(\mathbf{p}, z) \mapsto (\mathbf{p}/\lambda, z/\lambda)$ homothety: $\Theta^*(\langle,\rangle_{\lambda A}) = \frac{1}{\lambda^2}\langle,\rangle_A$
  - $A \to -A$: Change of orientation.

Classification of 3d MLG

G: n-dim Lie group. $X \in \mathfrak{g} \rightsquigarrow \mathrm{ad}_X = \mathrm{ad}(X)\colon \mathfrak{g} \to \mathfrak{g}$ endomorphism

$$Y \mapsto [X, Y]$$

$X \in \mathfrak{g} \overset{\varphi}{\mapsto} \mathrm{trace}(\mathrm{ad}_X) \in \mathbb{R}$ linear $\rightsquigarrow \mathfrak{u} := \ker(\varphi)$ unimodular kernel of G.

Taking trace in $\mathrm{ad}_{[X,Y]} = \mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_Y \circ \mathrm{ad}_X \ \forall X, Y \in \mathfrak{g}$ (Jacobi):

- $[X, Y] \in \mathfrak{u} \quad \forall X, Y \in \mathfrak{g} \ \Rightarrow \ \mathfrak{u}$ ideal of $\mathfrak{g}$.
- $\varphi\colon (\mathfrak{g}, [\cdot,\cdot]) \to (\mathbb{R}, [\cdot,\cdot] = 0)$ Lie algebra morphism.
- A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is unimodular if $\mathrm{trace}(\mathrm{ad}_X) = 0 \ \forall X \in \mathfrak{h}$. Hence $\mathfrak{u}$ is the largest unimodular Lie algebra of $\mathfrak{g}$.
- G unimodular $\Leftrightarrow \ \mathfrak{u} = \mathfrak{g}$.

From now on, $\dim G = 3$

$\Rightarrow \dim \mathfrak{u} = \dim \mathfrak{g} - \dim \mathrm{Im}(\varphi) = 2$ (non-unimodular), *or* 3 (unimodular).

- Classify the non-unimodular 3d MLG.
- Classify the unimodular 3d MLG.

Classification of 3d non-unimodular MLG

Assume $(G, \langle , \rangle)$ non-unimodular MLG $\rightsquigarrow \exists \{E_1, E_2, E_3\}$ ortho basis of $\mathfrak{g}$ s.t. $\mathfrak{u} = \text{Span}\{E_1, E_2\}$ and $\exists H \overset{\text{subgroup}}{\leq} G$ with Lie algebra $\mathfrak{u}$.

- $[E_1, E_2] = 0$, i.e., $H \cong \mathbb{R}^2$ (isomorphic).
- $\exists \alpha, \beta, \gamma, \delta \in \mathbb{R}$ s.t. $[E_3, E_1] = \alpha E_1 + \gamma E_2, \quad [E_3, E_2] = \beta E_1 + \delta E_2$ with $\text{trace}(\mathrm{ad}_{E_3}) = \alpha + \delta \neq 0 \;\; (E_3 \notin \mathfrak{u}) \quad \rightsquigarrow A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.
- A determines $[\cdot, \cdot] \Rightarrow$ A determines G up to group isomorphism.
- Given $\lambda > 0$, $(G, \lambda\langle , \rangle) \;\rightsquigarrow \frac{1}{\sqrt{\lambda}} A$.
- $(G, \langle , \rangle)$ isomorphic and isometric to $(\mathbb{R}^2 \rtimes_A \mathbb{R}, \langle , \rangle_A)$, where $\text{trace}(A) \neq 0$, $\mathbb{R}^2 \rtimes_A \{0\} \equiv \exp(\mathfrak{u})$ and $\{0\} \rtimes_A \mathbb{R} \equiv \exp(\mathfrak{u}^\perp)$.

Two possibilities:

1. $A = \alpha I_2 \;\; (\alpha \neq 0) \;\Rightarrow\; [X, Y] = l(X)Y - l(Y)X$ for $l \colon \mathfrak{g} \to \mathbb{R}$, $l(E_1) = l(E_2) = 0$, $l(E_3) = \alpha \;\Rightarrow\; (G, \langle , \rangle)$ CSC $-\alpha^2 < 0$ ($G \cong \mathbb{H}^3$).
2. A not multiple of I_2.

Classification of 3d non-unimodular MLG

Assume $(G, \langle , \rangle) \cong (\mathbb{R}^2 \rtimes_A \mathbb{R}, \langle , \rangle_A)$, trace$(A) \neq 0$, A not multiple of I_2.

1. $T = \text{trace}(A)$ and $D = \det(A)$ determine $\mathfrak{g}$ and G:
   Consider the endomorphism $L(X) = [E_3, X]$, $X \in \mathfrak{u}$.
   $\exists \widehat{E}_1 \in \mathfrak{u}$ s.t. $B = \{\widehat{E}_1, \widehat{E}_2 := L(\widehat{E}_1)\}$ basis $\Rightarrow M(L, B) = \begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix}$
2. Up to scaling $\langle , \rangle$ and orientation $\rightsquigarrow T = 2 \Rightarrow D$ invariant for G
3. $[E_3, E_1] = \alpha E_1 + \gamma E_2$, $[E_3, E_2] = \beta E_1 + \delta E_2$, $\alpha + \delta = 2$.

$$\left.\begin{array}{l} \text{Up to rotation in } \mathfrak{u} \rightsquigarrow \alpha\beta + \gamma\delta = 0 \\ \text{Up to } (E_1, E_2, E_3) \to (E_2, -E_1, E_3) \rightsquigarrow \alpha \geq \delta \\ \text{Up to } (E_1, E_2, E_3) \to (-E_1, E_2, E_3) \rightsquigarrow \gamma \geq \beta \end{array}\right\} \alpha, \beta, \delta, \gamma \text{ uniq determined}$$

   Milnor Lemma 6.5 $\Rightarrow \{E_1, E_2, E_3\}$ diagonalizes Ricci tensor of $\langle , \rangle$ with Ric(E_i) explicit polynomial expressions of α, β, δ.

4. Change of variables: $A = \begin{pmatrix} 1+a & -(1-a)b \\ (1+a)b & 1-a \end{pmatrix}$, $a, b \geq 0$.

$$\begin{aligned} \text{Ric}(E_1) &= -2\left(1 + a(1+b^2)\right) \\ \text{Ric}(E_2) &= -2\left(1 - a(1+b^2)\right) \\ \text{Ric}(E_3) &= -2\left(1 + a^2(1+b^2)\right). \end{aligned}$$

How many of these $A(a, b)$ give rise to distinct MLG $(\mathbb{R}^2 \rtimes_A \mathbb{R}, \langle , \rangle_A)$?

Classification of 3d non-unimodular MLG

Solve D-invariant $D = (1-a^2)(1+b^2)$ for a :

$a(b,D) = \sqrt{1 - \frac{D}{1+b^2}}$ for each $D \in \mathbb{R}$, $b \in [m(D), \infty)$, where

$$m(D) = \left\{ \begin{array}{cc} \sqrt{D-1} & \text{if } D > 1, \\ 0 & \text{otherwise.} \end{array} \right\} \rightsquigarrow A(b,D) \text{ with } D\text{-invariant } D.$$

$(D,b) = (1,0) \rightsquigarrow A(D,b) = I_2$ discarded.

$(D,b) \neq (1,0) \Rightarrow \forall b_1, b_2 \in [m(D), \infty)$, $\mathbb{R}^2 \rtimes_{A(D,b_1)} \mathbb{R} \cong \mathbb{R}^2 \rtimes_{A(D,b_2)} \mathbb{R}$

But $\langle , \rangle_{A(D,b_1)}$ not isometric to $\langle , \rangle_{A(D,b_2)}$:

$$\left.\begin{array}{rcl} \mathrm{Ric}(E_1) & = & -2\left(1 + \sqrt{x(x-D)}\right) \\ \mathrm{Ric}(E_2) & = & -2\left(1 - \sqrt{x(x-D)}\right) \\ \mathrm{Ric}(E_3) & = & -2(1 + x - D) \end{array}\right\} \text{where } x = x(b) = 1 + b^2$$

and $b \in [m(D), \infty) \mapsto \{\mathrm{Ric}(E_1), \mathrm{Ric}(E_2), \mathrm{Ric}(E_3)\}$ (unordered) is injective.
In summary:

Classification of 3d non-unimodular MLG

Theorem

- *$D = 1$:*
  - $A = I_2 \Rightarrow G \cong \mathbb{H}^3$ *& 1-parameter family of CSC metrics (homothetic).*
  - $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightsquigarrow$ *unique group structure & 2-parameter family of metrics:* $\{ \langle , \rangle_{A(b,D=1)} \ / \ b \in [0 = m(D), \infty)\}$ *plus homotheties.*
- *For each $D \neq 1$, $\exists^1$ group structure & 2-parameter family of metrics:* $\{ \langle , \rangle_{A(b,D)} \ / \ b \in [m(D), \infty)\}$ *plus homotheties.* ▸ Picture

If $D > 1 \Rightarrow m(D) = \sqrt{D-1}$, and for $a = a(D, b = \sqrt{D-1}) = 0 \rightsquigarrow$ $\mathrm{Ric}(E_1) = \mathrm{Ric}(E_2) = \mathrm{Ric}(E_3) = -2 \ \Rightarrow \langle , \rangle_{A(a,\sqrt{D-1})}$ has CSC -1.

Corollary

G 3d non-unimodular Lie group.
G admits left inv $\langle , \rangle$ with CSC $< 0 \ \Leftrightarrow \ G \cong \mathbb{H}^3$ or D-invariant is > 1.

Classification of 3d unimodular MLG

$(G, \langle,\rangle)$ 3d MLG, oriented $\rightsquigarrow \exists$ cross product $\times$ on $\mathfrak{g}$ and
$\exists^1 L\colon \mathfrak{g} \to \mathfrak{g}$ endomorphism s.t. $[X, Y] = L(X \times Y)$, $\quad X, Y \in \mathfrak{g}$.

Lemma

G unimodular $\Leftrightarrow$ L self-adjoint.

$$\mathrm{Trace}(\mathrm{ad}_{E_1}) = \langle [E_1, E_2], E_2\rangle + \langle [E_1, E_3], E_3\rangle = \langle L(E_3), E_2\rangle - \langle L(E_2), E_3\rangle$$

Assume from now G unimodular $\Rightarrow \exists$ positively oriented ortho basis E_1, E_2, E_3 of $\mathfrak{g}$ of L-eigenvectors (unimodular basis):

$$[E_2, E_3] = c_1 E_1, \quad [E_3, E_1] = c_2 E_2, \quad [E_1, E_2] = c_3 E_3,$$

$c_1, c_2, c_3 \in \mathbb{R}$ (structure constants).

- Change orientation $\Rightarrow (c_1, c_2, c_3) \to (-c_1, -c_2, -c_3)$.
- c_1, c_2, c_3 depend upon $\langle,\rangle$. How?
  Change $\langle,\rangle$ s.t. bcE_1, acE_2, abE_3 orthonormal $(a, b, c \neq 0)$
  $\Rightarrow (c_1, c_2, c_3) \to (a^2 c_1, b^2 c_2, c^2 c_3)$. $\Rightarrow$ only the signs of c_1, c_2, c_3 determine the underlying unimodular Lie algebra

Six cases for 3d unimodular Lie groups:

Classification of 3d unimodular MLG

G: 3d, simply-connected, unimodular MLG.

c_1, c_2, c_3	$\dim I(G) = 3$	$\dim I(G) = 4$	$\dim I(G) = 6$
$+, +, +$	$\mathrm{SU}(2)$	$\mathbb{S}^3_{\mathrm{Berger}} = \mathbb{E}(\kappa > 0, \tau)$	$\mathbb{S}^3(k)$
$+, +, -$	$\widetilde{\mathrm{SL}}(2, \mathbb{R})$	$\mathbb{E}(\kappa < 0, \tau)$	$\emptyset$
$+, +, 0$	$\widetilde{E}(2)$	$\emptyset$	Flat
$+, -, 0$	Sol_3	$\emptyset$	$\emptyset$
$+, 0, 0$	$\emptyset$	$\mathrm{Nil}_3 = \mathbb{E}(0, \tau)$	$\emptyset$
$0, 0, 0$	$\emptyset$	$\emptyset$	$\mathbb{R}^3$

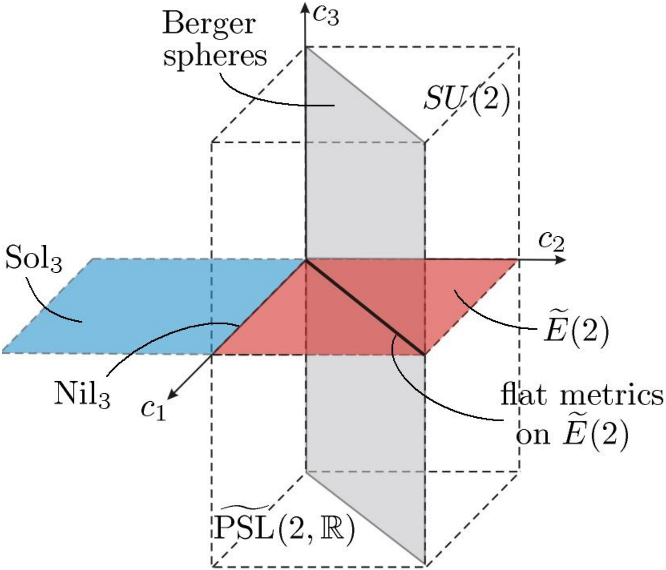
Metric properties (given c_1, c_2, c_3):

$\mu_1 = \frac{1}{2}(-c_1 + c_2 + c_3)$, $\mu_2 = \frac{1}{2}(c_1 - c_2 + c_3)$, $\mu_3 = \frac{1}{2}(c_1 + c_2 - c_3)$

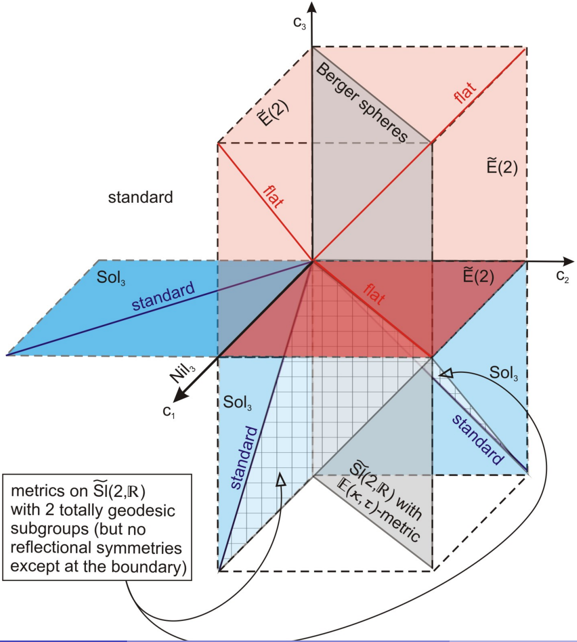
$\nabla_{E_1} E_1 = 0$	$\nabla_{E_1} E_2 = \mu_1 E_3$	$\nabla_{E_1} E_3 = -\mu_1 E_2$
$\nabla_{E_2} E_1 = -\mu_2 E_3$	$\nabla_{E_2} E_2 = 0$	$\nabla_{E_2} E_3 = \mu_2 E_1$
$\nabla_{E_3} E_1 = \mu_3 E_2$	$\nabla_{E_3} E_2 = -\mu_3 E_1$	$\nabla_{E_3} E_3 = 0$

- Ricci tensor diagonalizes in the basis $\{E_1, E_2, E_3\}$ with eigenvalues $\mathrm{Ric}(E_1) = 2\mu_2\mu_3$, $\mathrm{Ric}(E_2) = 2\mu_1\mu_3$, $\mathrm{Ric}(E_3) = 2\mu_1\mu_2$.

Classification of 3d unimodular MLG



Classification of 3d unimodular MLG



$SU(2)$

$$
\begin{aligned}
SU(2) &= \{A \in \mathcal{M}_2(\mathbb{C}) \mid \overline{A^{-1}} = A^t,\ \det A = 1\} \\
&= \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}) \ \mid\ |z|^2 + |w|^2 = 1 \right\} \\
&\overset{\phi}{\cong} \left\{ a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \;\middle|\; \begin{array}{l} a,b,c,d \in \mathbb{R} \\ a^2+b^2+c^2+d^2 = 1 \end{array} \right\}
\end{aligned}
$$

where $\begin{pmatrix} a-di & -b+ci \\ b+ci & a+di \end{pmatrix} \in SU(2) \overset{\phi}{\mapsto} a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$.

- Lie algebra $\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\lambda & a \\ -\overline{a} & -i\lambda \end{pmatrix} \ \mid\ \lambda \in \mathbb{R},\ a \in \mathbb{C} \right\}$
- $SU(2)$ diffeomorphic to $\mathbb{S}^3$ (is the unique simply-connected 3d Lie group not diffeo to $\mathbb{R}^3$).
- $SU(2)$ covers $2:1$ to $SO(3)$.
- The only normal subgroup of $SU(2)$ is its center $\mathbb{Z}_2 = \{\pm I_2\}$.
- 3-parameter family of left invariant metrics: change the lengths of E_1, E_2, E_3 but keep them orthogonal.
- Special metrics: Berger spheres (2 param), round spheres (1-param).

$\widetilde{\mathrm{SL}}(2,\mathbb{R})$

Projective special linear group: $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I_2\}$,
where $\mathrm{SL}(2,\mathbb{R}) = \{A \in \mathcal{M}_2(\mathbb{R}) \mid \det A = 1\}$ special linear group

- Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \{B \in \mathcal{M}_2(\mathbb{R}) \mid \mathrm{trace}(B) = 0\}$.
- $$\begin{aligned}\mathrm{SL}(2,\mathbb{R}) &\overset{\text{diffeo}}{\cong} \mathrm{SO}(2) \times \{P \in \mathcal{S}_2(\mathbb{R}) \mid P \text{ def } +,\ \det P = 1\} \\ &\overset{\text{diffeo}}{\cong} \mathbb{S}^1 \times \{e^B \mid B \in \mathcal{S}_2(\mathbb{R}),\ \mathrm{trace}(B) = 0\} \\ &\overset{\text{diffeo}}{\cong} \mathbb{S}^1 \times \{B \in \mathcal{S}_2(\mathbb{R}) \mid \mathrm{trace}(B) = 0\} \\ &\overset{\text{diffeo}}{\cong} \mathbb{S}^1 \times \mathbb{R}^2 \ \Rightarrow\ \pi_1(SL(2,\mathbb{R})) = \mathbb{Z},\ \ \pi_1(PSL(2,\mathbb{R})) = \mathbb{Z}.\end{aligned}$$
- $\mathrm{PSL}(2,\mathbb{R})$ is a simple group.
- $\mathrm{SL}(2,\mathbb{R})$, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ not simple: $\mathrm{Center}(\mathrm{SL}(2,\mathbb{R})) = \{\pm I_2\}$,

$\mathrm{Center}(\widetilde{\mathrm{SL}}(2,\mathbb{R})) = \ker\left(\widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})\right) \cong \pi_1(SL(2,\mathbb{R})) = \mathbb{Z}$

- $\mathrm{SL}(2,\mathbb{R}) = \{\text{orientation-preserving endom of } \mathbb{R}^2 \text{ that preserve area}\}$
$$\overset{\text{isom}}{\cong} SU^1(2) = \left\{ \begin{pmatrix} z & w \\ \overline{w} & \overline{z} \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}) \ \middle|\ |z|^2 - |w|^2 = 1 \right\}$$
(compare to $SU(2)$)

$\widetilde{\mathrm{SL}}(2,\mathbb{R})$

- $\mathrm{PSL}(2,\mathbb{R}) = \{$orientation-preserving isometries of $\mathbb{H}^2\}$
  $= \left\{ z \in \mathbb{H}^2 \equiv (\mathbb{R}^2)^+ \mapsto \frac{az+b}{cz+d} \in (\mathbb{R}^2)^+ \mid a,b,c,d \in \mathbb{R},\ ad-bc=1 \right\}$
  $= \left\{ w \in \mathbb{H}^2 \equiv \mathbb{D} \mapsto e^{i\theta}\frac{w+a}{\overline{a}w+1} \in \mathbb{D} \mid \theta \in \mathbb{R},\ a \in \mathbb{D} \right\}$
  $\underset{\Psi}{\cong} U\mathbb{H}^2$ (Unit tangent bundle), where $\Psi\colon \mathrm{PSL}(2,\mathbb{R}) \to U\mathbb{H}^2$
  $((p_0,v_0) \in U\mathbb{H}^2$ fixed$)$ $\qquad \phi \mapsto (\phi(p_0), d\phi_{p_0}(v_0))$
- $\mathbb{E}(-1,\tau)$-metrics on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$: $\quad \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R}) \xrightarrow{\mathbb{S}^1\text{-bundle}} \mathbb{H}^2$.
  Lift the canonical metric on $\mathbb{H}^2$ with length $\lambda > 0$ on $\mathbb{S}^1$-fibers.
- Elliptic, parabolic, hyperbolic elements in $\mathrm{SL}(2,\mathbb{R})$:
  $A \in \mathrm{SL}(2,\mathbb{R}) \rightsquigarrow \begin{cases} P_A(\lambda) = \lambda^2 - T\lambda + 1 = 0,\ T = \operatorname{trace}(A) \\ \lambda = \frac{1}{2}\left(T \pm \sqrt{T^2-4}\right) \end{cases}$
  1. Elliptic ($|T| < 2$) $\Rightarrow$ A no real eigenvalues (complex conjugate, on $\mathbb{S}^1$), $A = P^{-1}\mathrm{Rot}_\theta P$ for some $P \in \mathrm{Gl}(2,\mathbb{R})$.
  2. Parabolic ($|T| = 2$) $\Rightarrow$ A unique (double) eigenvalue $\lambda = \frac{T}{2} = \pm 1$.
     - A diagonalizable $\Rightarrow A = \pm I_2$.
     - A not diag $\Rightarrow A = \pm P^{-1}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} P$ (shear map) (squeeze map)
  3. Hyperbolic ($|T| > 2$) $\Rightarrow$ eigenvalues $\lambda \neq 1/\lambda$, $A = P^{-1}\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} P$

$\widetilde{\mathrm{SL}}(2,\mathbb{R})$

- Elliptic, parabolic, hyperbolic 1-parameter subgroups of $\mathrm{PSL}(2,\mathbb{R})$:
  1. Elliptic subgroups: Continuous rotations around any fixed point in $\mathbb{H}^2$.
     - ★ No fixed points in $\partial_\infty \mathbb{H}^2 = \mathbb{S}^1$.
     - ★ Any two are conjugate ($\Gamma_{p_1} = (p_1 p_2^{-1})\Gamma_{p_2}(p_1 p_2^{-1})^{-1}$).
     - ★ $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \rightsquigarrow \varphi_t(z) = \frac{z\cos t - \sin t}{z \sin t + \cos t}$, $z \in (\mathbb{R}^2)^+ \rightsquigarrow E_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
  2. Hyperbolic subgroups: Translations along any geodesic $\Gamma \subset \mathbb{H}^2$.
     - ★ Two fixed points at $\partial_\infty \mathbb{H}^2$ (end points of Γ).
     - ★ Any two hyperbolic subgroups are conjugate.
     - ★ $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \rightsquigarrow \varphi_t(z) = e^{2t}z$, $z \in (\mathbb{R}^2)^+ \rightsquigarrow E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  3. Parabolic subgroups: Rotations about any fixed point $\theta \in \partial_\infty \mathbb{H}^2$.
     - ★ They only fix θ; leave invariant all horocycles based at θ.
     - ★ Any two are conjugate (by an elliptic rotation).
     - ★ $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \rightsquigarrow \varphi_t(z) = z + t$, $z \in (\mathbb{R}^2)^+ \rightsquigarrow B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
     - ★ Every parabolic subgroup is a limit of elliptic subgroups and a limit of hyperbolic subgroups.

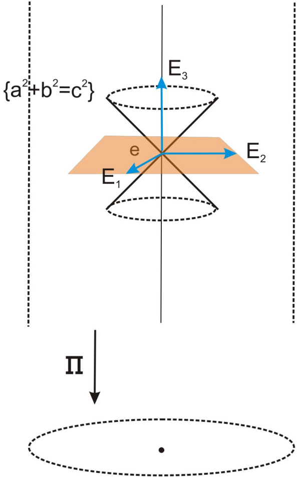
$[E_3, E_1] = 2E_3 + 4B_2 \Rightarrow \{E_1, B_2, E_3\}$ not an unimodular basis!

Exchange B_2 by $E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (coming from $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ hyperb)

$[E_1, E_2] = -2E_3, \quad [E_2, E_3] = 2E_1, \quad [E_3, E_1] = 2E_2 \leftarrow$ unimod basis

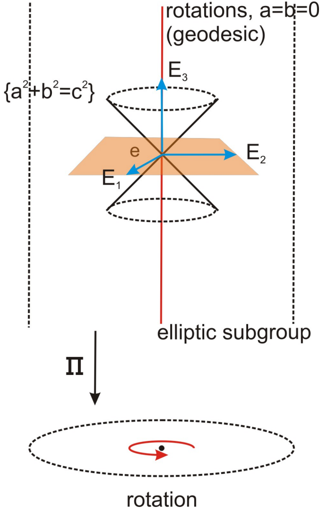
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$

(coordinates in $T_e\widetilde{\mathrm{SL}}(2,\mathbb{R})$ w.r.t. E_1, E_2, E_3)



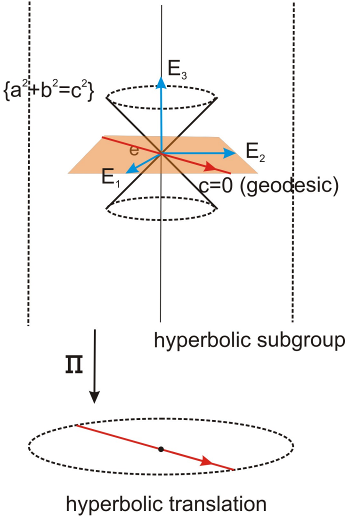
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$

(coordinates in $T_e\widetilde{\mathrm{SL}}(2,\mathbb{R})$ w.r.t. E_1, E_2, E_3)



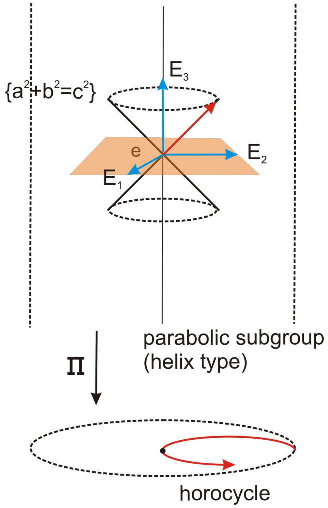
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$

(coordinates in $T_e\widetilde{\mathrm{SL}}(2,\mathbb{R})$ w.r.t. E_1, E_2, E_3)



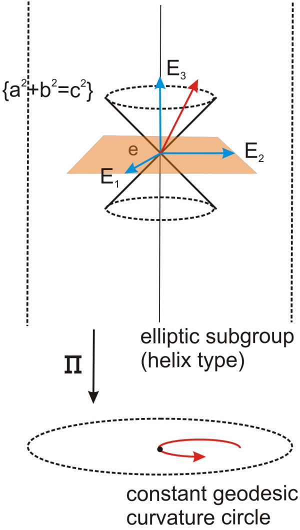
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$

(coordinates in $T_e\widetilde{\mathrm{SL}}(2,\mathbb{R})$ w.r.t. E_1, E_2, E_3)



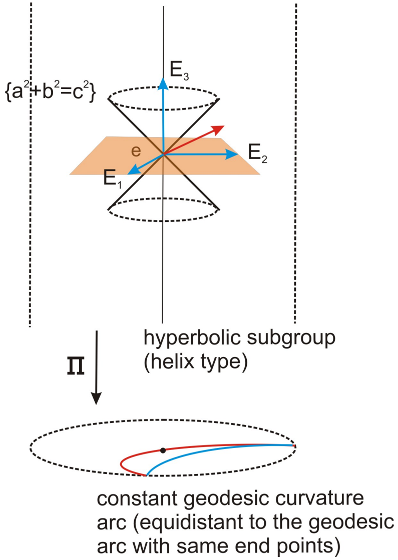
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$

(coordinates in $T_e\widetilde{\mathrm{SL}}(2,\mathbb{R})$ w.r.t. E_1, E_2, E_3)



$\widetilde{\mathrm{SL}}(2,\mathbb{R})$

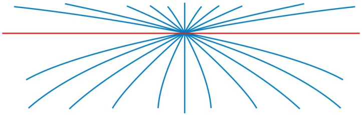
(coordinates in $T_e\widetilde{\mathrm{SL}}(2,\mathbb{R})$ w.r.t. E_1, E_2, E_3)

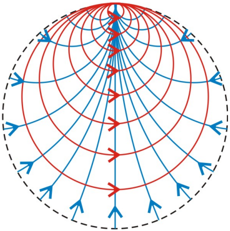


$\widetilde{\mathrm{SL}}(2,\mathbb{R})$

- 2-parameter subgroups of $\mathrm{PSL}(2,\mathbb{R})$:
  1. No subgroups of type $\mathbb{R}^2$.
  2. $\theta \in \partial_\infty \mathbb{H}^2 \;\rightsquigarrow\; \mathbb{H}^2_\theta = \{\phi \in I^+(\mathbb{H}^2) \mid \phi(\theta) = \theta\}$
     - ★ Rotations around θ (parabolic),
     - ★ Translations along geodesics with θ as end point (hyperbolic)

$\mathbb{H}^2_\theta$	parabolic horoc	hyperb transl geod	hyperbolic transl geod
$\mathbb{R} \rtimes_{(1)} \mathbb{R}$	$\mathbb{R} \times_{(1)} \{0\}$	$\{0\} \rtimes_{(1)} \mathbb{R}$	Γ_s





$$\Gamma_s = \{(s(e^t - 1), t) \mid t \in \mathbb{R}\},\ \forall s \in \mathbb{R}$$

▸ B

$\widetilde{E}(2)$

$\widetilde{E}(2) \overset{\text{univ.cover}}{\longrightarrow} E(2) = \{$ direct rigid motions of $\mathbb{R}^2$ $\}$

- $\widetilde{E}(2) \cong \mathbb{R}^2 \rtimes_A \mathbb{R}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- $E_1 = \cos z\partial_x + \sin z\partial_y$, $E_2 = -\sin z\partial_x + \cos z\partial_y$, $E_3 = \partial_z$ basis $\mathfrak{g}$
  $[E_1, E_2] = 0, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2$ (unimodular basis, $c_1 = c_2 = 1, c_3 = 0$)
- Left invariant metrics: Declare E_3 unitary (up to rescaling).
  $\varepsilon_1, \varepsilon_2 > 0 \rightsquigarrow$ declare $\{\varepsilon_1 E_1, \varepsilon_2 E_2, E_3\}$ orthonormal $\Rightarrow$
  structure constants $(c, \frac{1}{c}, 0)$ where $c = \frac{\varepsilon_2}{\varepsilon_1} \rightsquigarrow A(c) = \begin{pmatrix} 0 & -c \\ 1/c & 0 \end{pmatrix}$
  $\forall c \in [1, \infty)$

W.r.t. $A(c)$:

- $E_1^c = \cos z\partial_x + \frac{\sin z}{c}\partial_y$, $E_2^c = -c\sin z\partial_x + \cos z\partial_y$, $E_3^c = \partial_z$
- $\langle , \rangle_{A(c)} = [\cos^2 z + \frac{\sin^2 z}{c^2}]dx^2 + [\cos^2 z + c^2 \sin^2 z]dy^2 + dz^2$
  $+(c - \frac{1}{c})\sin z \cos z(dx \otimes dy + dy \otimes dx) \quad (c = 1 \Rightarrow \text{flat})$
- $F_1^c = \partial_x$, $F_2^c = \partial_y$, $F_3^c = -cy\partial_x + \frac{x}{c}\partial_y + \partial_z$ Killing basis
  $\|F_1^c\|^2_{A(c)} = \cos^2 z + \frac{\sin^2 z}{c^2}$, $\|F_2^c\|^2_{A(c)} = \cos^2 z + c^2 \sin^2 z$ (bounded)
  $\|F_3^c\|^2_{A(c)} = 1 + (x^2 + y^2)\sin^2 z + (\frac{x^2}{c^2} + c^2 y^2)\cos^2 z + xy(\frac{1}{c} - c)\sin(2z)$
  bounded in each solid vertical cylinder

Sol_3

$\mathrm{Sol}_3 = \mathbb{R}^2 \rtimes_A \mathbb{R}$ with $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (as groups)

- $E_1 = e^{-z}\partial_x,\ E_2 = e^z\partial_y,\ E_3 = \partial_z$ basis $\mathfrak{g}$
  $[E_1, E_2] = 0, [E_2, E_3] = -E_2, [E_3, E_1] = -E_1$ (not an unimod basis)
- Left invariant metrics: Declare E_3 unitary (up to rescaling).
  $\varepsilon_1, \varepsilon_2 > 0 \rightsquigarrow$ declare $\{\varepsilon_1(E_1 + E_2), \varepsilon_2(E_1 - E_2), E_3\}$ orthonormal $\Rightarrow$ structure constants $(c, \frac{-1}{c}, 0)$ where $c = \frac{\varepsilon_2}{\varepsilon_1} \rightsquigarrow A(c) = \begin{pmatrix} 0 & c \\ 1/c & 0 \end{pmatrix}$

W.r.t. $A(c)$: $\forall c \in [1, \infty)$

- $E_1^c = \cosh z\partial_x + \frac{\sinh z}{c}\partial_y,\ E_2^c = c\sinh z\partial_x + \cosh z\partial_y,\ E_3^c = \partial_z$
  unimod basis
- $\langle , \rangle_{A(c)} = [\cosh^2 z + \frac{\sinh^2 z}{c^2}]dx^2 + [\cosh^2 z + c^2\sinh^2 z]dy^2 + dz^2$
  $-(c + \frac{1}{c})\sinh z\cosh z(dx \otimes dy + dy \otimes dx)$ $(c = 1 \Rightarrow$ standard$)$
- $F_1^c = \partial_x,\ F_2^c = \partial_y,\ F_3^c = cy\partial_x + \frac{x}{c}\partial_y + \partial_z$ Killing basis
- $\widehat{F}_1^c = \frac{\sqrt{1+c^2}}{2}(-c\partial_x + \partial_y), \widehat{F}_2^c = \frac{\sqrt{1+c^2}}{2}(c\partial_x + \partial_y), \widehat{F}_3^c = F_3^c$ Killing b
  $\|\widehat{F}_1^c\|_{A(c)} = \frac{1+c^2}{2}e^z$ bounded in $\{z < 0\}$,
  $\|\widehat{F}_2^c\|_{A(c)} = \frac{1+c^2}{2e^z}$ bounded in $\{z > 0\}$.

The moduli space of 3d non-unimodular MLG

$(G,\langle,\rangle)$ 3d non-unimod MLG $\Rightarrow (G,\langle,\rangle)\cong(\mathbb{R}^2\rtimes_A\mathbb{R},\langle,\rangle_A)$ for some $A\in\mathcal{M}_2(\mathbb{R})$ with trace$(A)>0$. Rescale $\langle,\rangle$ to have trace$(A)=2$.

1. $A=\alpha I_2\Rightarrow(G,\langle,\rangle)\cong(\mathbb{H}^3,\ \mathrm{CSC}\ -\alpha^2)$.
2. $A\neq\alpha I_2\to A=A(D,b)=\begin{pmatrix}1+a & -(1-a)b\\ (1+a)b & 1-a\end{pmatrix}$, $\begin{cases} a(b,D)=\sqrt{1-\frac{D}{1+b^2}}\\ b\in[m(D),\infty)\end{cases}$
   - $D<1\Rightarrow A(D,b)$ diagonalizable.
     - ★ $A(D,b=0)\overset{(D\to1^-)}{\longrightarrow}I_2\Rightarrow(\mathbb{R}^2\rtimes_{A(D,0)}\mathbb{R},\langle,\rangle_A)\overset{(D\to1^-)}{\longrightarrow}(\mathbb{H}^3,\ \mathrm{CSC}\ -1)$.
     - ★ X: the non-unimod Lie group with D-inv $D=1$, $X\ncong\mathbb{H}^3$.

       Every left inv metric on X is limit of $(\mathbb{R}^2\rtimes_{A(D,b)}\mathbb{R},\langle,\rangle_A)$ as $D\to1^-$ ($b\in(0,\infty)$ fixed).
     - ★ $(\mathbb{R}^2\rtimes_{A(D,b)}\mathbb{R},\langle,\rangle_A)\overset{(D\to-\infty)}{\longrightarrow}$ (Sol_3, any left inv metric) (after blow-up):

$$\varepsilon>0,\ c_1\geq1\rightsquigarrow A_1(c_1,\varepsilon):=\begin{pmatrix}\varepsilon & c_1\\ 1/c_1 & \varepsilon\end{pmatrix}\ \begin{cases}\mathrm{trace}(A_1(c_1,\varepsilon))=2\varepsilon\neq0\\ D\text{-inv: }\frac{4\det(A_1(c_1,\varepsilon))}{\mathrm{trace}(A_1(c_1,\varepsilon))^2}=1-\frac{1}{\varepsilon^2}\overset{(\varepsilon\to0)}{\to}-\infty\end{cases}$$

   - $D>1\Rightarrow A(D,b)$, $b\in[\sqrt{D-1},\infty)$ not diagonalizable.
     - ★ $G(D):=(\mathbb{R}^2\rtimes_{A(D,\sqrt{D-1})}\mathbb{R},\langle,\rangle_A)$ CSC -1 (not isomorphic to $\mathbb{H}^3$).

       $G(D)\overset{(D\to1^+)}{\longrightarrow}(\mathbb{H}_3,\ \mathrm{CSC}\ -1)$

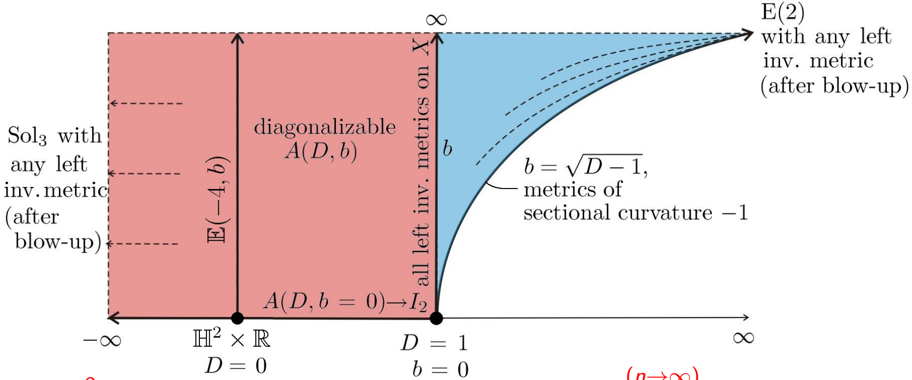
       Every left inv metric on X is limit of $(\mathbb{R}^2\rtimes_{A(D,b)}\mathbb{R},\langle,\rangle_A)$ as $D\to1^+$ ($b\in[\sqrt{D-1},\infty)$ fixed).
     - ★ $(\mathbb{R}^2\rtimes_{A(D,b)}\mathbb{R},\langle,\rangle_A)\overset{(D\to\infty)}{\longrightarrow}(\widetilde{\mathrm{E}}(2)$, any left inv metric) (after blow-up):

The moduli space of 3d non-unimodular MLG

$c_1 \in (0,1] \rightsquigarrow b(D) := \frac{1}{2c_1}\sqrt{(1+c_1^2)^2 D - 4c_1^2}$ (for D large enough in terms of c_1).

$b(D) \geq \sqrt{D-1} \rightsquigarrow a(D) := a(b(D), D) \in \mathbb{R}$, $A(D, b(D)) \in \mathcal{M}_2(\mathbb{R})$.

$$A_1(D, c_1) := \frac{c_1}{(1-a(D))b(D)} A(D, b(D)) = \begin{pmatrix} \frac{1}{c_1 b(D)} & -c_1 \\ \frac{1}{c_1} & \frac{c_1}{b(D)} \end{pmatrix} \overset{(D\to\infty)}{\to} \begin{pmatrix} 0 & -c_1 \\ 1/c_1 & 0 \end{pmatrix}$$



$\forall G \not\cong \mathbb{H}^3$ nonunim $\exists \langle,\rangle_n$ left inv on G s.t. $(G, \langle,\rangle_n) \overset{(n\to\infty)}{\longrightarrow} (\mathrm{Nil}_3, \text{standard})$:

$\left.\begin{matrix} \varepsilon > 0 \\ \delta \in \mathbb{R} \end{matrix}\right\} \rightsquigarrow B(\varepsilon, \delta) = \begin{pmatrix} \varepsilon & 1 \\ \delta & \varepsilon \end{pmatrix}$. Normalized D-inv: $\frac{4\det(B)}{\mathrm{trace}(B)^2} = 1 - \frac{\delta}{\varepsilon^2}$ covers all of $\mathbb{R}$

Take $(\varepsilon, \delta) \to (0,0)$. B

Isometries with fixed points (I)

X simply-connected, 3d MLG. $\text{Stab}_p = \{\phi \in I(X) \mid \phi(p)\}$, $p \in X$.

1. If $\dim I(X) = 3 \;\Rightarrow\; \text{Stab}_e \subset \text{Aut}(X)$.
2. If X unimodular and $v \in T_pX$ principal Ricci direction $\Rightarrow \exists \phi \in \mathrm{Stab}_p^+$ (order two) s.t. $d\phi_p(v) = v$. In particular: $\mathrm{Stab}_p^+ \supseteq \mathbb{Z}_2 \times \mathbb{Z}_2$ dihedral group.
3. If X unimodular and $\dim I(X) = 3 \Rightarrow \mathrm{Stab}_p^+ = \mathbb{Z}_2 \times \mathbb{Z}_2$.
4. If $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ non-unimodular and $\dim I(X) = 3 \;\Rightarrow$ $\mathrm{Stab}_p^+ = \{1_X, l_p \circ \psi \circ l_{p^{-1}}\} \cong \mathbb{Z}_2$, where $\psi(x, y, z) = (-x, -y, z)$.
5. If X unimodular and $\dim I(X) = 4 \Rightarrow \mathrm{Stab}_p^+$ contains an $\mathbb{S}^1$-subgroup of rotations about the principal Ricci direction which is SIMPLE.
6. If $\dim I(X) = 6 \;\Rightarrow\; \mathrm{Stab}_p \cong O(3)$ (orthogonal group).

Isometries with fixed points (II)

X simply-connected, 3d MLG. $\phi \in I(X) - \{\mathrm{Id}\}$ reflectional symmetry if $\exists\ \Sigma \looparrowright X$ s.t. $\Sigma \subset \mathrm{Fix}(\phi)$.

1. If $\exists$ orientation-reversing isometry $\Rightarrow$ X admits a reflectional symmetry.
2. If X admits a orient-revers isometry ϕ and $\dim I(X) = 4 \Rightarrow$ $X \cong \mathbb{H}^2(\kappa) \times \mathbb{R}$ (isomorphic and isometric). In this case, $\phi =$ reflection in $(\text{geodesic}_{\mathbb{H}^2}) \times \mathbb{R}$ or $\phi(p, z) = (\mathrm{Rot}_\theta p, -z + 2z_0)$.
3. If X admits a reflectional symmetry ϕ and $\dim I(X) = 3 \Rightarrow$ $X \cong \mathbb{R}^2 \rtimes_A \mathbb{R}$ (isomorphic and isometric) where $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a \neq b$. Furthermore:
   - If $a \neq -b$ (i.e., X non-unim, $D < 1$) $\Rightarrow \phi$ is left conjugate to $(x, y, z) \mapsto (-x, y, z)$ or $(x, y, z) \mapsto (x, -y, z)$.
   - If $a = -b$ (i.e., $X \cong \mathrm{Sol}_3$ $\Rightarrow \phi$ is left conjugate to $(x, y, z) \mapsto (-x, y, z)$, $(x, y, z) \mapsto (x, -y, z)$, $(x, y, z) \mapsto (y, -x, -z)$ or $(x, y, z) \mapsto (-y, x, -z)$.

Surface theory in 3d MLG

X 3d MLG, $\Sigma \looparrowright X$ oriented surface, $N\colon \Sigma \to TX$ unit normal.
Given $p \in \Sigma$, extend $N_p \in T_pX$ to a (unique) $G(p) \in \mathfrak{g}$, i.e., $G(p)|_p = N_p$
$\Rightarrow G\colon \Sigma \to \mathbb{S}^2 \subset T_eX$ left invariant Gauss map.

- $\Sigma, g\Sigma$ have the same G, $\forall g \in X$.

Sup. Σ connected.

- Σ has $G = \text{const} \Leftrightarrow \Sigma = g\Sigma_1$, Σ_1 2d subgroup of X.
- If Σ 2d-subgroup, $y \in X \Rightarrow \Sigma y$ equidistant surface to Σ. Recipr:

If $Y \overset{\text{comp}}{\subset} \{p \in X \mid d(p,\Sigma) = r > 0\} \Rightarrow Y = \Sigma y = y(y^{-1}\Sigma y), \forall y \in Y$.

Transversality Lemma

$f\colon S \looparrowright X$ sphere s.t. $G\colon S \to \mathbb{S}^2$ diffeo. Given Σ 2d subgroup of X,

1. $\{g\Sigma \text{ left coset} \mid (g\Sigma) \cap f(S) \neq \emptyset\} \cong [0,1]$.
2. $(g(0)\Sigma) \cap f(S)$, $(g(1)\Sigma) \cap f(S)$ consist of single points.
3. $\forall t \in (0,1)$, $(g(t)\Sigma) \pitchfork f(S)$ connected, immersed closed curve.

Proof. $X \ncong SU(2) \Rightarrow X \cong \mathbb{R}^3 \Rightarrow \{g\Sigma \mid g \in X\} \cong \mathbb{R}$.
$\Pi\colon X \to \mathbb{R} \equiv \{g\Sigma \mid g \in X\}$ has exactly 2 critical points. $\square$

Algebraic open book decompositions (AOBD)

X 3d Lie group, $\pi_1(X) = 0$. $\Gamma \subset X$ 1-param subgroup.
An AOBD of X with binding Γ is a foliation $\mathcal{B} = \{L(\theta)\}_{\theta\in[0,2\pi)}$ of $X - \Gamma$ s.t. $\forall\theta \in [0,\pi)$, $H(\theta) := L(\theta) \cup \Gamma \cup L(\pi+\theta)$ two-dim subgroup of X ($L(\theta)$: *leaves* of $\mathcal{B}$).

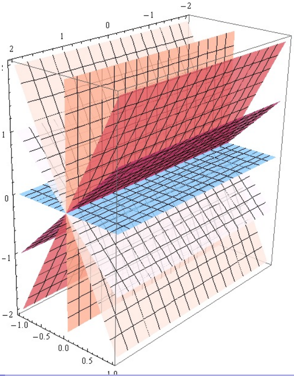
- $X = \mathbb{R}^3$. Planes containing any given line passing through $\vec{0}$.
- $X = \mathbb{H}^3$. $[X,Y] = l(X)Y - l(Y)X$, $\forall X, Y \in \mathfrak{g} \Rightarrow \forall\, \mathfrak{h} \leq \mathfrak{g}$ two-dim linear subspace, $\mathfrak{h}$ is $[,]$-closed $\Rightarrow$ $\{\text{AOBD}\} = \{\text{1-param subgroups}\}$.
- $X = \text{Nil}_3 = \mathbb{R}^2 \rtimes_A \mathbb{R}$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

  Binding: $\Gamma = \{(x,0,0) \mid x \in \mathbb{R}\}$

  Subgroups: $\begin{cases} H(\lambda) = \{(x,y,\lambda y) \mid x,y \in \mathbb{R}\} \\ H(\infty) = \{(x,0,z) \mid x,z \in \mathbb{R}\} \end{cases}$

  (abelian and normal)

  Vertical planes in the $\mathbb{E}(0,\tau)$-model.



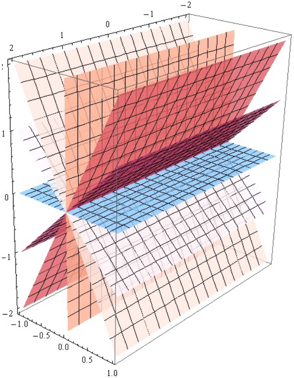
Algebraic open book decompositions

- $X = \mathbb{H}^2 \times \mathbb{R} = \mathbb{R}^2 \rtimes_A \mathbb{R}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

  Binding: $\Gamma_1 = \{(x, 0, 0) \mid x \in \mathbb{R}\}$

  Subgroups: $\begin{cases} H_1(\lambda) = \{(x, \lambda z, z) \mid x, y \in \mathbb{R}\} \\ H_1(\infty) = \mathbb{R}^2 \rtimes_A \{0\} \end{cases}$

  (only abelian for $\lambda = \infty$, all normal)

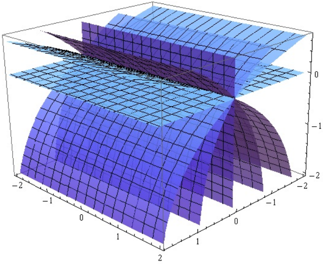


  Binding: $\Gamma_2 = \{(0, y, 0) \mid y \in \mathbb{R}\}$

  Subgroups $\begin{cases} H_2(\lambda) = \{(\lambda(e^z - 1), y, z) \mid \\ \qquad\qquad y, z \in \mathbb{R}\},\ \lambda \in \mathbb{R}, \\ H_2(\infty) = \mathbb{R}^2 \rtimes_A \{0\} \end{cases}$

  (all abelian and normal)

  Vertical planes in the $\mathbb{E}(-1, 0)$-model.



Algebraic open book decompositions

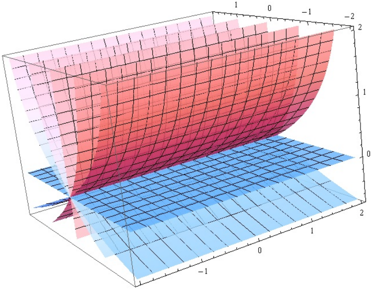
- $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$, $b \neq 0, 1$

i.e. $\begin{cases} \text{Sol}_3 \text{ when } b = -1 \\ \text{any non-unimod MLG with } D < 1 \end{cases}$

Binding: $\Gamma_1 = \{(x, 0, 0) \mid x \in \mathbb{R}\}$

Subgroups $\begin{cases} H_1(\lambda) = \{(x, \lambda(e^{bz} - 1), z) \mid \\ \qquad\qquad x, z \in \mathbb{R}\}, \ \lambda \in \mathbb{R}, \\ H_1(\infty) = \mathbb{R}^2 \rtimes_A \{0\} \end{cases}$

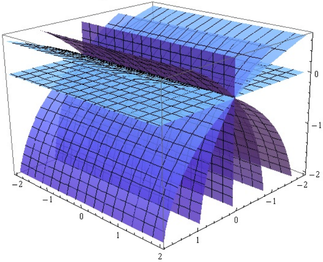
(only abelian for $\lambda = \infty$) B



Binding: $\Gamma_2 = \{(0, y, 0) \mid y \in \mathbb{R}\}$

Subgroups $\begin{cases} H_2(\lambda) = \{(\lambda(e^{z} - 1), y, z) \mid \\ \qquad\qquad y, z \in \mathbb{R}\}, \ \lambda \in \mathbb{R}, \\ H_2(\infty) = \mathbb{R}^2 \rtimes_A \{0\} \end{cases}$

(only abelian for $\lambda = \infty$)



Algebraic open book decompositions

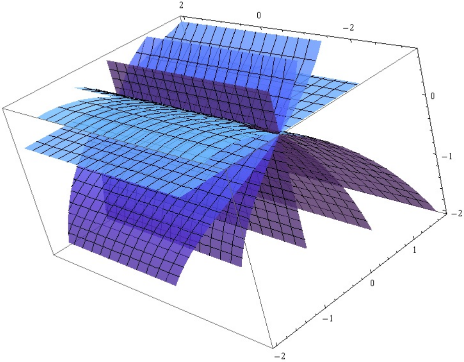
- $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

[1]non-unimod MLG with $D = 1$, $X \not\cong \mathbb{H}^3$

Binding: $\Gamma = \{(0, y, 0) \mid y \in \mathbb{R}\}$

$$\text{Subgroups} \begin{cases} H(\lambda) = \{(\lambda(e^z - 1), y + \lambda\left[e^z(z-1)+1\right], z) \mid \\ \qquad\qquad y, z \in \mathbb{R}\},\ \lambda \in \mathbb{R}, \\ H(\infty) = \mathbb{R}^2 \rtimes_A \{0\} \end{cases}$$

(only abelian for $\lambda = \infty$)



Theorem (classification of 2d subgroups and AOBD)

X 3d MLG, $\pi_1(X) = 0$. Then:

1. *$X = \mathrm{SU}(2)$ has no 2d subgroups.* ▸ def
2. *If $X = \widetilde{\mathrm{SL}}(2,\mathbb{R}) \;\Rightarrow \{$2d subgr$\} = \{\widetilde{\mathbb{H}^2_\theta} \mid \theta \in \partial_\infty \mathbb{H}^2\}$, no AOBD.*
3. *If $X = \widetilde{\mathrm{E}}(2) \;\Rightarrow \{$2d subgr$\} = \{\mathbb{R}^2 \rtimes_A \{0\}\}$, no AOBD.*
4. *If X non-unimod, $D > 1 \;\Rightarrow \{$2d subgr$\} = \{\mathbb{R}^2 \rtimes_A \{0\}\}$, no AOBD.*
5. *Any other case for X: the only AOBD are the examples above, and the only 2d subgroups are the leaves of these AOBD.*

Theorem (spheres with G diffeo are embedded if $\exists$ AOBD)

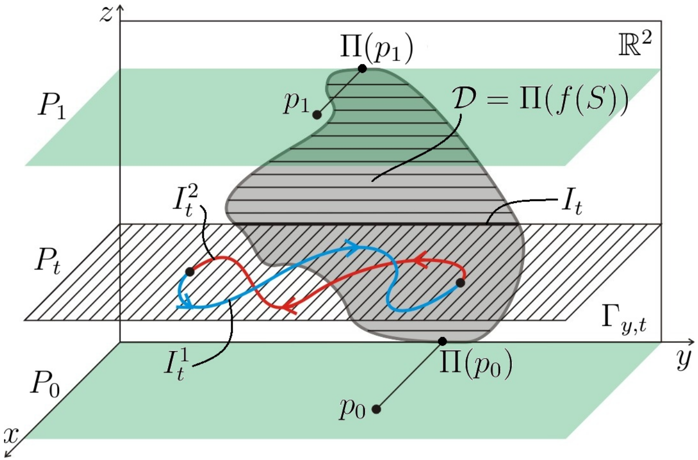
X 3d MLG, $\pi_1(X) = 0$, s.t. $\exists\ \mathcal{B}$ AOBD with binding Γ.
$f \colon S \looparrowright X$ immersed sphere s.t. $G \colon S \to \mathbb{S}^2$ diffeo. Then:
$f(S) = S_1 \cup S_2$, where each S_i is a graph over $\mathcal{D} = \Pi(f(S))$,
$\Pi \colon X \to \mathbb{R}^2 \equiv \{g\Gamma \mid g \in X\}$, and S is embedded.

Proof (for f analytic, which can be achieved by approximation).
$\exists \mathcal{B}$ AOBD $\Rightarrow$ $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, $A \in \mathcal{M}_2(\mathbb{R})$ metrically.
Normalize: $\Gamma = \{(x, 0, 0) \mid y \in \mathbb{R}\}$ $(A \to A^t)$.
$P_0 := \mathbb{R}^2 \rtimes_A \{0\}$ one of the subgroups of $\mathcal{B}$, $P_t := \mathbb{R}^2 \rtimes_A \{t\}$, $t \in \mathbb{R}$.
$f(S) \subset \mathbb{R}^2 \rtimes_A [0, 1]$, $f(S) \cap P_t = \{p_t\}$, $t = 0, 1$ (scale A, left translate f)

$(\star)$ For all $g \in X$, $f^{-1}(g\Gamma)$ contains at most two points.

By contrad, assume $\exists$ $g \in X$ s.t. $\#[g\Gamma \cap f(S)] \geq 3$. Left translate $f(S)$ s.t. $\Gamma = g\Gamma$. Move slightly s.t. $\Gamma \pitchfork f(S) = \{p_1, \ldots, p_{2n}\}$, $n \geq 2$ (analiticity)
$\partial_\theta \in \mathfrak{X}(X - \Gamma)$ normal to product foliation $\mathcal{B} = \{L(\theta) \mid \theta \in [0, 2\pi)\}$.
$\partial_\theta^T \in \mathfrak{X}(S - f^{-1}(\Gamma))$ pullback by f of tangential comp of $(\partial_\theta)|_{f(S)-\Gamma}$.
∂_θ^T no zeros in $S - f^{-1}(\Gamma)$: Fix $H(\theta)$ subgroup of $\mathcal{B}$.
A zero of ∂_θ^T in $f^{-1}(H(\theta))$ produces a tangential intersection $H(\theta) \cap f(S)$.
But $\#[H(\theta) \cap f(S)] \geq 2n > 1 \overset{\text{Transv Lemma}}{\Rightarrow} H(\theta) \cap f(S)$ transversal !!
$f(S)$ transverse to $\Gamma \Rightarrow \text{index}_{p_j}[\partial_\theta^T] = +1$, $\forall j = 1, \ldots, 2n$.
Hopf Index Theorem $\Rightarrow 2 = \chi(S) = \sum_{j=1}^{2n} \text{index}_{p_j}[\partial_\theta^T] = 2n \geq 4$!!

Now use $(\star)$ to express $f(S)$ as union of 2 graphical disks in the direction of the (left translations of) Γ, as in picture.



Each of the immersed curves $f(S) \cap P_t$ consists of two Jordan arcs I_t^1, I_t^2 with the same extrema.

$\bigcup_{t\in[0,1]} I_t^1$, $\bigcup_{t\in[0,1]} I_t^2$: graphical disks whose union is $f(S)$.

$f\colon S \looparrowright X$ is an embedding:
Suppose X admits two different AOBD.
Classif thm of AOBD $\Rightarrow X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, A diagonal, $\mathcal{B}_1, \mathcal{B}_2$ AOBD with bindings $\Gamma_1 = \{(x,0,0) \mid x \in \mathbb{R}\}$, $\Gamma_2 = \{(0,y,0) \mid y \in \mathbb{R}\}$.
$f(S) =$ two graphical disks w.r.t. two directions $\Rightarrow f(S)$ embedded.

Suppose X admits just one AOBD.
Classif thm of AOBD $\Rightarrow X = \mathrm{Nil}_3$ or $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
In both cases, $\forall \langle , \rangle$ left invariant metric on X, $\exists \{(X_n, \langle , \rangle_n)\}_n$ non-unimod 3d MLG with $D_n < 1$, s.t. $(X_n, \langle , \rangle_n) \to (X, \langle , \rangle)$ as $n \to \infty$.
The property "left invariant Gauss map of f is a diffeo" is an open property $\Rightarrow f\colon S \looparrowright (X_n, \langle , \rangle_n)$ has G_n diffeo for n large $\Rightarrow f$ embedding.

QED

CMC surfaces in a 3d MLG X

H-potential of X

Given $H \geq 0$, define $R \colon \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \to \overline{\mathbb{C}}$:

1. X non-unimod $\Rightarrow$ $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, $A = \begin{pmatrix} 1+a & -(1-a)b \\ (1+a)b & 1-a \end{pmatrix}$ $a, b \geq 0$.

$$R(q) = H\left(1+|q|^2\right)^2 - (1-|q|^4) - a\left(q^2 - \overline{q}^2\right) - ib\left[2|q|^2 - a\left(q^2+\overline{q}^2\right)\right]$$

2. X unimod $\Rightarrow$ $[E_i, E_j] = c_k E_k$, $\mu_i = \frac{1}{2}(-c_i + c_j + c_k)$.

$$R(q) = H\left(1+|q|^2\right)^2 - \frac{i}{2}\left[\mu_2|1+q^2|^2 + \mu_1|1-q^2|^2 + 4\mu_3|q|^2\right]$$

- $R \equiv 0$ only if $X = \mathbb{R}^3$ and $H = 0$.
- $R(\infty) = \infty$ except if X unimod and $(H, \mu_1 + \mu_2) = (0,0)$; i.e. for minimal surfaces in $\widetilde{\mathrm{E}}(2)$, Sol_3, Nil_3 or $\mathbb{R}^3$.
- $\frac{R(q)}{|q|^4} \overset{(q\to\infty)}{\longmapsto} \begin{cases} H+1 & \text{if } X \text{ non-unimod} \\ H - \frac{i}{2}(\mu_1+\mu_2) & \text{if } X \text{ unimod}, (H, \mu_1+\mu_2) \neq (0,0) \end{cases}$
- R has no zeros if
  - $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ non-unimod, $\forall\, H > 1$ (normalize trace$(A) = 2$).
  - X unimod, $\forall H > 0$.
  - $X \cong \mathrm{SU}(2)$, $\forall H \geq 0$.

Theorem (PDE for the left inv Gauss map)

$f\colon \Sigma \looparrowright X$ oriented H-surface, $G\colon \Sigma \to \mathbb{S}^2 \subset T_eX$ left inv Gauss map.

- *$g\colon \Sigma \to \overline{\mathbb{C}}$ (stereo projection of G from South pole) satisfies:*

  $$(\star) \quad g_{z\bar{z}} = \frac{R_q}{R}(g) g_z g_{\bar{z}} + \left(\frac{R_{\bar{q}}}{R} - \frac{\overline{R_q}}{\overline{R}} \right)(g)|g_z|^2 \qquad (R\colon H\text{-potential})$$

- *Weierstrass-type representation: $\exists\ E_1, E_2, E_3$ ortho basis of $\mathfrak{g}$ s.t.: Given z conf coord on Σ, then $f_z = \partial_z f = \sum_{i=1}^3 A_i (E_i)_f$ where*

  $$A_1 = \frac{\eta}{4}\left(\bar{g} - \frac{1}{\bar{g}}\right), \quad A_2 = \frac{i\eta}{4}\left(\bar{g} + \frac{1}{\bar{g}}\right), \quad A_3 = \frac{\eta}{2}, \quad \eta = \frac{4\bar{g} g_z}{R(g)}.$$

  Induced metric: $ds^2 = \lambda |dz|^2$, where $\lambda = \frac{4\left(1+|g|^2\right)^2}{|R(g)|^2} |g_z|^2$.

- *Recipr: Σ simply-conn Riemann surface, $g\colon \Sigma \to \overline{\mathbb{C}}$ sol'n of $(\star)$ for the H-potential R in some 3d MLG ($H \geq 0$). If R no zeros in $\overline{\mathbb{C}}$ & g nowhere antiholomorphic $\Rightarrow \exists^1$ (up to left transl) $f\colon \Sigma \looparrowright X$ with CMC H and left invariant Gauss map g.*

Gauss-Codazzi, Frobenius.

Corollary

$f:\Sigma \looparrowright X$ H-surface. $G:\Sigma\to\mathbb{S}^2$ constant $\Leftrightarrow f(\Sigma)=y\Sigma_1$, where Σ_1 2d subgroup of X ($\not\cong SU(2)$). Moreover:

1. *Σ is embedded.*
2. *If X unimod $\Rightarrow H=0$.*
3. *If X non-unimod $\Rightarrow H\in[0,1]$ $\quad$ ($X=\mathbb{R}^2\rtimes_A\mathbb{R}$, $trace(A)=2$)*

Example

$$X=\mathbb{R}^2\rtimes_A\mathbb{R},\qquad \tfrac{b+1}{2}A=\left(\begin{array}{cc}1 & 0\\ 0 & b\end{array}\right).$$

Absolute mean curvature of $H_1(\lambda)$: $|H|=\begin{cases}\frac{|\lambda|}{\sqrt{1+\lambda^2}}\in[0,1) & \text{if } \lambda\in\mathbb{R}\\ 1 & \text{if } \lambda=\infty\end{cases}$

$H_1(\lambda)$

Uniqueness of H-spheres whose Gauss map is a diffeo (DM)

Fix $\Sigma_1 \looparrowright X$ H-sphere s.t. $g_1 \colon \Sigma_1 \equiv \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ diffeo (left inv Gauss map)
$\Sigma \looparrowright X$ H-surface, with left inv Gauss map $g \colon \Sigma \to \overline{\mathbb{C}}$.

Define $\boxed{Q\,(dz)^2 = \left[L(g){g_z}^2 + M(g)g_z\overline{g}_z\right](dz)^2}$ (C^∞-quadr diff on Σ)

where $M, L \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by $\begin{cases} M = 1/R, \\ L(g_1(\xi)) = -\dfrac{M(g_1(\xi))(\overline{g_1})_\xi}{(g_1)_\xi} \end{cases}$

and $R \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the H-potential.

- Cauchy-Riemann inequality: $\frac{|Q_{\overline{z}}|}{|Q|}$ locally bded in Σ. In particular: either $Q\,(dz)^2 \equiv 0$ on Σ (this occurs for translations of Σ_1), or isolated zeros have negative index (Alencar, do Carmo & Tribuzy).
- EVERY $\Sigma \looparrowright X$ H-sphere satisfies $Q\,(dz)^2 = 0$ (Hopf index thm) $\overset{(\star)}{\Rightarrow}$ g local diffeo $\Rightarrow$ g diffeo $\Rightarrow$ $g = g_1 \circ$ (Möbius transf) $\Rightarrow \Sigma, \Sigma_1$ same left invariant Gauss map $\Rightarrow$ $\boxed{\Sigma = \Sigma_1 \text{ up to left translation.}}$

$$(\star)\ \left|\frac{\overline{g}_z}{g_z}\right| = \left|-\frac{L(g)}{M(g)}\right| = \left|\frac{(\overline{g}_1)_\xi}{(g_1)_\xi}\right| < 1$$

Theorem

$\Sigma \looparrowright X$ *index one H-sphere* $\Rightarrow G \colon \Sigma \to \mathbb{S}^2$ *is a diffeomorphism.*

Proof: G local diffeo? By contrad, we can assume $e \in \Sigma$ and $\exists v_1 \in U_e\Sigma$ s.t. $dG_e(v_1) = 0$. Consider $F \in \mathfrak{g}^*$ given by $F_e = v_1$ (Killing)

- Construct locally an F-invariant H-surface $\widehat{\Sigma} \looparrowright X$ $(H = H(\Sigma))$, with a 2nd order contact at e with Σ (reduce the PDE for g to an ODE and use the Weierstrass repr).
- $u = \langle N_\Sigma, F\rangle$ vanishes at e to 2nd order (because $\widehat{u} = \langle N_{\widehat{\Sigma}}, F\rangle \equiv 0$).
- Cheng-Rossman $\Rightarrow u^{-1}(0)$ analytic 1-dimensional set containing at least two transversely intersecting arcs at $e \Rightarrow \Sigma - u^{-1}(0)$ at least three components $\Rightarrow$ index$(\Sigma) \geq 2$ (Courant's nodal domain thm). $\square$

Corollary

- *Index* 1 *H-spheres in X are unique up to left translations.*
- *Given* $S \looparrowright X$ *index 1 H-sphere,* $\exists\, p \in X$ *s.t. S symmetric under every isometry of X that fixes p.*

Geometric constants associated to a MLG.

X 3d homogenous, $\pi_1(X) = 0$. Critical mean curvature:

$$H(X) = \inf\{H \geq 0 \ : \ \exists\, \Sigma \looparrowright X \text{ cpt oriented } H\text{-surface}\}$$

- $H(\mathbb{R}^3) = 0, \quad H(\mathbb{H}^3) = 1$ (foliation by horospheres)
- If $X = \mathbb{S}^2 \times \mathbb{R} \Rightarrow H(X) = 0$ ($\mathbb{S}^2 \times \{0\}$ minimal)
- If X diffeo to $\mathbb{S}^3 \Rightarrow H(X) = 0$ ($\exists\, \Sigma \looparrowright X$ minimal sphere: Simon-Smith)
- If $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, trace$(A) = 2$ (non-unimod) $\Rightarrow H(X) \geq 1$

Cheeger constant: Suppose X not cpt.

$$\mathrm{Ch}(X) = \inf\left\{ \frac{A(\partial\Omega)}{V(\Omega)} \mid \overline{\Omega} \subset X \text{ compact, } \partial\Omega \text{ smooth}\right\}$$

- For cpt Riem mnfds, $\mathrm{Ch}(X) = \inf \frac{A(\partial\Omega)}{\min(V(\Omega), V(X-\Omega))}$ (classical)
- Relation with isoper profile I:
  $\mathrm{Ch}(X) \leq \inf\{$slopes of straight lines intersecting $I\}$

Isoperimetric constant:

$$I(X) = \inf\{H \geq 0 \mid \exists\, \Sigma \subset X \text{ isoper surf with CMC H}\}$$

- $H(X) \leq I(X)$.

Theorem

X 3d MLG, diffeo to $\mathbb{R}^3$. Then: $H(X) \leq I(X) \leq \frac{1}{2}\mathrm{Ch}(X)$.
Furthermore, if $X = \mathbb{R}^2 \rtimes_A \mathbb{R} \;\Rightarrow\; H(X) = I(X) = \frac{1}{2}\mathrm{Ch}(X) = \frac{1}{2}|trace(A)|$

Proof: $2I(X) \leq \mathrm{Ch}(X)$: By contrad, assume that $\mathrm{Ch}(X) < 2I(X)$
$\Rightarrow\; \exists\, \Omega_0 \subset X$ with $\overline{\Omega_0}$ cpt & $\partial\Omega_0$ C^∞, s.t. $\mathrm{A}(\partial\Omega_0) < 2I(X) \cdot V(\Omega_0)$.
Isoper profile of X: $V > 0 \mapsto \mathfrak{I}(V) = \text{mín}\{\mathrm{A}(\partial\Omega) \mid \Omega \subset X, V(\Omega) = V\}$

- $\mathfrak{I}$ locally Lipschitz ($\Rightarrow \exists\, \mathfrak{I}'$ a.e. in $\{V > 0\}$)
- $\exists\, \mathfrak{I}'_-(V), \mathfrak{I}'_+(V)\ \forall V > 0$, and $\mathfrak{I}'_+(t) \leq 2H \leq \mathfrak{I}'_-(t)$ where $H =$ mean curvature of an isoper surface $\partial\Omega$ with $V(\Omega) = V$
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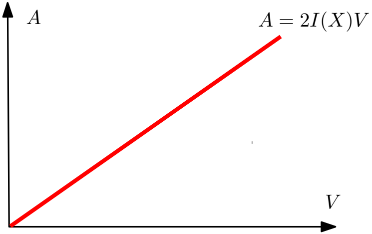

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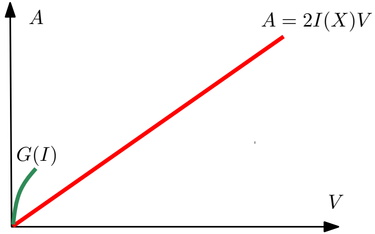

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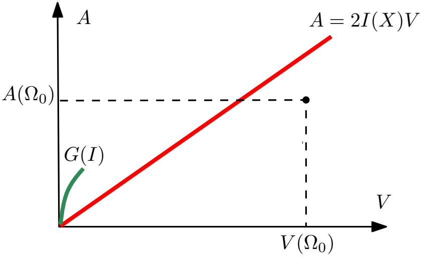

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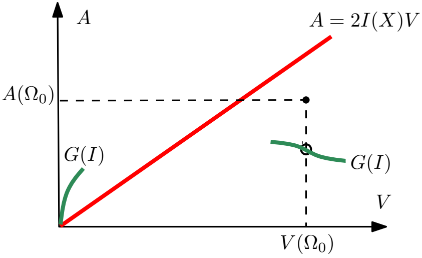

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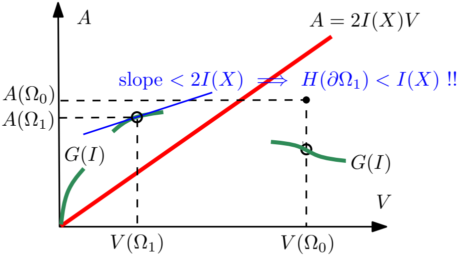

Assume $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, $A \in \mathcal{M}_2(\mathbb{R})$, $\mathrm{trace}(A) \geq 0$.
$\mathrm{trace}(A) \leq \mathrm{Ch}(X)$: Similar as before, exchanging $2I(X)$ by $\mathrm{trace}(A)$ (apply mean comparison principle to $\partial\Omega_1$ some leaf of $\{\mathbb{R}^2 \rtimes_a \{z\} \mid z \in \mathbb{R}\}$.

$\mathrm{Ch}(X) \leq \mathrm{trace}(A)$:
$D(R) = \{(x, y, 0) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid x^2 + y^2 < R\}$.
$C(a, R) = \{(x, y, z) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid (x, y, 0) \in D(R),\ z \in [0, a]\} \quad (a > 1)$

Can we estimate by above $\sup_{a,R} \frac{\mathrm{Volume}(C(a,R))}{\mathrm{Area}(\partial C(a,R))}$?

$\mathrm{div}_X(\partial_z) = -\mathrm{trace}(A) \Rightarrow -\mathrm{trace}(A) \cdot \mathrm{Volume}(C(a, R)) = \int_{\partial C(a,R)} \langle \partial_z, N\rangle$,
N: outward pointing unit normal vector to $\partial C(a, R)$.
$D^{\mathrm{Top}}(a, R) = \partial C(a, R) \cap \{z = a\}$, $S(a, R) = \partial C(a, R) \cap \{0 < z < a\}$.

$$\begin{aligned} \int_{\partial C(a,R)} \langle \partial_z, N\rangle &= \mathrm{Area}(D^{\mathrm{Top}}(a, R)) - \mathrm{Area}(D(R)) \\ &= 2\mathrm{Area}(D^{\mathrm{Top}}(a, R)) + \mathrm{Area}(S(a, R)) - \mathrm{Area}(\partial C(a, R)). \end{aligned}$$

$$\Rightarrow \mathrm{trace}(A) \frac{\mathrm{Volume}(C(a, R))}{\mathrm{Area}(\partial C(a, R))} = 1 - 2\frac{\mathrm{Area}(D^{\mathrm{Top}}(a, R))}{\mathrm{Area}(\partial C(a, R))} - \frac{\mathrm{Area}(S(a, R))}{\mathrm{Area}(\partial C(a, R))}.$$

Upper bound of $\sup_{a,R} \frac{\text{Volume}(C(a,R))}{\text{Area}(\partial C(a,R))}$ (no assumption on trace(A))

Area element on $\mathbb{R}^2 \rtimes_A \{z\}$: $dA_z = e^{-z\,\text{trace}(A)}\, dx \wedge dy$, hence

$$\text{Area}(D^{\text{Top}}(a,R)) = e^{-a\,\text{trace}(A)}\text{Area}(D(R)) \quad \left(= \pi R^2 e^{-a\,\text{trace}(A)}\right)$$ B1

Thus for $R > 0$ fixed,

$$\lim_{a\to\infty} \frac{\text{Area}(D^{\text{Top}}(a,R))}{\text{Area}(\partial C(a,R))} \leq \lim_{a\to\infty} \frac{\text{Area}(D^{\text{Top}}(a,R))}{\text{Area}(D(R))} = \begin{cases} 1 & \text{if trace}(A) = 0, \\ 0 & \text{if trace}(A) > 0. \end{cases}$$

Given $a > 1$, $\exists M(a) > 0$ s.t. $\forall t \in [0,a]$,

$$\text{Length}(S(a,R) \cap \{z = t\}) \leq M(a)\,\text{Length}(\partial D(R)).$$

$$\text{Coarea fla} \Rightarrow \text{Area}(S(a,R)) \leq a\,M(a)\cdot\text{Length}(\partial D(R)) = 2\pi R\,a\,M(a).$$

Then for $a > 1$ fixed,

$$\lim_{R\to\infty} \frac{\text{Area}(S(a,R))}{\text{Area}(\partial C(a,R))} \leq \lim_{R\to\infty} \frac{\text{Area}(S(a,R))}{\text{Area}(D(R))} \leq \lim_{R\to\infty} \frac{2\pi R\,a\,M(a)}{\pi R^2} = 0$$ B2

$$\text{As trace}(A)\frac{\text{Volume}(C(a,R))}{\text{Area}(\partial C(a,R))} = 1 - 2\frac{\text{Area}(D^{\text{Top}}(a,R))}{\text{Area}(\partial C(a,R))} - \frac{\text{Area}(S(a,R))}{\text{Area}(\partial C(a,R))},$$

$$\Rightarrow \text{trace}(A) \sup_{a,R} \frac{\text{Volume}(C(a,R))}{\text{Area}(\partial C(a,R))} = 1 \text{ if } X \text{ non-unim } (\Rightarrow \text{Ch}(X) \leq \text{trace}(A))$$

It remains to prove: $\mathrm{trace}(A) = 0 \ \Rightarrow\ \mathrm{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R}) = 0$.
As $\mathrm{Area}(D^{\mathrm{Top}}(a,R))$ does not depend on a ▸ $A(D^{\mathrm{Top}})$

$$0 = \mathrm{trace}(A)\frac{\mathrm{Volume}(C(a,R))}{\mathrm{Area}(\partial C(a,R))} = 1 - 2\frac{\mathrm{Area}(D^{\mathrm{Top}}(a,R))}{\mathrm{Area}(\partial C(a,R))} - \frac{\mathrm{Area}(S(a,R))}{\mathrm{Area}(\partial C(a,R))}$$

$$\Rightarrow\ \mathrm{Area}(\partial C(a,R)) = 2\mathrm{Area}(D(R)) + \mathrm{Area}(S(a,R))$$

$$\Rightarrow\ \lim_{R\to\infty} \frac{\mathrm{Area}(\partial C(a,R))}{\mathrm{Area}(D(R))} = 2 + \lim_{R\to\infty} \frac{\mathrm{Area}(S(a,R))}{\mathrm{Area}(D(R))} \overset{\text{▸ limit}}{=} 2. \quad (\star)$$

$$\mathrm{Volume}(C(a,R)) \overset{\text{Coarea}}{=} \int_0^a \mathrm{Area}(C(a,R)\cap\{z=t\})\,dt = a\,\mathrm{Area}(D(R))$$

$$\Rightarrow\ \lim_{R\to\infty} \frac{\mathrm{Volume}(C(a,R))}{\mathrm{Area}(\partial C(a,R))} = \lim_{R\to\infty} \frac{a\,\mathrm{Area}(D(R))}{\mathrm{Area}(\partial C(a,R))} \overset{(\star)}{=} \frac{a}{2}$$

$$\inf_{a,R>1} \frac{\mathrm{Area}(\partial C(a,R))}{\mathrm{Volume}(C(a,R))} = 0\ \Rightarrow \mathrm{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R}) = 0.$$

$$H(X) \le I(X) \le \tfrac{1}{2}\mathrm{Ch}(X) = \tfrac{1}{2}\mathrm{trace}(A) = H\left(\mathbb{R}^2 \rtimes_A \{z\}\right) \le H(X). \quad \text{QED}$$