Contemporary Mathematics

December 12, 2011

Constant mean curvature surfaces in metric Lie groups

William H. Meeks III and Joaquín Pérez

This paper is dedicated to the memory of Robert Osserman.

Abstract. In these notes we present some aspects of the basic theory on the geometry of a three-dimensional simply-connected Lie group X endowed with a left invariant metric. This material is based upon and extends some of the results of Milnor in [**Mil76**]. We then apply this theory to study the geometry of constant mean curvature $H \geq 0$ surfaces in X, which we call *H-surfaces*. The focus of these results on H-surfaces concerns our joint on going research project with Pablo Mira and Antonio Ros to understand the existence, uniqueness, embeddedness and stability properties of H-spheres in X. To attack these questions we introduce several new concepts such as the *H-potential* of X, the *critical mean curvature* $H(X)$ of X and the notion of an *algebraic open book decomposition* of X. We apply these concepts to classify the two-dimensional subgroups of X in terms of invariants of its metric Lie algebra, as well as classify the stabilizer subgroup of the isometry group of X at any of its points in terms of these invariants. We also calculate the Cheeger constant for X to be $\mathrm{Ch}(X) = \mathrm{trace}(A)$, when $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is a semidirect product for some 2×2 real matrix; this result is a special case of a more general theorem by Peyerimhoff and Samiou [**PS04**]. We also prove that in this semidirect product case, $\mathrm{Ch}(X) = 2H(X) = 2I(X)$, where $I(X)$ is the infimum of the mean curvatures of isoperimetric surfaces in X. In the last section, we discuss a variety of unsolved problems for H-surfaces in X.

Contents

1991 *Mathematics Subject Classification.* Primary 53A10; Secondary 49Q05, 53C42.

Key words and phrases. Minimal surface, constant mean curvature, H-surface, algebraic open book decomposition, stability, index of stability, nullity of stability, curvature estimates, CMC foliation, Hopf uniqueness, Alexandrov uniqueness, metric Lie group, critical mean curvature, H-potential, homogeneous three-manifold, left invariant metric, left invariant Gauss map, isoperimetric domain, Cheeger constant.

The first author was supported in part by NSF Grant DMS - 1004003. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

The second author was supported in part by MEC/FEDER grants no. MTM2007-61775 and MTM2011-22547, and Regional J. Andalucía grant no. P06-FQM-01642.

1. Introduction.

This manuscript covers some of the material given in three lectures by the first author at the RSME School Luis Santaló on Geometric Analysis which took place in the summer of 2010 at the University of Granada. The material covered in these lectures concerns an active branch of research in the area of surface geometry in simply-connected, three-dimensional homogeneous spaces, especially when the surface is two-sided and has constant mean curvature $H \in \mathbb{R}$. After appropriately orienting such a surface with constant mean curvature H, we will assume $H \geq 0$ and will refer to the surface as an *H-surface*.

We next briefly explain the contents of the three lectures in the course. The first lecture introduced the notation, definitions and examples, as well as the basic tools. Using the Weierstrass representation for minimal surfaces ($H = 0$) in Euclidean three-space $\mathbb{R}^3$, we explained how to obtain results about existence of complete, proper minimal immersions in domains of $\mathbb{R}^3$ with certain restrictions (this is known as the *Calabi-Yau problem*). We also explained how embeddedness influences dramatically the Calabi-Yau problem, with results such as the Minimal Lamination Closure Theorem. Other important tools covered in the first lecture were the curvature estimates of Meeks and Tinaglia for embedded H-disks away from their boundary when $H > 0$, the Dynamics Theorem due to Meeks, Pérez and Ros [**MIPRb, MIPR08**] and a different version of this last result due to Meeks and Tinaglia [**MIT10**], and the notion of a CMC foliation, which is a foliation of a Riemannian three-manifold by surfaces of constant mean curvature, where the mean curvature can vary from leaf to leaf.

The second lecture introduced complete, simply-connected, homogeneous three-manifolds and the closely related subject of three-dimensional Lie groups equipped with a left invariant metric; in short, *metric Lie groups*. We presented the basic examples and focused on the case of a metric Lie group that can be expressed as a semidirect product. These metric semidirect products comprise all non-unimodular ones, and in the unimodular family they consist of (besides the trivial case of $\mathbb{R}^3$) the Heisenberg group Nil_3, the universal cover $\widetilde{\mathrm{E}}(2)$ of the group of orientation preserving isometries of $\mathbb{R}^2$ and the solvable group Sol_3, each group endowed with an arbitrary left invariant metric. We then explained how to classify all simply-connected, three-dimensional metric Lie groups, their two-dimensional subgroups and their isometry groups in terms of algebraic invariants associated to their metric Lie algebras.

The third lecture was devoted to understanding H-surfaces, and especially H-spheres, in a three-dimensional metric Lie group X. Two questions of interest here are how to approach the outstanding problem of uniqueness up to ambient isometry for such an H-sphere and the question of when these spheres are embedded. With this aim, we develop the notions of the *H-potential* of X and of an *algebraic open book decomposition* of X, and described a recent result of Meeks, Mira, Pérez and Ros where they prove embeddedness of immersed spheres (with non-necessarily constant mean curvature) in such an X, provided that X admits an algebraic open book decomposition and that the left invariant Lie algebra Gauss map of the sphere is a diffeomorphism. This embeddedness result is closely related to the aforementioned problem of uniqueness up to ambient isometry for an H-sphere in X. We also explained a result which computes the Cheeger constant of any metric semidirect product in terms of invariants of its metric Lie algebra. The third lecture finished with a brief presentation of the main open problems and conjectures in this field of H-surfaces in three-dimensional homogeneous Riemannian manifolds.

These notes will cover in detail the contents of the second and third lectures. This material depends primarily on the classical work of Milnor [**Mil76**] on the classification of simply-connected, three-dimensional metric Lie groups X and on recent results concerning H-spheres in X by Daniel and Mira [**DM08**], Meeks [**MI**], and Meeks, Mira, Pérez and Ros [**MIMPRa, MIMPRb**].

2. Lie groups and homogeneous three-manifolds.

We first study the theory and examples of geometries of homogeneous n-manifolds.

DEFINITION 2.1.

(1) A Riemannian n-manifold X is *homogeneous* if the group $I(X)$ of isometries of X acts transitively on X.
(2) A Riemannian n-manifold X is *locally homogenous* if for each pair of points $p, q \in X$, there exists an $\varepsilon = \varepsilon(p, q) > 0$ such that the metric balls $B(p, \varepsilon)$, $B(q, \varepsilon) \subset X$ are isometric.

Clearly every homogeneous n-manifold is complete and locally homogeneous, but the converse of this statement fails to hold. For example, the hyperbolic plane $\mathbb{H}^2$ with a metric of constant curvature -1 is homogeneous but there exists a constant curvature -1 metric on any compact surface of genus $g > 1$ such that the related Riemannian surface M_g is locally isometric to $\mathbb{H}^2$. This M_g is a complete locally homogeneous surface but since the isometry group of M_g must be finite, then M is not homogeneous. In general, for $n \leq 4$, a locally homogeneous n-manifold X is locally isometric to a simply-connected homogeneous n-manifold $\widehat{X}$ (see Patrangenaru [**Pat96**]). However, this property fails to hold for $n \geq 5$ (see Kowalski [**Kow90**]). Still we have the following general result when X is complete and locally homogeneous, whose proof is standard.

THEOREM 2.2. *If X is a complete locally homogeneous n-manifold, then the universal cover $\widetilde{X}$ of X, endowed with the pulled back metric, is homogeneous. In particular, such an X is always locally isometric to a simply-connected homogeneous n-manifold.*

Many examples of homogeneous Riemannian n-manifolds arise as Lie groups equipped with a metric which is invariant under left translations.

DEFINITION 2.3.

(1) A *Lie group* G is a smooth manifold with an algebraic group structure, whose operation $*$ satisfies that $(x, y) \mapsto x^{-1} * y$ is a smooth map of the product manifold $G \times G$ into G. We will frequently use the multiplicative notation xy to denote $x * y$, when the group operation is understood.
(2) Two Lie groups, G_1, G_2 are *isomorphic* if there is a smooth group isomorphism between them.
(3) The respective *left* and *right multiplications* by $a \in G$ are defined by:

$$\begin{aligned} l_a \colon G &\to G, & \qquad r_a \colon G &\to G \\ x &\mapsto ax, & x &\mapsto xa. \end{aligned}$$

(4) A Riemannian metric on G is called *left invariant* if l_a is an isometry for every $a \in G$. The Lie group G together with a left invariant metric is called a *metric Lie group*.

In a certain generic sense [**LT93**], for each dimension $n \neq 2$, simply-connected Lie groups with left invariant metrics are "generic" in the space of simply-connected homogeneous n-manifolds. For example, in dimension one, $\mathbb{R}$ with its usual additive group structure and its usual metric is the unique example. In dimension two we have the product Lie group $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with its usual metric as well as a unique non-commutative Lie group all of whose left invariant metrics have constant negative curvature; we will denote this Lie group by $\mathbb{H}^2$ (in Example 2.8 below it is shown that the usual hyperbolic n-space $\mathbb{H}^n$ is isometric to the Lie group of similarities[1] of $\mathbb{R}^{n-1}$ endowed with some left invariant metric; motivated by the fact that this last Lie group only admits left invariant metrics of constant negative curvature, we let $\mathbb{H}^n$ denote this group of similarities of $\mathbb{R}^{n-1}$). On the other hand, the two-spheres $\mathbb{S}^2(k)$ with metrics of constant positive curvature k are examples of complete, simply-connected homogeneous surfaces which cannot be endowed with a Lie group structure (the two-dimensional sphere is not parallelizable). Regarding the case of dimension three, we shall see in Theorem 2.4 below that simply-connected, three-dimensional metric Lie groups are "generic" in the space of all simply-connected homogeneous three-manifolds. For the sake of completeness, we include a sketch of the proof of this result.

Regarding the following statement of Theorem 2.4, we remark that a simply-connected, homogeneous Riemannian three-manifold can be isometric to more than one Lie group equipped with a left invariant metric. In other words, non-isomorphic Lie groups might admit left invariant metrics which make them isometric as Riemannian manifolds. This non-uniqueness property can only occur in the following three cases:

- The Riemannian manifold is isometric to $\mathbb{R}^3$ with its usual metric: the universal cover $\widetilde{\mathrm{E}}(2)$ of the group of orientation-preserving rigid motions of the Euclidean plane, equipped with its standard metric, is isometric to the flat $\mathbb{R}^3$, see item (1-b) of Theorem 2.14.
- The Riemannian manifold is isometric to $\mathbb{H}^3$ with a metric of constant negative curvature: every non-unimodular three-dimensional Lie group with D-invariant $D > 1$ admits such a left invariant metric, see Lemma 2.13 and item (1-a) of Theorem 2.14.
- The Riemannian manifold is isometric to certain simply-connected homogeneous Riemannian three-manifolds $\mathbb{E}(\kappa, \tau)$ with isometry group of dimension four (with parameters $\kappa < 0$ and $\tau \neq 0$; these spaces will be explained in Section 2.6): the (unique) non-unimodular three-dimensional Lie group with D-invariant equal to zero admits left invariant metrics which are isometric to these $\mathbb{E}(\kappa, \tau)$-spaces, see item (2-a) of Theorem 2.14. These spaces $\mathbb{E}(k, \tau)$ are also isometric to the universal cover $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ of the special linear group $\mathrm{SL}(2, \mathbb{R})$ equipped with left invariant metrics, where two structure constants for its unimodular metric Lie algebra are equal, see Figure 3.

The proof of the next theorem can be modified to demonstrate that if X_1 and X_2 are two connected, isometric, n-dimensional metric Lie groups whose (common)

[1]By a *similarity* we mean the composition of a homothety and a translation of $\mathbb{R}^{n-1}$.

isometry group $I(X)$ is n-dimensional, and we denote by $I_0(X)$ the component of the identity in $I(X)$, then X_1 and X_2 are isomorphic to $I_0(X)$, and hence isomorphic to each other.

THEOREM 2.4. *Except for the product manifolds $\mathbb{S}^2(k) \times \mathbb{R}$, where $\mathbb{S}^2(k)$ is a sphere of constant curvature $k > 0$, every simply-connected, homogeneous Riemannian three-manifold is isometric to a metric Lie group.*

SKETCH OF THE PROOF. We first check that the homogeneous three-manifold $Y = \mathbb{S}^2(k) \times \mathbb{R}$ is not isometric to a three-dimensional metric Lie group. Arguing by contradiction, suppose Y has the structure of a three-dimensional Lie group with a left invariant metric. Since Y is a Riemannian product of a constant curvature two-sphere centered at the origin in $\mathbb{R}^3$ with the real line $\mathbb{R}$, then $\mathrm{SO}(3) \times \mathbb{R}$ is the identity component of the isometry group of Y, where $\mathrm{SO}(3)$ acts by rotation on the first factor and $\mathbb{R}$ acts by translation on the second factor. Let $F\colon \mathbb{S}^2(k) \times \mathbb{R} \to \mathrm{SO}(3) \times \mathbb{R}$ be the injective Lie group homomorphism defined by $F(y) = l_y$ and let $\Pi\colon \mathrm{SO}(3) \times \mathbb{R} \to \mathrm{SO}(3)$ be the Lie group homomorphism given by projection on the first factor. Thus, $(\Pi \circ F)(Y)$ is a Lie subgroup of $\mathrm{SO}(3)$. Since the kernel of Π is isomorphic to $\mathbb{R}$ and $F(\ker(\Pi \circ F))$ is contained in $\ker(\Pi)$, then $F(\ker(\Pi \circ F))$ is either the identity element of $\mathrm{SO}(3) \times \mathbb{R}$ or an infinite cyclic group. As F is injective, then we have that $\ker(\Pi \circ F)$ is either the identity element of Y or an infinite cyclic subgroup of Y. In both cases, the image $(\Pi \circ F)(Y)$ is a three-dimensional Lie subgroup of $\mathrm{SO}(3)$, hence $(\Pi \circ F)(Y) = \mathrm{SO}(3)$. Since Y is not compact and $\mathrm{SO}(3)$ is compact, then $\ker(\Pi \circ F)$ cannot be the identity element of Y. Therefore $\ker(\Pi \circ F)$ is an infinite cyclic subgroup of Y and $\Pi \circ F\colon Y \to \mathrm{SO}(3)$ is the universal cover of $\mathrm{SO}(3)$. Elementary covering space theory implies that the fundamental group of $\mathrm{SO}(3)$ is infinite cyclic but instead, $\mathrm{SO}(3)$ has finite fundamental group $\mathbb{Z}_2$. This contradiction proves that $\mathbb{S}^2(k) \times \mathbb{R}$ is not isometric to a three-dimensional metric Lie group.

Let X denote a simply-connected, homogeneous Riemannian three-manifold with isometry group $I(X)$. Since the stabilizer subgroup of a point of X under the action of $I(X)$ is isomorphic to a subgroup of the orthogonal group $O(3)$, and the dimensions of the connected Lie subgroups of $O(3)$ are zero, one or three, then it follows that the Lie group $I(X)$ has dimension three, four or six. If $I(X)$ has dimension six, then the metric on X has constant sectional curvature and, after homothetic scaling is $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$ with their standard metrics, all of which admit some Lie group structure that makes this standard metric a left invariant metric. If $I(X)$ has dimension four, then X is isometric to one of the Riemannian bundles $\mathbb{E}(\kappa, \tau)$ over a complete, simply-connected surface of constant curvature $\kappa \in \mathbb{R}$ and bundle curvature $\tau \in \mathbb{R}$, see e.g., Abresch and Rosenberg [**AR05**] or Daniel [**Dan07**] for a discussion about these spaces. Each of these spaces has the structure of some metric Lie group except for the case of $\mathbb{E}(\kappa, 0)$, $\kappa > 0$, which is isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$.

Now assume $I(X)$ has dimension three and denote its identity component by $I_0(X)$. Choose a base point $p_0 \in X$ and consider the map $\phi\colon I_0(X) \to X$ given by $\phi(h) = h(p_0)$. We claim that ϕ is a diffeomorphism. To see this, consider the stabilizer S of p_0 in $I_0(X)$, which is a discrete subgroup of $I_0(X)$. The quotient $I_0(X)/S$ is a three-dimensional manifold which is covered by $I_0(X)$ and the map ϕ factorizes through $I_0(X)/S$ producing a covering space $I_0(X)/S \to X$. Since X is simply-connected, then both of the covering spaces $I_0(X) \to I_0(X)/S$ and

$I_0(X)/S \to X$ are trivial and in particular, S is the trivial group. Hence, ϕ is a diffeomorphism and X can be endowed with a Lie group structure. Clearly, the original metric on X is nothing but the left invariant extension of the scalar product at the tangent space $T_{p_0}X$ and the point p_0 plays the role of the identity element. □

DEFINITION 2.5.

(1) Given elements a, p in a Lie group G and a tangent vector $v_p \in T_pG$, av_p (resp. v_pa) denotes the vector $(l_a)_*(v_p) \in T_{ap}G$ (resp. $v_pa = (r_a)_*(v_p) \in T_{pa}G$) where $(l_a)_*$ (resp. $(r_a)_*$) denotes the differential of l_a (resp. of r_a).
(2) A vector field X on G is called *left invariant* if $X = aX$, for every $a \in G$, or equivalently, for each $a, p \in G$, $X_{ap} = aX_p$. X is called *right invariant* if $X = Xa$ for every $a \in G$.
(3) $L(G)$ denotes the vector space of left invariant vector fields on G, which can be naturally identified with the tangent space T_eG at the identity element $e \in G$.
(4) $\mathfrak{g} = (L(G), [,])$ is a Lie algebra under the Lie bracket of vector fields, i.e., for $X, Y \in L(G)$, then $[X, Y] \in L(G)$. $\mathfrak{g}$ is called the *Lie algebra* of G. If G is simply-connected, then $\mathfrak{g}$ determines G up to isomorphism, see e.g., Warner [**War83**].

For each $X \in \mathfrak{g}$, the image set of the integral curve γ^X of X passing through the identity is a 1-parameter subgroup of G, i.e., there is a group homomorphism

$$\exp_{X_e} \colon \mathbb{R} \to \gamma^X(\mathbb{R}) \subset G,$$

which is determined by the property that the velocity vector of the curve $\alpha(t) = \exp_{X_e}(t)$ at $t = 0$ is X_e. Note that the image subgroup $\exp_{X_e}(\mathbb{R})$ is isomorphic to $\mathbb{R}$ when $\exp_{X_e}$ is injective or otherwise it is isomorphic to $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. When G is a subgroup of the general linear group[2] $Gl(n, \mathbb{R})$, then $\mathfrak{g}$ can be identified with some linear subspace of $\mathcal{M}_n(\mathbb{R}) = \{n \times n \text{ matrices with real entries}\}$ which is closed under the operation $[A, B] = AB - BA$ (i.e., the commutator of matrices is the Lie bracket), and in this case given $A = X_e \in T_eG$, one has

$$\exp_{X_e}(t) = \exp(tA) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \in \gamma^X(\mathbb{R}).$$

This explains the notation for the group homomorphism $\exp_{X_e} \colon \mathbb{R} \to G$. In general, we let $\exp \colon T_eG = \mathfrak{g} \to G$ be the related map $\exp(X) = \gamma^X(1)$.

Given an $X \in \mathfrak{g}$ with related 1-parameter subgroup $\gamma^X \subset G$, then X is the velocity vector field associated to the 1-parameter group of diffeomorphisms given by right translations by the elements $\gamma^X(t)$, i.e., the derivative at $t = 0$ of $p\,\gamma^X(t)$ is X_p, for each $p \in G$. Analogously, the derivative at $t = 0$ of $\gamma^X(t)\,p$ is the value at every $p \in G$ of the right invariant vector field Y on G determined by $Y_e = X_e$.

Recall that a Riemannian metric on G is called *left invariant* if for all $a \in G$, $l_a \colon G \to G$ is an isometry of G. In this case, $(G, \langle , \rangle)$ is called a *metric Lie group.* Each such left invariant metric on G is obtained by taking a inner product $\langle , \rangle_e$ on T_eG and defining for $a \in G$ and $v, w \in T_aG$, $\langle v, w\rangle_a = \langle a^{-1}v, a^{-1}w\rangle_e$.

[2]Every finite dimensional Lie algebra is isomorphic to a subalgebra of the Lie algebra $\mathcal{M}_n(\mathbb{R})$ of $Gl(n, \mathbb{R})$ for some n, where $Gl(n, \mathbb{R})$ denotes the group of $n \times n$ real invertible matrices (this is Ado's theorem, see e.g., Jacobson [**Jac62**]). In particular, every simply-connected Lie group G is a covering group of a Lie subgroup of $Gl(n, \mathbb{R})$.

The velocity field of the 1-parameter group of diffeomorphisms $\{l_{[\gamma^X(t)]} \mid t \in \mathbb{R}\}$ obtained by the left action of $\gamma^X(\mathbb{R})$ on G defines a right invariant vector field K^X, where $K^X_e = X_e$. Furthermore, the vector field K^X is a Killing vector field for **any** left invariant metric on G, since the diffeomorphisms $l_{[\gamma^X(t)]}$ are in this case isometries for all $t \in \mathbb{R}$.

Every left invariant metric on a Lie group is complete. Also recall that the fundamental group $\pi_1(G)$ of a connected Lie group G is always abelian. Furthermore, the universal cover $\widetilde{G}$ with the pulled back metric is a metric Lie group and the natural covering map $\Pi\colon \widetilde{G} \to G$ is a group homomorphism whose kernel can be naturally identified with the fundamental group $\pi_1(G)$. In this way, $\pi_1(G)$ can be considered to be an abelian normal subgroup of $\widetilde{G}$ and $G = \widetilde{G}/\pi_1(G)$ (compare this last result with Theorem 2.2).

We now consider examples of the simplest metric Lie groups.

EXAMPLE 2.6. **The Euclidean n-space.** The set of real numbers $\mathbb{R}$ with its usual metric and group operation $+$ is a metric Lie group. In this case, both $\mathfrak{g}$ and the vector space of right invariant vector fields are just the set of constant vector fields $v_p = (p,t)$, $p \in \mathbb{R}$, where we consider the tangent bundle of $\mathbb{R}$ to be $\mathbb{R} \times \mathbb{R}$. Note that by taking $v = (0,1) \in T_0\mathbb{R}$, $\exp_v = 1_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$ is a group isomorphism. In fact, $(\mathbb{R}, +)$ is the unique simply-connected one-dimensional Lie group.

More generally, $\mathbb{R}^n$ with its flat metric is a metric Lie group with trivial Lie algebra (i.e., $[,] = 0$). In this case, the same description of $\mathfrak{g}$ and $\exp = 1_{\mathbb{R}^n}$ holds as in the case $n = 1$.

EXAMPLE 2.7. **Two-dimensional Lie groups.** A homogeneous Riemannian surface is clearly of constant curvature. Hence a simply-connected, two-dimensional metric Lie group G must be isometric either to $\mathbb{R}^2$ or to the hyperbolic plane $\mathbb{H}^2(k)$ with a metric of constant negative curvature $-k$. This metric classification is also algebraic: since simply-connected Lie groups are determined up to isomorphism by their Lie algebras, this two-dimensional case divides into two possibilities: either the Lie bracket is identically zero (and the only example is $(\mathbb{R}^2, +)$) or $[,]$ is of the form

$$[X,Y] = l(X)Y - l(Y)X, \qquad X, Y \in \mathfrak{g}, \tag{2.1}$$

for some well-defined non-zero linear map $l\colon \mathfrak{g} \to \mathbb{R}$. In this last case, it is not difficult to check that the Gauss curvature of every left invariant metric on G is $-\|l\|^2 < 0$; here $\|l\|$ is the norm of the linear operator l with respect to the chosen metric. Note that although l does not depend on the left invariant metric, $\|l\|^2$ does. In fact, this property is independent of the dimension: if the Lie algebra $\mathfrak{g}$ of an n-dimensional Lie group G satisfies (2.1), then **every** left invariant metric on G has constant sectional curvature $-\|l\|^2 < 0$ (see pages 312-313 of Milnor [**Mil76**] for details).

EXAMPLE 2.8. **Hyperbolic n-space.** For $n \geq 2$, the hyperbolic n-space $\mathbb{H}^n$ is naturally a non-commutative metric Lie group: it can be seen as the group of similarities of $\mathbb{R}^{n-1}$, by means of the isomorphism

$$(\mathbf{a}, a_n) \in \mathbb{H}^n \mapsto \phi_{(\mathbf{a},a_n)}\colon \quad \begin{aligned} &\mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \\ &\mathbf{x} \mapsto a_n\mathbf{x} + \mathbf{a} \end{aligned}$$

where we have used the upper halfspace model $\{(\mathbf{a}, a_n) \mid \mathbf{a} \in \mathbb{R}^{n-1}, a_n > 0\}$ for $\mathbb{H}^n$. Since equation (2.1) can be shown to hold for the Lie algebra of $\mathbb{H}^n$, it follows that

every left invariant metric on $\mathbb{H}^n$ has constant negative curvature. We will revisit this example as a metric semidirect product later.

EXAMPLE 2.9. The **special orthogonal group.**

$$\mathrm{SO}(3) = \{A \in Gl(3,\mathbb{R}) \mid A \cdot A^T = I_3,\ \det A = 1\},$$

where I_3 is the 3×3 identity matrix, is the group of rotations about axes passing through the origin in $\mathbb{R}^3$, with the natural multiplicative structure. SO(3) is diffeomorphic to the real projective three-space and its universal covering group corresponds to the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4 = \{a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \mid a,b,c,d \in \mathbb{R}\}$, considered to be the set of unit length quaternions. Since left multiplication by a unit length quaternion is an isometry of $\mathbb{R}^4$ with its standard metric, the restricted metric on $\mathbb{S}^3$ with constant sectional curvature 1 is a left invariant metric. As SO(3) is the quotient of $\mathbb{S}^3$ under the action of the normal subgroup $\{\pm \mathrm{Id}_4\}$, then this metric descends to a left invariant metric on SO(3).

Let $T_1(\mathbb{S}^2) = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|x\| = \|y\| = 1,\ x \perp y\}$ be the unit tangent bundle of $\mathbb{S}^2$, which can be viewed as a Riemannian submanifold of $T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$. Given $\lambda > 0$, the metric $g_\lambda = \sum_{i=1}^3 dx_i^2 + \lambda \sum_{i=1}^3 dy_i^2$, where $x = (x_1,x_2,x_3)$, $y = (y_1,y_2,y_3)$ defines a Riemannian submersion from $(T_1(\mathbb{S}^2), g_\lambda)$ into $\mathbb{S}^2$ with its usual metric. Consider the diffeomorphism $F\colon \mathrm{SO}(3) \to T_1(\mathbb{S}^2)$ given by

$$F(c_1,c_2,c_3) = (c_1,c_2),$$

where $c_1, c_2, c_3 = c_1 \times c_2$ are the columns of the corresponding matrix in SO(3). Then g_λ lifts to a Riemannian metric on $\mathrm{SO}(3) \equiv \mathbb{S}^3/\{\pm \mathrm{Id}_4\}$ via F and then it also lifts to a Riemannian metric $\widetilde{g}_\lambda$ on $\mathbb{S}^3$. Therefore $(\mathbb{S}^3, \widetilde{g}_\lambda)$ admits a Riemannian submersion into the round $\mathbb{S}^2$ that makes $(\mathbb{S}^3, \widetilde{g}_\lambda)$ isometric to one of the Berger spheres (i.e., to one of the spaces $\mathbb{E}(\kappa,\tau)$ with $\kappa = 1$ to be explained in Section 2.6). $\widetilde{g}_\lambda$ produces the round metric on $\mathbb{S}^3$ precisely when the length with respect to g_λ of the $\mathbb{S}^1$-fiber above each $x \in \mathbb{S}^2$ is 2π, or equivalently, with $\lambda = 1$.

The 1-parameter subgroups of SO(3) are the circle subgroups given by all rotations around some fixed axis passing through the origin in $\mathbb{R}^3$.

2.1. Three-dimensional metric semidirect products. Generalizing direct products, a semidirect product is a particular way of cooking up a group from two subgroups, one of which is a normal subgroup. In our case, the normal subgroup H will be two-dimensional, hence H is isomorphic to $\mathbb{R}^2$ or $\mathbb{H}^2$, and the other factor V will be isomorphic to $\mathbb{R}$. As a set, a semidirect product is nothing but the cartesian product of H and V, but the operation is different. The way of gluing different copies of H is by means of a 1-parameter subgroup $\varphi\colon \mathbb{R} \to \mathrm{Aut}(H)$ of the automorphism group of H, which we will denote by

$$\varphi(z) = \varphi_z\colon \quad \begin{array}{ccc} H & \to & H \\ \mathbf{p} & \mapsto & \varphi_z(\mathbf{p}), \end{array}$$

for each $z \in \mathbb{R}$. The group operation $*$ of the semidirect product $H \rtimes_\varphi V$ is given by

$$(\mathbf{p}_1, z_1) * (\mathbf{p}_2, z_2) = (\mathbf{p}_1 \star \varphi_{z_1}(\mathbf{p}_2), z_1 + z_2), \tag{2.2}$$

where $\star, +$ denote the operations in H and V, respectively.

In the sequel we will focus on the commutative case for H, i.e., $H \equiv \mathbb{R}^2$ (see Corollary 3.7 for a justification). Then φ is given by exponentiating some matrix

$A \in \mathcal{M}_2(\mathbb{R})$, i.e., $\varphi_z(\mathbf{p}) = e^{zA}\mathbf{p}$, and we will denote the corresponding group by $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Let us emphasize some particular cases depending on the choice of A:

- $A = 0 \in \mathcal{M}_2(\mathbb{R})$ produces the usual direct product of groups, which in our case is $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ (analogously, if $H \equiv \mathbb{H}^2$ and we take the group morphism $\varphi\colon \mathbb{R} \to \mathrm{Aut}(H)$ to be identically $\varphi(z) = 1_H$, then one gets $\mathbb{H}^2 \times \mathbb{R}$).
- Taking $A = I_2$ where I_2 is the 2×2 identity matrix, then $e^{zA} = e^z I_2$ and one recovers the group $\mathbb{H}^3$ of similarities of $\mathbb{R}^2$. In one dimension less, this construction leads to $\mathbb{H}^2$ by simply considering A to be the identity 1×1 matrix (1), and the non-commutative operation $*$ on $\mathbb{H} = \mathbb{H}^2 = \mathbb{R} \rtimes_{(1)} \mathbb{R}$ is
$$(x, y) * (x', y') = (x + e^y x', y + y').$$
- The map
$$(x, y) \in \mathbb{R} \rtimes_{(1)} \mathbb{R} \overset{\Phi}{\mapsto} (x, e^y) \in (\mathbb{R}^2)^+ \tag{2.3}$$
gives an isomorphism between $\mathbb{R} \rtimes_{(1)} \mathbb{R}$ and the upper halfspace model for $\mathbb{H}^2$ with the group structure given in Example 2.8. This isomorphism is useful for identifying the orbits of 1-parameter subgroups of $\mathbb{R} \rtimes_{(1)} \mathbb{R}$. For instance, the orbits of points under left or right multiplication by the 1-parameter normal subgroup $\mathbb{R} \rtimes_{(1)} \{0\}$ are the horizontal straight lines $\{(x, y_0) \mid x \in \mathbb{R}\}$ for any $y_0 \in \mathbb{R}$, which correspond under Φ to parallel horocycles in $(\mathbb{R}^2)^+$ (horizontal straight lines). The orbits of points under right (resp. left) multiplication by the 1-parameter (not normal) subgroup $\{0\} \rtimes_{(1)} \mathbb{R}$ are the vertical straight lines $\{(x_0, y) \mid y \in \mathbb{R}\}$ (resp. the exponential graphs $\{(x_0 e^y, y) \mid y \in \mathbb{R}\}$) for any $x_0 \in \mathbb{R}$, which correspond under Φ to vertical geodesics in $(\mathbb{R}^2)^+$ (resp. into half lines starting at the origin $\vec{0} \in \mathbb{R}^2$).

  Another simple consequence of this semidirect product model of $\mathbb{H}^2$ is that $\mathbb{H}^2 \times \mathbb{R}$ can be seen as $(\mathbb{R} \rtimes_{(1)} \mathbb{R}) \times \mathbb{R}$. It turns out that the product group $\mathbb{H}^2 \times \mathbb{R}$ can also be constructed as the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ where $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The relation between these two models of $\mathbb{H}^2 \times \mathbb{R}$ is just a permutation of the second and third components, i.e., the map
$$(x, y, z) \in (\mathbb{R} \rtimes_{(1)} \mathbb{R}) \times \mathbb{R} \mapsto (x, z, y) \in \mathbb{R}^2 \rtimes_A \mathbb{R}$$
is a Lie group isomorphism.
- If $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $e^{zA} = \begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}$ and $\mathbb{R}^2 \rtimes_A \mathbb{R} = \widetilde{\mathrm{E}}(2)$, the universal cover of the group of orientation-preserving rigid motions of the Euclidean plane.
- If $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, then $e^{zA} = \begin{pmatrix} e^{-z} & 0 \\ 0 & e^z \end{pmatrix}$ and $\mathbb{R}^2 \rtimes_A \mathbb{R} = \mathrm{Sol}_3$ (a solvable group), also known as the group $E(1,1)$ of orientation-preserving rigid motions of the Lorentz-Minkowski plane.
- If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $e^{zA} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ and $\mathbb{R}^2 \rtimes_A \mathbb{R} = \mathrm{Nil}_3$, which is the Heisenberg group of nilpotent matrices of the form $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$.

2.2. Left and right invariant vector fields and left invariant metrics on a semidirect product. So far we have mainly focused on the Lie group structures rather than on the left invariant metrics that each group structure carries. Obviously, a left invariant metric is determined by declaring a choice of a basis of the Lie algebra as an orthonormal set, although different left invariant basis can give rise to isometric left invariant metrics.

Our next goal is to determine a canonical basis of the left invariant (resp. right invariant) vector fields on a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for any matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.4}$$

We first choose coordinates $(x, y) \in \mathbb{R}^2$, $z \in \mathbb{R}$. Then $\partial_x = \frac{\partial}{\partial x}$, ∂_y, ∂_z is a parallelization of $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$. Taking derivatives at $t = 0$ in the expression (2.2) of the left multiplication by $(\mathbf{p_1}, z_1) = (t, 0, 0) \in G$ (resp. by $(0, t, 0), (0, 0, t)$), we obtain the following basis $\{F_1, F_2, F_3\}$ of the right invariant vector fields on G:

$$F_1 = \partial_x, \quad F_2 = \partial_y, \quad F_3(x, y, z) = (ax + by)\partial_x + (cx + dy)\partial_y + \partial_z. \tag{2.5}$$

Analogously, if we take derivatives at $t = 0$ in the right multiplication by $(\mathbf{p_2}, z_2) = (t, 0, 0) \in G$ (resp. by $(0, t, 0), (0, 0, t)$), we obtain the following basis $\{E_1, E_2, E_3\}$ of the Lie algebra $\mathfrak{g}$ of G:

(2.6)

$$E_1(x, y, z) = a_{11}(z)\partial_x + a_{21}(z)\partial_y, \quad E_2(x, y, z) = a_{12}(z)\partial_x + a_{22}(z)\partial_y, \quad E_3 = \partial_z,$$

where we have denoted

$$e^{zA} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}. \tag{2.7}$$

Regarding the Lie bracket, $[E_1, E_2] = 0$ since $\mathbb{R}^2$ is abelian. Thus, $\mathrm{Span}\{E_1, E_2\}$ is a commutative two-dimensional Lie subalgebra of $\mathfrak{g}$. Furthermore, $E_1(y) = dy(E_1) = y(E_1) = a_{21}(z)$ and similarly,

$$\begin{array}{lll} E_1(x) = a_{11}(z), & E_1(y) = a_{21}(z), & E_1(z) = 0, \\ E_2(x) = a_{12}(z), & E_2(y) = a_{22}(z), & E_2(z) = 0, \\ E_3(x) = 0, & E_3(y) = 0, & E_3(z) = 1, \end{array}$$

from where we directly get

$$[E_3, E_1] = a_{11}'(z)\partial_x + a_{21}'(z)\partial_y. \tag{2.8}$$

Now, equation (2.6) implies that $\partial_x = a^{11}(z)E_1 + a^{21}(z)E_2$, $\partial_y = a^{12}(z)E_1 + a^{22}(z)E_2$, where $a^{ij}(z) = a_{ij}(-z)$ are the entries of e^{-zA}. Plugging these expressions in (2.8), we obtain

$$[E_3, E_1] = aE_1 + cE_2, \tag{2.9}$$

and analogously

$$[E_3, E_2] = bE_1 + dE_2. \tag{2.10}$$

Note that equations (2.9) and (2.10) imply that the linear map $\mathrm{ad}_{E_3} \colon \mathrm{Span}\{E_1, E_2\} \to \mathrm{Span}\{E_1, E_2\}$ given by $\mathrm{ad}_{E_3}(Y) = [E_3, Y]$ has matrix A with respect to the basis $\{E_1, E_2\}$.

$\mathrm{Span}\{E_1, E_2\}$ is an integrable two-dimensional distribution whose leaf passing through the identity element is the normal subgroup $\mathbb{R}^2 \rtimes_A \{0\} = \ker(\Pi)$, where Π is the group morphism $\Pi(x, y, z) = z$. Clearly, the integral surfaces of this distribution define the foliation $\mathcal{F} = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Since the Lie bracket restricted to the Lie algebra of $\ker(\Pi)$ vanishes, then every left invariant metric on $\mathbb{R}^2 \times_A \mathbb{R}$ restricts to $\ker(\Pi)$ as a flat metric. This implies that each of the leaves of $\mathcal{F}$ is intrinsically flat, regardless of the left invariant metric that we consider on $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Nevertheless, we will see below than the leaves of $\mathcal{F}$ may be extrinsically curved (they are not totally geodesic in general).

2.3. Canonical left invariant metric on a semidirect product. Given a matrix $A \in \mathcal{M}_2(\mathbb{R})$, we define the *canonical left invariant metric* on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ to be that one for which the left invariant basis $\{E_1, E_2, E_3\}$ given by (2.6) is orthonormal.

Equations (2.9) and (2.10) together with the classical Koszul formula give the Levi-Civita connection ∇ for the canonical left invariant metric of $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$:

(2.11)
$$\begin{array}{l|l|l}
\nabla_{E_1} E_1 = a\, E_3 & \nabla_{E_1} E_2 = \frac{b+c}{2}\, E_3 & \nabla_{E_1} E_3 = -a\, E_1 - \frac{b+c}{2}\, E_2 \\
\nabla_{E_2} E_1 = \frac{b+c}{2}\, E_3 & \nabla_{E_2} E_2 = d\, E_3 & \nabla_{E_2} E_3 = -\frac{b+c}{2}\, E_1 - d\, E_2 \\
\nabla_{E_3} E_1 = \frac{c-b}{2}\, E_2 & \nabla_{E_3} E_2 = \frac{b-c}{2}\, E_1 & \nabla_{E_3} E_3 = 0.
\end{array}$$

In particular, $z \mapsto (x_0, y_0, z)$ is a geodesic in G for every $(x_0, y_0) \in \mathbb{R}^2$. We next emphasize some other metric properties of the canonical left invariant metric $\langle , \rangle$ on G:

- The mean curvature of each leaf of the foliation $\mathcal{F} = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ with respect to the unit normal vector field E_3 is the constant $H = \text{trace}(A)/2$. In particular, if we scale A by $\lambda > 0$ to obtain λA, then H changes into λH (the same effect as if we were to scale the ambient metric by $1/\lambda$).
- The change from the orthonormal basis $\{E_1, E_2, E_3\}$ to the basis $\{\partial_x, \partial_y, \partial_z\}$ given by (2.6) produces the following expression for the metric $\langle , \rangle$:

(2.12)
$$\begin{aligned}
\langle , \rangle \;&=\; \left[a_{11}(-z)^2 + a_{21}(-z)^2\right] dx^2 + \left[a_{12}(-z)^2 + a_{22}(-z)^2\right] dy^2 + dz^2 \\
&+\; \left[a_{11}(-z)a_{12}(-z) + a_{21}(-z)a_{22}(-z)\right] (dx \otimes dy + dy \otimes dx) \\
&=\; e^{-2\text{trace}(A)z} \left\{ \left[a_{21}(z)^2 + a_{22}(z)^2\right] dx^2 + \left[a_{11}(z)^2 + a_{12}(z)^2\right] dy^2 \right\} + dz^2 \\
&-\; e^{-2\text{trace}(A)z} \left[a_{11}(z)a_{21}(z) + a_{12}(z)a_{22}(z)\right] (dx \otimes dy + dy \otimes dx)\,.
\end{aligned}$$

In particular, given $(x_0, y_0) \in \mathbb{R}^2$, the map $(x, y, z) \overset{\phi}{\mapsto} (-x + 2x_0, -y + 2y_0, z)$ is an isometry of $(\mathbb{R}^2 \rtimes_A \mathbb{R}, \langle , \rangle)$ into itself. Note that ϕ is the rotation by angle π around the line $l = \{(x_0, y_0, z) \mid z \in \mathbb{R}\}$, and the fixed point set of ϕ is the geodesic l.

Remark 2.10. As we just observed, the vertical lines in the (x, y, z)-coordinates of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ are geodesics of its canonical metric, which are the axes or fixed point sets of the isometries corresponding to rotations by angle π around them. For any line L in $\mathbb{R}^2 \rtimes_A \{0\}$ let P_L denote the vertical plane $\{(x, y, z) \mid (x, y, 0) \in L,\ z \in \mathbb{R}\}$ containing the set of vertical lines passing though L. It follows that the plane P_L is ruled by vertical geodesics and furthermore, since rotation by angle π around any vertical line in P_L is an isometry that leaves P_L invariant, then P_L has zero mean curvature. Thus, every metric Lie group which can be expressed as a semidirect product of the form $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric has many minimal foliations by parallel vertical planes, where by parallel we mean that the related lines in $\mathbb{R}^2 \rtimes_A \{0\}$ for these planes are parallel in the intrinsic metric.

A natural question to ask is: Under which conditions are $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and $\mathbb{R}^2 \rtimes_B \mathbb{R}$ isomorphic (and if so, when are their canonical metrics isometric), in terms of the defining matrices $A, B \in \mathcal{M}_2(\mathbb{R})$? Regarding these questions, we make the following comments.

(1) Assume A, B are similar, i.e., there exists $P \in Gl(2,\mathbb{R})$ such that $B = P^{-1}AP$. Then, $e^{zB} = P^{-1}e^{zA}P$ from where it follows easily that the map $\psi\colon \mathbb{R}^2 \rtimes_A \mathbb{R} \to \mathbb{R}^2 \rtimes_B \mathbb{R}$ given by $\psi(\mathbf{p}, t) = (P^{-1}\mathbf{p}, t)$ is a Lie group isomorphism.

(2) Now consider the canonical left invariant metrics on $\mathbb{R}^2 \rtimes_A \mathbb{R}$, $\mathbb{R}^2 \rtimes_B \mathbb{R}$. If we assume that A, B are congruent (i.e., $B = P^{-1}AP$ for some orthogonal matrix $P \in O(2)$), then the map ψ defined in (1) above is an isometry between the canonical metrics on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and $\mathbb{R}^2 \rtimes_B \mathbb{R}$.

(3) What is the effect of scaling the matrix A on $\mathbb{R}^2 \rtimes_A \mathbb{R}$? If $\lambda > 0$, then obviously

$$e^{z(\lambda A)} = e^{(\lambda z)A}. \tag{2.13}$$

Hence the mapping $\psi_\lambda(x, y, z) = (x, y, z/\lambda)$ is a Lie group isomorphism from $\mathbb{R}^2 \rtimes_A \mathbb{R}$ into $\mathbb{R}^2 \rtimes_{\lambda A} \mathbb{R}$. Equation (2.13) also gives that the entries $a^\lambda_{i,j}(z)$ of the matrix $e^{z(\lambda A)}$ in equation (2.7) satisfy

$$a^\lambda_{i,j}(z) = a_{ij}(\lambda z),$$

which implies that the left invariant vector fields $E^\lambda_1, E^\lambda_2, E^\lambda_3$ given by applying (2.6) to the matrix λA satisfy

$$E^\lambda_i(x, y, z) = E_i(x, y, \lambda z), \quad i = 1, 2, 3.$$

The last equality leads to $(\psi_\lambda)_*(E_i) = E^\lambda_i$ for $i = 1, 2$ while $(\psi_\lambda)_*(E_3) = \frac{1}{\lambda}E^\lambda_3$. That is, ψ_λ is not an isometry between the canonical metrics $\langle,\rangle_A$ on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and $\langle,\rangle_{\lambda A}$ on $\mathbb{R}^2 \rtimes_{\lambda A} \mathbb{R}$, although it preserves the metric restricted to the distribution spanned by E_1, E_2. Nevertheless, the diffeomorphism $(p, z) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \stackrel{\phi_\lambda}{\to} \left(\frac{1}{\lambda}p, \frac{1}{\lambda}z\right) \in \mathbb{R}^2 \rtimes_{\lambda A} \mathbb{R}$ can be proven to satisfy $\phi^*_\lambda(\langle,\rangle_{\lambda A}) = \frac{1}{\lambda^2}\langle,\rangle_A$ (ϕ_λ is not a group homomorphism). Thus, $\langle,\rangle_A$ and $\langle,\rangle_{\lambda A}$ are homothetic metrics. We will prove in Sections 2.5 and 2.6 that *$\mathbb{R}^2 \rtimes_A \mathbb{R}$ and $\mathbb{R}^2 \rtimes_{\lambda A} \mathbb{R}$ are isomorphic groups.*

2.4. Unimodular and non-unimodular Lie groups. A Lie group G is called *unimodular* if its left invariant Haar measure is also right invariant. This notion based on measure theory can be simply expressed in terms of the adjoint representation as follows.

Each element $g \in G$ defines an *inner automorphism* $a_g \in \mathrm{Aut}(G)$ by the formula $a_g(h) = ghg^{-1}$. Since the group homomorphism $g \in G \mapsto a_g \in \mathrm{Aut}(G)$ satisfies $a_g(e) = e$ (here e denotes the identity element in G), then its differential $d(a_g)_e$ at e is an automorphism of the Lie algebra $\mathfrak{g}$ of G. This defines the so-called *adjoint representation,*

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g}), \quad \mathrm{Ad}(g) = \mathrm{Ad}_g := d(a_g)_e.$$

Since $a_{gh} = a_g \circ a_h$, the chain rule insures that $\mathrm{Ad}(gh) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h)$, i.e., Ad is a homomorphism between Lie groups. Therefore, its differential is a linear mapping $\mathrm{ad} := d(\mathrm{Ad})$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \mathrm{ad}\ } & \mathrm{End}(\mathfrak{g}) \\ \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\ \mathrm{Ad}\ } & \mathrm{Aut}(\mathfrak{g}) \end{array}$$

It is well-known that for any $X \in \mathfrak{g}$, the endomorphism $\mathrm{ad}_X = \mathrm{ad}(X) \colon \mathfrak{g} \to \mathfrak{g}$ is given by $\mathrm{ad}_X(Y) = [X, Y]$ (see e.g., Proposition 3.47 in [**War83**]).

It can be proved (see e.g., Lemma 6.1 in [**Mil76**]) that G is unimodular if and only if $\det(\mathrm{Ad}_g) = 1$ for all $g \in G$. After taking derivatives, this is equivalent to:

$$\text{For all } X \in \mathfrak{g}, \quad \mathrm{trace}(\mathrm{ad}_X) = 0. \tag{2.14}$$

The kernel $\mathfrak{u}$ of the linear mapping $X \in \mathfrak{g} \overset{\varphi}{\mapsto} \mathrm{trace}(\mathrm{ad}_X) \in \mathbb{R}$ is called the *unimodular kernel* of G. If we take the trace in the *Jacobi identity*

$$\mathrm{ad}_{[X,Y]} = \mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_Y \circ \mathrm{ad}_X \quad \text{for all } X, Y \in \mathfrak{g},$$

then we deduce that

$$[X, Y] \in \ker(\varphi) = \mathfrak{u} \quad \text{ for all } X, Y \in \mathfrak{g}. \tag{2.15}$$

In particular, φ is a homomorphism of Lie algebras from $\mathfrak{g}$ into the commutative Lie algebra $\mathbb{R}$, and $\mathfrak{u}$ is an ideal of $\mathfrak{g}$. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called *unimodular* if $\mathrm{trace}(\mathrm{ad}_X) = 0$ for all $X \in \mathfrak{h}$. Hence $\mathfrak{u}$ is itself a unimodular Lie algebra.

2.5. Classification of three-dimensional non-unimodular metric Lie groups. Assume that G is a three-dimensional, non-unimodular Lie group and let $\langle , \rangle$ be a left invariant metric on G. Since the unimodular kernel $\mathfrak{u}$ is a two-dimensional subalgebra of $\mathfrak{g}$, we can find an orthonormal basis E_1, E_2, E_3 of $\mathfrak{g}$ such that $\mathfrak{u} = \mathrm{Span}\{E_1, E_2\}$ and there exists a related subgroup H of G whose Lie algebra is $\mathfrak{u}$. Using (2.15) we have that $[E_1, E_3], [E_2, E_3]$ are orthogonal to E_3, hence $0 = \mathrm{trace}(\mathrm{ad}_{E_1}) = \langle [E_1, E_2], E_2 \rangle$ and $0 = \mathrm{trace}(\mathrm{ad}_{E_2}) = \langle [E_2, E_1], E_1 \rangle$, from where $[E_1, E_2] = 0$, i.e., H is isomorphic to $\mathbb{R}^2$. Furthermore, there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} [E_3, E_1] &= \alpha E_1 + \gamma E_2, \\ [E_3, E_2] &= \beta E_1 + \delta E_2, \end{aligned} \tag{2.16}$$

with $\mathrm{trace}(\mathrm{ad}_{E_3}) = \alpha + \delta \neq 0$ since $E_3 \notin \mathfrak{u}$.

Note that the matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{2.17}$$

determines the Lie bracket on $\mathfrak{g}$ and thus, it also determines completely the Lie group G, in the sense that two simply-connected, non-unimodular metric Lie groups with the same matrix A as in (2.17) are isomorphic. In fact:

If we keep the group structure and change the left invariant metric $\langle , \rangle$ by a homothety of ratio $\lambda > 0$, then the related matrix A in (2.17) associated to $\lambda\langle , \rangle$ changes into $(1/\sqrt{\lambda})A$.

Furthermore, comparing (2.16) with (2.9), (2.10) we deduce:

LEMMA 2.11. *Every simply-connected, three-dimensional, non-unimodular metric Lie group is isomorphic and isometric to a semidirect product* $\mathbb{R}^2 \rtimes_A \mathbb{R}$ *with its canonical metric, where the normal subgroup* $\mathbb{R}^2 \rtimes_A \{0\}$ *of* $\mathbb{R}^2 \rtimes_A \mathbb{R}$ *is the abelian two-dimensional subgroup* $\exp(\mathfrak{u})$ *associated to the unimodular kernel* $\mathfrak{u}$*,* $\{0\} \rtimes_A \mathbb{R} = \exp(\mathfrak{u}^\perp)$ *and* A *is given by (2.16), (2.17) with* $\mathrm{trace}(A) \neq 0$.

We now consider two different possibilities.

Case 1. SUPPOSE $A = \alpha I_2$. Then the Lie bracket satisfies equation (2.1) for the non-zero linear map $l\colon \mathfrak{g} \to \mathbb{R}$ given by $l(E_1) = l(E_2) = 0$, $l(E_3) = \alpha$ and thus, $(G, \langle,\rangle)$ has constant sectional curvature $-\alpha^2 < 0$. Recall that $\alpha = 1$ gives the hyperbolic three-space $\mathbb{H}^3$ with its usual Lie group structure. Since scaling A does not change the group structure but only scales the left invariant canonical metric, then $\mathbb{H}^3$ is the unique Lie group in this case.

Case 2. SUPPOSE A IS NOT A MULTIPLE OF I_2. In this case, the trace and the determinant

$$\begin{array}{l} T = \mathrm{trace}(A) = \alpha + \delta, \\ D = \det(A) = \alpha\delta - \beta\gamma \end{array}$$

of A are enough to determine $\mathfrak{g}$ (resp. G) up to a Lie algebra (resp. Lie group) isomorphism. To see this fact, consider the linear transformation $L(X) = [E_3, X]$, $X \in \mathfrak{u}$. Since A is not proportional to I_2, there exists $\widehat{E}_1 \in \mathfrak{u}$ such that $\widehat{E}_1$ and $\widehat{E}_2 := L(\widehat{E}_1)$ are linearly independent. Then the matrix of L with respect to the basis $\{\widehat{E}_1, \widehat{E}_2\}$ of $\mathfrak{u}$ is

$$\begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix}.$$

Since scaling the matrix A by a positive number corresponds to changing the left invariant metric by a homothety and scaling it by -1 changes the orientation, we have that in this case of A not being a multiple of the identity, the following property holds.

If we are allowed to identify left invariant metrics under rescaling, then we can assume $T = 2$ *and then* D *gives a complete invariant of the group structure of* G*, which we will call the Milnor* D*-invariant of* G*.*

In this Case 2, we can describe the family of non-unimodular metric Lie groups as follows. Fix a group structure and a left invariant metric $\langle,\rangle$. Rescale the metric so that $\mathrm{trace}(A) = 2$. Pick an orthonormal basis E_1, E_2, E_3 of $\mathfrak{g}$ so that the unimodular kernel is $\mathfrak{u} = \mathrm{Span}\{E_1, E_2\}$, $[E_1, E_2] = 0$ and the Lie bracket is given by (2.16) with $\alpha + \delta = 2$. After a suitable rotation in $\mathfrak{u}$ (this does not change the metric), we can also assume that $\alpha\beta + \gamma\delta = 0$. After possibly changing E_1, E_2 by $E_2, -E_1$ we can assume $\alpha \geq \delta$ and then possibly replacing E_1 by $-E_1$, we can also assume $\gamma \geq \beta$. It then follows from Lemma 6.5 in [**Mil76**] that the orthonormal basis $\{E_1, E_2, E_3\}$ diagonalizes the Ricci tensor associated to $\langle,\rangle$, with principal Ricci curvatures being

$$\begin{array}{rcl} \mathrm{Ric}(E_1) & = & -\alpha(\alpha+\delta) + \frac{1}{2}(\beta^2 - \gamma^2), \\ \mathrm{Ric}(E_2) & = & -\delta(\alpha+\delta) + \frac{1}{2}(\gamma^2 - \beta^2), \\ \mathrm{Ric}(E_3) & = & -\alpha^2 - \delta^2 - \frac{1}{2}(\beta+\delta)^2. \end{array} \tag{2.18}$$

The equation $\alpha\beta + \gamma\delta = 0$ allows us to rewrite A as follows:

$$A = \begin{pmatrix} 1+a & -(1-a)b \\ (1+a)b & 1-a \end{pmatrix}, \tag{2.19}$$

where $a = \alpha - 1$ and $b = \left\{ \begin{array}{cl} \frac{\gamma}{\alpha} = \frac{\beta}{\alpha-2} & \text{if } \alpha \neq 0,2 \\ -\beta/2 & \text{if } \alpha = 0 \\ \gamma/2 & \text{if } \alpha = 2 \end{array} \right\}$. Our assumptions $\alpha \geq \delta$ and $\gamma \geq \beta$ imply that $a, b \geq 0$, which means that the related matrix A for $\mathrm{ad}_{E_3} : \mathfrak{u} \to \mathfrak{u}$ given in (2.19) with respect to the basis $\{E_1, E_2\}$ is now **uniquely determined**. The Milnor D-invariant of the Lie group in this language is given by

$$D = (1-a^2)(1+b^2). \tag{2.20}$$

Given $D \in \mathbb{R}$ we define

$$m(D) = \left\{ \begin{array}{cl} \sqrt{D-1} & \text{if } D > 1, \\ 0 & \text{otherwise.} \end{array} \right. \tag{2.21}$$

Thus we can solve in (2.20) for $a = a(b)$ in the range $b \in [m(D), \infty)$ obtaining

$$a(b) = \sqrt{1 - \frac{D}{1+b^2}}. \tag{2.22}$$

Note that we can discard the case $(D, b) = (1, 0)$ since (2.22) leads to the matrix $A = I_2$ which we have already treated. So from now on we assume $(D, b) \neq (1, 0)$. For each $b \in [m(D), \infty)$, the corresponding matrix $A = A(D, b)$ given by (2.19) for $a = a(b)$ defines (up to isomorphism) the same group structure on the semidirect product $\mathbb{R}^2 \rtimes_{A(D,b)} \mathbb{R}$, and it is natural to ask if the corresponding canonical metrics on $\mathbb{R}^2 \rtimes_{A(D,b)} \mathbb{R}$ for a fixed value of D are non-isometric. The answer is affirmative: the Ricci tensor in (2.18) can be rewritten as

$$\begin{array}{rcl} \mathrm{Ric}(E_1) & = & -2\left(1 + a(1+b^2)\right) \\ \mathrm{Ric}(E_2) & = & -2\left(1 - a(1+b^2)\right) \\ \mathrm{Ric}(E_3) & = & -2\left(1 + a^2(1+b^2)\right). \end{array} \tag{2.23}$$

Plugging (2.22) in the last formula we have

$$\begin{array}{rcl} \mathrm{Ric}(E_1) & = & -2\left(1 + \sqrt{x(x-D)}\right) \\ \mathrm{Ric}(E_2) & = & -2\left(1 - \sqrt{x(x-D)}\right) \\ \mathrm{Ric}(E_3) & = & -2(1 + x - D), \end{array}$$

where $x = x(b) = 1 + b^2$. It is not difficult to check that the map that assigns to each $b \in [m(D), \infty)$ the unordered triple $\{\mathrm{Ric}(E_1), \mathrm{Ric}(E_2), \mathrm{Ric}(E_3)\}$ is injective, which implies that for D fixed, different values of b give rise to non-isometric left invariant metrics on the same group structure $\mathbb{R}^2 \rtimes_{A(D,b)} \mathbb{R}$. This family of metrics, together with the rescaling process to get $\mathrm{trace}(A) = 2$, describe the 2-parameter family of left invariant metrics on a given non-unimodular group in this Case 2. We summarize these properties in the following statement.

LEMMA 2.12. *Let $A \in \mathcal{M}_2(\mathbb{R})$ be a matrix as in (2.19) with $a, b \geq 0$ and let $D = \det(A)$. Then:*

(1) If $A = I_2$, then $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is isomorphic to $\mathbb{H}^3$ and there is only one left invariant metric on G (up to scaling), the standard one with constant sectional

curvature -1. Furthermore, this choice of A is the only one which gives rise to the group structure of $\mathbb{H}^3$.

(2) If $A \neq I_2$, then the family of left invariant metrics on $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is parameterized (up to scaling the metric) by the values $b \in [m(D), \infty)$, by means of the canonical metric on $\mathbb{R}^2 \rtimes_{A_1} \mathbb{R}$, where $A_1 = A_1(D, b)$ given by (2.19) and (2.22). Furthermore, the group structure of G is determined by its Milnor D-invariant, i.e., different matrices $A \neq I_2$ with the same (normalized) Milnor D-invariant produce isomorphic Lie groups.

Recall from Section 2.3 that each of the integral leaves $\mathbb{R}^2 \rtimes_A \{z\}$ of the distribution spanned by E_1, E_2 has unit normal vector field $\pm E_3$, and the Gauss equation together with (2.19) imply that the mean curvature of these leaves (with respect to E_3) is $\frac{1}{2}\text{trace}(A) = 1$.

We finish this section with a result by Milnor [**Mil76**] that asserts that if we want to solve a purely geometric problem in a metric Lie group $(G, \langle , \rangle)$ (for instance, classifying the H-spheres in G for any value of the mean curvature $H \geq 0$), then one can sometimes have different underlying group structures to attack the problem.

Lemma 2.13. *A necessary and sufficient condition for a non-unimodular three-dimensional Lie group G to admit a left invariant metric with constant negative curvature is that $G = \mathbb{H}^3$ or its Milnor D-invariant is $D > 1$. In particular, there exist non-isomorphic metric Lie groups which are isometric.*

Proof. First assume G admits a left invariant metric with constant negative curvature. If G is in Case 1, i.e., its associated matrix A in (2.17) is a multiple of I_2, then item (1) of Lemma 2.12 gives that G is isomorphic to $\mathbb{H}^3$. If G is in Case 2, then (2.23) implies that $a = 0$ and (2.20) gives $D \geq 1$. But $D = 1$ would give $b = 0$ which leads to the Case 1 for G.

Reciprocally, we can obviously assume that G is not isomorphic to $\mathbb{H}^3$ and $D > 1$. In particular, G is in Case 2. Pick a left invariant metric $\langle , \rangle$ on G so that $\text{trace}(A) = 2$ and use Lemma 2.12 to write the metric Lie group $(G, \langle , \rangle)$ as $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $A = A(D, b)$ as in (2.19) and (2.22). Now, taking $b = \sqrt{D-1}$ gives $a(b) = 0$ in (2.22). Hence (2.23) gives $\text{Ric} = -2$ and the sectional curvature of the corresponding metric on G is -1. □

2.6. Classification of three-dimensional unimodular Lie groups. Once we have picked an orientation and a left invariant metric $\langle , \rangle$ on a three-dimensional Lie group G, the cross product operation makes sense in its Lie algebra $\mathfrak{g}$: given $X, Y \in \mathfrak{g}$, $X \times Y$ is the unique element in $\mathfrak{g}$ such that

$$\langle X \times Y, Z \rangle = \det(X, Y, Z) \quad \text{for all } X, Y, Z \in \mathfrak{g},$$

where det denotes the oriented volume element on $(G, \langle , \rangle)$. Thus, $\|X \times Y\|^2 = \|X\|^2\|Y\|^2 - \langle X, Y \rangle^2$ and if $X, Y \in \mathfrak{g}$ are linearly independent, then the triple $\{X, Y, X \times Y\}$ is a positively oriented basis of $\mathfrak{g}$. The Lie bracket and the cross product are skew-symmetric bilinear forms, hence related by a unique endomorphism $L \colon \mathfrak{g} \to \mathfrak{g}$ by

$$[X, Y] = L(X \times Y), \quad X, Y \in \mathfrak{g}.$$

It is straightforward to check that G is unimodular if and only if L is self-adjoint (see Lemma 4.1 in [**Mil76**]).

Assume in what follows that G is unimodular. Then there exists a positively oriented orthonormal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{g}$ consisting of eigenvectors of L, i.e.,

$$[E_2, E_3] = c_1 E_1, \quad [E_3, E_1] = c_2 E_2, \quad [E_1, E_2] = c_3 E_3, \tag{2.24}$$

for certain constants $c_1, c_2, c_3 \in \mathbb{R}$ usually called the *structure constants* of the unimodular metric Lie group. Note that a change of orientation forces $\times$ to change sign, and so it also produces a change of sign to all of the c_i. The structure constants depend on the chosen left invariant metric, but only their signs determine the underlying unimodular Lie algebra as follows from the following fact. If we change the left invariant metric by changing the lengths of E_1, E_2, E_3 (but we keep them orthogonal), say we declare bcE_1, acE_2, abE_3 to be orthonormal for a choice of non-zero real numbers a, b, c (note that the new basis is always positively oriented), then the new structure constants are a^2c_1, b^2c_2, c^2c_3. This implies that a change of left invariant metric does not affect the signs of the structure constants c_1, c_2, c_3 but only their lengths, and that we can multiply c_1, c_2, c_3 by arbitrary positive numbers without changing the underlying Lie algebra.

Now we are left with exactly six cases, once we have possibly changed the orientation so that the number of negative structure constants is at most one. Each of these six cases is realized by exactly one simply-connected unimodular Lie group, listed in the following table. These simply-connected Lie groups will be studied in some detail later.

Signs of c_1, c_2, c_3	simply-connected Lie group
$+, +, +$	$\mathrm{SU}(2)$
$+, +, -$	$\widetilde{\mathrm{SL}}(2, \mathbb{R})$
$+, +, 0$	$\widetilde{\mathrm{E}}(2)$
$+, -, 0$	Sol_3
$+, 0, 0$	Nil_3
$0, 0, 0$	$\mathbb{R}^3$

Table 1: Three-dimensional, simply-connected unimodular Lie groups.

The six possibilities in Table 1 correspond to non-isomorphic unimodular Lie groups, since their Lie algebras are also non-isomorphic: an invariant which distinguishes them is the signature of the (symmetric) *Killing form*

$$X, Y \in \mathfrak{g} \mapsto \beta(X, Y) = \mathrm{trace}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y)\,.$$

Before describing the cases listed in Table 1, we will study some curvature properties for unimodular metric Lie groups, which can be expressed in a unified way. To do this, it is convenient to introduce new constants $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ by

$$\mu_1 = \frac{1}{2}(-c_1 + c_2 + c_3), \quad \mu_2 = \frac{1}{2}(c_1 - c_2 + c_3), \quad \mu_3 = \frac{1}{2}(c_1 + c_2 - c_3). \tag{2.25}$$

The Levi-Civita connection ∇ for the metric associated to these constants μ_i is given by

$$\begin{array}{l|l|l} \nabla_{E_1} E_1 = 0 & \nabla_{E_1} E_2 = \mu_1 E_3 & \nabla_{E_1} E_3 = -\mu_1 E_2 \\ \nabla_{E_2} E_1 = -\mu_2 E_3 & \nabla_{E_2} E_2 = 0 & \nabla_{E_2} E_3 = \mu_2 E_1 \\ \nabla_{E_3} E_1 = \mu_3 E_2 & \nabla_{E_3} E_2 = -\mu_3 E_1 & \nabla_{E_3} E_3 = 0. \end{array} \tag{2.26}$$

The symmetric Ricci tensor associated to the metric diagonalizes in the basis $\{E_1, E_2, E_3\}$, with eigenvalues

$$\mathrm{Ric}(E_1) = 2\mu_2\mu_3, \quad \mathrm{Ric}(E_2) = 2\mu_1\mu_3, \quad \mathrm{Ric}(E_3) = 2\mu_1\mu_2. \tag{2.27}$$

At this point, it is natural to consider several different cases.

(1) If $c_1 = c_2 = c_3$ (hence $\mu_1 = \mu_2 = \mu_3$), then $(G, \langle,\rangle)$ has constant sectional curvature $\mu_1^2 \geq 0$. This leads to $\mathbb{R}^3$ and $\mathbb{S}^3$ with their standard metrics (the hyperbolic three-space $\mathbb{H}^3$ is non-unimodular as a Lie group).

(2) If $c_3 = 0$ and $c_1 = c_2 > 0$ (hence $\mu_1 = \mu_2 = 0$, $\mu_3 = c_1 > 0$), then $(G, \langle,\rangle)$ is flat. This leads to $\widetilde{\mathrm{E}}(2)$ with its standard metric.

(3) If exactly two of the structure constants c_i are equal and no c_i is zero, after possibly reindexing we can assume $c_1 = c_2$. Then rotations about the axis with direction E_3 are isometries of the metric, and we find a standard $\mathbb{E}(\kappa, \tau)$-space, i.e., a simply-connected homogeneous space that submerses over the complete simply-connected surface $\mathbb{M}^2(\kappa)$ of constant curvature κ, with bundle curvature τ and four dimensional isometry group. If we identify E_3 with the unit Killing field that generates the kernel of the differential of the Riemannian submersion $\Pi\colon \mathbb{E}(\kappa, \tau) \to \mathbb{M}^2(\kappa)$, then it is well-known that the symmetric Ricci tensor has eigenvalues $\kappa - 2\tau^2$ (double, in the plane $\langle E_3\rangle^{\perp}$) and $2\tau^2$. Hence $\mu_1^2 = \tau^2$ recovers the bundle curvature, and the base curvature κ is c_1c_3, which can be positive, zero or negative. There are two types of $\mathbb{E}(\kappa, \tau)$-spaces in this setting, both with $\tau \neq 0$: Berger spheres, which occur when both $c_1 = c_2$ and c_3 are positive (hence we have a 2-parameter family of metrics, which can be reduced to just one parameter after rescaling) and the universal cover $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ of the special linear group, which occurs when $c_1 = c_2 > 0$ and $c_3 < 0$ (hence with a 1-parameter family of metrics after rescaling). For further details, see Daniel [**Dan07**].

(4) If $c_1 = c_2 = 0$ and $c_3 > 0$, then similar arguments lead to the Heisenberg group Nil_3, (the left invariant metric on $\mathrm{Nil}_3 = \mathbb{E}(\kappa = 0, \tau)$ is unique modulo homotheties). The other two $\mathbb{E}(\kappa, \tau)$-spaces not appearing in this setting or in the previous setting of item (3) are $\mathbb{S}^2 \times \mathbb{R}$, which is not a Lie group, and $\mathbb{H}^2 \times \mathbb{R}$, which is a non-unimodular Lie group.

(5) If all three structure constants c_1, c_2, c_3 are different, then the isometry group of $(G, \langle,\rangle)$ is three-dimensional. In this case, we find the special unitary group $\mathrm{SU}(2)$ (when all the c_i are positive), the universal cover $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ of the special linear group (when two of the constants c_i are positive and one is negative), the universal cover $\widetilde{\mathrm{E}}(2)$ of the group of rigid motions of the Euclidean plane (when two of the constants c_i are positive and the third one vanishes) and the solvable group Sol_3 (when one of the c_i is positive, other is negative and the third one is zero), see Figure 3 for a pictorial representation of these cases (1)–(4).

The following result summarizes how to express the metric semidirect products with isometry groups of dimension four or six.

THEOREM 2.14 (Classification of metric semidirect products with 4 or 6 dimensional isometry groups).
Let $(G, \langle,\rangle)$ be a metric Lie group which is isomorphic and isometric to a non-trivial semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric for some $A \in \mathcal{M}_2(\mathbb{R})$.

(1) Suppose that the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ has isometry group of dimension six.

(1) If G is non-unimodular, then up to rescaling the metric, A is similar to $\begin{pmatrix} 1 & -b \\ b & 1 \end{pmatrix}$ for some $b \in [0, \infty)$. These groups are precisely those non-unimodular groups that are either isomorphic to $\mathbb{H}^3$ or have Milnor D-invariant $D = \det(A) > 1$ and the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ has constant sectional curvature -1. Furthermore $(G, \langle , \rangle)$ is isometric to the hyperbolic three-space, and under the left action of G on itself, G is isomorphic to a subgroup of the isometry group of the hyperbolic three-space.

(2) If G is unimodular, then either $A = 0$ and $(G, \langle , \rangle)$ is $\mathbb{R}^3$ with its flat metric, or, up to rescaling the metric, A is similar to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Here the underlying group is $\widetilde{\mathrm{E}}(2)$ and the canonical metric given by A on $\mathbb{R}^2 \rtimes_A \mathbb{R} = \widetilde{\mathrm{E}}(2)$ is flat.

(2) Suppose that the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ has isometry group of dimension four.

(1) If G is non-unimodular, then up to rescaling the metric, A is similar to $\begin{pmatrix} 2 & 0 \\ 2b & 0 \end{pmatrix}$ for some $b \in \mathbb{R}$. Furthermore, when A has this expression, then the underlying group structure is that of $\mathbb{H}^2 \times \mathbb{R}$, and $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric is isometric to the $\mathbb{E}(\kappa, \tau)$-space with $b = \tau$ and $\kappa = -4$.

(2) If G is unimodular, then up to scaling the metric, A is similar to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and the group is Nil_3.

PROOF. WE WILL START BY ANALYZING THE NON-UNIMODULAR CASE. Suppose that A is a non-zero multiple of the identity. As we saw in Case 1 just after Lemma 2.11, the Lie bracket satisfies equation (2.1) and thus, $\mathbb{R}^2 \rtimes_A \mathbb{R}$ has constant negative sectional curvature. In particular, its isometry group has dimension six and we are in case (1-a) of the theorem with $b = 0$.

Now assume that A is not a multiple of the identity. By the discussion in Case 2 just after Lemma 2.11, after rescaling the metric so that $\mathrm{trace}(A) = 2$, in a new orthonormal basis E_1, E_2, E_3 of the Lie algebra $\mathfrak{g}$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$, we can write A as in equation (2.19) in terms of constants $a, b \geq 0$ with either $a > 0$ or $b > 0$ and such that the Ricci tensor acting on these vector fields is given by (2.23). We now discuss two possibilities.

(1) If the dimension of the isometry group of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is six, then $\mathrm{Ric}(E_1) = \mathrm{Ric}(E_2) = \mathrm{Ric}(E_3)$ from where one deduces that $a = 0$. Plugging this equality in (2.19) we obtain the matrix in item (1-a) of the theorem. The remaining properties stated in item (1-a) are easy to check.

(2) If the isometry group of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ has dimension four, then two of the numbers $\mathrm{Ric}(E_1)$, $\mathrm{Ric}(E_2)$, $\mathrm{Ric}(E_3)$ are equal and the third one is different from the other one (this follows since the Ricci tensor diagonalizes in the basis E_1, E_2, E_3). Now (2.23) implies that $a > 0$, hence $\mathrm{Ric}(E_2)$ is different from both $\mathrm{Ric}(E_1)$, $\mathrm{Ric}(E_3)$ and so, $\mathrm{Ric}(E_1) = \mathrm{Ric}(E_3)$. Then (2.23) implies that $a = 1$ and (2.19) gives that $A = \begin{pmatrix} 2 & 0 \\ 2b & 0 \end{pmatrix}$.

Since trace$(A) = 2$ and the Milnor D-invariant for $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is zero, then $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is isomorphic to $\mathbb{H}^2 \times \mathbb{R}$. Note that

$$e^{zA} = \left(\begin{array}{cc} e^{2z} & 0 \\ b(e^{2z}-1) & 1 \end{array} \right),$$

from where (2.5) and (2.6) imply that $E_2 = \partial_y = F_2$. In particular, E_2 is a Killing vector field. Let $H = \exp(\mathrm{Span}(E_2))$ be the 1-parameter subgroup of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ generated by E_2. Since E_2 is Killing, then the canonical metric $\langle , \rangle_A$ on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ descends to the quotient space $M = (\mathbb{R}^2 \rtimes_A \mathbb{R})/H$, making it a homogeneous surface. Since every integral curve of $E_2 = \partial_y$ intersects the plane $\{(x, 0, z) \mid x, z \in \mathbb{R}\}$ in a single point, the quotient surface M is diffeomorphic to $\mathbb{R}^2$. Therefore, up to homothetic scaling, M is isometric to $\mathbb{R}^2$ or $\mathbb{H}^2$ with their standard metrics and $\mathbb{R}^2 \rtimes_A \mathbb{R} \to M$ is a Riemannian submersion. This implies that $(\mathbb{R}^2 \rtimes_A \mathbb{R}, \langle , \rangle_A)$ is isometric to an $\mathbb{E}(\kappa, \tau)$-space with $k \leq 0$. Since the eigenvalues of the Ricci tensor for this last space are $\kappa - 2\tau^2$ (double) and $2\tau^2$, then $b = \tau$ and $\kappa = -4$. Now item (2-a) of the theorem is proved.

Now assume that G is unimodular. We want to use equations (2.9) and (2.10) together with (2.24), although note that they are expressed in two basis which *a priori* might not be the same. This little problem can be solved as follows. Consider the orthonormal left invariant basis $\{E_1, E_2, E_3\}$ for the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ given by (2.6). For the orientation on G defined by declaring this basis to be positive, let $L \colon \mathfrak{g} \to \mathfrak{g}$ be the self-adjoint endomorphism of the Lie algebra $\mathfrak{g}$ given by $[X, Y] = L(X \times Y)$, $X, Y \in \mathfrak{g}$, where $X \times Y$ is the cross product associated to the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and to the chosen orientation. Since $[E_1, E_2] = 0$, then E_3 is an eigenvector of L with associated eigenvalue zero. As L is self-adjoint, then L leaves invariant $\mathrm{Span}(E_3)^\perp = \mathrm{Span}\{E_1, E_2\}$, and thus, there exists a positive orthonormal basis $\{E_1', E_2'\}$ of $\mathrm{Span}\{E_1, E_2\}$ (with the induced orientation and inner product) which diagonalizes L. Obviously, the matrix of change of basis between $\{E_1, E_2\}$ and $\{E_1', E_2'\}$ is orthogonal, hence item (2) at the end of Section 2.3 shows that the corresponding metric semidirect products associated to A and to the diagonal form of L are isomorphic and isometric. This property is equivalent to the desired property that the basis used in equations (2.9), (2.10) and (2.24) can be chosen to be the same.

Now using the notation in equations (2.9), (2.10) and (2.24), we have $0 = [E_1, E_2] = c_3 E_3$ hence $c_3 = 0$, $c_2 E_2 = [E_3, E_1] = aE_1 + cE_2$ hence $a = 0$ and $c = c_2$, $-c_1 E_1 = [E_3, E_2] = bE_1 + dE_2$, hence $d = 0$ and $b = -c_1$. On the other hand, using (2.25) we have $-\mu_1 = \mu_2 = \frac{1}{2}(c_1 - c_2)$, $\mu_3 = \frac{1}{2}(c_1 + c_2)$ from where (2.27) reads as

$$\mathrm{Ric}(E_1) = \frac{1}{2}(c_1^2 - c_2^2) = -\mathrm{Ric}(E_2), \quad \mathrm{Ric}(E_3) = -\frac{1}{2}(c_1 - c_2)^2. \tag{2.28}$$

As before, we discuss two possibilities.

(1) If the dimension of the isometry group of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is six, then $\mathrm{Ric}(E_1) = \mathrm{Ric}(E_2) = \mathrm{Ric}(E_3)$ from where (2.28) gives $c_1 = c_2$. If $c_1 = 0$, then $A = 0$. If $c_1 \neq 0$, then up to scaling the metric we can assume $c_1 = 1$ and we arrive to item (1-b) of the theorem.

(2) If the isometry group of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ has dimension four, then two of the numbers $\mathrm{Ric}(E_1)$, $\mathrm{Ric}(E_2)$, $\mathrm{Ric}(E_3)$ are equal and the third one is different from the other one

(again because the Ricci tensor diagonalizes in the basis E_1, E_2, E_3). If $\text{Ric}(E_1) = \text{Ric}(E_2)$, then $c_1^2 = c_2^2$ and $\text{Ric}(E_1) = \text{Ric}(E_2) = 0$. Since $\text{Ric}(E_3)$ cannot be zero, then it is strictly negative. This is impossible, since the Ricci eigenvalues in a standard $\mathbb{E}(\kappa, \tau)$-space are $\kappa - 2\tau^2$ (double, which in this case vanishes), and $2\tau^2 \geq 0$. Thus we are left with only two possible cases: either $\text{Ric}(E_1) = \text{Ric}(E_3)$ (hence (2.28) gives $c_1 = 0$) or $\text{Ric}(E_2) = \text{Ric}(E_3)$ (and then $c_2 = 0$). These two cases lead, after rescaling and a possible change of orientation, to the matrices

$$A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

which are congruent. Hence we have arrived to the description in item (2-b) of the theorem. This finishes the proof. $\square$

2.7. The unimodular groups in Table 1 and their left invariant metrics. Next we will study in more detail the unimodular groups listed in Table 1 in the last section, focusing on their metric properties when the corresponding isometry group has dimension three.

The special unitary group. This is the group

$$\begin{aligned} \text{SU}(2) &= \{A \in \mathcal{M}_2(\mathbb{C}) \mid \overline{A^{-1}} = A^t, \ \det A = 1\} \\ &= \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}) \ \mid \ |z|^2 + |w|^2 = 1 \right\}, \end{aligned}$$

with the group operation of matrix multiplication. SU(2) is isomorphic to the group of quaternions $a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ (here $a, b, c, d \in \mathbb{R}$) of absolute value 1, by means of the group isomorphism

$$\begin{pmatrix} a - di & -b + ci \\ b + ci & a + di \end{pmatrix} \in \text{SU}(2) \mapsto a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k},$$

and thus, SU(2) is diffeomorphic to the three-sphere. SU(2) covers the special orthogonal group SO(3) with covering group $\mathbb{Z}_2$, see Example 2.9. SU(2) is the unique simply-connected three-dimensional Lie group which is not diffeomorphic to $\mathbb{R}^3$. The only normal subgroup of SU(2) is its center $\mathbb{Z}_2 = \{\pm I_2\}$.

The family of left invariant metrics on SU(2) has three parameters, which can be realized by changing the lengths of the left invariant vector fields E_1, E_2, E_3 in (2.24) but keeping them orthogonal. For instance, assigning the same length to all of them produces the 1-parameter family of standard metrics with isometry group of dimension six; assigning the same length to two of them (say E_1, E_2) different from the length of E_3 produces the 2-parameter family of Berger metrics with isometry group of dimension four; finally, assigning different lengths to the three orthogonal vector fields produces the more general 3-parameter family of metrics with isometry group of dimension three.

The universal covering of the special linear group, $\widetilde{\text{SL}}(2, \mathbb{R})$. The *projective special linear group* is

$$\text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\{\pm I_2\},$$

where $\text{SL}(2, \mathbb{R}) = \{A \in \mathcal{M}_2(\mathbb{R}) \mid \det A = 1\}$ is the *special linear group* (with the operation given by matrix multiplication). Obviously, both groups $\text{SL}(2, \mathbb{R}), \text{PSL}(2, \mathbb{R})$ have the same universal cover, which we denote by $\widetilde{\text{SL}}(2, \mathbb{R})$ (the notation $\widetilde{\text{PSL}}(2, \mathbb{R})$

is also commonly used in the literature). The Lie algebra of any of the groups $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{PSL}(2,\mathbb{R})$, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \{B \in \mathcal{M}_2(\mathbb{R}) \mid \mathrm{trace}(B) = 0\}.$$

$\mathrm{PSL}(2,\mathbb{R})$ is a simple group, i.e., it does not contain normal subgroups except itself and the trivial one. Since the universal cover $\widetilde{G}$ of a connected Lie group G with $\pi_1(G) \neq 0$ contains an abelian normal subgroup isomorphic to $\pi_1(G)$, then $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is not simple. In fact, the center of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is isomorphic to $\pi_1(\mathrm{PSL}(2,\mathbb{R})) = \mathbb{Z}$.

It is sometimes useful to have geometric interpretations of these groups. In the case of $\mathrm{SL}(2,\mathbb{R})$, we can view it either as the group of orientation-preserving linear transformations of $\mathbb{R}^2$ that preserve the (oriented) area, or as the group of complex matrices

$$\mathrm{SU}^1(2) = \left\{ \left(\begin{array}{cc} z & w \\ \overline{w} & \overline{z} \end{array} \right) \in \mathcal{M}_2(\mathbb{C}) \;\mid\; |z|^2 - |w|^2 = 1 \right\},$$

with the multiplication as its operation. This last model of $\mathrm{SL}(2,\mathbb{R})$ is useful since it mimics the identification of $\mathrm{SU}(2)$ with the unitary quaternions (simply change the standard Euclidean metric $dx_1^2+dx_2^2+dx_3^2+dx_4^2$ on $\mathbb{C}^2 \equiv \mathbb{R}^4$ by the non-degenerate metric $dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$). The map

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}(2,\mathbb{R}) \mapsto \frac{1}{2} \left(\begin{array}{cc} a+d+i(b-c) & b+c+i(a-d) \\ b+c-i(a-d) & a+d-i(b-c) \end{array} \right) \in \mathrm{SU}^1(2)$$

is an isomorphism of groups, and the Lie algebra of $\mathrm{SU}^1(2)$ is

$$\mathfrak{su}^1(2) = \left\{ \left(\begin{array}{cc} i\lambda & a \\ \overline{a} & -i\lambda \end{array} \right) \;\mid\; \lambda \in \mathbb{R}, a \in \mathbb{C} \right\}.$$

Regarding the projective special linear group $\mathrm{PSL}(2,\mathbb{R})$, we highlight four useful models isomorphic to it:

(1) The group of orientation-preserving isometries of the hyperbolic plane. Using the upper half-plane model for $\mathbb{H}^2$, these are transformations of the type

$$z \in \mathbb{H}^2 \equiv (\mathbb{R}^2)^+ \mapsto \frac{az+b}{cz+d} \in (\mathbb{R}^2)^+ \qquad (a,b,c,d \in \mathbb{R},\ ad-bc=1).$$

(2) The group of conformal automorphisms of the unit disc, i.e., Möbius transformations of the type $\phi(z) = e^{i\theta}\frac{z+a}{\overline{a}z+1}$, for $\theta \in \mathbb{R}$ and $a \in \mathbb{C}$, $|a| < 1$.

(3) The unit tangent bundle of the hyperbolic plane. This representation occurs because an isometry of $\mathbb{H}^2$ is uniquely determined by the image of a base point and the image under its differential of a given unitary vector tangent at that point. This point of view of $\mathrm{PSL}(2,\mathbb{R})$ as an $\mathbb{S}^1$-bundle over $\mathbb{H}^2$ (and hence of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ as an $\mathbb{R}$-bundle over $\mathbb{H}^2$) defines naturally the one-parameter family of left invariant metrics on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with isometry group of dimension four, in a similar manner as the Berger metrics in the three-sphere starting from metrics g_λ, $\lambda > 0$, on the unit tangent bundle of $\mathbb{S}^2$, see Example 2.9.

The characteristic polynomial of a matrix $A \in \mathrm{SL}(2,\mathbb{R})$ is $\lambda^2 - T\lambda + 1 = 0$, where $T = \mathrm{trace}(A)$. Its roots are given by $\lambda = \frac{1}{2}\left(T \pm \sqrt{T^2-4}\right)$. The sign of the discriminant T^2-4 allows us to classify the elements $A \in \mathrm{SL}(2,\mathbb{R})$ in three different types:

(1) **Elliptic.** In this case $|T| < 2$, A has no real eigenvalues (its eigenvalues are complex conjugate and lie on the unit circle). Thus, A is of the form $P^{-1}\mathrm{Rot}_\theta P$ for some $P \in Gl(2,\mathbb{R})$, where Rot_θ denotes the rotation of some angle $\theta \in [0, 2\pi)$.
(2) **Parabolic.** Now $|T| = 2$ and A has a unique (double) eigenvalue $\lambda = \frac{T}{2} = \pm 1$. If A is diagonalizable, then $A = \pm I_2$. If A is not diagonalizable, then $A = \pm P^{-1}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} P$ for some $t \in \mathbb{R}$ and $P \in Gl(2,\mathbb{R})$, i.e., A is similar to a *shear mapping.*
(3) **Hyperbolic.** Now $|T| > 2$ and A has two distinct real eigenvalues, one inverse of the other: $A = P^{-1}\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} P$ for some $\lambda \neq 0$ and $P \in Gl(2,\mathbb{R})$, i.e., A is similar to a *squeeze mapping.*

Since the projective homomorphism $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \varphi(z) = \frac{az+b}{cz+d}$, $z \in \mathbb{H}^2 \equiv (\mathbb{R}^2)^+$ with kernel $\{\pm I_2\}$ relates matrices in $\mathrm{SL}(2,\mathbb{R})$ with Möbius transformations of the hyperbolic plane, we can translate the above classification of matrices to this last language. For instance, the rotation $\mathrm{Rot}_\theta \in \mathrm{SL}(2,\mathbb{R})$ of angle $\theta \in [0, 2\pi)$ produces the Möbius transformation $z \in (\mathbb{R}^2)^+ \mapsto \frac{\cos(\theta)z-\sin(\theta)}{\sin(\theta)z+\cos(\theta)}$, which corresponds in the Poincaré disk model of $\mathbb{H}^2$ to the rotation of angle -2θ around the origin. This idea allows us to list the three types of 1-parameter subgroups of $\mathrm{PSL}(2,\mathbb{R})$:

(1) **Elliptic subgroups.** Elements of these subgroups correspond to continuous rotations around any fixed point in $\mathbb{H}^2$. In the Poincaré disk model, these 1-parameter subgroups fix no points in the boundary at infinity $\partial_\infty\mathbb{H}^2 = \mathbb{S}^1$. If $\Gamma_{p_1}, \Gamma_{p_2}$ are two such elliptic subgroups where each Γ_{p_i} fixes the point p_i, then $\Gamma_{p_1} = (p_1p_2^{-1})\Gamma_{p_2}(p_1p_2^{-1})^{-1}$.
(2) **Hyperbolic subgroups.** These are translations along any fixed geodesic Γ in $\mathbb{H}^2$. In the Poincaré disk model, the hyperbolic subgroup associated to a geodesic Γ fixes the two points at infinity corresponding to the end points of Γ. In the upper halfplane model $(\mathbb{R}^2)^+$ for $\mathbb{H}^2$, we can assume that the invariant geodesic Γ is the positive imaginary half-axis, and then the corresponding 1-parameter subgroup is $\{\varphi_t(z) = e^t z\}_{t\in\mathbb{R}}$, $z \in (\mathbb{R}^2)^+$. As in the elliptic case, every two 1-parameter hyperbolic subgroups are conjugate.
(3) **Parabolic subgroups.** In the Poincaré disk model, these are the rotations about any fixed point $\theta \in \partial_\infty\mathbb{H}^2$. They only fix this point θ at infinity, and leave invariant the 1-parameter family of horocycles based at θ. As in the previous cases, parabolic subgroups are all conjugate by elliptic rotations of the Poincaré disk. In the upper halfplane model $(\mathbb{R}^2)^+$ for $\mathbb{H}^2$, we can place the point θ at ∞ and then the corresponding 1-parameter subgroup is $\{\varphi_t(z) = z + t\}_{t\in\mathbb{R}}$, $z \in (\mathbb{R}^2)^+$. Every parabolic subgroup is a limit of elliptic subgroups (simply consider a point $\theta \in \partial_\infty\mathbb{H}^2$ as a limit of centers of rotations in $\mathbb{H}^2$). Also, every parabolic subgroup can be seen as a limit of hyperbolic subgroups (simply consider a point $\theta \in \mathbb{S}^1$ as a limit of suitable geodesics of $\mathbb{H}^2$), see Figure 1.

Coming back to the language of matrices in $\mathrm{SL}(2,\mathbb{R})$, it is worth while computing a basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ in which the Lie bracket adopts the form (2.24). A matrix in $\mathfrak{sl}(2,\mathbb{R})$ which spans the Lie subalgebra of the particular elliptic subgroup $\{\mathrm{Rot}_\theta \mid \theta \in \mathbb{R}\}$ is $E_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Similarly, the parabolic subgroup

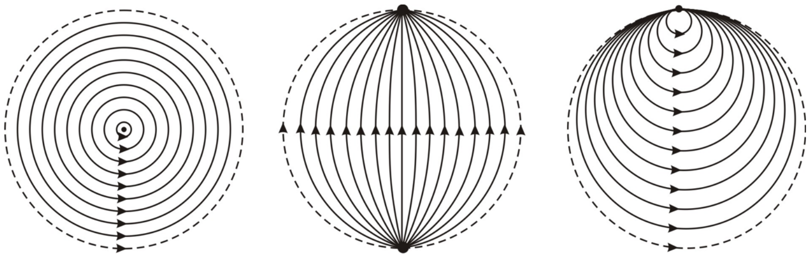

FIGURE 1. Orbits of the actions of 1-parameter subgroups of $\mathrm{PSL}(2,\mathbb{R})$. Left: Elliptic. Center: Hyperbolic. Right: Parabolic.

associated to $\{\varphi_t(z) = z+t\}_{t\in\mathbb{R}}$, $z \in (\mathbb{R}^2)^+$, produces the left invariant vector field $B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})$. In the hyperbolic case, the subgroup $\{\varphi_t(z) = e^{2t}z\}_{t\in\mathbb{R}}$, $z \in (\mathbb{R}^2)^+$, has related 1-parameter in $\mathrm{SL}(2,\mathbb{R})$ given by the matrices $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, with associated left invariant vector field $E_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})$. The Lie bracket in $\mathfrak{sl}(2,\mathbb{R})$ is given by the commutator of matrices. It is elementary to check that $[B_2, E_3] = E_1$, $[E_1, B_2] = 2B_2$, $[E_3, E_1] = 2E_3 + 4B_2$, which does not look like the canonical expression (2.24) valid in any unimodular group. Note that E_1, E_3 are orthogonal in the usual inner product of matrices, but B_2 is not orthogonal to E_3. Exchanging B_2 by $E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}) \cap \mathrm{Span}\{E_1, E_3\}^\perp$, we have

$$[E_1, E_2] = -2E_3, \quad [E_2, E_3] = 2E_1, \quad [E_3, E_1] = 2E_2, \tag{2.29}$$

which is of the form (2.24). Note that E_2 corresponds to the 1-parameter hyperbolic subgroup of Möbius transformations $\varphi_t(z) = \frac{\cosh(t)z+\sinh(t)}{\sinh(t)z+\cosh(t)}$, $z \in (\mathbb{R}^2)^+$.

We now describe the geometry of the 1-parameter subgroups $\Gamma_v = \exp(\{tv \mid t \in \mathbb{R}\})$ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ in terms of the coordinates of a tangent vector $v \neq 0$ at the identity element e of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, with respect to the basis $\{E_1, E_2, E_3\}$ of $\mathfrak{sl}(2,\mathbb{R})$. Consider the left invariant metric $\langle , \rangle$ that makes $\{E_1, E_2, E_3\}$ an orthonormal basis. Using (2.29), (2.25) and (2.27), we deduce that the metric Lie group $(\widetilde{\mathrm{SL}}(2,\mathbb{R}), \langle , \rangle)$ is isometric to an $\mathbb{E}(\kappa,\tau)$-space with $\kappa = -4$ and $\tau^2 = 1$ (recall that the eigenvalues of the Ricci tensor on $\mathbb{E}(\kappa,\tau)$ are $\kappa - 2\tau^2$ double and $2\tau^2$ simple). Let

$$\Pi \colon \mathbb{E}(-4,1) \to \mathbb{H}^2(-4)$$

be a Riemannian submersion onto the hyperbolic plane endowed with the metric of constant curvature -4. Consider the cone $\{(a,b,c) \mid a^2 + b^2 = c^2\}$ in the tangent space at the identity of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, where the (a,b,c)-coordinates refer to the coordinates of vectors with respect to the basis $\{E_1, E_2, E_3\}$ at $T_e\widetilde{\mathrm{SL}}(2,\mathbb{R})$. If $a = b = 0$, then $\Gamma_v = \exp(\{tv \mid t \in \mathbb{R}\})$ is the lift to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of the elliptic subgroup of rotations of $\mathbb{H}^2(-4)$ around the point $\Pi(e)$. If $c = 0$, then Γ_v is the hyperbolic subgroup obtained after horizontal lift of the translations along a geodesic in $\mathbb{H}^2(-4)$ passing through $\Pi(e)$, and Γ_v is a geodesic in the space $\mathbb{E}(-4,1)$. If $a^2 + b^2 = c^2$, then $\Pi(\Gamma_v)$ is a horocycle in $\mathbb{H}^2(-4)$, and it is the orbit of $\Pi(e)$ under the action of the

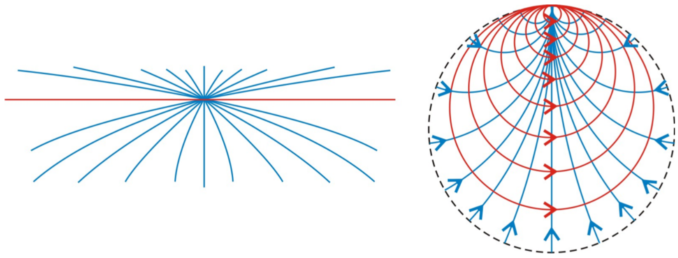

FIGURE 2. Two representations of the two-dimensional subgroup $\mathbb{H}^2_\theta$ of $\mathrm{PSL}(2,\mathbb{R})$, $\theta \in \partial_\infty \mathbb{H}^2$. Left: As the semidirect product $\mathbb{R} \rtimes_{(1)} \mathbb{R}$. Right: As the set of orientation-preserving isometries of $\mathbb{H}^2$ which fix θ. Each of the curves in the left picture corresponds to the orbit of a 1-parameter subgroup of $\mathbb{R} \rtimes_{(1)} \mathbb{R}$.

parabolic subgroup $\Pi(\Gamma_v)$ on $\mathbb{H}^2(-4)$. If $a^2+b^2<c^2$ then $\Pi(\Gamma_v)$ is a constant geodesic curvature circle passing through $\Pi(e)$ and completely contained in $\mathbb{H}^2(-4)$, and $\Pi(\Gamma_v)$ is the orbit of $\Pi(e)$ under the action of the elliptic subgroup $\Pi(\Gamma_v)$ on $\mathbb{H}^2(-4)$. Finally, if $a^2+b^2>c^2$ then $\Pi(\Gamma_v)$ is a constant geodesic curvature arc passing through $\Pi(e)$ with two end points in the boundary at infinity of $\mathbb{H}^2(-4)$, and $\Pi(\Gamma_v)$ is the orbit of $\Pi(e)$ under the action of the hyperbolic subgroup $\Pi(\Gamma_v)$ on $\mathbb{H}^2(-4)$. In this last case, $\Pi(\Gamma_v)$ is the set of points at fixed positive distance from a geodesic γ in $\mathbb{H}^2(-4)$, and Γ_v is the lift to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of the set of hyperbolic translations of $\mathbb{H}^2(-4)$ along γ.

Regarding the two-dimensional subgroups of $\mathrm{PSL}(2,\mathbb{R})$, equation (2.29) easily implies that $\mathfrak{sl}(2,\mathbb{R})$ has no two-dimensional commutative subalgebras. Thus, $\mathrm{PSL}(2,\mathbb{R})$ has no two-dimensional subgroups of type $\mathbb{R}^2$: all of them are of $\mathbb{H}^2$-type. For each $\theta \in \partial_\infty \mathbb{H}^2$, we consider the subgroup

$$\mathbb{H}^2_\theta = \{\text{orientation-preserving isometries of } \mathbb{H}^2 \text{ which fix } \theta\}. \tag{2.30}$$

Elements in $\mathbb{H}^2_\theta$ are rotations around θ (parabolic) and translations along geodesics one of whose end points is θ (hyperbolic). As we saw in Section 2.1, $\mathbb{H}^2 = \mathbb{H}^2_\theta$ is isomorphic to $\mathbb{R} \rtimes_{(1)} \mathbb{R}$. It is worth relating the 1-parameter subgroups of both two-dimensional groups. The subgroup $\mathbb{R} \rtimes_{(1)} \{0\}$ is normal in $\mathbb{R} \rtimes_{(1)} \mathbb{R}$ (this is not normal as a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ since this last one is simple) and corresponds to the parabolic subgroup of $\mathrm{PSL}(2,\mathbb{R})$ fixing θ, while the subgroup $\{0\} \rtimes_{(1)} \mathbb{R}$ of $\mathbb{R} \rtimes_{(1)} \mathbb{R}$ corresponds to a hyperbolic 1-parameter subgroup of translations along a geodesic one of whose end points is θ. The other 1-parameter subgroups of $\mathbb{R}^2 \rtimes_{(1)} \mathbb{R}$ are $\Gamma_s = \{(s(e^t-1), t) \mid t \in \mathbb{R}\}$ for each $s \in \mathbb{R}$, each of which corresponds to the hyperbolic 1-parameter subgroup of translations along one of the geodesics in $\mathbb{H}^2$ with common end point θ, see Figure 2.

The family of left invariant metrics on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ has three parameters, which can be realized by changing the lengths of the left invariant vector fields E_1, E_2, E_3 defined just before (2.29), but keeping them orthogonal. Among these metrics we have a 2-parameter family, each one having an isometry group of dimension four; these special metrics correspond to the case where one changes the lengths of E_1 and E_2 by the same factor. The generic case of a left invariant metric on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ has a three-dimensional group of isometries.

The universal cover of the group of orientation-preserving rigid motions of the Euclidean plane, $\widetilde{\mathrm{E}}(2)$. The universal cover $\widetilde{\mathrm{E}}(2)$ of the group $\mathrm{E}(2)$ of orientation-preserving rigid motions of the Euclidean plane is isomorphic to the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\widetilde{\mathrm{E}}(2)$ carries a 2-parameter family of left invariant metrics, which can be described as follows. Using coordinates (x, y, z) in $\widetilde{\mathrm{E}}(2)$ so that (x, y) are standard coordinates in $\mathbb{R}^2 \equiv \mathbb{R}^2 \rtimes_A \{0\}$ and z parametrizes $\mathbb{R} \equiv \{0\} \rtimes_A \mathbb{R}$, then $e^{zA} = \begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}$ and so, (2.2) gives the group operation as

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2 \cos z_1 - y_2 \sin z_1, y_1 + x_2 \sin z_1 + y_2 \cos z_1, z_1 + z_2).$$

A basis of the Lie algebra $\mathfrak{g}$ of $\widetilde{\mathrm{E}}(2)$ is given by (2.6):

$$E_1(x, y, z) = \cos z\, \partial_x + \sin z\, \partial_y, \quad E_2(x, y, z) = -\sin z\, \partial_x + \cos z\, \partial_y, \quad E_3 = \partial_z.$$

A direct computation (or equations (2.9), (2.10)) gives the Lie bracket as

$$[E_1, E_2] = 0, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2,$$

compare with equation (2.24).

To describe the left invariant metrics on $\widetilde{\mathrm{E}}(2)$, we first declare $E_3 = \partial_z$ to have length one (equivalently, we will determine the left invariant metrics up to rescaling). Given $\varepsilon_1, \varepsilon_2 > 0$, we declare the basis $\{E_1' = \varepsilon_1 E_1, E_2' = \varepsilon_2 E_2, E_3' = E_3\}$ to be orthonormal. This defines a left invariant metric $\langle , \rangle$ on $\widetilde{\mathrm{E}}(2)$. Then,

$$[E_2', E_3'] = \varepsilon_2 [E_2, E_3] = \varepsilon_2 E_1 = \frac{\varepsilon_2}{\varepsilon_1} E_1', \qquad [E_3', E_1'] = \varepsilon_1 [E_3, E_1] = \varepsilon_1 E_2 = \frac{\varepsilon_1}{\varepsilon_2} E_2'.$$

Hence the basis $\{E_1', E_2', E_3'\}$ satisfies equation (2.24) with $c_1 = \frac{\varepsilon_2}{\varepsilon_1}$ and $c_2 = \frac{\varepsilon_1}{\varepsilon_2} = \frac{1}{c_1}$ (c_3 is zero). Now we relabel the E_i' as E_i, obtaining that $\{E_1, E_2, E_3\}$ is an orthonormal basis of $\langle , \rangle$ and $[E_1, E_2] = 0$, $[E_3, E_1] = \frac{1}{c_1} E_2$, $[E_2, E_3] = c_1 E_1$. Comparing these equalities with (2.9), (2.10) we conclude that $(\widetilde{\mathrm{E}}(2), \langle , \rangle)$ is isomorphic and isometric to the Lie group $\mathbb{R}^2 \rtimes_{A(c_1)} \mathbb{R}$ endowed with its canonical metric, where

$$A(c_1) = \begin{pmatrix} 0 & -c_1 \\ 1/c_1 & 0 \end{pmatrix}, \qquad c_1 > 0. \tag{2.31}$$

In fact, the matrices $A(c_1)$, $A(1/c_1)$ given by (2.31) are congruent, hence we can restrict the range of values of c_1 to $[1, \infty)$. Now we have obtained an explicit description of the family of left invariant metrics on $\widetilde{\mathrm{E}}(2)$ (up to rescaling).

The solvable group Sol_3. As a group, Sol_3 is the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. As in the case of $\widetilde{\mathrm{E}}(2)$, Sol_3 carries a 2-parameter family of left invariant metrics, which can be described in a very similar way as in the case of the universal cover $\widetilde{\mathrm{E}}(2)$ of the Euclidean group, so we will only detail the differences between both cases. Using standard coordinates (x, y, z) in $\mathrm{Sol}_3 = \mathbb{R}^2 \rtimes_A \mathbb{R}$, a basis of the Lie algebra $\mathfrak{g}$ of Sol_3 is given by

$$E_1(x, y, z) = e^{-z} \partial_x, \qquad E_2(x, y, z) = e^{z} \partial_y, \qquad E_3 = \partial_z,$$

and the Lie bracket is determined by the equations

$$[E_1, E_2] = 0, \quad [E_3, E_2] = E_2, \quad [E_3, E_1] = -E_1. \tag{2.32}$$

To describe the left invariant metrics on Sol_3, we declare the basis $\{E_1' = \varepsilon_1(E_1+E_2), E_2' = \varepsilon_2(E_1-E_2), E_3' = E_3\}$ to be orthonormal, for some $\varepsilon_1, \varepsilon_2 > 0$ (again we are working up to rescaling). Then,

$$[E_2', E_3'] = \varepsilon_2(E_1+E_2) = \frac{\varepsilon_2}{\varepsilon_1}E_1', \qquad [E_3', E_1'] = \varepsilon_1(-E_1+E_2) = -\frac{\varepsilon_1}{\varepsilon_2}E_2'$$

and thus, the basis $\{E_1', E_2', E_3'\}$ satisfies equation (2.24) with $c_1 = \frac{\varepsilon_2}{\varepsilon_1}$ and $c_2 = -\frac{1}{c_1}$ and $c_3 = 0$. After relabeling the E_i' as E_i, we find that $\{E_1, E_2, E_3\}$ is an orthonormal basis with $[E_1, E_2] = 0$, $[E_3, E_1] = -\frac{1}{c_1}E_2$, $[E_2, E_3] = c_1E_1$. From here and (2.9), (2.10), we deduce that up to rescaling, the metric Lie groups supported by Sol_3 are just $\mathbb{R}^2 \rtimes_{A(c_1)} \mathbb{R}$ endowed with its canonical metric, where

$$A(c_1) = \begin{pmatrix} 0 & c_1 \\ 1/c_1 & 0 \end{pmatrix}, \qquad c_1 > 0. \tag{2.33}$$

(Note that we have used that a change of sign in the matrix $A = A(c_1)$ just corresponds to a change of orientation). Finally, since the matrices $A(c_1)$, $A(1/c_1)$ in (2.33) are congruent, we can restrict the range of c_1 to $[1, \infty)$ in this last description of left invariant metrics on Sol_3.

2.8. Moduli spaces of unimodular and non-unimodular three-dimensional metric Lie groups. The moduli space of unimodular, three-dimensional metric Lie groups can be understood with the following pictorial representation that uses the numbers c_1, c_2, c_3 in (2.24), see also Table 1 in Section 2.6. In these (c_1, c_2, c_3)-coordinates, $\mathrm{SU}(2)$ corresponds to the open positive quadrant $\{c_1 > 0, c_2 > 0, c_3 > 0\}$ (three-dimensional, meaning that the space of left invariant metrics on $\mathrm{SU}(2)$ is three-parametric) and $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ to the open quadrant $\{c_1 > 0, c_2 > 0, c_3 < 0\}$ (also three-dimensional). Both three-dimensional quadrants have a common part of their boundaries which corresponds to the set of left invariant metrics on $\widetilde{\mathrm{E}}(2)$, which is represented by the two-dimensional quadrant $\{c_1 > 0, c_2 > 0, c_3 = 0\}$. Sol_3 corresponds to the two-dimensional quadrant $\{c_1 > 0, c_2 < 0, c_3 = 0\}$. The two-dimensional quadrants corresponding to $\widetilde{\mathrm{E}}(2)$ and Sol_3 have in their common boundaries the half line $\{c_1 > 0, c_2 = c_3 = 0\}$, which represents the 1-parameter family of metrics on Nil_3 (all the same after rescaling). The origin $\{c_1 = c_2 = c_3\}$ corresponds to $\mathbb{R}^3$ with its usual metric, see Figure 3.

A description of the moduli space of non-unimodular, three-dimensional metric Lie groups is as follows. By Lemma 2.11, any such metric Lie group is isomorphic and isometric to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some matrix $A \in \mathcal{M}_2(\mathbb{R})$ with $\mathrm{trace}(A) > 0$. The space of such matrices is four-dimensional, but the corresponding moduli space of metric Lie groups is 2-parametric after scaling, as follows from Lemma 2.12 (recall that the condition $\mathrm{trace}(A) = 2$, which is assumed in Lemma 2.12, is equivalent to scaling the metric by the multiplicative factor $\frac{1}{4}\mathrm{trace}(A)^2$).

(1) The cases $D < 1$ produce diagonalizable matrices $A = A(D, b)$ given by (2.19), with $a = a(b)$ defined by (2.22) for any $b \in [0, \infty)$, also see (2.21). The matrix A has two different eigenvalues adding up to 2 (the discriminant of the characteristic equation of A is $4(1-D) > 0$). The matrices $A(D, b = 0)$ converge as $D \to 1^-$ to I_2. Hence, the corresponding metric Lie groups $\mathbb{R}^2 \rtimes_{A(D,0)} \mathbb{R}$ limit as $D \to 1^-$ to $\mathbb{H}^3$ with its usual metric. The remaining metric Lie groups with Milnor D-invariant equal to 1 can be also obtained as a limit of appropriately

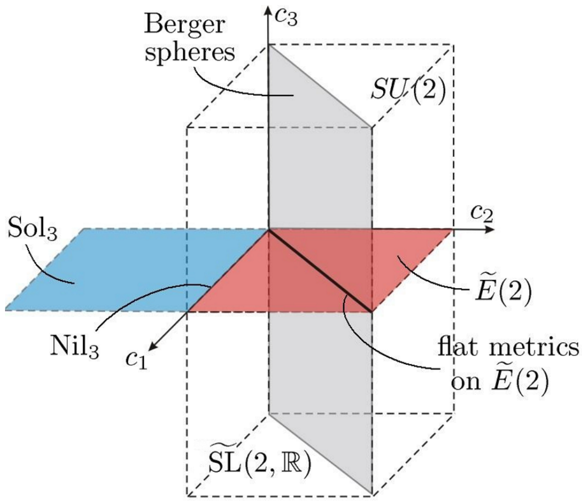

FIGURE 3. Pictorial representation of the three-dimensional unimodular metric Lie groups. The upper quarter of plane $\{c_1 = c_2, c_3 > 0\}$ corresponds to the Berger spheres, while the lower quarter $\{c_1 = c_2, c_3 < 0\}$ corresponds to the $\mathbb{E}(\kappa,\tau)$-spaces with $\kappa < 0$, $\tau \neq 0$, which are isometric to $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ with certain related left invariant metrics. In rigor, $\widetilde{E}(2)$ is also in the boundary 2-dimensional quadrants $\{c_1 > 0, c_2 = 0, c_3 > 0\} \cup \{c_1 = 0, c_2 > 0, c_3 > 0\}$ (this is just a permutation of the roles of the subindexes in the c_i), and Sol_3 is also in the boundary 2-dimensional quadrants $\{c_1 > 0, c_2 = 0, c_3 < 0\} \cup \{c_1 = 0, c_2 > 0, c_3 < 0\}$.

chosen metric Lie groups of the form $\mathbb{R}^2 \rtimes_{A(D_n,b)} \mathbb{R}$ for any fixed value $b > 0$ and $D_n \to 1^-$.

Regarding the limit of the non-unimodular metric Lie groups as $D \to -\infty$, they limit to the unimodular group Sol_3 equipped with any of its left invariant metrics: this follows by considering, given $\varepsilon > 0$ and $c_1 \geq 1$, the matrix

$$A_1(c_1,\varepsilon) = \begin{pmatrix} \varepsilon & c_1 \\ 1/c_1 & \varepsilon \end{pmatrix}.$$

Since $\mathrm{trace}(A_1(c_1,\varepsilon)) = 2\varepsilon \neq 0$, then $\mathbb{R}^2 \rtimes_{A_1} \mathbb{R}$ with $A_1 = A_1(c_1,\varepsilon)$ produces a non-unimodular Lie group. Its Milnor D-invariant (note that $A_1(c_1,\varepsilon)$ is not normalized to have trace $= 2$) is

$$\frac{4\det(A_1(c_1,\varepsilon))}{\mathrm{trace}(A_1(c_1,\varepsilon))^2} = 1 - \frac{1}{\varepsilon^2},$$

which limits to $-\infty$ as $\varepsilon \to 0^+$. Finally, the limit of $A_1(c_1,\varepsilon)$ as $\varepsilon \to 0^+$ is the matrix $A(c_1)$ defined by (2.33), which, after scaling, describes an arbitrary left invariant metric on Sol_3.

(2) As we explained above, the case $D = 1$ only produces two group structures, that of $\mathbb{H}^3$ when the matrix A is a multiple of the identity matrix, and a second group X non-isomorphic to $\mathbb{H}^3$. $\mathbb{H}^3$ can only be equipped with a 1-parameter family of left invariant metrics, all the same up to scaling to the standard metric

of constant curvature -1. X has a 2-parameter family of left invariant metrics: one parameter is the scaling factor to get the condition $\mathrm{trace}(A)=2$, and the other one is given by the equation $1=D=(1-a^2)(1+b^2)$ in the matrix representation (2.19).

(3) In the case $D>1$ we have for each value of D a unique group structure, which supports a 2-parameter family of left invariant metrics by Lemma 2.12. One of these parameters is the scaling factor to get $\mathrm{trace}(A)=2$ and the other one is $b\in[\sqrt{D-1},\infty)$ so that $A=A(D,b)$ given by (2.19) and (2.22) is the matrix whose canonical metric on $\mathbb{R}^2\rtimes_A\mathbb{R}$ is the desired metric. As we saw in the proof of Lemma 2.13, the particular case $A(D,b=\sqrt{D-1})$ produces $a=0$ in (2.22), and the canonical metric on the Lie group $G(D)=\mathbb{R}^2\rtimes_{A(D,\sqrt{D-1})}\mathbb{R}$ has constant sectional curvature -1 by (2.23) for any $D>1$. These are metric Lie groups not isomorphic to $\mathbb{H}^3$ but isometric to this space form. Clearly, the limit as $D\to 1^+$ of $G(D)$ with its canonical metric is $\mathbb{H}^3$ with its standard metric, but with other choices of $b\in[\sqrt{D-1},\infty)$ and then taking limits as $D\to 1^+$ in $\mathbb{R}^2\rtimes_{A(D,b)}\mathbb{R}$ produces all possible different metric Lie groups $\mathbb{R}^2\rtimes_{A(1,b)}\mathbb{R}$ with underlying group structure X described in item (2) above.

To compute the limit of $\mathbb{R}^2\rtimes_{A(D,b)}\mathbb{R}$ as $D\to\infty$, take $c_1\in(0,1]$ and define $b(D)=\frac{1}{2c_1}\sqrt{(1+c_1^2)^2D-4c_1^2}$, which makes sense if D is large enough in terms of c_1. It is elementary to check that $b(D)\geq\sqrt{D-1}$, hence it defines $a(D)=a(b(D))$ by equation (2.22), and that $\frac{1-a(D)}{1+a(D)}=c_1^2$. Now consider the matrix

$$A_1(D,c_1)=\frac{c_1}{(1-a(D))b(D)}A(D,b(D))\overset{(2.19)}{=}\begin{pmatrix}\frac{1}{c_1b(D)} & -c_1\\ \frac{1}{c_1} & \frac{c_1}{b(D)}\end{pmatrix}.$$

Since $b(D)$ tends to ∞ as $D\to\infty$, then the limit of $A_1(D,c_1)$ as $D\to\infty$ is the matrix $A(c_1)$ defined in (2.31). This means that the limit as $D\to\infty$ of the non-unimodular metric Lie groups is, after a suitable rescaling, $\widetilde{\mathrm{E}}(2)$ with any of its left invariant metrics (note that we have considered an arbitrary value $c_1\in(0,1]$, which covers all possible left invariant metrics on $\widetilde{\mathrm{E}}(2)$, see the paragraph which contains (2.31)). Of course, the limit of $G(D)$ as $D\to\infty$ is, after homothetic blow-up, the flat $\widetilde{\mathrm{E}}(2)$. See Figure 4.

Finally, given $\varepsilon>0$ and $\delta\in\mathbb{R}$, consider the matrix $B(\varepsilon,\delta)=\begin{pmatrix}\varepsilon & 1\\ \delta & \varepsilon\end{pmatrix}$. The normalized Milnor D-invariant of $B=B(\varepsilon,\delta)$ is

$$\frac{4\det(B)}{\mathrm{trace}(B)^2}=1-\frac{\delta}{\varepsilon^2},$$

which covers all possible real values (in fact, we can restrict to values of δ in any arbitrarily small interval around $0\in\mathbb{R}$). In particular, any non-unimodular Lie group structure different from $\mathbb{H}^3$ can be represented as $\mathbb{R}^2\rtimes_{B(\varepsilon,\delta)}\mathbb{R}$ for appropriate ε,δ. Clearly, the limit of $B(\varepsilon,\delta)$ as $(\varepsilon,\delta)\to(0,0)$ is $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, which corresponds to Nil_3. This implies that for every non-unimodular Lie group different from $\mathbb{H}^3$, there exists a sequence of left invariant metrics on it such that the corresponding sequence of metric Lie groups converges to Nil_3 with its standard metric.

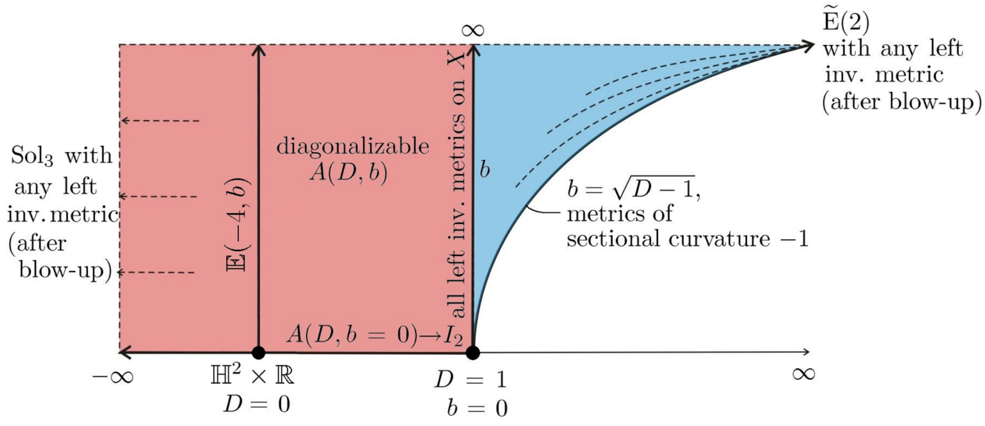

FIGURE 4. Representation of the moduli space of three-dimensional, non-unimodular metric Lie groups in terms of points in the (D,b)-plane, so that the group is $\mathbb{R}^2\rtimes_{A(D,b)}\mathbb{R}$ with $A(D,b)\in\mathcal{M}_2(\mathbb{R})$ scaled to have trace $=2$. X denotes the unique three-dimensional, non-unimodular Lie group with Milnor D-invariant $D=1$ which is not isomorphic to $\mathbb{H}^3$. All points in the line $D=0$ correspond to the Lie group $\mathbb{H}^2\times\mathbb{R}$ although metrically only $(D,b)=(0,0)$ corresponds to the product homogeneous manifold; the other points in the line $D=0$ represent metrically the $\mathbb{E}(\kappa,\tau)$ space with $\kappa=-4$ and $\tau=b$. The dotted lines in the interior of the region $\{(D,b)\mid D>1,\sqrt{D-1}\leq b\}$, correspond to the curves $D\mapsto(D,\frac{1}{2c_1}\sqrt{(1+c_1^2)^2D-4c_1^2})$ for $c_1\in(0,1]$ fixed, whose corresponding metric Lie groups converge after rescaling to $\widetilde{\mathrm{E}}(2)$ with any left invariant metric depending on c_1.

2.9. Three-dimensional unimodular metric semidirect products. Among the list of three-dimensional unimodular Lie groups, those which can be expressed as a semidirect product $\mathbb{R}^2\rtimes_A\mathbb{R}$ for some matrix $A\in\mathcal{M}_2(\mathbb{R})$, are just $\widetilde{\mathrm{E}}(2)$, Sol_3, Nil_3 and $\mathbb{R}^3$ ($\mathrm{SU}(2)$ is excluded since it is compact and $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is excluded because its only normal subgroup is its center which is infinite cyclic). Now we can summarize some of the results obtained so far in the following description of all possible left invariant metrics on these groups, in terms of the matrix A.

THEOREM 2.15 (Classification of unimodular metric semidirect products). *Let $(G,\langle,\rangle)$ be a unimodular metric Lie group which can be expressed as a semidirect product $\mathbb{R}^2\rtimes_A\mathbb{R}$, $A\neq 0$. Then, there exists an orthonormal basis E_1,E_2,E_3 for the Lie algebra $\mathfrak{g}$ of G so that $[E_1,E_2]=0$, and:*

(1) Each of the integral leaves $\mathbb{R}^2\rtimes_A\{z\}$ of the distribution spanned by E_1,E_2 has unit normal vector field $\pm E_3$ and its mean curvature is equal to $\frac{1}{2}\mathrm{trace}(A)=0$.
(2) After scaling the metric, the matrix A can be chosen uniquely as:

$$A=\begin{pmatrix}0 & \pm\frac{1}{a}\\ a & 0\end{pmatrix}\ \textit{for } a\in[1,\infty),\quad \textit{or}\quad A=\begin{pmatrix}0&1\\0&0\end{pmatrix},$$

in the sense that $(G, \langle , \rangle)$ *is isomorphic and isometric to* $\mathbb{R}^2 \rtimes_A \mathbb{R}$ *with its canonical metric.*

(3) *If* $\det(A) = -1$ *(resp.* $\det(A) = 1$, $\det(A) = 0$*), then the group is* Sol_3 *(resp.* $\widetilde{\mathrm{E}}(2)$, Nil_3*) and the corresponding matrices* A *produce all its left invariant metrics (up to scaling).*

2.10. The exponential map. Given an n-dimensional Lie group X, the exponential map $\exp\colon \mathfrak{g} \to X$ gives a diffeomorphism from a neighborhood of the origin in the Lie algebra $\mathfrak{g}$ of X onto a neighborhood of the identity element e in X. In this section we will show that if X is three-dimensional and simply-connected, then exp is a global diffeomorphism except in the cases $X = \mathrm{SU}(2)$, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and $\widetilde{\mathrm{E}}(2)$ where this property fails to hold.

Since $\mathrm{SU}(2)$ is compact and connected, then $\exp\colon \mathfrak{g} = \mathfrak{su}(2) \to \mathrm{SU}(2)$ is onto[3] but it cannot be injective; here

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\lambda & a \\ -\overline{a} & -i\lambda \end{pmatrix} \mid \lambda \in \mathbb{R}, a \in \mathbb{C} \right\}.$$

Regarding $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, it suffices to show that the exponential map of $\mathrm{SL}(2,\mathbb{R})$ is not onto. The Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ is $\mathfrak{sl}(2,\mathbb{R}) = \{B \in \mathcal{M}_2(\mathbb{R}) \mid \mathrm{trace}(B) = 0\}$. A straightforward computation gives that

$$\mathrm{trace}(e^B) = 2\cosh(\sqrt{-\det(B)}).$$

If $\det(B) > 0$, then $\cosh(\sqrt{-\det(B)}) = \cos(\sqrt{\det(B)})$ and hence trace $(e^B) \geq -2$. On the other hand, if $\det(B) \leq 0$, then $\sqrt{-\det(B)} \in \mathbb{R}$ and $\mathrm{trace}(e^B) \geq 2$. Thus $\exp(\mathfrak{sl}(2,\mathbb{R})) \subset \{A \in \mathrm{SL}(2,\mathbb{R}) \mid \mathrm{trace}(A) \geq -2\}$ which proves that $\exp\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ is not onto.

Finally, in the case of $\widetilde{\mathrm{E}}(2)$, exp is neither injective nor onto, see the proof of the next result and also see Remark 2.17 for details.

Proposition 2.16. *Let* X *be a three-dimensional, simply-connected Lie group with Lie algebra* $\mathfrak{g}$. *Suppose that* X *is not isomorphic to* $\mathrm{SU}(2)$, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *or* $\widetilde{\mathrm{E}}(2)$. *Then, the exponential map* $\exp\colon \mathfrak{g} \to X$ *is a diffeomorphism.*

Proof. Suppose X is not isomorphic to either $\mathrm{SU}(2)$ or $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ (for the moment, we do not exclude the possibility of X being isomorphic to $\widetilde{\mathrm{E}}(2)$). By the results in Sections 2.5 and 2.6, X is isomorphic to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some $A \in \mathcal{M}_2(\mathbb{R})$.

Recall that the basis E_1, E_2, E_3 of $\mathfrak{g}$ given by (2.6) satisfies $E_i(0,0,0) = e_i$, where e_1, e_2, e_3 is the usual basis of $\mathbb{R}^3$. Given $(\alpha, \beta, \lambda) \in \mathbb{R}^3$, the image by exp of $\alpha E_1 + \beta E_2 + \lambda E_3$ is the value at $t = 1$ of the 1-parameter subgroup $\gamma\colon \mathbb{R} \to \mathbb{R}^2 \rtimes_A \mathbb{R}$ such that $\gamma'(0) = \alpha e_1 + \beta e_2 + \lambda e_3$. We next compute such a subgroup.

Writing $\gamma(t) = (\mathbf{p}(t), z(t))$ (here $\mathbf{p} = (x, y)$ are the usual coordinates in $\mathbb{R}^2$), then (2.2) implies

$$\left(\mathbf{p}(t) + e^{Az(t)}\mathbf{p}(s), z(t) + z(s)\right) = \gamma(t) * \gamma(s) = \gamma(t+s) = (\mathbf{p}(t+s), z(t+s)).$$

It follows that $z\colon \mathbb{R} \to \mathbb{R}$ is a group homomorphism, hence $z(t) = \mu t$ for some $\mu \in \mathbb{R}$. Obviously $\mu = \lambda$ as $\gamma'(0) = \alpha e_1 + \beta e_2 + \lambda e_3$. Taking derivatives in

[3]The exponential map of a compact Lie group is always surjective onto the identity component.

$\mathbf{p}(t+s) = \mathbf{p}(t) + e^{Az(t)}\mathbf{p}(s)$ with respect to t and evaluating at $t=0$, we obtain

$$\mathbf{p}'(s) = \mathbf{p}'(0) + z'(0)Ae^{Az(0)}\mathbf{p}(s) = \mathbf{p}'(0) + \lambda A\,\mathbf{p}(s),$$

which is a linear ODE of first order with constant coefficients and initial condition $\mathbf{p}(0) = 0 \in \mathbb{R}^2$. Integrating this initial value problem we have

$$\mathbf{p}(s) = B(s,\lambda)\,\mathbf{p}'(0) \qquad \text{where } B(s,\lambda) = e^{\lambda sA}\int_0^s e^{-\lambda\tau A}d\tau,$$

from where

$$\exp(\alpha E_1 + \beta E_2 + \lambda E_3) = \gamma(1) = (\mathbf{p}(1), z(1)) = (B(1,\lambda)\,\mathbf{p}'(0), \lambda) \in \mathbb{R}^2 \rtimes_A \mathbb{R},$$

where $\mathbf{p}'(0) = (\alpha, \beta)$. Since $\lambda \in \mathbb{R} \mapsto B(1,\lambda)$ is smooth, then the property that $\exp\colon \mathfrak{g} \to \mathbb{R}^2 \rtimes_A \mathbb{R}$ is a diffeomorphism is equivalent to the invertibility of $B(1,\lambda)$ for all $\lambda \in \mathbb{R}$.

If $\lambda = 0$, then $B(1,0) = I_2$, hence we can assume $\lambda \neq 0$ for the remainder of this proof.

If $A = \delta I_2$ for some $\delta \in \mathbb{R}$ (we can assume $\delta \neq 0$ since otherwise $\mathbb{R}^2 \rtimes_A \mathbb{R} = \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^3$ and there is nothing to prove), then

$$B(s,\lambda) = e^{\lambda\delta s}\int_0^s e^{-\lambda\delta\tau}d\tau I_2 = \frac{e^{\lambda\delta s}}{\lambda\delta}(1 - e^{-\lambda\delta s})I_2,$$

which is invertible for all $s \in \mathbb{R} - \{0\}$. Hence in the sequel we will assume that A is not a multiple of I_2.

Since A is not a multiple of I_2, there exists $P \in Gl(2,\mathbb{R})$ such that $A = P^{-1}A_1P$ where $A_1 = \begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix}$. Thus,

$$B(s,\lambda) = P^{-1}e^{\lambda sA_1}P\int_0^s P^{-1}e^{-\lambda\tau A_1}P\,d\tau = P^{-1}\left(e^{\lambda sA_1}\int_0^s e^{-\lambda\tau A_1}\,d\tau\right)P,$$

which implies that we may assume $A = A_1$ in our study of the invertibility of $B(1,\lambda)$. We now distinguish two cases.

Case 1: $D = 0$. In this case,

$$B(s,\lambda) = \begin{cases} \begin{pmatrix} s & 0 \\ \lambda s^2/2 & s \end{pmatrix} & \text{if } T = 0, \\ \begin{pmatrix} s & 0 \\ \frac{e^{\lambda Ts} - T\lambda s - 1}{\lambda T^2} & \frac{e^{\lambda Ts} - 1}{\lambda T} \end{pmatrix} & \text{if } T \neq 0, \end{cases}$$

hence $B(s,\lambda)$ is invertible for all $s \in \mathbb{R} - \{0\}$.

Case 2: $D \neq 0$. A long but straightforward computation gives

$$\det(B(1,\lambda)) = \frac{1}{D\lambda^2}\left(1 + e^{\lambda T} - 2e^{\frac{\lambda T}{2}}\cos(\lambda w)\right),$$

where $\frac{T}{2} \pm iw$ are the complex eigenvalues of A (here $w \in \mathbb{R}$). Now, estimating $\cos(\lambda w) \leq 1$ we have $1 + e^{\lambda T} - 2e^{\frac{\lambda T}{2}}\cos(\lambda I) \geq (e^{\frac{\lambda T}{2}} - 1)^2$. Hence,

$$|\det(B(1,\lambda))| \geq \frac{(e^{\frac{\lambda T}{2}} - 1)^2}{|D|\lambda^2},$$

from where the desired invertibility of $B(1,\lambda)$ for all $\lambda \in \mathbb{R}$ holds whenever $T \neq 0$.

Finally, if $T=0$ (we still assume $D\neq 0$) then

$$\lambda\, B(s,\lambda)=\begin{cases}\begin{pmatrix}\frac{\sin(\sqrt{D}\lambda s)}{\sqrt{D}} & \cos(\sqrt{D}\lambda s)-1\\ \frac{1-\cos(\sqrt{D}\lambda s)}{D} & \frac{\sin(\sqrt{D}\lambda s)}{\sqrt{D}}\end{pmatrix} & \text{if } D>0,\\ \begin{pmatrix}\frac{\sinh(\sqrt{-D}\lambda s)}{\sqrt{-D}} & \cosh(\sqrt{-D}\lambda s)-1\\ \frac{1-\cosh(\sqrt{-D}\lambda s)}{D} & \frac{\sinh(\sqrt{-D}\lambda s)}{\sqrt{-D}}\end{pmatrix} & \text{if } D<0.\end{cases}$$

In particular,

$$\frac{\lambda^2 D}{2}\det B(s,\lambda)=\begin{cases}1-\cos(\sqrt{D}\lambda s) & \text{if } D>0,\\ 1-\cosh(\sqrt{D}\lambda s) & \text{if } D<0.\end{cases}$$

In the case $D<0$ we conclude that $B(s,\lambda)$ is invertible for all $s\in\mathbb{R}-\{0\}$, while in the case $D>0$ the invertibility of $B(s,\lambda)$ fails to hold exactly when $\sqrt{D}\lambda s\in 2\pi\mathbb{Z}-\{0\}$ (in fact, $B(s,0)=sI_2$ after applying the L'Hopital rule, and $B(s,\lambda)=0\in\mathcal{M}_2(\mathbb{R})$ if $\sqrt{D}\lambda s\in 2\pi\mathbb{Z}-\{0\}$). Finally, note that since scaling the matrix A does not affect the Lie group structure of $\mathbb{R}^2\rtimes_A\mathbb{R}$, we can reduce the case $D>0$, $T=0$ to the matrix $A=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, which corresponds to $\mathbb{R}^2\rtimes_A\mathbb{R}=\widetilde{\mathrm{E}}(2)$. □

Remark 2.17. The last proof shows that when X is isomorphic to $\widetilde{\mathrm{E}}(2)$, then $B(1,\lambda)=\frac{1}{\lambda}\begin{pmatrix}\sin\lambda & -1+\cos\lambda\\ 1-\cos\lambda & \sin\lambda\end{pmatrix}$ whenever $\lambda\neq 0$. This formula extends to $\lambda=0$ with $B(1,0)=I_2$. Therefore, $\exp\colon\mathfrak{g}\to\widetilde{\mathrm{E}}(2)$ maps the horizontal slab $S(-2\pi,2\pi)=\{(\alpha,\beta,\lambda)\mid -2\pi<\lambda<2\pi\}\subset\mathfrak{g}$ (here we are using coordinates w.r.t. E_1,E_2,E_3) diffeomorphically onto $\mathbb{R}^2\rtimes_A(-2\pi,2\pi)$, where $A=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. Each of the boundary planes $\Pi(\pm 2\pi)=\{\lambda=\pm 2\pi\}$ of $S(-2\pi,2\pi)$ is mapped under exp to one of the points $(0,0,\pm 2\pi)$. Hence $\exp|_{\Pi(2\pi)}$ is not injective and the differential of exp at every point in $\Pi(2\pi)$ is zero (the same property holds for $\Pi(-2\pi)$; in fact, the behavior of exp is periodic in the vertical variable λ with period 2π). In particular, $\exp(\mathfrak{g})$ consists of the complement of the union of the punctured horizontal planes $[\mathbb{R}^2\rtimes_A\{2k\pi\}]-\{(0,0,2k\pi)\}$, $k\in\mathbb{Z}-\{0\}$.

Regarding the 1-parameter subgroups of $\widetilde{\mathrm{E}}(2)$, their description is as follows. Let $\gamma\colon\mathbb{R}\to\widetilde{E}(2)$ be the 1-parameter subgroup determined by the initial condition $\gamma'(0)=(\mathbf{p}'(0),\lambda)\neq(\mathbf{0},0)\in\mathbb{R}^2\times\mathbb{R}$ where $\mathbf{p}'(0)\in\mathbb{R}^2$.

- If $\mathbf{p}'(0)=0$, then $\gamma(\mathbb{R})$ is the z-axis in $\mathbb{R}^2\rtimes_A\mathbb{R}$.
- If $\lambda=0$, then $\gamma(\mathbb{R})$ is the horizontal straight line $\{(t\,\mathbf{p}'(0),0)\mid t\in\mathbb{R}\}\subset\mathbb{R}^2\rtimes_A\{0\}$.
- If $\lambda\neq 0$ and $\mathbf{p}'(0)\neq\mathbf{0}$, then

$$\gamma(t\,\mathbf{p}'(0),t\lambda)=\left(\frac{1}{\lambda}\begin{pmatrix}\sin(t\lambda) & -1+\cos(t\lambda)\\ 1-\cos(t\lambda) & \sin(t\lambda)\end{pmatrix}\cdot\mathbf{p}'(0)^T,\ t\lambda\right),$$

  where $\mathbf{p}'(0)$ is considered as a column vector in the right-hand-side. It is straightforward to check that $\gamma(\mathbb{R})$ parameterizes the vertical circular helix with pitch λ over the circle of center $\frac{1}{\lambda}A\,\mathbf{p}'(0)$ that contains the z-axis. Each of these 1-parameter subgroups passes through all the points $(0,0,2k\pi)$, $k\in\mathbb{Z}$, see Figure 5.

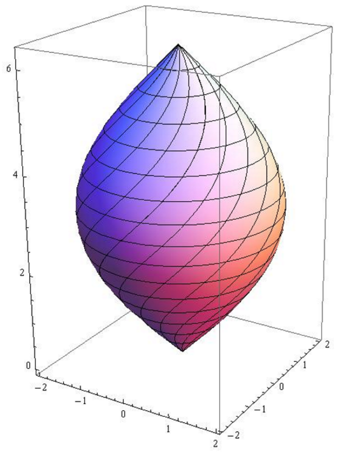

FIGURE 5. The 1-parameter subgroups of $\widetilde{\mathrm{E}}(2)$ with initial condition $\gamma'(0) = (e^{i\theta}, \lambda) \in \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^2 \times \mathbb{R}$ ($\lambda \neq 0$ fixed) foliate a surface invariant under vertical translation by $(0, 0, 2\pi)$ minus infinitely many peak singularities occurring at the points $(0, 0, 2k\pi)$, $k \in \mathbb{Z}$, one of which fundamental domains S_λ has been represented here. S_λ is a surface invariant under revolution around the z-axis in the natural (x, y, z)-coordinates and it has two cusp singularities, at the origin (lower horizontal plane) and at $(0, 0, 2\pi)$ (upper horizontal plane). Each of the 1-parameter subgroups that foliate the periodic surface $\cup_{k\in\mathbb{Z}}[S_\lambda + (0, 0, 2k\pi)] - \{\text{cusps}\}$ is a vertical circular helix with pitch λ over a circle of radius $\frac{1}{\lambda}$ which contains the z-axis. The surfaces S_λ obtained for different values of λ differ in the angle of the cusps, but not in the cusps themselves.

2.11. Isometries with fixed points. In this section we study the isometries of a simply-connected, three-dimensional metric Lie group X which fix some point in X, in terms of the group structure and metric on X. For an isometry $\phi\colon X \to X$ fixing the identity element e of X, we will denote by $\mathrm{Fix}_0(\phi)$ the *component* of the fixed point set of ϕ which contains e.

PROPOSITION 2.18. *Let X be an n-dimensional metric Lie group (not necessarily simply-connected), whose isometry group $I(X)$ is also n-dimensional. If $\phi\colon X \to X$ is an isometry such that $\phi(e) = e$, then ϕ is a Lie group isomorphism of X. Furthermore, the following statements hold:*

(1) Given a 1-parameter subgroup Γ of X, then $\phi(\Gamma)$ is a 1-parameter subgroup and $\phi\colon \Gamma \to \phi(\Gamma)$ is a group isomorphism.
(2) $\mathrm{Fix}_0(\phi)$ is a subgroup of X, which is a totally geodesic submanifold in X.
(3) If $d\phi_e(v) = v$ for some $v \in T_eX - \{0\}$ and 1 is simple as an eigenvalue of $d\phi_e$, then the geodesic γ in X with initial conditions $\gamma(0) = e$, $\gamma'(0) = v$ is a 1-parameter subgroup of X and $\gamma(\mathbb{R}) \subset \mathrm{Fix}_0(\phi)$.

PROOF. As ϕ is an isometry, its differential ϕ_* preserves the vector space of Killing vector fields on X. Since the isometry group of X is assumed to have the same dimension as X, then every Killing vector field is right invariant; in particular,

ϕ_* is a Lie algebra automorphism of the space of right invariant vector fields on X. By integration, ϕ is an isomorphism of the opposite group structure $\star$ of X: $\phi(x \star y) = \phi(x) \star \phi(y)$, where $x \star y = yx$, $x, y \in X$. Then $\phi(xy) = \phi(y \star x) = \phi(y) \star \phi(x) = \phi(x)\phi(y)$, which proves that ϕ is a Lie group automorphism of X.

The properties in items (1), (2) and (3) follow immediately from the main statement and the fact that the fixed point set of an isometry of a Riemannian manifold is always totally geodesic. For example, to prove item (2) recall that the fixed point set of an automorphism of a group is always a subgroup. □

COROLLARY 2.19. *Let X be an n-dimensional metric Lie group (not necessarily simply-connected), whose isometry group $I(X)$ is also n-dimensional. If $\phi\colon X \to X$ is an isometry, then each component Σ of the fixed point set of ϕ is of the form pH for some subgroup H of X and some $p \in X$.*

PROOF. Let Σ be a component of the fixed point set of ϕ, and take $p \in \Sigma$. Then the isometry $\psi = l_p^{-1} \circ \phi \circ l_p$ satisfies $\psi(e) = e$ and has $p^{-1}\Sigma$ as the component of its fixed point set passing through e. By Proposition 2.18, $p^{-1}\Sigma$ is a subgroup of X from which the corollary follows. □

DEFINITION 2.20. Given a point p in a Riemannian n-manifold X, we say that a tangent vector $v \in T_pX - \{0\}$ is a *principal Ricci curvature direction* at p if v is an eigenvector of the Ricci tensor at p. The corresponding eigenvalue of the Ricci tensor at p is called a *principal Ricci curvature.*

If the Riemannian n-manifold X is homogeneous, then clearly the principal Ricci curvatures are constant. The usefulness of the concept of principal Ricci curvature direction in our setting where X is a three-dimensional metric Lie group is that when the three principal Ricci curvatures of X are distinct, then there is an orthonormal basis E_1, E_2, E_3 of the Lie algebra of X such that for any isometry ϕ of X, the differential of ϕ satisfies $\phi_*(E_i) = \pm E_i$. In particular, in this case $\phi_*\colon \mathfrak{g} \to \mathfrak{g}$ is an Lie algebra isomorphism.

Given a Riemannian manifold X and a point $p \in X$, the *stabilizer* of p is $\mathrm{Stab}_p = \{\phi \in I(X) \mid \phi(p) = p\}$, which is a subgroup of the isometry group $I(X)$ of X. We will denote by Stab_p^+ the subgroup of Stab_p of orientation-preserving isometries.

PROPOSITION 2.21. *Let X be a simply-connected, three-dimensional metric Lie group. Given any element $p \in X$ and a unitary principal Ricci curvature direction v at p, the following properties are true:*

(1) If X is unimodular and v goes in the direction of one of the vectors[4] *$(E_i)_p$ with $E_i \in \mathfrak{g}$ given by (2.24), $i = 1, 2, 3$, then there exists an element $\phi \in \mathrm{Stab}_p^+$ of order two such that $d\phi_p(v) = v$. In particular, Stab_p^+ contains a dihedral group $D_2(p) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, if the isometry group $I(X)$ of X is of dimension three, then $\mathrm{Stab}_p^+ = D_2(p)$.*

[4]This condition for v is necessary: consider the left invariant metric on Sol_3 given by the structure constants $c_1 = -c_2 = 1$, $c_3 = 0$. Then, (2.25) produces $-\mu_1 = \mu_2 = 1$, $\mu_3 = 0$ and (2.27) gives $\mathrm{Ric}(E_1) = \mathrm{Ric}(E_2) = 0$, $\mathrm{Ric}(E_3) = -2$, hence all vectors v in the two-dimensional linear subspace of $\mathfrak{g} = T_e\mathrm{Sol}_3$ spanned by $\{(E_1)_e, (E_2)_e\}$ are principal Ricci curvature directions, although those which admit the desired $\phi \in \mathrm{Stab}_e^+$ of order two such that $d\phi_p(v) = v$ reduce to v in the direction of either E_1 or E_2.

(2) If the isometry group $I(X)$ of X is of dimension four, then there exists a unique principal Ricci direction w at p whose Ricci eigenvalue is different from the other Ricci eigenvalue (which has multiplicity two), and Stab_p^+ *contains an* $\mathbb{S}^1$*-subgroup, all whose elements have differentials at p which fix w.*

(3) If $I(X)$ has dimension six, then Stab_p *is naturally isomorphic to the orthogonal group $O(3)$.*

(4) If $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is non-unimodular and $I(X)$ has dimension three (here $A \in \mathcal{M}_2(\mathbb{R})$), then $\mathrm{Stab}_p^+ = \{1_X, l_p \circ \psi \circ l_{p^{-1}}\} \cong \mathbb{Z}_2$*, where $\psi\colon X \to X$ is the isometry $(x,y,z) \mapsto (-x,-y,z)$.*

Proof. First suppose that the isometry group of X has dimension six. In this case X has constant curvature and item (3) is well-known to hold.

Next suppose that the isometry group of X has dimension four. It is also well-known that in this case, X is isometric to an $\mathbb{E}(\kappa,\tau)$-space, for which the statement in item (2) can be directly checked.

Assume that X is unimodular with isometry group of dimension three, and let us denote its left invariant metric by g. As explained in Section 2.6, there exists a g-orthonormal basis E_1, E_2, E_3 of $\mathfrak{g}$ which satisfies (2.24) and this basis diagonalizes the Ricci tensor of X. By hypothesis v is one of these directions E_1, E_2, E_3, say $v = E_3$. Recall that changing the left invariant metric of X corresponds to changing the lengths of E_1, E_2, E_3 while keeping them orthogonal (this corresponds to changing the values of the constants c_1, c_2, c_3 in (2.24) without changing their signs). During this deformation, these vector fields continue to be principal Ricci directions of the deformed metric. Perform such a deformation until arriving to a special left invariant metric g_0 on the same Lie group, which is determined depending on the Lie group as indicated in the following list, see the paragraph in which equation (2.24) lies.

(1) If X is isomorphic to $\mathrm{SU}(2)$, then g_0 is a metric of constant sectional curvature.
(2) If X is isomorphic to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, then g_0 is the $\mathbb{E}(\kappa,\tau)$-metric given by the structure constants $c_1 = c_2 = 1$, $c_3 = -1$ (up to a permutation of the indexes of the c_i).
(3) If X is isomorphic to $\widetilde{\mathrm{E}}(2)$, then g_0 is the flat metric given by the structure constants $c_1 = c_2 = 1$, $c_3 = 0$ (up to a permutation).
(4) If X is isomorphic to Sol_3, then g_0 is the metric given by the structure constants $c_1 = -c_2 = 1$, $c_3 = 0$ (up to a permutation).
(5) If X is isomorphic to Nil_3, then $g_0 = g$.

In each of the metric Lie groups (X, g_0) listed above, there exists an order two, orientation-preserving isometry ϕ of (X, g_0) such that $\phi(p) = p$, $d\phi_p(v) = v$ and whose differential leaves invariant the Lie algebra $\mathfrak{g}$ of left invariant vector fields of X. We claim that $\phi^* g = g$, i.e., ϕ is the desired isometry for (X, g). This follows from the fact that as an endomorphism of $\mathfrak{g}$, the eigenvalues of the differential map ϕ_* are $-1, -1, 1$, and hence,

$$(\phi^* g)(E_i, E_i) = g(\phi_*(E_i), \phi_*(E_i)) = g(\pm E_i, \pm E_i) = g(E_i, E_i),$$

for $i = 1, 2, 3$. Hence $\phi^* g = g$. To complete the proof of item (1) of the proposition it remains to prove that any $\sigma \in \mathrm{Stab}_p^+ - \{1_X\}$ is one of these elements ϕ. Clearly we can assume $p = e$. Since we are presently assuming that the isometry group of X is three-dimensional, then one of the principal Ricci curvature directions has a distinct principal Ricci curvature value, say E_3, and so satisfies $\sigma_*(E_3) = \pm E_3$. After possibly composing σ with the order two isometry $\phi \in \mathrm{Stab}_e^+$ of X corresponding to rotation of angle π around $E_1(e) \in T_eX$, we may assume that $\sigma_*(E_3) = E_3$. By Proposition 2.18, σ is a group isomorphism, and so $\sigma_*(E_1) = \cos\theta\, E_1 + \sin\theta\, E_2$ and $\sigma_*(E_2) = -\sin\theta\, E_1 + \cos\theta\, E_2$ for some $\theta \in [0, 2\pi)$ and it suffices to check that $\theta = \pi$. Calculating we find that

$$-c_2 \sin\theta\, E_1 + c_2 \cos\theta\, E_2 = c_2\, \sigma_*(E_2) = \sigma_*(c_2 E_2) = \sigma_*([E_3, E_1])$$
$$= [\sigma_*(E_3), \sigma_*(E_1)] = [E_3, \cos\theta\, E_1 + \sin\theta\, E_2] = c_2 \cos\theta\, E_2 - c_1 \sin\theta\, E_1.$$

Since the isometry group of X is assumed to be three-dimensional, then $c_1 \neq c_2$ and so the above two equations imply that $\sin\theta = 0$, which means that $\theta = \pi$. Hence, σ is a rotation by angle π around E_3 as desired, which completes the proof of item (1).

To finish the proof, suppose X is a non-unimodular metric Lie group with isometry group of dimension three. By Lemma 2.11, X is isomorphic and isometric to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical metric, for some matrix $A \in \mathcal{M}_2(\mathbb{R})$. We can assume without loss of generality that A is not a multiple of the identity (otherwise X is isomorphic and homothetic to $\mathbb{H}^3$ with its standard metric, which is covered in item (3) of this proposition). Hence, up to scaling the metric we can assume that A has trace 2 and it is given by (2.19). Recall that in the second possibility just after Lemma 2.11, we constructed an orthonormal[5] basis $\{E_1, E_2, E_3\}$ of the Lie algebra $\mathfrak{g}$ of X which diagonalizes the Ricci tensor of X, and the corresponding Ricci eigenvalues are given by (2.23), from where one easily deduces that if exactly two of these eigenvalues coincide then the matrix A is given by (2.19) with $a = 1$. In this case, the vector field $E_2 = \partial_y$ given by (2.6) is both left and right invariant, from where one deduces that the isometry group of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric is four-dimensional (see the proof of Theorem 2.14 for details), a contradiction. Hence the principal Ricci curvatures of X are distinct.

Let $\phi \in \mathrm{Stab}_p^+ - \{1_X\}$. Since the principal Ricci curvatures of X are pairwise distinct, then ϕ maps each principal Ricci curvature direction E_i into itself (up to sign) and ϕ has order two. As an endomorphism of $\mathfrak{g}$, the differential ϕ_* of ϕ must have eigenvalues 1 and -1, with -1 of multiplicity two since ϕ is orientation-preserving. In particular, the set of fixed points of ϕ is a geodesic passing through p which is an integral curve of one of the vector fields E_1, E_2, E_3. If $\phi_*(E_1) = E_1$, then $\nabla_{E_1} E_1 = 0$. By equations (2.11) and (2.19), this implies $1 + a = 0$. But in the representation of A given in (2.19) we had $a \geq 0$, which gives a contradiction. A similar argument in the case $\phi_*(E_2) = E_2$ shows $1 - a = 0$, but we have already excluded the value $a = 1$ for a. Hence, ϕ satisfies $\phi_*(E_3) = E_3$ and $\phi_*(E_i) = -E_i$, $i = 1, 2$. After left translating p to e, to prove item (4) it suffices to check that $\phi(x, y, z) = (-x, -y, z)$. This follows from the usual uniqueness result of local isometries in terms of their value and differential at a given point. This completes the proof of the proposition. □

Definition 2.22. We will say that a non-trivial isometry of a three-dimensional metric Lie group X is a *reflectional symmetry* if its fixed point set contains a component which is a surface.

Our next goal is to characterize which simply-connected, three-dimensional metric Lie groups admit a reflectional symmetry (apart from the trivial case of six

[5]With respect to the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

dimensional isometry group) or more generally, which ones admit an orientation-reversing isometry. Closely related to this problem is to know which of these metric Lie groups admit a two-dimensional subgroup which is totally geodesic. A particularly interesting example in this situation is the Lie group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with some special left invariant metrics, which we next study in some detail.

EXAMPLE 2.23. Consider the simply-connected, unimodular Lie group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. As explained in Section 2.6, any left invariant metric g on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is determined by the structure constants c_1, c_2, c_3 appearing in (2.24). In our case, two of the structure constants of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ can be assumed to be positive and the third one is negative. We can order these structure constants so that $c_1 < 0 < c_2 < c_3$ (we could have scaled the left invariant metric g so that one of these constants has absolute value 1, but we will not do it in order to keep symmetry of the notation). By (2.25), μ_1 is clearly positive while μ_3 is negative. We will consider the case $\mu_2 = 0$. Therefore,

$$c_2 = c_1 + c_3 > 0, \quad \mu_1 = c_3, \quad \mu_3 = c_1.$$

Consider the equation

$$\tan^2\theta = -c_1/c_3, \tag{2.34}$$

which has two solutions $\theta_0, -\theta_0 \in (-\pi,\pi)$ (note that θ depends on the structure constants and thus it depends on the left invariant metric on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, but it is independent of a rescaling of the metric). For each of these values of θ, we define the left invariant vector field

$$X = \cos\theta\, E_3 + \sin\theta\, E_1$$

(here E_1, E_2, E_3 is a positively oriented g-orthonormal basis of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ given by (2.24), with associated structure constants $c_1, c_2 = c_1 + c_3, c_3$). We claim that the two-dimensional linear subspace $\Pi_\theta = \mathrm{Span}\{E_2, X\}$ of $T_e\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is a subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ and that the corresponding two-dimensional subgroup $\Sigma_\theta = \exp(\Pi_\theta)$ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is totally geodesic. To see this, first note that

$$[E_2, X] = \cos\theta[E_2,E_3] + \sin\theta[E_2,E_1] = c_1\cos\theta E_1 - c_3\sin\theta E_3 = \lambda X,$$

where $\lambda = c_1\cot\theta \overset{(2.34)}{=} -c_3\tan\theta \neq 0$. Therefore, Π_θ is closed under Lie bracket and it defines under exponentiation an $\mathbb{H}^2$-type subgroup Σ_θ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Clearly the unit length vector field $N = E_2 \times X = \cos\theta\, E_1 - \sin\theta\, E_3 \in \mathfrak{sl}(2,\mathbb{R})$ is normal to Σ_θ. Thus, the Weingarten operator of Σ_θ is determined by

$$\begin{aligned}
\nabla_{E_2}N &= \cos\theta\,\nabla_{E_2}E_1 - \sin\theta\,\nabla_{E_2}E_3 \overset{(2.26)}{=} -\mu_2\cos\theta E_2 - \mu_2\sin\theta E_1 = 0,\\
\nabla_X N &= \cos^2\theta\,\nabla_{E_3}E_1 - \sin^2\theta\,\nabla_{E_1}E_3 = (\mu_3\cos^2\theta + \mu_1\sin^2\theta)E_2\\
&= (c_1\cos^2\theta + c_3\sin^2\theta)E_2 \overset{(2.34)}{=} 0,
\end{aligned}$$

from where we deduce that Σ_θ is totally geodesic. The group Σ_θ depends on $\theta = \theta(\frac{-c_1}{c_3}) \in (0,\pi)$, and $-c_1/c_3$ also parametrizes (up to homothety) the left invariant metric.

Next we prove that none of these subgroups Σ_θ defines a reflective symmetry ϕ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with the corresponding left invariant metric[6]. Arguing by contradiction, suppose that ϕ exists. First note that if F is the Killing vector field on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ determined by $F_e = N_e$, then $\phi_*(F)$ is again Killing and thus $\phi_*(F)$ is right invariant. Therefore, $\phi_*(F)$ is determined by its value at e, and since $[\phi_*(F)]_e = d\phi_e(F_e) = d\phi_e(N_e) = -N_e = -F_e$, then we conclude that $\phi_*(F) = -F$ (globally on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$). In particular, if $p \in \Sigma_\theta \subset \mathrm{Fix}(\phi)$, then $d\phi_p(F_p) = -F_p$. Since the eigenspace associated to the eigenvalue -1 of $d\phi_p$ is generated by N_p, then we deduce that F_p, N_p are collinear for every $p \in \Sigma_\theta$ (the same arguments can be applied to any n-dimensional metric Lie group with n-dimensional isometry group and a orientation-reversing isometry whose fix point set is a hypersurface). This is impossible in our setting, as we next demonstrate. Clearly it suffices to prove this property downstairs, i.e., in $\mathrm{SL}(2,\mathbb{R})$. Then the right invariant vector field F is given by $F_A = F_e \cdot A = N_e \cdot A$ for all $A \in \mathrm{SL}(2,\mathbb{R})$ while the left invariant vector field N satisfies $N_A = A \cdot N_e$ for all $A \in \mathrm{SL}(2,\mathbb{R})$. We now compute N_e and $\exp(\Pi_\theta) \subset \mathrm{SL}(2,\mathbb{R})$ explicitly.

First observe that

$$E_1 = \frac{\sqrt{c_2c_3}}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \frac{\sqrt{-c_1c_3}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_3 = \frac{\sqrt{-c_1c_2}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the orthonormal basis of $\mathfrak{sl}(2,\mathbb{R})$ we have been working with. To check this, note that E_1, E_2, E_3 are multiples of the matrices appearing in (2.29), so we just need to check the Lie brackets give the desired structure constants c_1, c_2, c_3. This is a direct computation that only uses (2.29). Thus,

$$N_e = \cos\theta \frac{\sqrt{c_2c_3}}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \sin\theta \frac{\sqrt{-c_1c_2}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Using (2.34) one has $\cos\theta = \sqrt{\frac{c_3}{c_3-c_1}}$, $\sin\theta = \sqrt{\frac{-c_1}{c_3-c_1}}$, hence

$$N_e = \frac{1}{2}\sqrt{\frac{c_3+c_1}{c_3-c_1}}\begin{pmatrix} 0 & c_1-c_3 \\ c_1+c_3 & 0 \end{pmatrix}. \tag{2.35}$$

Now we compute the subgroup $\exp(\Pi_\theta) \subset \mathrm{SL}(2,\mathbb{R})$. Given $\lambda, \delta \in \mathbb{R}$,

$$\lambda E_2 + \delta X = \frac{\sqrt{-c_1c_3}}{2}\left[\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \delta\sqrt{\frac{c_1+c_3}{c_3-c_1}}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\right],$$

which after exponentiation gives

$$\exp(\Pi_\theta) = \left\{ A(\lambda,\delta) = \begin{pmatrix} e^{\frac{\sqrt{-c_1c_3}}{2}\lambda} & 0 \\ 2\frac{\delta}{\lambda}\sqrt{\frac{c_1+c_3}{c_3-c_1}}\sinh\left(\frac{\sqrt{-c_1c_3}}{2}\lambda\right) & e^{-\frac{\sqrt{-c_1c_3}}{2}\lambda} \end{pmatrix} \ : \ \lambda, \delta \in \mathbb{R} \right\}$$

The mapping $(\lambda, \delta) \mapsto A(\lambda, \delta)$ is injective, hence $\exp(\Pi_\theta)$ is topologically a plane in $\mathrm{SL}(2,\mathbb{R})$, and it lifts to countably many copies on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, exactly one of which passes through the origin (this is the subgroup Σ_θ).

[6]Interestingly, when the parameter $c_1 < 0$ degenerates to 0, then the corresponding metric Lie groups converge to $\widetilde{\mathrm{E}}(2)$ with its flat metric; the other "degenerate limit" of these left invariant metrics on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ occurs when $c_2 > 0$ converges to 0, in which case we get the standard metric on Sol_3. In both cases, the totally geodesic subgroups $\Sigma_\theta, \Sigma_{-\theta}$ limit to related totally geodesic 2-dimensional subgroups of either $\widetilde{\mathrm{E}}(2)$ or Sol_3, and these limiting totally geodesic 2-dimensional subgroups are the fixed point set of reflectional isometries of these metric Lie groups.

Finally, one easily checks that the a_{11}-element of the matrix $A(\lambda,\delta)\cdot N_e$ is zero for all $\lambda,\delta\in\mathbb{R}$, while the a_{11}-element of $N_e\cdot A(\lambda,\delta)$ is

$$-(c_1+c_3)\frac{\delta}{\lambda}\sinh\left(\frac{\sqrt{-c_1c_3}}{2}\lambda\right),$$

which can only vanish if $\delta=0$. This clearly implies that $N_e\cdot A(\lambda,\delta)$, $A(\lambda,\delta)\cdot N_e$ cannot be collinear everywhere in $\exp(\Pi_\theta)$, and thus the reflectional symmetry of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ fixing Σ_θ does not exist (similarly for $\Sigma_{-\theta}$).

PROPOSITION 2.24. *Let X be a simply-connected, three-dimensional metric Lie group with a three or four-dimensional isometry group. If $\phi\colon X\to X$ is a orientation-reversing isometry, then:*

(1) X admits a reflectional symmetry.

(2) If X has a four-dimensional isometry group, then X is isomorphic to $\mathbb{H}^2\times\mathbb{R}$ and after scaling, the left invariant metric on X is isometric to the standard product metric on $\mathbb{H}^2\times\mathbb{R}$. In this case, ϕ is either a reflectional symmetry with respect to a vertical plane[7] *or it is the composition of a reflectional symmetry with respect to some level set $\mathbb{H}^2\times\{t\}$ with a rotation around a vertical line.*

(3) If X has a three-dimensional isometry group, then, after scaling, X is isomorphic and isometric to $\mathbb{R}^2\rtimes_A\mathbb{R}$ with its canonical metric, where

$$A(b)=\begin{pmatrix}1&0\\0&b\end{pmatrix}, \tag{2.36}$$

for some unique $b\in\mathbb{R}-\{1\}$. Furthermore,

(a) If $b\neq-1$ (i.e., X is not isomorphic to Sol_3*), then ϕ is conjugate by a left translation to one of the reflectional symmetries $(x,y,z)\mapsto(-x,y,z)$ or $(x,y,z)\mapsto(x,-y,z)$.*

(b) If $b=-1$, then ϕ is conjugate by a left translation to one of the reflectional symmetries $(x,y,z)\mapsto(-x,y,z)$, $(x,y,z)\mapsto(x,-y,z)$ or ϕ is one of the isometries $(x,y,z)\mapsto(y,-x,-z)$ or $(x,y,z)\mapsto(-y,x,-z)$.

PROOF. Since $(l_{(\phi(e)^{-1})}\circ\phi)(e)=e$ and left translation by $\phi(e)^{-1}$ is orientation preserving, without loss of generality we may assume that the orientation-reversing isometry $\phi\colon X\to X$ fixes the origin e of X. We divide the proof in four cases.

CASE I-A: Suppose X is non-unimodular with four-dimensional isometry group. By Theorem 2.14, X is isomorphic to $\mathbb{H}^2\times\mathbb{R}$ and, after rescaling the metric, X is isometric to $\mathbb{R}^2\rtimes_A\mathbb{R}$ with its canonical metric, where

$$A=\begin{pmatrix}2&0\\2b&0\end{pmatrix}.$$

We claim that $b=0$ from where item (2) of the proposition follows directly. First note that since the matrix A is written in the form (2.19) with $a=1$, then the vector field $E_2=\partial_y$ is a principal Ricci curvature direction and $\mathrm{Ric}(E_2)$ is different from $\mathrm{Ric}(E_1)=\mathrm{Ric}(E_3)$ (here E_1,E_2,E_3 is the left invariant basis given by (2.6)), from where one has $\phi_*(E_2)=\pm E_2$. After possibly composing ϕ with a rotation by angle π around the z-axis (which is an orientation-preserving isometry of the

[7]A vertical plane in $\mathbb{H}^2\times\mathbb{R}$ is the cartesian product $\gamma\times\mathbb{R}$ of some geodesic γ in $\mathbb{H}^2$ with $\mathbb{R}$, and the reflectional symmetry is the product of the geodesic reflection in $\mathbb{H}^2$ with respect to γ with the identity map in $\mathbb{R}$.

canonical metric, see the comment just after equation (2.12)), we may assume that $\phi_*(E_2) = E_2$, and so the differential $d\phi_e$ is a reflection with respect to a two-dimensional linear subspace Π of T_eX which contains $(E_2)_e$.

Let $\mathrm{Fix}(\phi) \subset X$ be the fixed point set of ϕ. Since ϕ is an isometry, then $\mathrm{Fix}(\phi)$ is a (possibly non-connected) totally geodesic submanifold of X. The chain rule clearly implies that the tangent space $T_e\mathrm{Fix}(\phi)$ is contained in Π. We claim that $T_e\mathrm{Fix}(\phi) = \Pi$; to see this, take a vector $v \in \Pi$ and let Γ_v be the geodesic of X with initial conditions $\Gamma_v(0) = e$, $\Gamma_v'(0) = v$. Since the isometry ϕ satisfies $\phi(e) = e$ and $d\phi_e(v) = v$, then $\phi(\Gamma_v) = \Gamma_v$ and Γ_v is contained in $\mathrm{Fix}(\phi)$. As this occurs for every $v \in \Pi$, then $T_e\mathrm{Fix}(\phi) = \Pi$ as desired. Since (2.11) gives $\nabla_{E_2}E_2 = 0$ (here ∇ is the Levi-Civita connection in X), then the 1-parameter subgroup Γ associated to E_2 is a geodesic of X and thus, Γ is contained in $\mathrm{Fix}(\phi)$. Since X is isometric to an $\mathbb{E}(\kappa, \tau)$-space with E_2 playing the role of the vertical direction (kernel of the Riemannian submersion), then there exist rotational isometries of X about Γ of any angle. Composing ϕ with a rotation by a suitable angle about Γ we may assume E_3 is tangent at e to $\mathrm{Fix}(\phi)$. Since $\mathrm{Fix}(\phi)$ is totally geodesic, then $\nabla_{(E_3)_e}E_2$ is tangent to $\mathrm{Fix}(\phi)$. Using again (2.11) we now deduce that $b = 0$, as desired.

Case I-B: Suppose that X is unimodular with four-dimensional isometry group.
After scaling, X is isomorphic and isometric to either $\mathrm{SU}(2)$, Nil_3 or $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ with an $\mathbb{E}(\kappa, \tau)$-metric. The same argument as in the previous paragraph works by exchanging E_2 by the left invariant, unit length vertical field E which is a principal Ricci curvature direction corresponding to the multiplicity one Ricci principal curvature, and using the well-known formula $\nabla_v E = \tau v \times E$, where τ is the bundle curvature. Note that $\tau = 0$ leads to a contradiction since in this case $\mathbb{E}(\kappa, 0) = \mathbb{H}^2 \times \mathbb{R}$ which cannot be endowed with a unimodular Lie group structure. Item (2) now follows.

Case II-A: Suppose that X is unimodular with three-dimensional isometry group.
Pick an orientation for X and let E_1, E_2, E_3 be an orthonormal basis of the Lie algebra $\mathfrak{g}$ of X such that (2.24) holds, for certain structure constants $c_1, c_2, c_3 \in \mathbb{R}$. Recall that the basis E_1, E_2, E_3 diagonalizes the Ricci tensor of X, with principal Ricci curvatures given by (2.27). Since the isometry group of X is not six dimensional, then there exists a principal Ricci curvature which is different from the other two. Let $i \in \{1, 2, 3\}$ be the index so that E_i is the principal Ricci direction whose associated principal Ricci curvature is simple. As ϕ is an isometry and $\phi(e) = e$, then $\phi_*(E_i) = \pm E_i$.

We claim that we can assume $\phi_*(E_i) = E_i$: Assume $\phi_*(E_i) = -E_i$. Pick an index $j \in \{1, 2, 3\} - \{i\}$. By item (1) of Proposition 2.21 (see also its proof), X admits an orientation-preserving, order two isometry ψ_j which fixes e, whose differential at e fixes $(E_j)_e$ and such that ψ_j induces a well-defined automorphism of $\mathfrak{g}$. Then $\psi_j \circ \phi$ is an orientation-preserving isometry, it fixes e, its differential at e fixes $(E_i)_e$ and $(\psi_j \circ \phi)_*(E_i) = (\psi_j)_*\phi_*(E_i) = (\psi_j)_*(-E_i) = E_i$, from where our claim holds.

Since $d\phi_e$ fixes $(E_i)_e$, then $d\phi_e$ leaves invariant $\mathrm{Span}\{(E_j)_e, (E_k)_e\}$, where i, j, k is a cyclic permutation of $1, 2, 3$. Furthermore the restriction of $d\phi_e$ to $\mathrm{Span}\{(E_j)_e, (E_k)_e\}$ is a linear isometry with determinant -1, hence a symmetry with respect to a line $L = \mathrm{Span}\{V_e\}$ for certain unitary left invariant vector field $V \in \mathrm{Span}\{E_j, E_k\} \subset \mathfrak{g}$. Thus $d\phi_e \colon T_eX \to T_eX$ is the linear reflection with respect to the two-dimensional subspace $\Pi = \mathrm{Span}\{(E_i)_e, V_e\}$ of T_eX. By Proposition 2.18, the component $\mathrm{Fix}_0(\phi)$ of the set of fixed points of ϕ which passes through e is a subgroup and a totally geodesic submanifold of X. Furthermore, $T_e\mathrm{Fix}_0(\phi) = \Pi$.

Clearly we can write $V = \cos\theta E_j + \sin\theta E_k$ for certain $\theta \in [0, \pi/2]$. Since $E_i \times V$ is a unit normal vector field along the totally geodesic submanifold $\mathrm{Fix}_0(\phi)$, then

$$0 = \nabla_{E_i}(E_i \times V) = \cos\theta\, \nabla_{E_i} E_k - \sin\theta\, \nabla_{E_i} E_j \overset{(2.26)}{=} -\mu_i \cos\theta\, E_j - \mu_i \sin\theta\, E_k,$$

from where we deduce that

$$\mu_i = 0. \tag{2.37}$$

Arguing analogously,

$$0 = \nabla_V(E_i \times V) = \cos\theta\, \nabla_V E_k - \sin\theta\, \nabla_V E_j \overset{(2.26)}{=} (\mu_j \cos^2\theta + \mu_k \sin^2\theta)\, E_i,$$

which implies that

$$\mu_j \cos^2\theta + \mu_k \sin^2\theta = 0. \tag{2.38}$$

Note that $\theta \neq 0, \frac{\pi}{2}$ (otherwise μ_j or μ_k vanish, which is impossible by (2.37) since two of the constants μ_1, μ_2, μ_3 being equal implies that two of the constants c_1, c_2, c_3 are equal, which in turns implies that the isometry group of X is at least four dimensional). In particular, (2.38) implies that the constants μ_1, μ_2, μ_3 of X are one positive, one negative and one zero. This rules out the possibility of X being isomorphic to either $\mathrm{SU}(2)$ or $\widetilde{\mathrm{E}}(2)$ (both cases have two of the constants μ_1, μ_2, μ_3 positive). In summary, we have deduced that the Lie group X is isomorphic either to Sol_3 or to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Next we will study both cases separately.

Suppose X is isomorphic to Sol_3. Then one of the constants c_1, c_2, c_3 in (2.24) is zero, other is negative and the last one is positive. Since $\mu_i = 0$ then $c_i = c_j + c_k$, which implies that $c_i = 0$ and $c_j = -c_k$. After rescaling the metric, we can assume $c_j = 1$ and the corresponding left invariant metric on Sol_3 is the standard one, i.e., the one for which the left invariant basis in (2.32) is orthonormal. The isometry group of Sol_3 with the standard metric is well-known: in the usual coordinates (x, y, z) for the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, the connected component of the identity is generated by the following three families of isometries:

$$(x, y, z) \mapsto (x + t, y, z), \quad (x, y, z) \mapsto (x, y + t, z), \quad (x, y, z) \mapsto (e^{-t}x, e^t y, z + t),$$

and the stabilizer of the origin is a dihedral group D_4 generated by the following two orientation-reversing isometries:

$$(x, y, z) \overset{\sigma}{\mapsto} (y, -x, -z), \quad (x, y, z) \overset{\tau}{\mapsto} (-x, y, z).$$

Here, σ has order four and τ is a reflective symmetry (order two) with respect to the totally geodesic plane $\{x = 0\}$. Note that $\sigma^2 \circ \tau$ is another reflective symmetry, now with respect to the totally geodesic plane $\{y = 0\}$.

Next assume that X is isomorphic to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Hence, two of the structure constants of X are positive and the third one is negative. We can order these structure constants so that $c_1 < 0 < c_2 < c_3$. Now equations (2.37) and (2.38) lead to the special situation studied in Example 2.23 (compare with equation (2.34) and note that with the notation in that example, $i = 2$, $j = 3$ and $k = 1$). We can discard this case since the corresponding totally geodesic subgroup $\mathrm{Fix}_0(\phi)$ coincides with Σ_θ given in Example 2.23, which does not give rise to a reflectional symmetry of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

Case II-B: Suppose that X is non-unimodular with three-dimensional isometry group.
Since X is non-unimodular, then $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ with A as in (2.19) with $a, b \geq 0$ and we have the related natural basis E_1, E_2, E_3 of principal Ricci curvature directions. Our assumption that the isometry group of X is three-dimensional implies that $a \notin \{0, 1\}$ (recall that if $a = 0$ then the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ has constant sectional curvature, while if $a = 1$ then the isometry group of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric is four dimensional, see Theorem 2.14). It follows that the principal Ricci curvature values given by equation (2.23) are distinct, and so $\phi_*(E_i) = \pm E_i$ for $i = 1, 2, 3$. Thus ϕ has order two and, after possibly composing ϕ with the rotation by angle π around the z-axis, X admits an orientation-reversing isometry τ with $\tau_*(E_1) = E_1$ and $\tau_*(E_i) = \pm E_i$ for $i = 1, 2$.

We claim that $\tau_*(E_3) = E_3$. Otherwise, $\tau_*(E_3) = -E_3$ and one easily finds that the fixed point set of τ is $\mathbb{R}^2 \rtimes_A \{0\}$ (use again the uniqueness of geodesics with given initial conditions to show that every geodesic γ of X with $\gamma(0) = e$ and $\gamma'(0) \in \mathrm{Span}\{(E_1)_e, (E_2)_e\}$ is contained in the fixed point set of τ, where $e = (0, 0, 0) \in X$), which is totally geodesic and has E_3 as its unit normal field. This case is easily seen to be impossible by the formulas in (2.11). Hence, $\tau_*(E_3) = E_3$.

Since $\tau_*(E_1) = E_1$ and $\tau_*(E_3) = E_3$, then $\tau_*(E_2) = -E_2$. The same arguments as before give that E_1 and E_3 are tangent to the totally geodesic fixed point set of τ (thus τ is a reflectional symmetry). Therefore $[E_3, E_1]$ is a linear combination of E_3 and E_1, which after scaling again the metric, implies $[E_3, E_1] = E_1$. Then we can use (2.9) to compute the first column of A. After composing the reflectional symmetry τ with the rotation by angle π around the z-axis, we obtain another reflectional symmetry σ with $\sigma_*(E_1) = -E_1$ and $\sigma_*(E_i) = E_i$ for $i = 2, 3$, from which we conclude by the arguments in the previous sentence that $[E_3, E_1] = bE_2$ for some $b \in \mathbb{R}$. Now (2.10) gives the second column of the matrix A, or equivalently, the matrix for ad_{E_3} restricted to the unimodular kernel $\mathrm{Span}\{E_1, E_2\}$ is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

Note that τ corresponds to the reflection $(x, y, z) \mapsto (x, -y, z)$ and σ corresponds to the reflection $(x, y, z) \mapsto (-x, y, z)$. From here the statement in item (3-a) of Proposition 2.24 is straightforward, and the proof is complete. $\square$

Remark 2.25. Let X be a simply-connected, three-dimensional metric Lie group with six-dimensional isometry group $I(X)$.

(1) If the sectional curvature of X is positive, then X is isomorphic to $\mathrm{SU}(2)$ and X embeds isometrically in $\mathbb{R}^4$ as a round sphere centered at the origin. In this case, $I(X)$ can be identified with the group $O(4)$.
(2) If the sectional curvature of X is zero, then $I(X)$ can be identified with the group of rigid motions of $\mathbb{R}^3$ with its usual flat metric. Clearly $I(X)$ is isomorphic to the semidirect product $\mathbb{R}^3 \rtimes_\varphi O(3)$, where $\varphi \colon O(3) \to \mathrm{Aut}(\mathbb{R}^3)$ is the group morphism given by matrix multiplication.

(3) If the sectional curvature of X is negative, then, after scaling, X is isometric to the hyperbolic three-space $\mathbb{H}^3$ with its standard metric of constant curvature -1. Thus, $I(X)$ is just the group of isometries of $\mathbb{H}^3$, which can be identified with the group of conformal diffeomorphisms of the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, whose elements (after stereographic projection) are the holomorphic and antiholomorphic Möbius transformations:

$$z \mapsto \frac{az+b}{cz+d}, \qquad z \mapsto \frac{a\overline{z}+b}{c\overline{z}+d}, \quad z \in \mathbb{C} \cup \{\infty\},$$

where $a, b, c, d \in \mathbb{C}$ and $ad-bc = 1$. In this case, the group $I(X)$ is isomorphic to the semidirect product $\mathrm{PSL}(2,\mathbb{C}) \rtimes_{\varphi} \mathbb{Z}_2$, where $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I_2\}$, $\mathrm{SL}(2,\mathbb{C}) = \{A \in \mathcal{M}_2(\mathbb{C}) \mid \det A = 1\}$ and $\varphi\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathrm{SL}(2,\mathbb{C}))$ is the group morphism given by $\varphi(0) = 1_{\mathrm{PSL}(2,\mathbb{C})}$, $\varphi(1) = \varphi_1$ being the automorphism

$$\varphi_1([A]) = [\overline{A}], \quad [A] = \{\pm A\}, \ A \in \mathrm{SL}(2,\mathbb{C}).$$

3. Surface theory in three-dimensional metric Lie groups.

In the sequel we will study surfaces immersed in a simply-connected metric Lie group X.

3.1. Algebraic open book decompositions of X.

Definition 3.1. Let X be a simply-connected, three-dimensional Lie group and $\Gamma \subset X$ a 1-parameter subgroup. An *algebraic open book decomposition of X with binding* Γ is a foliation $\mathcal{B} = \{L(\theta)\}_{\theta \in [0,2\pi)}$ of $X - \Gamma$ such that the sets

$$H(\theta) = L(\theta) \cup \Gamma \cup L(\pi + \theta)$$

are two-dimensional subgroups of X, for all $\theta \in [0, \pi)$. We will call $L(\theta)$ the *leaves* and $H(\theta)$ the *subgroups* of the algebraic open book decomposition $\mathcal{B}$.

We now list many examples of algebraic open book decompositions of simply-connected, three-dimensional Lie groups. Theorem 3.6 at the end of this section states that this list is complete.

Example 3.2. **The cases of $\mathbb{R}^3$ and $\mathbb{H}^3$.** The most familiar examples of algebraic open book decompositions occur when the metric Lie group is $X = \mathbb{R}^3$. In this case every two-dimensional subgroup of X is a plane in $\mathbb{R}^3$. Clearly each algebraic open book decomposition of X has as binding a line passing through the origin and has as subgroups all of the planes containing this line. The set of algebraic open book decompositions of $X = \mathbb{R}^3$ are parameterized by its 1-parameter subgroups.

In the case $X = \mathbb{H}^3 = \mathbb{R}^2 \rtimes_{I_2} \mathbb{R}$, equation (2.1) implies that every two-dimensional subspace of the Lie algebra of $\mathbb{R}^2 \rtimes_{I_2} \mathbb{R}$ is closed under Lie bracket. It follows that the algebraic open book decompositions of X are parameterized by their bindings, which are all of the 1-parameter subgroups of X. These 1-parameter subgroups can be easily proven to be of the form

$$\{(tp_0, 0) \in \mathbb{R}^2 \rtimes_{I_2} \mathbb{R} \mid t \in \mathbb{R}\} \quad \text{or} \quad \left\{ \left(\frac{1}{\lambda}(e^{\lambda t} - 1)p_0, \lambda t \right) \mid t \in \mathbb{R} \right\},$$

where $p_0 \in \mathbb{R}^2$ and $\lambda \in \mathbb{R} - \{0\}$. Note that the two-dimensional subgroups of $\mathbb{H}^3$ are not necessarily planes passing through the origin in the above $\mathbb{R}^3$-coordinates, although they are topological planes.

EXAMPLE 3.3. **The Heisenberg group Nil_3.** Recall that Nil_3 is isomorphic to $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. An interesting example of an algebraic open book decomposition of Nil_3 is that one given by considering, for each $\lambda \in \mathbb{R}$, the two-dimensional abelian subgroup

$$H(\lambda) = \{(x, y, \lambda y) \quad x, y \in \mathbb{R}\}.$$

The (commutative) Lie algebra of $H(\lambda)$ is spanned by E_1 and $-zE_1 + E_2 + \lambda E_3$, where $E_1(x, y, z) = \partial_x$, $E_2(x, y, z) = z\,\partial_x + \partial_y$, $E_3 = \partial_z$ are given by (2.6). We can extend this definition to $\lambda = \infty$ by letting

$$H(\infty) = \{(x, 0, z) \mid x, z \in \mathbb{R}\}.$$

Clearly for $\lambda \neq \lambda'$, the subgroups $H(\lambda)$, $H(\lambda')$ only intersect along the 1-parameter subgroup $\Gamma = \{(x, 0, 0) \mid x \in \mathbb{R}\}$, and the family $\mathcal{B} = \{H(\lambda) - \Gamma \mid \lambda \in \mathbb{R} \cup \{\infty\}\}$ foliates $(\mathbb{R}^2 \rtimes_A \mathbb{R}) - \Gamma$. Hence, $\mathcal{B}$ defines an algebraic open book decomposition of Nil_3 with binding Γ. In the usual representation of Nil_3 as an $\mathbb{E}(\kappa = 0, \tau)$-space, each of the subgroups $H(\lambda)$ corresponds to a vertical plane, i.e., the preimage under the Riemannian submersion $\Pi\colon \mathbb{E}(0, \tau) \to \mathbb{R}^2$ of a straight line in $\mathbb{R}^2$ that passes through the origin.

EXAMPLE 3.4. **Semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $A \in \mathcal{M}_2(\mathbb{R})$ diagonal.** Consider the semidirect product $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$, where $A(b) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ and $b \in \mathbb{R}$. For $b = 0$ (resp. $b = -1$, $b = 1$), $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$ with its canonical metric is isomorphic and isometric to $\mathbb{H}^2 \times \mathbb{R}$ (resp. to Sol_3, $\mathbb{H}^3$) with its standard metric. For $b \neq \pm 1$, the Lie groups $X = \mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$ produce non-unimodular Lie groups whose normalized Milnor D-invariants cover all possible values less than 1. The case of $b = 1$, which is the same as $X = \mathbb{H}^3$, was already covered in Example 3.2.

Let E_1, E_2, E_3 the usual basis for the Lie algebra $\mathfrak{g}$ of $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$ in the sense of (2.16), (2.17), that is

$$[E_1, E_2] = 0, \quad [E_3, E_1] = E_1, \quad [E_3, E_2] = bE_2.$$

The horizontal distribution spanned by $\{E_1, E_2\}$, generates the normal, commutative Lie subgroup $\mathbb{R}^2 \rtimes_{A(b)} \{0\}$ of $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$. It is worth while considering interesting subgroups of type $\mathbb{H}^2$, which together with $\mathbb{R}^2 \rtimes_{A(b)} \{0\}$ form an algebraic open book decomposition:

(1) Suppose $b \neq 0$. Given $\lambda \in \mathbb{R}$, consider the distribution $\Delta_1(\lambda)$ generated by $\{E_1, E_3 + \lambda E_2\}$ (which is integrable since $[E_1, E_3 + \lambda E_2] = -E_1$). Since the Lie bracket restricted to $\Delta_1(\lambda)$ does not vanish identically, $\Delta_1(\lambda)$ produces an $\mathbb{H}^2$-type subgroup $H_1(\lambda) = \exp(\Delta_1(\lambda))$ of $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$. Using (x, y, z)-coordinates as in Section 2.2, the generators of the tangent bundle to $H_1(\lambda)$ are $\{e^z \partial_x, \lambda e^{bz} \partial_y + \partial_z\}$. From here we deduce that

$$H_1(\lambda) = \left\{ \left(x, \frac{\lambda}{b}(e^{bz} - 1), z \right) \mid x, z \in \mathbb{R} \right\}.$$

We can extend the above definition to $\lambda = \infty$, letting $H_1(\infty) = \mathbb{R}^2 \rtimes_{A(b)} \{0\}$. Clearly for $\lambda \neq \lambda'$, the subgroups $H_1(\lambda)$, $H_1(\lambda')$ only intersect along the 1-parameter subgroup $\Gamma_1 = \{(x, 0, 0) \mid x \in \mathbb{R}\}$, and the family $\mathcal{B}_1 = \{H_1(\lambda) - \Gamma_1 \mid \lambda \in \mathbb{R} \cup \{\infty\}\}$ foliates $(\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}) - \Gamma_1$. Hence, $\mathcal{B}_1$ defines an algebraic open book decomposition of $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$ with binding Γ_1.

In the same Lie group $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$, we can exchange the roles of E_1, E_2 and define, given $\lambda \in \mathbb{R}$, the integrable distribution $\Delta_2(\lambda)$ generated by $\{E_2, E_3 + \lambda E_1\}$, which produces an $\mathbb{H}^2$-type subgroup of $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$ when $b \neq 0$ and an $\mathbb{R}^2$-type subgroup when $b = 0$:

$$H_2(\lambda) = \exp(\Delta_2(\lambda)) = \{(\lambda(e^z - 1), y, z) \mid y, z \in \mathbb{R}\}.$$

Letting $H_2(\infty) = \mathbb{R}^2 \rtimes_{A(b)} \{0\}$ and $\Gamma_2 = \{(0, y, 0) \mid y \in \mathbb{R}\}$, we have that $\mathcal{B}_2 = \{H_2(\lambda) - \Gamma_2 \mid \lambda \in \mathbb{R} \cup \{\infty\}\}$ is an algebraic open book decomposition of $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$ with binding Γ_2.

Note that in the usual (x, y, z)-coordinates, the leaves of the algebraic open book decomposition $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) are products with the x-factor (resp. with the y-factor) of the exponential graphs $z \in \mathbb{R} \mapsto (0, \frac{\lambda}{b}(e^{bz} - 1), z)$ (resp. $z \mapsto (\lambda(e^z - 1), 0, z)$) except for $\lambda = \infty$. Furthermore, $H_i(\lambda)$ with $\lambda = 0, \infty$ and $i = 1, 2$ are the only subgroups of the algebraic open book decompositions $\mathcal{B}_1, \mathcal{B}_2$ that are genuine planes in these (x, y, z)-coordinates.

(2) Now assume $b = 0$. We will follow the arguments in case (1) above, focusing only on the differences. Given $\lambda \in \mathbb{R}$, the distribution $\Delta_1(\lambda) = \mathrm{Span}\{E_1, E_3 + \lambda E_2\}$ is again integrable and non-commutative, defining an $\mathbb{H}^2$-type subgroup $H_1(\lambda) = \exp(\Delta_1(\lambda))$ of $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$, which can be written in (x, y, z)-coordinates as

$$H_1(\lambda) = \{(x, \lambda z, z) \mid x, z \in \mathbb{R}\}.$$

Defining $H_1(\infty) = \mathbb{R}^2 \rtimes_{A(b)} \{0\}$ and $\mathcal{B}_1 = \{H_1(\lambda) - \Gamma_1 \mid \lambda \in \mathbb{R} \cup \{\infty\}\}$, we have that $\mathcal{B}_1$ is an algebraic open book decomposition of $\mathbb{R}^2 \rtimes_{A(0)} \mathbb{R} \cong \mathbb{H}^2 \times \mathbb{R}$ with binding $\Gamma_1 = \{(x, 0, 0) \mid x \in \mathbb{R}\}$. Note that the leaves of $\mathcal{B}_1$ are now planes in the coordinates (x, y, z). Also note that (2.12) implies that the canonical metric on $\mathbb{R}^2 \rtimes_{A(0)} \mathbb{R}$ is

$$\langle , \rangle = (e^{-2z} dx^2 + dz^2) + dy^2 = ds^2_{\mathbb{H}^2} + dy^2,$$

where $ds^2_{\mathbb{H}^2}$ stands for the standard hyperbolic metric with constant curvature -1 in the (x, z)-plane. The $\mathbb{R}$-factor in $\mathbb{H}^2 \times \mathbb{R}$ corresponds to the y-axis in $\mathbb{R}^2 \rtimes_{A(0)} \mathbb{R}$ (obviously we could exchange y by z in this discussion).

Given $\lambda \in \mathbb{R}$, consider the integrable distribution $\Delta_2(\lambda) = \mathrm{Span}\{E_2, E_3 + \lambda E_1\}$, which is commutative and generates the $\mathbb{R}^2$-type subgroup

$$H_2(\lambda) = \exp(\Delta_2(\lambda)) = \{(\lambda(e^z - 1), y, z) \mid y, z \in \mathbb{R}\}$$

of $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$. Together with $H_2(\infty) = \mathbb{R}^2 \rtimes_{A(0)} \{0\}$, we can consider the algebraic open book decomposition $\mathcal{B}_2 = \{H_2(\lambda) - \Gamma_2 \mid \lambda \in \mathbb{R} \cup \{\infty\}\}$ of $\mathbb{R}^2 \rtimes_{A(0)} \mathbb{R}$ with binding $\Gamma_2 = \{(0, y, 0) \mid y \in \mathbb{R}\}$. In the standard (x, y, z)-coordinates, the leaves of $\mathcal{B}_2$ are products with the y-factor of the exponential graphs $z \in \mathbb{R} \mapsto (\lambda(e^z - 1), 0, z)$, except for $\lambda = \infty$. $H_2(0), H_2(\infty)$ are the only subgroups of $\mathcal{B}_2$ which are genuine planes in these (x, y, z)-coordinates. In the model $\mathbb{H}^2 \times \mathbb{R}$ for $\mathbb{R}^2 \rtimes_{A(0)} \mathbb{R}$, the algebraic open book decomposition $\mathcal{B}_2$ corresponds to the pencil of totally geodesic planes $\{\gamma \times \mathbb{R} \mid \gamma \in C\}$, where C is the collection of geodesics of $\mathbb{H}^2$ based at a given point.

Example 3.5. **The three-dimensional simply-connected Lie group X with Milnor D-invariant $D = 1$ which is not isomorphic to $\mathbb{H}^3$.** Recall that this Lie group X can be expressed as the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where

$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. We define, given $\lambda \in \mathbb{R}$, the integrable distribution $\Delta(\lambda)$ generated by $\{E_2, E_3 + \lambda E_1\}$, which produces the $\mathbb{H}^2$-type subgroup $H(\lambda) = \exp(\Delta(\lambda))$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ given by

$$H(\lambda) = \{(\lambda(e^z - 1), y + \lambda\, [e^z(z-1) + 1]\,, z) \mid y, z \in \mathbb{R}\}\,.$$

Defining $H(\infty) = \mathbb{R}^2 \rtimes_A \{0\}$ and $\Gamma = \{(0, y, 0) \mid y \in \mathbb{R}\}$, then we have that $\mathcal{B} = \{H(\lambda) - \Gamma \mid \lambda \in \mathbb{R} \cup \{\infty\}\}$ is an algebraic open book decomposition of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with binding Γ.

The next theorem classifies all two-dimensional subgroups and all algebraic open book decompositions of a three-dimensional simply-connected Lie group. In particular, it shows that the possible algebraic open book decompositions are precisely the ones described in the above examples. Since the proof of the following result requires the notions of left invariant Gauss map and H-potential which will be explained in Sections 3.2 and 3.3 below, we postpone its proof to Section 3.4.

THEOREM 3.6. *Let X be a three-dimensional, simply-connected Lie group. Then:*

(1) If $X = \mathrm{SU}(2)$, then X has no two-dimensional subgroups.

(2) If $X = \widetilde{\mathrm{SL}}(2, \mathbb{R})$, then its connected two-dimensional subgroups are the lifts to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ of the subgroups $\mathbb{H}^2_\theta$ described in (2.30). In particular, since the intersection of any three distinct such subgroups of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is the trivial subgroup, then $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ does not admit any algebraic open book decompositions.

(3) If $X = \widetilde{\mathrm{E}}(2)$ expressed as a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some matrix $A \in \mathcal{M}_2(\mathbb{R})$, then X has no two-dimensional subgroups other than $\mathbb{R}^2 \rtimes_A \{0\}$.

(4) If $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is a non-unimodular group with Milnor D-invariant $D > 1$ (here $A \in \mathcal{M}_2(\mathbb{R})$ is some matrix), then X has no two-dimensional subgroups other than $\mathbb{R}^2 \rtimes_A \{0\}$.

(5) Suppose X is not in one of the cases 1, 2, 3, 4 above. Then X admits an algebraic open book decomposition and every such decomposition is listed in one of the examples above. Furthermore, every two-dimensional subgroup of X is a subgroup in one of these algebraic open book decompositions.

(6) If X is unimodular, then every two-dimensional subgroup of X is minimal.

(7) Suppose that $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is non-unimodular with $\mathrm{trace}(A) = 2$ and let T be a two-dimensional subgroup. Then the mean curvature H of T satisfies $H \in [0, 1]$. Furthermore:

   (a) If $D = \det(A) > 1$, then $T = \mathbb{R}^2 \rtimes_A \{0\}$, and so the mean curvature of T is 1.

   (b) If $D = \det(A) \leq 1$, then for each $H \in [0, 1]$, there exists a two-dimensional subgroup $T(H)$ with mean curvature H.

Theorem 3.11 of the next section will illustrate the usefulness of having algebraic open book decompositions, as a tool for understanding the embeddedness of certain spheres.

Recall that in Section 2.1 we constrained the study of three-dimensional semidirect products $H \rtimes_\varphi \mathbb{R}$ (here $\varphi \colon \mathbb{R} \to \mathrm{Aut}(H)$ is a group homomorphism) to the case of H being commutative, except in the case $H = \mathbb{H}^2$ and $\varphi(z) = 1_H$, which produces $\mathbb{H}^2 \times \mathbb{R}$. With Theorem 3.6 in hand, we can give the following justification of our constraint. Note that for arbitrary φ, the subgroup $H \rtimes_\varphi \{0\}$ of $H \rtimes_\varphi \mathbb{R}$ is always normal.

COROLLARY 3.7. *Suppose that X is a simply-connected, three-dimensional Lie group which admits a non-commutative, two-dimensional normal subgroup H. Then, X is isomorphic to $\mathbb{H}^2 \times \mathbb{R}$ and after representing X as the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then H is one of the subgroups $H_1(\lambda)$, $\lambda \in \mathbb{R}$, of the algebraic open book decomposition $\mathcal{B}_1$ given in Example 3.4 (2).*

PROOF. Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of X (resp. of H). It is an elementary exercise to prove that H is a normal subgroup of X if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

We will work up to isomorphism. By Theorem 3.6 X cannot be SU(2), and if $X = \widetilde{\mathrm{SL}}(2,\mathbb{R})$ then H is the lift to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of one of the subgroups $\mathbb{H}^2_\theta$ described in (2.30), which is clearly not normal in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ (its conjugate subgroups are the liftings of $\mathbb{H}^2_{\theta'}$ with θ' varying in $\partial_\infty \mathbb{H}^2$). Hence we can assume X is a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some matrix $A \in \mathcal{M}_2(\mathbb{R})$. In particular, there exists a basis E_1, E_2, E_3 such that $[E_1, E_2] = 0$. If $\mathfrak{h}$ is generated by $X = \alpha E_1 + \beta E_2 + \gamma E_3$ and $Y = \alpha' E_1 + \beta' E_2 + \gamma' E_3 \in \mathfrak{g}$, then

$$[X, Y] = \begin{vmatrix} \alpha & \gamma \\ \alpha' & \gamma' \end{vmatrix} [E_1, E_3] + \begin{vmatrix} \beta & \gamma \\ \beta' & \gamma' \end{vmatrix} [E_2, E_3]. \tag{3.1}$$

Since $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $[X, E_1] \in \mathfrak{h}$ and so

$$0 = \det([X, E_1], X, Y) = \gamma \det([E_3, E_1], X, Y), \tag{3.2}$$

where det denotes the determinant 3-form associated to a previously chosen orientation on $\mathfrak{g}$. Analogously,

$$\gamma' \det([E_3, E_1], X, Y) = \gamma \det([E_3, E_2], X, Y) = \gamma' \det([E_3, E_2], X, Y) = 0. \tag{3.3}$$

Note that $(\gamma, \gamma') \neq (0, 0)$ (otherwise $\mathfrak{h}$ would be commutative, a contradiction). Thus (3.2) and (3.2) imply that

$$[E_3, E_1], [E_3, E_2] \in \mathfrak{h}. \tag{3.4}$$

If $\det(A) \neq 0$ then (2.9) and (2.10) imply that $[E_3, E_1], [E_3, E_2]$ are linearly independent, hence (3.4) allows us to choose X as $[E_3, E_1]$ and Y as $[E_3, E_2]$. This gives $(\gamma, \gamma') = (0, 0)$, which we have checked is impossible. Thus, $\det(A) = 0$.

Note that A is not a multiple of the identity (otherwise $A = 0$ and $X = \mathbb{R}^3$, which is commutative). We claim that X is non-unimodular. Arguing by contradiction, if X is unimodular then necessarily $X = \mathrm{Nil}_3$ (the cases $X = \mathrm{Sol}_3$ and $X = \widetilde{\mathrm{E}}(2)$ are discarded since both have related matrix A which is regular). Thus $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and thus, $E_1 = [E_3, E_2] \in \mathfrak{h}$ can be chosen as X in (3.1) (in particular, $\beta = \gamma = 0$). Now (3.1) gives $[X, Y] = [E_1, Y] = 0$, i.e., $\mathfrak{h}$ is commutative, a contradiction. Thus X is non-unimodular, from where the Milnor D-invariant $D = \det(A) = 0$ is a complete invariant of the group structure of X. This implies that X is isomorphic to $\mathbb{H}^2 \times \mathbb{R}$.

Finally, represent X by $\mathbb{R}^2 \rtimes_A \mathbb{R}$ where $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then (2.9) and (3.4) give $E_1 = [E_3, E_1] \in \mathfrak{h}$, which means that X can be chosen as E_1 and $\mathfrak{h} = \mathrm{Span}\{E_1, Y\} = \mathrm{Span}\{E_1, \alpha' E_1 + \beta' E_2 + \gamma' E_3\} = \mathrm{Span}\{E_1, \beta' E_2 + \gamma' E_3\} =$

$\mathrm{Span}\{E_1, \lambda E_2 + E_3\}$, where $\lambda = \beta'/\gamma'$ (note that $\gamma' \neq 0$ since $-\gamma' E_1 = \gamma'[E_1, E_3] = [E_1, \beta' E_2 + \gamma' E_3] \neq 0$ because $\mathfrak{h}$ is non-commutative). Now the proof is complete. $\square$

COROLLARY 3.8. *Suppose that X is a simply-connected, three-dimensional Lie group which admits a commutative, two-dimensional normal subgroup $H = \mathbb{R}^2$. Then, X is isomorphic to $(H = \mathbb{R}^2) \rtimes_A \mathbb{R}$, for some $A \in \mathcal{M}_2(\mathbb{R})$.*

Furthermore, if X admits a second $\mathbb{R}^2$-type subgroup, then:

(1) X is isomorphic to $\mathbb{R}^3$, Nil_3 or $\mathbb{H}^2 \times \mathbb{R}$.
(2) Every $\mathbb{R}^2$-type subgroup in $\mathbb{R}^3$, Nil_3 or $\mathbb{H}^2 \times \mathbb{R}$ is normal.

In particular, by Theorem 3.6 and Corollary 3.7, every two-dimensional subgroup of $\mathbb{R}^3$, Nil_3 or $\mathbb{H}^2 \times \mathbb{R}$ is normal and these groups are the only simply-connected three-dimensional Lie groups which admit more than one normal two-dimensional subgroup.

PROOF. Note that the $\mathbb{R}^2$-type subgroups of any of the groups $\mathbb{R}^3$, Nil_3 or $\mathbb{H}^2 \times \mathbb{R}$ are normal. To see this property it suffices to check that the Lie subalgebras of their $\mathbb{R}^2$-type subgroups described by Theorem 3.6 are ideals, which is a straightforward calculation. With this normal subgroup property proved, the proofs of the remaining statements of the corollary are straightforward and the details are left to the reader. $\square$

3.2. The left invariant Gauss map and the embeddedness of certain spheres in X. Given an oriented surface Σ immersed in a three-dimensional metric Lie group X, we denote by $N\colon \Sigma \to TX$ the unit normal vector field to Σ (TX stands for the tangent bundle to X). Given any point $p \in \Sigma$, we extend the vector $N_p \in T_pX$ to a (unique) left invariant vector field $G(p) \in \mathfrak{g}$, i.e., $G(p)|_p = N_p$. This is equivalent to

$$G(p) = \sum_{i=1}^{3} \langle N_p, (E_i)_p \rangle E_i,$$

where E_1, E_2, E_3 is an orthonormal basis of the Lie algebra $\mathfrak{g}$ of X. We will call $G\colon \Sigma \to \mathfrak{g}$ the *left invariant Gauss map of* Σ. Clearly, G takes values in the unit sphere of the metric Lie algebra $\mathfrak{g} \equiv \mathbb{R}^3$.

The definition of left invariant Gauss map makes sense for any hypersurface in a metric Lie group, not just in the case where the Lie group has dimension three. The next lemma describes some useful elementary facts about hypersurfaces in a metric Lie group with constant left invariant Gauss map.

LEMMA 3.9.

(1) A connected oriented hypersurface in an $(n+1)$-dimensional metric Lie group has constant left invariant Gauss map if and only if it is a left coset of some n-dimensional subgroup.
(2) If Σ is a connected, codimension one subgroup in a metric Lie group X, then each component Y of the set of points at any fixed constant distance from Σ is a right coset of Σ. For any $y \in Y$, Y is also the left coset yH of the codimension one subgroup $H = y^{-1}\Sigma y$ of X.
(3) If Σ is a connected oriented hypersurface of X with constant left invariant Gauss map, then each component of the set of points at any fixed constant distance from Σ is a left coset of some codimension one subgroup Δ of X and a right coset of some conjugate subgroup of Δ.

Proof. Let X be an n-dimensional metric Lie group. Since a connected n-dimensional subgroup Σ in X is closed under multiplication by elements in Σ, then Σ is orientable, and after choosing an orientation, its left invariant Gauss map is clearly constant. Since the left invariant Gauss map of a left coset of Σ coincides with that of Σ, we have the desired constancy of the left invariant Gauss map of every left coset of Σ.

Reciprocally, suppose Σ is an oriented hypersurface in an $(n+1)$-dimensional metric Lie group X, whose left invariant Gauss map is constant. Let e be the identity element of X. After left translation, we can assume e belongs to Σ. Consider the unit normal $N_e \in T_eX$ of Σ at e. Let $N \in \mathfrak{g}$ be the left invariant vector field corresponding to N_e. Note that N can be viewed as the left invariant Gauss map of Σ at every point. Let $N^\perp$ denote the distribution orthogonal to N. We claim that the distribution $N^\perp$ is integrable, and that the integral leaf of $N^\perp$ passing through e is Σ. Let $\gamma^N(\mathbb{R})$ be the associated 1-parameter subgroup. For $\varepsilon > 0$ sufficiently small, the surfaces $\{\gamma^N(t)B_\Sigma(e,\varepsilon) \mid t \in (-\varepsilon,\varepsilon)\}$ obtained by translating the intrinsic disk $B_\Sigma(e,\varepsilon) \subset \Sigma$ centered at e with radius ε by left multiplication by $\gamma^N(t)$, foliate a small neighborhood of e in X and are tangent to the analytic distribution $N^\perp$ at every point in this neighborhood. It follows that $N^\perp$ is integrable on all of X and that Σ is the integral leaf of $N^\perp$ passing through e, which proves the claim.

By construction, for any $x \in \Sigma$, $x\Sigma$ and $x^{-1}\Sigma$ are integral leaves of the distribution $N^\perp$ and since each of these leaves intersects the integral leaf Σ, we must have $\Sigma = x\Sigma = x^{-1}\Sigma$. Elementary group theory now implies Σ is a subgroup of X, which completes the proof of the first statement in the lemma.

We now prove item (2). Let Σ be a connected codimension one subgroup in a metric Lie group X. To see that the second statement in the lemma holds, first observe that a component Y of the set of points at a fixed distance $\varepsilon > 0$ from Σ is equal to the right coset Σy, for any $y \in Y$. To see this, take an element $y \in Y$ and we will show that $\Sigma y = Y$. Given $x \in \Sigma$, clearly $\Sigma = x\Sigma$. Then the left multiplication by x is an isometry l_x of X which leaves Σ invariant. Therefore, l_x leaves invariant the set of points at distance ε. Now a connectedness argument gives that l_x leaves Y invariant. In particular, $xy = l_x(y) \in Y$ and since x is arbitrary in Σ, we conclude that $\Sigma y \subset Y$. By the connectedness of Σ, it follows that $\Sigma y = Y$. Finally, note that $Y = y(y^{-1}\Sigma y)$, which is a left coset of the subgroup $y^{-1}\Sigma y$. This gives the second statement in the lemma.

The proof of item (3) follows directly from items (1) and (2) in the lemma and details are left to the reader. □

The following result is a special case of the Transversality Lemma in Meeks, Mira, Pérez and Ros [**MIMPRb**].

Lemma 3.10 (Transversality Lemma). *Let S be an immersed sphere in a simply-connected, three-dimensional metric Lie group X, whose left invariant Gauss map G is a diffeomorphism. Let Σ be a two-dimensional subgroup of X. Then:*

(1) The set of left cosets $\{g\Sigma \mid g \in X\}$ which intersect S can be parametrized by the interval $[0,1]$, i.e., $\{g(t)\Sigma \mid t \in [0,1]\}$ are these cosets.
(2) Each of the left cosets $g(0)\Sigma$ and $g(1)\Sigma$ intersects S at a single point.
(3) For every $t \in (0,1)$, $g(t)\Sigma$ intersects S transversely in a connected, immersed closed curve.

PROOF. First note that since X admits a two-dimensional subgroup, then X is not isomorphic to SU(2) and so it is diffeomorphic to $\mathbb{R}^3$. In this case the set of left cosets $\{g\Sigma \mid g \in X\}$ can be smoothly parameterized by $\mathbb{R}$. Let $\Pi\colon X \to \mathbb{R} \equiv \{g\Sigma \mid g \in X\}$ be the related smooth quotient map. The critical points of $\Pi|_S$ are those points of S where the value of the left invariant Gauss map of S coincides (up to sign) with the one of Σ. Since G is bijective, then $\Pi|_S$ has at most two critical points. On the other hand, $\Pi|_S$ has at least two critical points: a local maximum and a local minimum. From here, the proof of each statement in the lemma is elementary. □

The above Transversality Lemma is a key ingredient in the proof of the next result, which also uses the Hopf index theorem for vector fields. For details about this proof, see [**MIMPRb**].

THEOREM 3.11 (Meeks-Mira-Pérez-Ros [**MIMPRb**]). *Let X be a simply-connected, three-dimensional metric Lie group which admits an algebraic open book decomposition $\mathcal{B}$ with binding Γ. Let $\Pi\colon X \to \mathbb{R}^2 \equiv X/\Gamma$ be the related quotient map to the space of left cosets of Γ. If $f\colon S \looparrowright X$ is an immersion of a sphere whose left invariant Gauss map is a diffeomorphism, then:*

(1) $\mathcal{D} = \Pi(f(S))$ is a smooth, compact embedded disk in $\mathbb{R}^2$ and $\Pi^{-1}(\mathrm{Int}(\mathcal{D}))$ consists of two components F_1, F_2 such that $\Pi|_{F_i}\colon F_i \to \mathrm{Int}(\mathcal{D})$ is a diffeomorphism.

(2) $f(S)$ is an embedded sphere (i.e., f is an injective immersion).

3.3. Surfaces with constant mean curvature. In this section we begin our study of surfaces whose mean curvature is a constant $H \in [0,\infty)$ (briefly, *H-surfaces*). Our first goal will be to find a PDE satisfied by the left invariant Gauss map of any H-surface in a given three-dimensional metric Lie group, which generalizes the well-known fact that the Gauss map of an H-surface in $\mathbb{R}^3$ is harmonic as a map between the surface and the unit two-sphere. An important ingredient in this PDE is the notion of H-potential, which only depends on the metric Lie group X, and that we now describe. We will distinguish cases depending on whether or not X is unimodular.

DEFINITION 3.12. Let X be a three-dimensional, <u>non-unimodular</u> metric Lie group. Rescale the metric on X so that X is isometric and isomorphic to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric, where $A \in \mathcal{M}_2(\mathbb{R})$ is given by (2.19) for certain constants $a, b \geq 0$. Given $H \geq 0$, we define the *H-potential* for X as the map $R\colon \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \to \overline{\mathbb{C}}$ given by

$$R(q) = H\left(1+|q|^2\right)^2 - (1-|q|^4) - a\left(q^2 - \overline{q}^2\right) - ib\left[2|q|^2 - a\left(q^2+\overline{q}^2\right)\right], \tag{3.5}$$

where $\overline{q}$ denotes the conjugate complex of $q \in \overline{\mathbb{C}}$.

DEFINITION 3.13. let X be a three-dimensional, <u>unimodular</u> metric Lie group with structure constants c_1, c_2, c_3 defined by equation (2.24) and let $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ be the related numbers defined in (2.25) in terms of c_1, c_2, c_3. Given $H \geq 0$, we define the *H-potential* for X as the map $R\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by

$$R(q) = H\left(1+|q|^2\right)^2 - \frac{i}{2}\left[\mu_2|1+q^2|^2 + \mu_1|1-q^2|^2 + 4\mu_3|q|^2\right]. \tag{3.6}$$

The H-potential of X only vanishes identically if $X = \mathbb{R}^3$ and $H = 0$. Otherwise, the H-potential R is non-zero at some point and extends continuously to $q = \infty$ with $R(\infty) = \infty$ except in the case X unimodular and $(H, \mu_1 + \mu_2) = (0, 0)$; this particular case corresponds to minimal surfaces in $\widetilde{\mathrm{E}}(2)$, Sol_3, Nil_3 or in $\mathbb{R}^3$, in which case the $(H = 0)$-potential is $R(q) = 2i[(\mu_1 - \mu_3)x^2 - (\mu_1 + \mu_3)y^2]$ where $q = x + iy$, $x, y \in \mathbb{R}$. The behavior at $q = \infty$ of the H-potential is

$$\frac{R(q)}{|q|^4} \overset{(q\to\infty)}{\longmapsto} \begin{cases} H + 1 & \text{if } X \text{ is non-unimodular,} \\ H - \frac{i}{2}(\mu_1 + \mu_2) & \text{if } X \text{ is unimodular and } (H, \mu_1 + \mu_2) \neq (0, 0). \end{cases}$$

Lemma 3.14. *The H-potential for X has no zeros in $\overline{\mathbb{C}}$ if any of the following holds:*

(1) $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ with A satisfying (2.19) and $H > 1$ (non-unimodular case).
(2) X is unimodular and $H > 0$.
(3) X is isomorphic to $\mathrm{SU}(2)$.

Proof. Observe that the real part of $R(q)$ is $H\left(1 + |q|^2\right)^2 - (1 - |q|^4)$ if X is non-unimodular, and $H\left(1 + |q|^2\right)^2$ if X is unimodular. From here one deduces statements (1) and (2) of the lemma. Now assume that X is isomorphic to $\mathrm{SU}(2)$. A direct computation gives that the imaginary part of the H-potential $R(q)$ for X is

$$\Im(R(q)) = c_3(|q|^2 - 1)^2 + 4(c_1 \Re(q)^2 + c_2 \Im(q)^2),$$

where $c_1, c_2, c_3 > 0$ are the structure constants for the left invariant metric on $\mathrm{SU}(2)$ defined in (2.24). In particular, $\Im(R)$ does not have zeros in $\overline{\mathbb{C}}$ and the proof is complete. $\square$

As usual, we will represent by $R_q = \partial_q R$, $R_{\overline{q}} = \partial_{\overline{q}} R$, $q \in \mathbb{C}$. The next result is a consequence of the fundamental equations of a surface in a three-dimensional metric Lie group.

Theorem 3.15 (Meeks-Mira-Pérez-Ros [**MIMPRb**]). *Let $f\colon \Sigma \looparrowright X$ be an immersed H-surface in a three-dimensional metric Lie group X, with left invariant Gauss map $G\colon \Sigma \to \mathbb{S}^2$. Suppose that the H-potential R of X does not vanish on $G(\Sigma)$. Then:*

(1) Structure equation for the left invariant Gauss map.
The stereographic projection $g\colon \Sigma \to \overline{\mathbb{C}}$ of G from the South pole of $\mathbb{S}^2$ satisfies the elliptic PDE

$$g_{z\overline{z}} = \frac{R_q}{R}(g) g_z g_{\overline{z}} + \left(\frac{R_{\overline{q}}}{R} - \frac{\overline{R_q}}{\overline{R}}\right)(g)|g_z|^2. \tag{3.7}$$

(2) Weierstrass-type representation.
Let $\{E_1, E_2, E_3\}$ be the orthonormal basis of the Lie algebra $\mathfrak{g}$ of X given by (2.24) if X is unimodular, and by (2.16) and (2.19) if X is non-unimodular. Given a conformal coordinate z on Σ, we can express $f_z = \partial_z f = \sum_{i=1}^3 A_i (E_i)_f$ where

$$A_1 = \frac{\eta}{4}\left(\overline{g} - \frac{1}{\overline{g}}\right), \quad A_2 = \frac{i\eta}{4}\left(\overline{g} + \frac{1}{\overline{g}}\right), \quad A_3 = \frac{\eta}{2}, \quad \eta = \frac{4\overline{g} g_z}{R(g)}, \tag{3.8}$$

and $R\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *is the* H*-potential for* X*. Moreover, the induced metric by* f *on* Σ *is given by* $ds^2 = \lambda|dz|^2$*, with*

$$\lambda = \frac{4\left(1+|g|^2\right)^2}{|R(g)|^2}|g_z|^2. \tag{3.9}$$

PROOF. Take a conformal coordinate $z = x + iy$ on Σ, and write the induced metric by f as $ds^2 = \lambda|dz|^2$, where $\lambda = |\partial_x|^2 = |\partial_y|^2$. We will use brackets to express coordinates of a vector field with respect to the orthonormal basis E_1, E_2, E_3. For instance,

$$f_z = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \frac{1}{1+|g|^2}\begin{bmatrix} g+\overline{g} \\ i(\overline{g}-g) \\ 1-|g|^2 \end{bmatrix}$$

are the coordinates of the tangent vector f_z and of the unit normal vector field to f (i.e., $G = (N_1, N_2, N_3)$ is the left invariant Gauss map), where in the last equality we have stereographically projected G from the South pole or equivalently,

$$g = \frac{N_1 + iN_2}{1+N_3} \colon \Sigma \to \overline{\mathbb{C}}.$$

Consider the locally defined complex function $\eta = 2A_3 = 2\langle f_z, E_3\rangle$. Clearly, $\eta\, dz$ is a global complex 1-form on Σ. Define $B_1 = \frac{\eta}{4}\left(\overline{g} - \frac{1}{g}\right)$ and $B_2 = \frac{i\eta}{4}\left(\overline{g} + \frac{1}{g}\right)$. Note that $B_1 - iB_2 = \overline{g}A_3$, $B_1 + iB_2 = -\frac{1}{g}A_3$, from where $B_1^2 + B_2^2 = (B_1 - iB_2)(B_1 + iB_2) = -A_3^2 = A_1^2 + A_2^2$. Also, $B_1N_1 + B_2N_2 = -A_3N_3$. From here is not hard to prove that $A_i = B_i$, $i = 1, 2$ (after possibly a change of orientation in Σ), which are first three formulas in (3.8).

Using the already proven first three formulas in (3.8), we get

$$\frac{\lambda}{2} = \langle f_z, f_{\overline{z}}\rangle = \sum_{i=1}^{3}|A_i|^2 = \frac{|\eta|^2}{8}\left(|g| + \frac{1}{|g|}\right)^2. \tag{3.10}$$

In order to get the last equation in (3.8) we need to use the Gauss equation, which relates the Levi-Civita connections ∇ on X and ∇^Σ on Σ. For instance,

$$\nabla_{f_z} f_z = \nabla^\Sigma_{\partial_z}\partial_z + \sigma(\partial_z, \partial_z)N,$$

where σ is the second fundamental form of f. Now, $\langle f_z, f_z\rangle = 0$ and $\langle f_z, f_{\overline{z}}\rangle = \frac{\lambda}{2}$ since z is a conformal coordinate and f is an isometric immersion. The first equation gives $\langle \nabla^\Sigma_{\partial_z}\partial_z, \partial_z\rangle = \langle \nabla^\Sigma_{\partial_{\overline{z}}}\partial_z, \partial_z\rangle = 0$, while the second one implies

$$\langle \nabla^\Sigma_{\partial_z}\partial_z, \partial_{\overline{z}}\rangle = \frac{\lambda_z}{2} - \langle \partial_z, \nabla^\Sigma_{\partial_z}\partial_{\overline{z}}\rangle \overset{(\star)}{=} \frac{\lambda_z}{2} - \langle \partial_z, \nabla^\Sigma_{\partial_{\overline{z}}}\partial_z\rangle = \frac{\lambda_z}{2},$$

where in $(\star)$ we have used that $[\partial_z, \partial_{\overline{z}}] = 0$. Thus, $\nabla^\Sigma_{\partial_z}\partial_z = \frac{\lambda_z}{\lambda}\partial_z$ and

$$\nabla_{f_z} f_z = \frac{\lambda_z}{\lambda} f_z + \sigma(\partial_z, \partial_z)N. \tag{3.11}$$

Arguing similarly, $\nabla^\Sigma_{\partial_{\overline{z}}}\partial_z = 0$ and

$$\nabla_{f_{\overline{z}}} f_z = \sigma(\partial_z, \partial_{\overline{z}})N = \frac{\lambda H}{2}N. \tag{3.12}$$

On the other hand, $\langle \nabla_{f_z} N, f_z\rangle = -\langle N, \nabla_{f_z} f_z\rangle \overset{(3.11)}{=} -\sigma(\partial_z,\partial_z)$ and $\langle \nabla_{f_z} N, f_{\overline{z}}\rangle \overset{(3.12)}{=} -\frac{H\lambda}{2}$, from where we obtain

$$\nabla_{f_z} N = -Hf_z - \frac{2}{\lambda}\sigma(\partial_z,\partial_z)f_{\overline{z}}. \tag{3.13}$$

Expressing (3.11) with respect to the basis $\{E_1,E_2,E_3\}$, we have

$$\left[\begin{array}{c}(A_1)_z\\(A_2)_z\\(A_3)_z\end{array}\right] + \sum_{i,j=1}^3 A_iA_j\nabla_{E_i}E_j = \frac{\lambda_z}{\lambda}\left[\begin{array}{c}A_1\\A_2\\A_3\end{array}\right] + \sigma(\partial_z,\partial_z)\left[\begin{array}{c}N_1\\N_2\\N_3\end{array}\right]. \tag{3.14}$$

Working similarly with (3.12) and (3.13), we get

$$\left[\begin{array}{c}(A_1)_{\overline{z}}\\(A_2)_{\overline{z}}\\(A_3)_{\overline{z}}\end{array}\right] + \sum_{i,j=1}^3 \overline{A_i}A_j\nabla_{E_i}E_j = \frac{\lambda H}{2}\left[\begin{array}{c}N_1\\N_2\\N_3\end{array}\right], \tag{3.15}$$

$$\left[\begin{array}{c}(N_1)_z\\(N_2)_z\\(N_3)_z\end{array}\right] + \sum_{i,j=1}^3 A_iN_j\nabla_{E_i}E_j = -H\left[\begin{array}{c}A_1\\A_2\\A_3\end{array}\right] - \frac{2}{\lambda}\sigma(\partial_z,\partial_z)\left[\begin{array}{c}\overline{A_1}\\\overline{A_2}\\\overline{A_3}\end{array}\right]. \tag{3.16}$$

From this point in the proof we need to distinguish between the unimodular and non-unimodular case. We will assume X is unimodular, leaving the non-unimodular case for the reader. Plugging (2.26) into the left-hand-side of (3.14), (3.15) and (3.16) we get the following three first order systems of PDE:

$$\left[\begin{array}{c}(A_1)_z+(\mu_2-\mu_3)A_2A_3\\(A_2)_z+(\mu_3-\mu_1)A_1A_3\\(A_3)_z+(\mu_1-\mu_2)A_1A_2\end{array}\right] = \frac{\lambda_z}{\lambda}\left[\begin{array}{c}A_1\\A_2\\A_3\end{array}\right] + \sigma(\partial_z,\partial_z)\left[\begin{array}{c}N_1\\N_2\\N_3\end{array}\right], \tag{3.17}$$

$$\left[\begin{array}{c}(A_1)_{\overline{z}}-\mu_3A_2\overline{A_3}+\mu_2A_3\overline{A_2}\\(A_2)_{\overline{z}}+\mu_3A_1\overline{A_3}-\mu_1A_3\overline{A_1}\\(A_3)_{\overline{z}}+\mu_1A_2\overline{A_1}-\mu_2A_1\overline{A_2}\end{array}\right] = \frac{\lambda H}{2}\left[\begin{array}{c}N_1\\N_2\\N_3\end{array}\right], \tag{3.18}$$

$$\left[\begin{array}{c}(N_1)_z-\mu_3A_3N_2+\mu_2A_2N_3\\(N_2)_z+\mu_3A_3N_1-\mu_1A_1N_3\\(N_3)_z+\mu_1A_1N_2-\mu_2A_2N_1\end{array}\right] = -H\left[\begin{array}{c}A_1\\A_2\\A_3\end{array}\right] - \frac{2}{\lambda}\sigma(\partial_z,\partial_z)\left[\begin{array}{c}\overline{A_1}\\\overline{A_2}\\\overline{A_3}\end{array}\right]. \tag{3.19}$$

Plugging the three first formulas of (3.8) in the first component of (3.18), we obtain

$$\eta_{\overline{z}}\left(\overline{g}-\tfrac{1}{g}\right)+\eta\overline{g_z}\left(1+\tfrac{1}{\overline{g}^2}\right) = H|\eta|^2\left(1+\tfrac{1}{|g|^2}\right)\Re(g)+\tfrac{i|\eta|^2}{2}\left[\mu_3\left(\overline{g}+\tfrac{1}{\overline{g}}\right)+\mu_2\left(g+\tfrac{1}{g}\right)\right], \tag{3.20}$$

where $\Re$ stands for real part. If we work similarly in the second and third components of (3.18), we have respectively

$$i\eta_{\overline{z}}\left(\overline{g}+\tfrac{1}{g}\right)+i\eta\overline{g_z}\left(1-\tfrac{1}{\overline{g}^2}\right) = H|\eta|^2\left(1+\tfrac{1}{|g|^2}\right)\Im(g)+\tfrac{|\eta|^2}{2}\left[\mu_3\left(\tfrac{1}{\overline{g}}-\overline{g}\right)+\mu_1\left(g-\tfrac{1}{g}\right)\right], \tag{3.21}$$

where $\Im$ denotes imaginary part, and

$$\eta_{\overline{z}} = \tfrac{H|\eta|^2(1-|g|^4)}{4|g|^2}-\tfrac{i|\eta|^2}{8}\left[\mu_1\left(\tfrac{1}{\overline{g}}+\overline{g}\right)\left(g-\tfrac{1}{g}\right)-\mu_2\left(\tfrac{1}{\overline{g}}-\overline{g}\right)\left(g+\tfrac{1}{g}\right)\right]. \tag{3.22}$$

Now, multiplying (3.20)+i(3.21) by $\overline{g}$ and using (3.22) we can solve for η, finding the fourth formula in (3.8). Substituting this formula in (3.10) we get (3.9), which proves item (2) of the theorem in the unimodular case.

Regarding item (1), first note that

$$\frac{\eta_{\overline{z}}}{\eta} = (\log \eta)_{\overline{z}} \overset{(3.8)}{=} \left(\log \frac{\overline{g}g_z}{R(g)}\right)_{\overline{z}} = \frac{\overline{g_z}}{\overline{g}} + \frac{g_{z\overline{z}}}{g_z} - \frac{(R(g))_{\overline{z}}}{R(g)}. \tag{3.23}$$

We again suppose that X is unimodular, leaving the non-unimodular case for the reader. Equation (3.22) gives

$$\frac{\eta_{\overline{z}}}{\eta} = \frac{\overline{\eta}}{4}\Theta \overset{(3.8)}{=} \frac{g\overline{g_z}}{\overline{R(g)}}\Theta, \tag{3.24}$$

where $\Theta = H\left(\frac{1}{|g|^2} - |g|^2\right) - \frac{i}{2}\left[\mu_1\left(\frac{1}{\overline{g}} + \overline{g}\right)\left(g - \frac{1}{g}\right) - \mu_2\left(\frac{1}{\overline{g}} - \overline{g}\right)\left(g + \frac{1}{g}\right)\right]$.

Finally, matching the right-hand-side of (3.23) and (3.24) and using that $(R(g))_{\overline{z}} = R_q(g)g_{\overline{z}} + R_{\overline{q}}(g)\overline{g_z}$, we obtain the desired PDE equation for g. □

One can view (3.7) as a necessary condition for the left invariant Gauss map of an immersed H-surface in a three-dimensional metric Lie group. This condition is also sufficient, in the following sense.

Corollary 3.16. *Let Σ be a simply-connected Riemann surface and $g\colon \Sigma \to \overline{\mathbb{C}}$ a smooth function satisfying (3.7) for the H-potential R in some three-dimensional metric Lie group X (here $H \geq 0$ is fixed). If R has no zeros in $g(\Sigma) \subset \overline{\mathbb{C}}$ and g is nowhere antiholomorphic*[8]*, then there exists a unique (up to left translations) immersion $f\colon \Sigma \looparrowright X$ with constant mean curvature H and left invariant Gauss map g.*

Proof. Assume $g \neq \infty$ around a point in Σ, which can be done by changing the point in the unit sphere from where we stereographically project. One first defines η and then A_1, A_2, A_3 by means of the formulas in (3.8). Then a direct computation shows that

$$(A_1)_{\overline{z}} = \frac{g_{z\overline{z}}(\overline{g}^2 - 1)}{R(g)} - \frac{g_z\,[R(g)]_{\overline{z}}\,(\overline{g}^2 - 1)}{R(g)^2} + 2\frac{\overline{g}|g_z|^2}{R(g)}.$$

As g satisfies (3.7), one can write the above equation as

$$(A_1)_{\overline{z}} = \frac{|g_z|^2}{|R(g)|^2}\overline{(2gR(g) - R_q(g)(g^2 - 1))}. \tag{3.25}$$

As in the proof of Theorem 3.15, we will assume in the sequel that X is unimodular, leaving the details of the non-unimodular case to the reader.

Equation (3.6) implies that

$$2gR(g) - R_q(g)(g^2 - 1) = -2i|g|^2\left[\left(g + \frac{1}{g}\right)(\mu_3 + iH) + \left(\overline{g} + \frac{1}{\overline{g}}\right)(\mu_2 + iH)\right], \tag{3.26}$$

where μ_1, μ_2, μ_3 are given by (2.25) in terms of the structure constants c_1, c_2, c_3 of the unimodular metric Lie group X. Substituting (3.26) into (3.25) and using again (3.8) we arrive to

$$(A_1)_{\overline{z}} = A_2\overline{A_3}(\mu_3 - iH) - A_3\overline{A_2}(\mu_2 - iH). \tag{3.27}$$

[8]In other words, $g_z \neq 0$ at every point of Σ.

Analogously,

$$\begin{cases} (A_2)_{\overline{z}} = A_3\overline{A_1}(\mu_1 - iH) - A_1\overline{A_3}(\mu_3 - iH), \\ (A_3)_{\overline{z}} = A_1\overline{A_2}(\mu_2 - iH) - A_2\overline{A_1}(\mu_1 - iH). \end{cases} \tag{3.28}$$

Now (3.27), (3.28) imply that

$$\begin{cases} (A_1)_z - (\overline{A_1})_z = c_1(A_2\overline{A_3} - A_3\overline{A_2}), \\ (A_2)_z - (\overline{A_2})_z = c_2(A_3\overline{A_1} - A_1\overline{A_3}), \\ (A_3)_z - (\overline{A_3})_z = c_3(A_1\overline{A_2} - A_2\overline{A_1}), \end{cases} \tag{3.29}$$

where we have used (2.25) to express the constants μ_i in terms of the c_j.

We now study the integrability conditions of $f\colon \Sigma \looparrowright X$. It is useful to consider X to be locally a subgroup of $Gl(n,\mathbb{R})$ and its Lie algebra $\mathfrak{g}$ to be a subalgebra of $\mathcal{M}_n(\mathbb{R})$ for some n, which we can always assume by Ado's theorem (this is not strictly necessary, but allows us to simplify the notation since left translation in X becomes left multiplication of matrices, while the Lie bracket in $\mathfrak{g}$ becomes the usual commutator of matrices).

Assume for the moment that $f\colon \Sigma \to X$ exists with left invariant Gauss map g. If $z = x + iy$ is a conformal coordinate in Σ, then $f_z = \sum_{i=1}^3 A_i(E_i)_f$, where $\{E_1, E_2, E_3\}$ is a orthonormal basis of $\mathfrak{g}$ given by (2.24). Define

$$\mathcal{A} := f^{-1}f_z = \sum_{i=1}^3 A_i e_i, \tag{3.30}$$

where $e_i = (E_i)_e$ and e is the identity element in X. Then, $\mathcal{A}$ is a smooth map from Σ into the complexified space of T_eX, which we can view as a complex subspace of $\mathcal{M}_n(\mathbb{C})$. Now,

$$\begin{aligned} \mathcal{A}_{\overline{z}} - (\overline{\mathcal{A}})_z &= (f^{-1}f_z)_{\overline{z}} - (f^{-1}f_{\overline{z}})_z = (f^{-1})_{\overline{z}}f_z - (f^{-1})_z f_{\overline{z}} + f^{-1}(f_{z\overline{z}} - f_{\overline{z}z}) \\ &= -(f^{-1}f_{\overline{z}})(f^{-1}f_z) + (f^{-1}f_z)(f^{-1}f_{\overline{z}}) + f^{-1}(f_{z\overline{z}} - f_{\overline{z}z}) \\ &= -\overline{\mathcal{A}}\cdot\mathcal{A} + \mathcal{A}\cdot\overline{\mathcal{A}} + f^{-1}(f_{z\overline{z}} - f_{\overline{z}z}) = [\mathcal{A},\overline{\mathcal{A}}] + f^{-1}(f_{z\overline{z}} - f_{\overline{z}z}). \end{aligned}$$

As $2(f_{z\overline{z}} - f_{\overline{z}z}) = i(f_{xy} - f_{yx})$, then the usual integrability conditions $f_{xy} = f_{yx}$ for f amount to the first order PDE for $\mathcal{A}$:

$$\mathcal{A}_{\overline{z}} - (\overline{\mathcal{A}})_z = [\mathcal{A},\overline{\mathcal{A}}] = \sum_{i,j} A_i\overline{A_j}[e_i,e_j] = \sum_{i,j} A_i\overline{A_j}c_{k(i,j)}e_i \times e_j,$$

where $i, j, k(i,j)$ is a permutation of $1, 2, 3$ (provided that $i \neq j$).

Now (3.29) just means that the last integrability condition is satisfied whenever $A_1, A_2, A_3\colon \Sigma \to \mathbb{C}$ are given by (3.8) in terms of a solution g of (3.7). By the classical Frobenius theorem, this implies that given a solution g of (3.7), there exists a smooth map $f\colon \Sigma \to X$ such that (3.30) holds. The pullback of the ambient metric on X through f is $ds^2 = \lambda|dz|^2$ with λ given by (3.9), which is smooth without zeros on Σ provided that the H-potential R does not vanish in $g(\Sigma)$ and g is nowhere antiholomorphic. Thus, f is an immersion. The fact that $f(\Sigma)$ is an H-surface follows directly from the Gauss and Codazzi equations for f, see the proof of Theorem 3.15. Finally, the Frobenius theorem implies that the solution f to (3.30) is unique if we prescribe an initial condition, say $f(p_0) = e$ where p_0 is any point in Σ. This easily implies the uniqueness property in the statement of the corollary. □

By examining the proof of Theorem 3.15, it can be shown that if $f\colon \Sigma \looparrowright X$ has constant mean curvature H and constant left invariant Gauss map $g = q_0 \in \overline{\mathbb{C}}$, then the H-potential R of X vanishes at q_0. From here we get, using Lemmas 3.9 and 3.14, the next corollary.

COROLLARY 3.17. *The left invariant Gauss map of a connected immersed H-surface Σ in a three-dimensional metric Lie group X is constant if and only if the surface is a left coset of a two-dimensional subgroup of X. Such a Σ exists if and only if X is not isomorphic to* SU(2). *Furthermore, if such a Σ exists then:*
(1) Σ is embedded.
(2) If X is unimodular, then $H = 0$. In particular, a two-dimensional subgroup of a unimodular X is always minimal.
(3) If X is non-unimodular[9]*, then $0 \le H \le 1$. In particular, a two-dimensional subgroup of X has constant mean curvature $H \in [0, 1]$.*

REMARK 3.18. In fact, Lemma 3.14 implies that Corollary 3.17 can be stated (and its proof extends without changes) in a slightly stronger and more technical version to be used later on. This new version asserts that for an immersed H-surface Σ in a three-dimensional metric Lie group X such that one of the following conditions holds,

$$\left\{\begin{array}{l} X \text{ is isomorphic to } \mathrm{SU}(2), \\ H > 0 \text{ and } X \text{ is unimodular}, \\ H > 1 \text{ and } X = \mathbb{R}^2 \rtimes_A \mathbb{R} \text{ is non-unimodular scaled to } \mathrm{trace}(A) = 2, \end{array}\right. \tag{3.31}$$

then the stereographically projected left invariant Gauss map $g\colon \Sigma \to \overline{\mathbb{C}}$ of Σ satisfies $g_z \neq 0$ everywhere in Σ, where z is any conformal coordinate in Σ, i.e., g is nowhere antiholomorphic in Σ.

The next corollary gives another application of the H-potential $R(q)$.

COROLLARY 3.19. *Consider the non-unimodular Lie group $X_1 = \mathbb{R}^2 \rtimes_B \mathbb{R}$, where $B = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$. By item (2a) of Theorem 2.14, the metric Lie group X_1 equipped with its canonical metric is isometric (but not isomorphic) to the metric Lie group X_2 given by the unimodular group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ endowed with an $\mathbb{E}(\kappa,\tau)$-metric; here the bundle curvature is $\tau = 1$ and the base curvature is $\kappa = -4$. Consider X_1 and X_2 to be subgroups of the four dimensional isometry group $I(X_1)$ of X_1, both acting by left translation and with identity elements satisfying $e_1 = e_2$. Then, the connected component Δ of $X_1 \cap X_2$ passing through the common identity element is the two-dimensional non-commutative subgroup of X_1 given by*

$$\Delta = \{(x, x, z) \mid x, z \in \mathbb{R}\}, \tag{3.32}$$

with associated Lie subalgebra $\{\alpha(E_1 + E_2) + \beta E_3 \mid \alpha, \beta \in \mathbb{R}\}$ where E_1, E_2, E_3 are defined by (2.6) for the above matrix B.

PROOF. Clearly, the connected component Δ of $X_1 \cap X_2$ passing through the common identity element is a two-dimensional subgroup of both X_1 and X_2. We want to deduce the equality (3.32) for Δ. Applying Corollary 3.17 to Δ as a subgroup of the unimodular group X_2, we deduce that Δ has zero mean curvature.

[9]Recall that we have normalized the metric so that X is isometric and isomorphic to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $\mathrm{trace}(A) = 2$.

Since this last property is invariant under ambient isometries, then the mean curvature of Δ viewed as a subgroup of X_1 is also zero. Using the H-potential formula for $H = 0$ in the non-unimodular group X_1 where $a = 1 = b$, we find that the (constant) stereographic projection $g\colon \Delta \to \overline{\mathbb{C}}$ from the South pole of the left invariant Gauss map of $\Delta \looparrowright X_1$ satisfies

$$0 = R(g) = |g|^4 - 1 - (g^2 - \overline{g}^2) - i[2|g|^2 - (g^2 + \overline{g}^2)]. \tag{3.33}$$

Since the real part of the last right-hand-side is $|g|^4 - 1$, we have $g = e^{i\theta}$ for some $\theta \in (-\pi, \pi]$. Then (3.33) becomes

$$0 = -4i \sin\theta(\cos\theta + \sin\theta),$$

hence θ is one of the values $0, \pi, \frac{3\pi}{4}, -\frac{\pi}{4}$. The cases $\theta = 0$ or $\theta = \pi$ can be discarded since in this case the tangent bundle to Δ would be generated by $E_2 = \partial_y$ and $E_3 = \partial_z$ (recall that the E_i are given by equation (2.6)), which would give that Δ is commutative; but X_2 does not admit any commutative two-dimensional subgroups. Therefore $\theta = \frac{3\pi}{4}$ up to a change of orientation, which implies that the tangent bundle to Δ is spanned by $E_1 + E_2 = e^{2z}(\partial_x + \partial_y), E_3 = \partial_z$. Now the description of Δ in (3.32) follows directly. $\square$

3.4. The proof of Theorem 3.6 and some related corollaries. In this section we will prove the earlier stated Theorem 3.6.

Proof of Theorem 3.6. We claim that the Lie algebra $\mathfrak{su}(2)$ of SU(2) does not admit any two-dimensional subalgebras: Choose a basis E_1, E_2, E_3 of $\mathfrak{su}(2)$ such that $[E_i, E_{i+1}] = E_{i+2}$ (indices are mod 3). Then for every $X, Y \in \mathfrak{su}(2)$, it holds $[X, Y] = X \times Y$ where x is the cross product defined by the left invariant metric and orientation in SU(2) which make E_1, E_2, E_3 a positive orthonormal basis (this is just the standard round metric on the three-sphere). In particular, $[X, Y]$ is not in the span of X, Y provided that X, Y are linearly independent, from where our claim follows. Therefore, SU(2) cannot have a two-dimensional subgroup and item (1) of Theorem 3.6 is proved.

Next assume $X = \widetilde{\mathrm{SL}}(2, \mathbb{R})$. It suffices to demonstrate that the projection of every two-dimensional subgroup Σ of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ under the covering map $\widetilde{\mathrm{SL}}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})$ is one of the subgroups $\mathbb{H}^2_\theta$ defined in (2.30). Recall that $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is the three-dimensional unimodular Lie group which admits a left invariant metric with associated structure constants $(c_1, c_2, c_3) = (1, 1, -1)$ as explained in Section 2.6. Plugging these values in equation (2.25), we obtain $(\mu_1, \mu_2, \mu_3) = (-1/2, -1/2, 3/2)$. Consider the H-potential $R = R(q)$ defined in (3.6) for these values of μ_1, μ_2, μ_3. Since both the left invariant Gauss map G of Σ and its mean curvature are constant, and the induced metric on Σ is non-degenerate, then equation (3.9) implies that $R(g)$ vanishes identically on Σ, where g is the stereographic projection from the South pole of $\mathbb{S}^2$ of G. In particular, Σ is minimal (see also Corollary 3.17) and

$$(1 + g^2)(1 + \overline{g}^2) + (1 - g^2)(1 - \overline{g}^2) - 12|g|^2 = 0 \quad \text{on } \Sigma. \tag{3.34}$$

Solving (3.34) we find $|g|^2 = 3 \pm 2\sqrt{2}$ on Σ. Note that $\{q \in \overline{\mathbb{C}} \mid |q|^2 = 3 \pm 2\sqrt{2}\}$ represents two horizontal antipodal circles in $\mathbb{S}^2$.

Let $\mathbb{H}^2_\theta$ be one of the subgroups of $\mathrm{PSL}(2, \mathbb{R})$ described in (2.30). The arguments in the last paragraph prove that the (constant) left invariant Gauss map of $\mathbb{H}^2_\theta$ lies in $\{|q|^2 = 3 \pm 2\sqrt{2}\}$. Note that if we conjugate $\mathbb{H}^2_\theta$ by elements in the one-dimensional elliptic subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of rotations around the origin in the Poincaré disk, then we obtain the collection $\{\mathbb{H}^2_{\theta'} \mid \theta' \in \mathbb{S}^1\}$ and the corresponding Gauss images of these $\mathbb{H}^2_{\theta'}$ cover all possible values in $\{|q|^2 = 3 \pm 2\sqrt{2}\}$. In particular, the projection of Σ under the covering map $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ produces a two-dimensional subgroup of $\mathrm{PSL}(2,\mathbb{R})$ which is tangent at the identity to one of the conjugates $\mathbb{H}^2_{\theta'}$ of $\mathbb{H}^2_{\theta}$, which implies this projected subgroup is $\mathbb{H}^2_{\theta'}$ for some θ'.

To prove item (3) of the theorem, express $X = \widetilde{\mathrm{E}}(2)$ as a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Using equations (2.9) and (2.10), we obtain the values $(c_1, c_2, c_3) = (1, 1, 0)$ for the structure constants defined in (2.24). Plugging these values in equation (2.25), we obtain $(\mu_1, \mu_2, \mu_3) = (0, 0, 1)$. Consider the H-potential $R = R(q)$ defined in (3.6) for these values of μ_1, μ_2, μ_3:

$$R(q) = H(1 + |q|^2)^2 - 2i|q|^2, \quad q \in \overline{\mathbb{C}}. \tag{3.35}$$

Let Σ be a two-dimensional subgroup of $\widetilde{\mathrm{E}}(2)$. Arguing as in the last paragraph, we have that $R(g) = 0$, where g is the stereographic projection from the South pole of the left invariant Gauss map of Σ. Thus, (3.35) implies that Σ is minimal and $g = 0$. This clearly implies $\Sigma = \mathbb{R}^2 \rtimes_A \{0\}$ as desired.

We now prove item (4) of the theorem. Suppose X is a non-unimodular Lie group with Milnor D-invariant $D > 1$. Then X is isomorphic to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for a matrix $A = A(b) \in \mathcal{M}_2(\mathbb{R})$ defined by (2.19) with $a = 0$ and $b > 0$. Consider the H-potential $R = R(q)$ defined in (3.5) for these values of a, b:

$$R(q) = H(1 + |q|^2)^2 - (1 - |q|^2) - 2bi|q|^2, \quad q \in \overline{\mathbb{C}}. \tag{3.36}$$

Let Σ be a two-dimensional subgroup of X. With the same notation and arguments as before, we have $R(g) = 0$ so (3.36) implies $g = 0$ and $H = 1$. Then, $\Sigma = \mathbb{R}^2 \rtimes_A \{0\}$ and (4) is proved.

We next prove item (5) of the theorem. The case $X = \mathbb{R}^3$ was explained in Example 3.2. The remaining cases to consider are precisely $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, where A is one of the following matrices:

(a) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which produces Nil_3.

(b) $A = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ where $b \in \mathbb{R}$, which produces Sol_3 (for $b = -1$), $\mathbb{H}^3$ (for $b = 1$) and all non-unimodular groups with normalized Milnor D-invariant $\frac{4b}{(1+b)^2} < 1$.

(c) $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, which produces the non-unimodular group with Milnor D-invariant $D = 1$ not isomorphic to $\mathbb{H}^3$.

We first consider case **(a)**. Applying the same arguments as before, we conclude that in the case of Nil_3 the structure constants can be taken as $(c_1, c_2, c_3) = (-1, 0, 0)$, hence $(\mu_1, \mu_2, \mu_3) = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ and the H-potential is

$$R(q) = H(1 + |q|^2)^2 + i\left(\Re(q^2) + |q|^2\right), \quad q \in \overline{\mathbb{C}}. \tag{3.37}$$

Then, the Gauss map of a two-dimensional subgroup of Nil_3 has its value in the circle on $\mathbb{S}^2$ corresponding to the imaginary axis after stereographic projection from the South pole. Since these are the same values as the subgroups in the algebraic open book decomposition described in Example 3.3, we conclude that the only two-dimensional subgroups of Nil_3 are the leaves of this algebraic open book decomposition.

For case **(b)**, first note the possibility $b = 1$ (so $X = \mathbb{H}^3$) was explained in Example 3.2. So assume $b \neq 1$. In Example 3.4 we described two algebraic open book decompositions of $\mathbb{R}^2 \rtimes_A \mathbb{R}$, whose two-dimensional subgroups have Gauss map images contained in the two great circles of $\mathbb{S}^2$ corresponding to the closures of the real and imaginary axes of $\mathbb{C} \cup \{\infty\}$ after stereographic projection from the South pole. Thus it suffices to show that $\mathbb{R}^2 \rtimes_A \mathbb{R}$ does not admit any two-dimensional subgroup Σ whose Gauss map image is a point outside of the union of the closures of the real and imaginary axes. Arguing by contradiction, suppose such a Σ exists. Intersecting Σ with $\mathbb{R}^2 \rtimes_A \{0\}$ we obtain a one-dimensional subgroup of the commutative group $\mathbb{R}^2 \rtimes_A \{0\} = \mathbb{R}^2$, hence a straight line l. Using the notation E_1, E_2, E_3 in (2.6), we have that l is spanned by some vector of the form $u = \lambda e_1 + \mu e_2$ for some $\lambda, \mu \in \mathbb{R} - \{0\}$, where $e_i = E_i(\vec{0})$, $i = 1, 2$, and $\vec{0} = (0, 0, 0)$. Then we can take a second vector v in the tangent space to Σ at the origin, of the form $v = \mu e_1 - \lambda e_2 + \delta e_3$, where $\delta \in \mathbb{R} - \{0\}$ and $e_3 = E_3(\vec{0})$. Thus $\lambda E_1 + \mu E_2, \mu E_1 - \lambda E_2 + \delta E_3$ generate the Lie algebra of Σ and so, we have $[\lambda E_1 + \mu E_2, \mu E_1 - \lambda E_2 + E_3](\vec{0}) \in T_{\vec{0}}\Sigma$. But $[\lambda E_1 + \mu E_2, \mu E_1 - \lambda E_2 + E_3](\vec{0}) = -\delta(\lambda e_1 + \mu b e_2)$. Since this last vector must be a linear combination of u, v and v has a non-zero component in the e_3-direction, then $\delta(\lambda e_1 + \mu b e_2)$ is a multiple of u. Using that $b \neq 1$ we easily obtain that either $\lambda = 0$ or $\mu = 0$, which is a contradiction.

Arguing in a similar manner as in case **(b)**, one shows that every two-dimensional subgroup in X for case **(c)** is in the algebraic open book decomposition described in Example 3.5.

Finally we prove items (6) and (7) of the theorem. Item (6) and the first statement of item (7) follow immediately from Corollary 3.17. Item (7a) follows from item (4). Item (7b) follows from item (5) and the fact that each of the open book decompositions contains the subgroup $\mathbb{R}^2 \rtimes_A \{0\}$ with constant mean curvature 1 and it also contains the minimal subgroup corresponding to the (x, z) or (y, z)-plane. This completes the proof of Theorem 3.6. □

We finish this section with three useful corollaries to Theorem 3.6.

COROLLARY 3.20. *Let Σ be a compact immersed surface in a three-dimensional, simply-connected metric Lie group different from* $\mathrm{SU}(2)$*. Then, the maximum value of the absolute mean curvature function of Σ is strictly bigger than the mean curvature of any of its two-dimensional subgroups; in particular, Σ is not minimal.*

PROOF. This property follows from applying the usual maximum principle to Σ and to the leaves of the foliation of left cosets of a given subgroup. □

COROLLARY 3.21. *Suppose that X is a three-dimensional, simply-connected metric Lie group such that in the case X is a non-unimodular group of the form $\mathbb{R}^2 \rtimes_A \mathbb{R}$, then* $\mathrm{trace}(A) = 2$*. Then, the H-potential of X vanishes at $q_0 \in \overline{\mathbb{C}}$ for some value H_0 of H if and only if there exists a two-dimensional Lie subgroup $\Sigma \subset X$ with constant mean curvature H_0 and whose left invariant Gauss map is constant of value q_0.*

PROOF. By the comment just before Corollary 3.17, the constant value of the left invariant Gauss map of any two-dimensional subgroup $\Sigma \subset X$ is a zero of the H-potential of the ambient metric Lie group.

A careful reading of the proof of Theorem 3.6 demonstrates that if the H-potential of X vanishes at some point $q_0 \in \overline{\mathbb{C}}$ for the value H_0 of H, then X contains a two-dimensional subgroup Σ with constant mean curvature H_0 whose left invariant Gauss map is q_0 after appropriately orienting Σ. $\square$

As a direct consequence of Theorem 3.6 and Corollary 3.21 we have the following statement.

COROLLARY 3.22. *Let X be a three-dimensional, simply-connected metric Lie group and $H \geq 0$. Then, the H-potential for X is everywhere non-zero if and only if:*

(1) X is isomorphic to SU(2)*, or*
(2) X is not isomorphic to SU(2)*, is unimodular and $H > 0$, or*
(3) $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is non-unimodular with trace*(A)= 2, Milnor D-invariant $D \leq 1$ and $H > 1$, or*
(4) X is non-unimodular with trace*(A)= 2, Milnor D-invariant $D > 1$ and $H \neq 1$.*

3.5. A Hopf-type quadratic differential for surfaces of constant mean curvature in three-dimensional metric Lie groups. Next we will see how the PDE (3.7) for the left invariant Gauss map of an H-surface in a three-dimensional metric Lie group X allows us to construct a complex quadratic differential $Q\,(dz)^2$ for any H-surface in X, which will play the role of the classical Hopf differential when proving uniqueness of H-spheres in X. The quadratic differential $Q\,(dz)^2$ is semi-explicit, in the sense that it is given in terms of data on the H-surface together with an auxiliary solution $g_1 \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of (3.7) which is assumed to be a diffeomorphism. We have already described properties which require the assumption of diffeomorphism on the left invariant Gauss map of an immersed sphere (Lemma 3.10, Theorem 3.11, see also Theorem 3.24 below). We will see conditions under which this assumption holds (Theorems 3.27 and 3.30). This definition of $Q\,(dz)^2$ and the results in this section are inspired by the work of Daniel and Mira [**DM**] for H-surfaces in Sol_3 endowed with its most symmetric left invariant metric.

Let $f \colon \Sigma \looparrowright X$ be an immersed H-surface in a three-dimensional metric Lie group X, where the value of H satisfies (3.31). Choose an orthonormal basis $\{E_1, E_2, E_3\}$ of the Lie algebra $\mathfrak{g}$ of X given by (2.24) if X is unimodular, and by (2.16) and (2.19) if X is non-unimodular. Let $G \colon \Sigma \to \mathbb{S}^2$ be the left invariant Gauss map of f, let $g \colon \Sigma \to \overline{\mathbb{C}}$ denote its projection from the South pole of $\mathbb{S}^2$ and let $R \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the H-potential for X. Assume that the following condition holds:

(3.38) *There exists a solution $g_1 \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of (3.7) which is a diffeomorphism.*

Then, the formula

$$Q\,(dz)^2 = \left[L(g) g_z^2 + M(g) g_z \overline{g}_z\right] (dz)^2 \tag{3.39}$$

defines a global[10] complex, smooth quadratic differential on Σ, where $M(q) = 1/R(q)$ for all $q \in \overline{\mathbb{C}}$ and $L\colon \overline{\mathbb{C}} \to \mathbb{C}$ is implicitly given by

$$L(g_1(\xi)) = -\frac{M(g_1(\xi))(\overline{g_1})_\xi}{(g_1)_\xi}. \tag{3.40}$$

Note that L is finite-valued since R does not vanish at any point of $\overline{\mathbb{C}}$ by (3.31) and Lemma 3.14. The fact that $Q\,(dz)^2$ is well-defined outside points of Σ where $g = \infty$ is clear. For points where $g = \infty$, simply note that $q^4 L(q)$ is bounded and smooth around $q = \infty$, which implies $L(g)g_z g_{\overline{z}}$ is bounded and smooth around a point in Σ where $g = \infty$; the second term in (3.39) can be treated in the same way.

A crucial property of $Q\,(dz)^2$, which depends on equation (3.7), is that it satisfies the following *Cauchy-Riemann inequality:*

$$\frac{|Q_{\overline{z}}|}{|Q|} \text{ is locally bounded in } \Sigma. \tag{3.41}$$

Inequality (3.41) implies that either $Q\,(dz)^2$ is identically zero on Σ, or it has only isolated zeros of negative index, see e.g., Alencar, do Carmo and Tribuzy [**AdCT07**]. By the classical Hopf index theorem, we deduce the next proposition (note that the condition (3.31) holds for H-spheres by Corollary 3.20).

THEOREM 3.23 (Meeks-Mira-Pérez-Ros [**MIMPRb**]). *Let X be a three-dimensional metric Lie group. Suppose that there exists an immersed H-sphere S_H in X whose left invariant Gauss map is a diffeomorphism. Then, every immersed H-sphere in X satisfies $Q\,(dz)^2 = 0$.*

We now investigate the condition $Q\,(dz)^2 = 0$ locally on an immersed H-surface $\Sigma \looparrowright X$ for a value of H such that condition (3.31) holds and for which there exists an immersed H-sphere S_H in X whose left invariant Gauss map is a diffeomorphism (we follow the same notation as above). By Remark 3.18, the stereographically projected Gauss map g of Σ is nowhere antiholomorphic. By (3.39), we have

$$\frac{\overline{g}_z}{g_z} = -\frac{L(g)}{M(g)} \overset{(3.40)}{=} \frac{(\overline{g}_1)_\xi}{(g_1)_\xi}. \tag{3.42}$$

On the other hand, a direct computation gives

$$0 \overset{(A)}{\leq} \left|\frac{g_{\overline{z}}}{g_z}\right|^2 = \frac{|dg|^2 - 2\,\mathrm{Jac}(g)}{|dg|^2 + 2\,\mathrm{Jac}(g)} \overset{(B)}{\leq} 1, \tag{3.43}$$

where dg is the differential of g and $\mathrm{Jac}(g)$ its Jacobian. Moreover, equality in (A) holds if and only if $g_{\overline{z}} = 0$ while equality in (B) occurs if and only if $\mathrm{Jac}(g) = 0$. Since g_1 is a diffeomorphism, then equality in (B) cannot hold for g_1. Hence (3.42) implies that equality in (B) cannot hold for g and thus, g is a local diffeomorphism.

We can now prove the main result of this section.

THEOREM 3.24 (Meeks-Mira-Pérez-Ros [**MIMPRb**]). *Let X be a three-dimensional metric Lie group and $H \geq 0$ be a value for which there exists an immersed H-sphere S_H in X whose left invariant Gauss map is a diffeomorphism. Then, S_H is the unique (up to left translations) H-sphere in X.*

[10]This means that $Q\,(dz)^2$ does not depend on the conformal coordinate z in Σ.

PROOF. Suppose $f : \overline{\mathbb{C}} \looparrowright X$ is an H-sphere and let $g\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be its stereographically projected left invariant Gauss map. By Theorem 3.23, the complex quadratic differential $Q\,(dz)^2$ associated to f vanishes identically. By the discussion just before the statement of this theorem, g is a (global) diffeomorphism. Hence g and the stereographically projected left invariant Gauss map g_1 of S_H are related by $g_1 = g \circ \varphi$ for some diffeomorphism $\varphi\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Now φ can be proved to be holomorphic by the local arguments in the proof of Lemma 4.6 in [**DM08**]. Hence, up to conformally reparametrizing S_H, we conclude that both S_H and $f(\overline{\mathbb{C}})$ are H-surfaces in X with the same left invariant Gauss map. Then, Corollary 3.16 insures that S_H and $f(\overline{\mathbb{C}})$ differ by a left translation. $\square$

3.6. Index-one H-spheres in simply-connected three-dimensional metric Lie groups. Let Σ be a compact (orientable) immersed H-surface in a simply-connected, three-dimensional metric Lie group X. Its *Jacobi operator* is the linearization of the mean curvature functional,

$$\mathcal{L} = \Delta + |\sigma|^2 + \mathrm{Ric}(N),$$

where Δ is the Laplacian in the induced metric, $|\sigma|^2$ the square of the norm of the second fundamental form of Σ and $N\colon \Sigma \to TX$ a unit normal vector field along Σ. It is well-known that $\mathcal{L}$ is (L^2) self-adjoint and its spectrum consists of a sequence

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \leq \ldots$$

of real eigenvalues (that is, for each λ_k there exists a non-zero smooth function $u_k\colon \Sigma \to \mathbb{R}$ such that $\mathcal{L}u_k + \lambda_k u_k = 0$ and each eigenvalue appears in the sequence counting its multiplicity) with $\lambda_k \nearrow \infty$ as $k \to \infty$. The number of negative eigenvalues of $\mathcal{L}$ is called the *index* of Σ, which we denote by $\mathrm{Ind}(\Sigma)$.

A function $u\colon \Sigma \to \mathbb{R}$ is called a *Jacobi function* if $\mathcal{L}u = 0$. Since Killing fields in X produce 1-parameter subgroups of ambient isometries, moving the surface Σ through these 1-parameter subgroups and then taking inner product with N, we produce Jacobi functions on Σ (some of which might vanish identically). Given a point $p \in \Sigma$, we can choose a right invariant vector field (hence Killing) F on X such that $F_p \notin T_p\Sigma$. Then the related Jacobi function $u = \langle F, N\rangle$ is not identically zero on Σ. Also, since X is homeomorphic to $\mathbb{S}^3$ or $\mathbb{R}^3$, Σ is homologous to zero. An application of the divergence theorem to a three-chain with boundary Σ implies that u changes sign. From here we can extract several consequences:

(1) 0 is always an eigenvalue of $\mathcal{L}$. The multiplicity of 0 as an eigenvalue is called the *nullity* of Σ.
(2) The Jacobi function u is not the *first* eigenfunction of $\mathcal{L}$ or equivalently, $\lambda_1 < 0$.
(3) $\mathrm{Ind}(\Sigma) \geq 1$, and $\mathrm{Ind}(\Sigma) = 1$ if and only if $\lambda_2 = 0$.

The next lemma shows that the index of Σ is usually at least three.

LEMMA 3.25. *Let Σ be a compact oriented H-surface in a simply-connected, three-dimensional metric Lie group. The nullity of Σ is at least three unless Σ is an immersed torus and X is* $\mathrm{SU}(2)$ *with a left invariant metric.*

PROOF. Since X has three linearly independent right invariant vector fields F_1, F_2, F_3, then their inner products with N produce three Jacobi functions on Σ. If these functions are linearly independent, then the nullity of Σ is at least three. Otherwise, there exists a linear combination F of F_1, F_2, F_3 which is tangent everywhere along Σ. Note that F is everywhere non-zero in X. By the Hopf index theorem, the Euler characteristic of Σ is zero, i.e., Σ must be an immersed torus. It remains to show that in this case, $X = \mathrm{SU}(2)$. Otherwise, X is diffeomorphic to $\mathbb{R}^3$ and the integral curves of F are proper non-closed curves in X. Since the integral curves of F in Σ do not have this property because Σ is compact, then X must be $\mathrm{SU}(2)$ with a left invariant metric. □

COROLLARY 3.26. *Let Σ be an immersed H-sphere of index one in a simply-connected, three-dimensional metric Lie group. Then, the nullity of Σ is three.*

PROOF. By Theorem 3.4 in Cheng [**Che76**] (who studied the particular case when operator is the Laplacian, see e.g., Rossman [**Ros02**] for a proof for a general operator of the form $\Delta + V$, V being a function), the space of Jacobi functions on an index-one H-sphere in a Riemannian three-manifold has dimension at most three. Hence, Lemma 3.25 completes the proof. □

The classical *isoperimetric problem* in a three-dimensional, simply-connected metric Lie group X consists of finding, given a finite positive number $t \leq V(X)$ (here $V(X)$ denotes the volume of X, which is infinite unless $X = \mathrm{SU}(2)$), those compact surfaces Σ in X which enclose a region $\Omega \subset X$ of volume t and minimize the area of $\partial\Omega = \Sigma$. Note that solutions of the isoperimetric problem are *embedded.* It is well-known that given $t \in (0, V(X)]$, there exist solutions of the isoperimetric problem, and they are smooth surfaces. The first variation of area gives that every such a solution Σ has constant mean curvature, and the second variation of area insures that the second derivative of the area functional for a normal variation with variational vector field fN, $f \in C^\infty(\Sigma)$, is given by

$$\mathcal{Q}(f,f) := \left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Area}(\Sigma + fN) = -\int_\Sigma f\,\mathcal{L}f.$$

The quadratic form $\mathcal{Q}$ defined above is called the *index form* for Σ. For a variation of Σ with vector field fN, the condition to preserve infinitesimally the enclosed volume can be equivalently stated by the fact that f has mean zero along Σ. Thus if Σ is a solution of the isoperimetric problem (briefly, a *isoperimetric surface*), then

$$\mathcal{Q}(f,f) \geq 0 \quad \text{for all } f \in C^\infty(M) \text{ such that } \int_\Sigma f = 0. \tag{3.44}$$

Compact oriented (not necessarily embedded) surfaces Σ in X satisfying (3.44) are called *weakly stable*[11]. Hence solutions of the isoperimetric problem are weakly stable surfaces, but the converse is not true for certain left-invariant metrics on $\mathbb{S}^3$ (see Torralbo and Urbano [**TU09**]); however there are no known compact weakly stable H-surfaces which are not solutions to the isoperimetric problem when X is a metric Lie group diffeomorphic to $\mathbb{R}^3$. If a compact, orientable immersed H-surface Σ in X has $\lambda_2 < 0$, then the eigenfunctions of $\mathcal{L}$ corresponding to λ_1, λ_2 generate a subspace W of $C^\infty(\Sigma)$ with dimension at least two, where $\mathcal{Q}$ is negative definite. Then we can find a non-zero solution $f \in W$ of $\int_\Sigma f = 0$, which implies by (3.44) that Σ is not weakly stable. Therefore,

(3.45) *If Σ is a weakly stable immersed H-surface in X, then* $\mathrm{Ind}(\Sigma) = 1$.

[11] Beware: This notion of weak stability for an H-surface M in a Riemannian three-manifold is different from the stronger one where one requires that $\mathcal{Q}(f,f) \geq 0$ for all $f \in C^\infty(M)$; we will refer to this last condition as the *stability* of the H-surface M.

We now relate the index-one property for an H-sphere with the property that its left invariant Gauss map is a diffeomorphism. See Daniel-Mira [**DM**] for the case when X is Sol_3 with the left invariant metric associated to the structure constants $(c_1, -c_1, 0)$.

THEOREM 3.27 (Meeks-Mira-Pérez-Ros [**MIMPRb**]). *Let S_H be an index-one H-sphere immersed in a three-dimensional metric Lie group X. Then, the left invariant Gauss map of S_H is a diffeomorphism.*

PROOF. Let $G\colon S_H \to \mathbb{S}^2$ be the left invariant Gauss map of S_H. By elementary covering theory, it suffices to check that G is a local diffeomorphism. Arguing by contradiction, assume this condition fails at a point $p_0 \in S_H$. Since G is invariant under left translation, we may assume that p_0 is the identity element e of X. Thus, there is a unit vector $v_1 \in T_eS_H$ that lies in the kernel of the differential $dG_e\colon T_eS_H \to T_{G(e)}\mathbb{S}^2$. Let F be the right invariant vector field in X such that $F_e = v_1$. Since F is right invariant, then it is a Killing field for the left invariant metric of X.

We claim that dG_e cannot be the zero linear map. Arguing by contradiction, if $dG_e = 0$, then choose a local conformal coordinate $z = x + iy$, $|z| < \varepsilon$ (here $\varepsilon > 0$) in S_H so that $z = 0$ corresponds to $e \in S_H$. Thus, the stereographic projection g of G from the South pole of $\mathbb{S}^2$ satisfies $g_z(0) = 0$. Since the induced metric on S_H is unbranched at e, then (3.9) implies that $R(g(e)) = 0$, where R denotes the H-potential for X. By Corollary 3.21, there exists a two-dimensional subgroup Σ of X with constant mean curvature H whose left invariant Gauss map is constant of value $g(e)$ (in particular, X is not isomorphic to SU(2)). This is contrary to Corollary 3.20, and our claim follows.

We next prove that if $N\colon S_H \to TX$ denotes the unit normal field to S_H, then the Jacobi function $u = \langle N, F\rangle$ vanishes to at least second order at e (note that $u(e) = 0$). To do this, choose a local conformal coordinate $z = x+iy$, $|z| < \varepsilon$ in S_H so that $z = 0$ corresponds to $e \in S_H$ and $\partial_x(0) = v_1 \in T_eS_H$. Hence $\{\partial_x(0), \partial_y(0)\}$ is an orthonormal basis of T_eS_H and $G_x(0) = 0$ where as usual, $G_x = \frac{\partial G}{\partial x}$. Consider the second order ODE given by particularizing (3.7) to functions of the real variable y, i.e.,

$$\widehat{g}_{yy} = \frac{R_q}{R}(\widehat{g})(\widehat{g}_y)^2 + \left(\frac{R_{\overline{q}}}{R} - \frac{\overline{R_q}}{\overline{R}}\right)(\widehat{g})|\widehat{g}_y|^2. \tag{3.46}$$

Let $\widehat{g} = \widehat{g}(y)$ be the (unique) solution of (3.46) with initial conditions $\widehat{g}(0) = g(0)$, $\widehat{g}_y(0) = g_y(0)$. We want to use Corollary 3.16 with this function $\widehat{g}\colon \{|z| < \varepsilon\} \to \overline{\mathbb{C}}$. To do this, we must check that the H-potential does not vanish in $\overline{\mathbb{C}}$ and $\widehat{g}_y$ does not vanish on $\{|z| < \varepsilon\}$; the first property follows from the arguments in the last paragraph, while the second condition holds (after possibly choosing a smaller $\varepsilon > 0$) since $\widehat{g}_y(0) = g_y(0) \neq 0$ because $dG_e \neq 0$. By Corollary 3.16, there exists a exists an immersion $\widehat{f}\colon \{|z| < \varepsilon\} \looparrowright X$ with constant mean curvature H and stereographically projected left invariant Gauss map $\widehat{g}$. The uniqueness of $\widehat{f}$ up to left translations and the fact that $\widehat{g}$ only depends on y imply that $\widehat{f}$ is invariant under the 1-parameter group of ambient isometries $\{\phi_t = l_{\exp(tF_e)}\}_{t\in\mathbb{R}}$ which generate the right invariant (hence Killing) vector field F; recall that F is determined by the equation $F_e = \partial_x(0)$. In particular, the function $\widehat{u} = \langle \widehat{N}, F\rangle$ vanishes identically, where $\widehat{N}$ is the unit normal vector field to $\Sigma = \widehat{f}(\{|z| < \varepsilon\})$

(note that we can choose $\widehat{N}$ so that $\widehat{N}_e = N_e$). Given $v \in T_eS_H = T_e\Sigma$,

$$du_e(v) = v(\langle N, F\rangle) = \langle \nabla_v N, F_e\rangle + \langle N_e, \nabla_v F\rangle,$$

and analogously,

$$0 = d\widehat{u}_e(v) = \langle \nabla_v \widehat{N}, F_e\rangle + \langle \widehat{N}_e, \nabla_v F\rangle = \langle \nabla_v \widehat{N}, F_e\rangle + \langle N_e, \nabla_v F\rangle.$$

Subtracting the last two equations we get

$$du_e(v) = \langle \nabla_v N - \nabla_v \widehat{N}, F_e\rangle. \tag{3.47}$$

On the other hand, $N = \sum_{i=1}^3 N_i E_i$ where E_1, E_2, E_3 is an orthonormal basis of the Lie algebra $\mathfrak{g}$ of X. Thus $G = (N_1, N_2, N_3)$ in coordinates with respect to $(E_1)_e, (E_2)_e, (E_3)_e$ and

$$\nabla_v N = \sum_i (dN_i)_e(v)(E_i)_e + \sum_i N_i(e)\nabla_v E_i = dG_e(v) + \sum_i N_i(e)\nabla_v E_i.$$

Arguing in the same way with $\widehat{N}$, subtracting the corresponding equations and using (3.47) we obtain

$$du_e(v) = \langle dG_e(v) - d\widehat{G}_e(v), F_e\rangle, \tag{3.48}$$

where $\widehat{G}$ is the $\mathbb{S}^2$-valued left invariant Gauss map of Σ. Now, if we take $v = v_1$ in (3.48), then the right-hand-side vanishes since v_1 lies in the kernel of dG_e and $\widehat{G}_x(0) = 0$. If we take $v = \partial_y(0)$ in (3.48), then the right-hand-side again vanishes since $\widehat{g}_y(0) = g_y(0)$. Therefore u vanishes at e at least to second order, as desired.

By Theorem 2.5 in Cheng [**Che76**], the nodal set $u^{-1}(0)$ of $u = \langle N, F\rangle$ is an analytic 1-dimensional set (u changes sign on S_H since $u(e) = 0$ and u being identically zero on S_H would imply that S_H is a torus) containing at least two transversely intersecting arcs at the point e. Since such an analytic set of a sphere separates the sphere into at least three domains, then S_H cannot have index one by the Courant's nodal domain theorem (see Proposition 1.1 in [**Che76**] when the operator is the Laplacian and see e.g., Rossman [**Ros02**] for a proof for a general operator of the form $\Delta + V$, V being a function). This contradiction completes the proof. □

COROLLARY 3.28. *Let X be a three-dimensional metric Lie group and suppose that there exists an immersed index-one H-sphere S_H in X. Then:*

(1) S_H is the unique H-sphere in X up to left translations. Furthermore, some left translation of S_H inherits all possible isometries of X which fix the origin.
(2) S_H is round when X is isometric to $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$, and S_H is rotationally invariant in the cases X is isometric to an $\mathbb{E}(\kappa,\tau)$-space with $\kappa \leq 0$.
(3) If X has constant sectional curvature, X is an $\mathbb{E}(\kappa,\tau)$-space with $\kappa \leq 0$ or X is algebraically isomorphic either to Sol_3 or to a three-dimensional non-unimodular Lie group with Milnor D-invariant $D \leq 1$, then S_H is embedded.

PROOF. To prove the first statement in item (1) of the corollary, just apply Theorems 3.24 and 3.27. The second statement in item (1) requires more work and we just refer the reader to the paper [**MIMPRb**]. Item (2) is a direct consequence of the last sentence in (1) (item (2) also holds in the case $X = \mathbb{S}^2 \times \mathbb{R}$ which is not a Lie group, see Abresch and Rosenberg [**AR04**]).

The embeddedness property for any H-sphere in the case that the curvature of X is constant follows from their roundedness. Similarly, the embeddedness of spheres in the ambient $\mathbb{E}(\kappa,\tau)$ with $\kappa \leq 0$ follows from the fact that in these spaces all examples are rotational and by classification they are embedded [**AR05**] (nevertheless, some H-spheres fail to be embedded in certain Berger spheres $\mathbb{E}(\kappa,\tau)$ with $\kappa > 0$, see Torralbo [**Tor10**]). By item (5) of Theorem 3.6 and item (2) of Theorem 3.11, H-spheres in a simply-connected, three-dimensional non-unimodular metric Lie group with Milnor D-invariant $D \leq 1$ or in Sol_3 whose left invariant Gauss maps are diffeomorphisms are embedded. It now follows by Theorem 3.27 that an immersed index-one H-sphere S_H in such a space is embedded. □

Corollary 3.28 is a particular case of the following expected conjecture.

CONJECTURE 3.29 (Hopf Uniqueness Conjecture, Meeks-Mira-Pérez-Ros). *Let X be a simply-connected, three-dimensional homogeneous Riemannian manifold. For every $H \geq 0$, any two H-spheres immersed in X differ by an ambient isometry of X.*

Conjecture 3.29 is known to hold if $X = \mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$ (Hopf [**Hop89**]), if the isometry group of X is four-dimensional (Abresch and Rosenberg [**AR04, AR05**]) and if X is the Lie group Sol_3 with its standard metric given as the canonical metric in $\mathbb{R}^2 \rtimes_{A(1)} \mathbb{R}$, where $A(1) \in \mathcal{M}_2(\mathbb{R})$ is defined in (2.33) with $c_1 = 1$ (Daniel and Mira [**DM08**], Meeks [**MI**]). We will sketch in the next section the proof of the validity of Conjecture 3.29 when $X = \mathrm{SU}(2)$ (Meeks, Mira, Pérez and Ros). We also remark that the same authors are in the final stages of writing a complete proof of Conjecture 3.29.

3.7. Classification of H-spheres in three-dimensional metric Lie groups. Let X be a simply-connected, three-dimensional homogeneous Riemannian manifold. Given a compact surface Σ immersed in X, we will denote by $\|H\|_\infty(\Sigma)$ the maximum value of the absolute mean curvature function $|H|\colon \Sigma \to \mathbb{R}$ of Σ. Associated to X we have the following non-negative constant, which we will call the *critical mean curvature* of X:

$$H(X) = \inf\{\|H\|_\infty(\Sigma) \mid \Sigma \text{ is compact surface immersed in } X\}. \tag{3.49}$$

We next illustrate this notion of critical mean curvature with examples. It is well known that $H(\mathbb{R}^3) = 0$ and $H(\mathbb{H}^3) = 1$. Recall that every simply-connected, three-dimensional homogeneous Riemannian manifold is either $\mathbb{S}^2(k)\times\mathbb{R}$ or a metric Lie group (Theorem 2.4). In the case $X = \mathbb{S}^2(k) \times \mathbb{R}$, there exist minimal spheres immersed in X and so, $H(X) = 0$. Existence of minimal spheres is also known to hold when X is a Lie group not diffeomorphic to $\mathbb{R}^3$, i.e., $X = \mathrm{SU}(2)$ with some left invariant metric, hence in this case $H(X)$ is again zero. In fact, Simon [**Sim85**] proved that for any Riemannian metric on $\mathbb{S}^3$, there exists an embedded, minimal, index-one two-sphere in this manifold. If X is non-unimodular, then, after rescaling the metric, X is isomorphic and isometric to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some matrix $A \in \mathcal{M}_2(\mathbb{R})$ with $\mathrm{trace}(A) = 2$; in this case, $H(X) \geq 1$ by the mean comparison principle applied to the foliation $\{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$, all whose have leaves mean curvature 1.

Consider again a simply-connected, three-dimensional homogeneous Riemannian manifold X, which must be either $\mathbb{S}^2(k) \times \mathbb{R}$ or a metric Lie group. In the first case, uniqueness of H-spheres is known to hold [**AR04**] (i.e., Conjecture 3.29 is a theorem for $X = \mathbb{S}^2(k) \times \mathbb{R}$). We now extend this classification result of H-spheres to any simply-connected, three-dimensional, compact metric Lie group.

THEOREM 3.30 (Meeks-Mira-Pérez-Ros [**MIMPRb**]). *Let X be a simply-connected, three-dimensional metric Lie group diffeomorphic to $\mathbb{S}^3$. Then:*

(1) The moduli space of H-spheres in X (up to left translations) is an analytic curve parameterized by the mean curvatures of the surfaces, which take on all values in $[0,\infty)$.
(2) If S_H is an immersed H-sphere in X, then S_H has index one and nullity three. In particular, the left invariant Gauss map of S_H is a diffeomorphism.

SKETCH OF THE PROOF. Consider the space $\mathcal{M}(X)$ of immersed H-spheres of index one in X (up to left translations), for all possible values of $H \geq 0$. Recall that Theorem 3.27 insures that for every $S \in \mathcal{M}(X)$, the left invariant Gauss map of S is a diffeomorphism.

It is well-known that for $t > 0$ sufficiently small, solutions to the isoperimetric problem in X for volume t exist and geometrically are small, almost-round balls with boundary spheres $S(t)$ of constant mean curvature approximately $\left(\frac{4\pi}{3t}\right)^{1/3}$. Since every such $S(t)$ is area-minimizing for its enclosed volume, then $S(t)$ is weakly stable and by (3.45), $\mathrm{Ind}(S(t)) = 1$. Hence $S(t) \in \mathcal{M}(X)$ and we deduce that there exists an $H_0 > 0$ such that for any $H \in [H_0, \infty)$, $\mathcal{M}(X)$ contains an embedded H-sphere.

The next step in the proof consists of demonstrating that $\mathcal{M}(X)$ is an analytic one-manifold locally parameterized by its mean curvature values. This is a standard application of the Implicit Function Theorem that uses the already proven property in item (2) of Theorem 3.27 that the nullity of each $S \in \mathcal{M}(X)$ is three, see for instance the works of Koiso [**Koi02**], Souam [**Sou10**] and Daniel and Mira [**DM08**] for this type of argument.

Next we consider the embedded index-one H_0-sphere $S_{H_0} \in \mathcal{M}(X)$ and start deforming S_{H_0} in the set of immersed spheres in X with constant mean curvature by decreasing its mean curvature, producing an analytic curve $H \overset{\Gamma}{\mapsto} S_H$ as indicated in the last paragraph. In fact, the image of the curve Γ lies entirely in $\mathcal{M}(X)$, i.e., the spheres $S_H = \Gamma(H)$ all have index one since otherwise, an intermediate value argument would lead to an H-value for which S_H has nullity four, which is impossible by Corollary 3.26.

Our goal is to show that the maximal interval of H-values in which such a deformation curve Γ can be defined is of the form $[0, H_0]$. To do this we argue by contradiction, assuming that the maximal interval of H-values is of the form $(H_\infty, H_0]$ for some $H_\infty > 0$. We want to study what possible problems can occur at H_∞ in order to stop the deformation process. After left translation, we can assume that all spheres $S_H = \Gamma(H)$ with $H \in (H_\infty, H_0]$ pass through the identity element $e = I_2 \in \mathrm{SU}(2)$. A standard compactness argument shows that in order for the process of deforming spheres to stop, there must exist a sequence $H_n \searrow H_\infty$ such that, after possibly passing to a subsequence, one of the following cases occurs:

(1) The second fundamental forms σ_n of the S_{H_n} blow-up at points $p_n \in S_{H_n}$ with $\|\sigma_n\|(p_n) \geq n$.
(2) The areas of the S_{H_n} are greater than n.

Next we will indicate why the first possibility cannot occur. One way of proving uniform bounds for the second fundamental form of the S_H is by mimicking the arguments in Proposition 5.2 in Daniel and Mira [**DM08**], which can be extended to our situation $X \cong \mathrm{SU}(2)$ (actually, these arguments work in *every* three-dimensional metric Lie group X since a bound on the norm of the second fundamental form of an H-surface can be found in terms of the H-potential for X, which in turn can be bounded in terms of H and the structure constants of X provided that the left invariant Gauss map is a diffeomorphism). A more geometric way of proving uniform bounds for the second fundamental form of the S_H is as follows. Arguing by contradiction, if the second fundamental forms of the spheres S_{H_n} are not uniformly bounded, then one can left translate and rescale S_{H_n} on the scale of the maximum norm of its second fundamental form, thereby producing a limit surface which is a non-flat, complete immersed minimal[12] surface M_∞ in $\mathbb{R}^3$. Under the limit process, the index of the Jacobi operator cannot increase; hence M_∞ has index zero or one. Index zero for M_∞ cannot occur since otherwise M_∞ would be stable[13], hence flat. Thus, M_∞ has index one. The family of such complete minimal surfaces is classified (López and Ros [**LR89**]), with the only possibilities being the catenoid and the Enneper minimal surface. The catenoid can be ruled out by flux arguments (its flux is non-zero, but the CMC flux of a sphere S_H is zero since it is simply-connected). The Enneper minimal surface limit E can be ruled out in a number of ways. One way of doing it consists of showing that the Alexandrov embedded balls bounded by the rescaled spheres S_{H_n} limits to a complete, non-compact, simply-connected three-manifold Y with connected boundary, and that Y submerses into $\mathbb{R}^3$ with the image of ∂Y being the Enneper surface E. Since E admits a rotational isometry ψ by angle π around one of the straight lines contained in E, then one can abstractly glue Y via ψ with a copy Y^* of Y and create an isometric submersion of the complete three-manifold $Y \cup_\psi Y^*$ into $\mathbb{R}^3$. Since $\mathbb{R}^3$ is simply-connected, such submersion must be a diffeomorphism and thus, E is embedded. This contradiction eliminates the Enneper surface as a limit of the rescaled spheres. In [**MIMPRb**] we provide a different proof of this last property.

Therefore, in order for the deformation process to stop at $H_\infty \geq 0$, the areas of the S_H are unbounded as $H \searrow H_\infty$ while the second fundamental form of S_H remains uniformly bounded. Thus, there exists a sequence $S_{H_n} \subset \mathcal{M}(X)$ with $H_n \searrow H_\infty$, $\mathrm{Area}(S_{H_n}) \geq n$ for all $n \in \mathbb{N}$ and $\|\sigma_n\|$ uniformly bounded. Since the left invariant Gauss maps G_n of the surfaces S_{H_n} are diffeomorphisms, it is possible to find open domains $\Omega_n \subset S_{H_n}$ with $e \in \Omega_n$, having larger and larger area, whose images $G_n(\Omega_n)$ have arbitrarily small spherical area. Carrying out this process carefully and using the index-one property for the S_{H_n}, one can produce a subsequence of such domains Ω_n which converge as mappings as $n \to \infty$ to a stable limit which is a complete immersion $f\colon M_\infty \looparrowright X$ with constant mean curvature H_∞ with $e \in f(M_\infty)$, bounded second fundamental form and with degenerate left invariant Gauss map G_∞, in the sense that the spherical area of G_∞ is zero. Since X is isomorphic to SU(2), then Corollary 3.17 implies that G_∞ cannot have rank zero at any point. Hence, G_∞ has rank one or two at every point. Since the spherical area of M_∞ is zero, then G_∞ has rank one at every point of M_∞. It follows that the image of G_∞ is an immersed curve $C \subset \mathbb{S}^2$ and M_∞ fibers over C via G_∞. In particular, M_∞ is either simply-connected or it is a cylinder. With a little more

[12]Minimality of the limit surface follows since the original mean curvatures H_n are bounded by above.

[13]Recall that an H-surface Σ is called *stable* if its index form is non-negative on the space of all compactly supported smooth functions on Σ; the difference between stability and weak stability is that in the second one only imposes non-negativity of the index form on compactly supported functions *whose mean is zero*.

work, it can be proven that C is a closed embedded curve in $\mathbb{S}^2$, the image of M_∞ is invariant under the left action of a 1-parameter subgroup of X (where the fibers of $G_\infty\colon M_\infty \to C$ project to the related orbits of this action), and that M_∞ has at most quadratic area growth. In particular, the underlying conformal structure of M_∞ is parabolic. This last property together with its stability imply that the space of *bounded* Jacobi functions on M_∞ is one-dimensional and coincides with the space of Jacobi functions with *constant sign* (Manzano, Pérez and Rodríguez [**MPR11**]). Let V_1, V_2 be a pair of linearly independent right invariant vector fields on X which are tangent to $f(M_\infty)$ at e. Since these vector fields are Killing and bounded, then they must be globally tangent to M_∞, otherwise they would produce a bounded Jacobi function on the surface which changes sign. But since the structure constants of SU(2) are non-zero, we obtain a contradiction to the fact that $[V_1, V_2]_e$ is a linear combination of V_1 and V_2; here we are using the fact that the space of right invariant vector fields on a Lie group is isomorphic as a Lie algebra to its Lie algebra of left invariant vector fields. This contradiction finishes the sketch of the proof that the curve Γ is defined for all values $[0, H_0]$.

Once we know that $\mathcal{M}(X)$ contains an H-sphere S_H for every value $H \in [0,\infty)$, Corollary 3.28 insures that for every $H > 0$, S_H is the unique H-sphere in X up to left translations. The same uniqueness property extends to $H = 0$ since every minimal sphere Σ in X has nullity three (Corollary 3.26) and thus it can be deformed to H-spheres with $H > 0$ by the Implicit Function theorem; this implies that $\Sigma = S_0 = \Gamma(0)$. The remaining properties in the statement of the theorem follow from Theorem 3.27. □

Theorem 3.30 is the SU(2)-version of a work in progress by Meeks, Mira, Pérez and Ros [**MIMPRb**] whose goal is to generalize it to every simply-connected, three-dimensional metric Lie group. We next state this expected result as a conjecture.

CONJECTURE 3.31 (Meeks-Mira-Pérez-Ros [**MIMPRb**]). *Let X be a simply-connected, three-dimensional metric Lie group diffeomorphic to $\mathbb{R}^3$. Then:*

(1) The moduli space of H-spheres in X (up to left translations) is parameterized by $H \in (H(X), \infty)$, where $H(X)$ is the critical mean curvature of X defined in (3.49).

(2) If S_H is an immersed H-sphere in X, then S_H is embedded, has index one and nullity three.

3.8. Calculating the Cheeger constant for a semidirect product. Let X be a complete Riemannian three-manifold of infinite volume. The *Cheeger constant* of X is defined by

$$\mathrm{Ch}(X) = \inf\left\{ \frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Volume}(\Omega)} \mid \overline{\Omega} \subset X \text{ compact, } \partial\Omega \text{ smooth} \right\}. \tag{3.50}$$

Classically, the Cheeger constant is defined for compact Riemannian manifolds, or at least for Riemannian manifolds of finite volume. In this case, the denominator in (3.50) should be replaced by the minimum between the volume of Ω and the volume of its complement. In the case of infinite ambient volume, definition (3.50) clearly generalizes the classical setting. We remark that Hoke [**III89**] proved that a simply-connected, non-compact, n-dimensional metric Lie group has Cheeger constant zero if and only if it is unimodular and amenable.

Consider a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some $A \in \mathcal{M}_2(\mathbb{R})$. An elementary computation using (2.6) and (2.7) gives that the volume element dV for the canonical metric in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is

$$dV = e^{-z\operatorname{trace}(A)}\, dx \wedge dy \wedge dz, \tag{3.51}$$

from where one has that the volume of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric is infinite. The next result calculates the Cheeger constant for this Riemannian manifold; it is a special case of a more general result of Peyerimhoff and Samiou [**PS04**], who proved the result for the case of an ambient simply-connected, n-dimensional solvable[14] Lie group.

THEOREM 3.32. *Let $A \in \mathcal{M}_2(\mathbb{R})$ be a matrix with* $\operatorname{trace}(A) \geq 0$. *Then,*

$$\operatorname{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R}) = \operatorname{trace}(A).$$

PROOF. We first prove that $\operatorname{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R}) \geq \operatorname{trace}(A)$. Arguing by contradiction, assume that $\operatorname{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R}) < \operatorname{trace}(A)$. Consider the isoperimetric profile of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric, defined as the function $I\colon (0,\infty) \to \mathbb{R}$ given by

$$I(t) = \min\{\operatorname{Area}(\partial\Omega) \mid \Omega \subset \mathbb{R}^2 \rtimes_A \mathbb{R} \text{ region with } \operatorname{Volume}(\Omega) = t\}.$$

Note that the minimum above is attained for every value of t due to the fact that $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is homogeneous.

The isoperimetric profile has been extensively studied in much more generality. We will emphasize here some basic properties of it, see e.g., Bavard and Pansu [**BP86**], Gallot [**Gal88**] and the survey paper by Ros [**Ros05**]:

(1) I is locally Lipschitz. In particular, its derivative I' exists almost everywhere in $(0,\infty)$.
(2) I has left and right derivatives $I'_-(t)$ and $I'_+(t)$ for any value of $t \in (0,\infty)$. Moreover if H is the mean curvature of an isoperimetric surface $\partial\Omega$ with $\operatorname{Volume}(\Omega) = t$ (with the notation above), then $I'_+(t) \leq 2H \leq I'_-(t)$.
(3) The limit as $t \to 0^+$ of $\frac{I(t)}{(36\pi t^2)^{1/3}}$ is 1.

Since we are assuming $\operatorname{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R}) < \operatorname{trace}(A)$, there exists a domain $\Omega_0 \subset \mathbb{R}^2 \rtimes_A \mathbb{R}$ with compact closure and smooth boundary, such that $\operatorname{Area}(\partial\Omega_0)$ is strictly less than $\operatorname{trace}(A)\cdot\operatorname{Volume}(\Omega_0)$. Consider in the $(\mathcal{V},\mathcal{A})$-plane (here $\mathcal{V}$ means volume and $\mathcal{A}$ area) the representation of the isoperimetric profile, i.e., the graph $G(I)$ of the function I, together with the straight half-line $r = \{\mathcal{A} = \operatorname{trace}(A)\mathcal{V}\}$. Then the pair $(\operatorname{Volume}(\Omega_0), \operatorname{Area}(\Omega_0))$ is a point in the first quadrant of the $(\mathcal{V},\mathcal{A})$-plane lying strictly below r. Furthermore by definition of isoperimetric profile, $G(I)$ intersects the vertical segment $\{\operatorname{Volume}(\Omega_0)\} \times (0, \operatorname{Area}(\partial\Omega_0)]$ at some point B. Property (3) above implies that $G(I)$ lies strictly above r for $t > 0$ sufficiently small. Since $G(I)$ passes through the point B, then there exists some intermediate value $V_1 \in (0, \operatorname{Volume}(\Omega_0))$ such that I has first derivative at $\mathcal{V} = V_1$ and the slope of $G(I)$ at $(V_1, I(V_1))$ is strictly smaller than the one of r. By property (2) above, if Ω_1 is an isoperimetric domain for volume V_1, then $\partial\Omega_1$ has constant mean curvature H where $I'(V_1) = 2H$. In particular, $\partial\Omega_1$ is a compact, embedded H-surface in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ whose mean curvature is strictly smaller than $\operatorname{trace}(A)/2$, the

[14]A Lie group X is called *solvable* if it admits a series of subgroups $\{e\} = X_0 \leq X_1 \leq \cdots \leq X_k = X$ such that X_{j-1} is normal in X_j and X_j/X_{j-1} is abelian, for all $j = 1,\ldots,k$. Taking $X_1 = \mathbb{R}^2 \rtimes_A \{0\}$ and $k = 2$ we deduce that every three-dimensional semidirect product is a solvable group.

mean curvature of the planes $\mathbb{R}^2 \rtimes_A \{z\}$, $z \in \mathbb{R}$ (see Section 2.3). Since these planes form a foliation of $\mathbb{R}^2 \rtimes_A \mathbb{R}$, we deduce that $\partial\Omega$ touches and lies on the mean convex side of some plane $\mathbb{R}^2 \rtimes_A \{z_0\}$, contradicting the mean comparison principle. Therefore, $\mathrm{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R}) \geq \mathrm{trace}(A)$.

Next we prove that $\mathrm{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R}) \leq \mathrm{trace}(A)$. Consider the disk $D(R)$ of radius R centered at the origin in $\mathbb{R} \rtimes_A \{0\}$. For $a > 1$, let

$$C(a,R) = \{(x,y,z) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid (x,y,0) \in D(R),\ z \in [0,a]\}$$

be the "vertical cylinder" in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ over $D(R)$, between heights 0 and a. From (2.11), we deduce that the divergence of the vector field ∂_z is $-\mathrm{trace}(A)$ on the whole space $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Applying the Divergence Theorem to ∂_z in $C(a,R)$ we have

$$-\mathrm{trace}(A) \cdot \mathrm{Volume}(C(a,R)) = \int_{\partial C(a,R)} \langle \partial_z, N\rangle, \tag{3.52}$$

where N is the outward pointing unit normal vector field to $\partial C(a,R)$. Note that if we call $D^{\mathrm{Top}}(a,R) = \partial C(a,R) \cap \{z = a\}$ and $S(a,R) = \partial C(a,R) \cap \{0 < z < a\}$, then $N|_{D^{\mathrm{Top}}(a,R)} = \partial_z$ and $N|_{D(R)} = -\partial_z$. Hence the right-hand-side of (3.52) can be written as

$$\begin{aligned} \int_{\partial C(a,R)} \langle \partial_z, N\rangle &= \mathrm{Area}(D^{\mathrm{Top}}(a,R)) - \mathrm{Area}(D(R)) \\ &= 2\mathrm{Area}(D^{\mathrm{Top}}(a,R)) + \mathrm{Area}(S(a,R)) - \mathrm{Area}(\partial C(a,R)). \end{aligned} \tag{3.53}$$

An elementary computation using (2.6) and (2.7) gives that the area element dA_z for the restriction of the canonical metric to the plane $\mathbb{R}^2 \rtimes_A \{z\}$ is $dA_z = e^{-z\mathrm{trace}(A)}\, dx \wedge dy$, which implies that

$$\mathrm{Area}(D^{\mathrm{Top}}(a,R)) = e^{-a\mathrm{trace}(A)}\mathrm{Area}(D(R)) = \pi R^2 e^{-a\mathrm{trace}(A)}.$$

This implies that for $R > 0$ fixed,

$$\lim_{a\to\infty} \frac{\mathrm{Area}(D^{\mathrm{Top}}(a,R))}{\mathrm{Area}(\partial C(a,R))} \leq \lim_{a\to\infty} \frac{\mathrm{Area}(D^{\mathrm{Top}}(a,R))}{\mathrm{Area}(D(R))} = \begin{cases} 1 & \text{if } \mathrm{trace}(A) = 0, \\ 0 & \text{if } \mathrm{trace}(A) > 0. \end{cases} \tag{3.54}$$

On the other hand, for each a there is a constant $M(a)$ such that for every $t \in [0,a]$,

$$\mathrm{Length}(S(a,R) \cap \{z = t\}) \leq M(a)\mathrm{Length}(\partial D(R)).$$

Then, by the coarea formula, we obtain:

$$\mathrm{Area}(S(a,R)) \leq a\, M(a) \cdot \mathrm{Length}(\partial D(R)) = 2\pi R\, a\, M(a). \tag{3.55}$$

Then for $a > 1$ fixed,

$$\lim_{R\to\infty} \frac{\mathrm{Area}(S(a,R))}{\mathrm{Area}(\partial C(a,R))} \leq \lim_{R\to\infty} \frac{\mathrm{Area}(S(a,R))}{\mathrm{Area}(D(R))} \leq \lim_{R\to\infty} \frac{2a\, M(a)}{R} = 0. \tag{3.56}$$

(In particular, inequalities in (3.56) become equalities). By equations (3.52) and (3.53) we have

$$\mathrm{trace}(A)\frac{\mathrm{Volume}(C(a,R))}{\mathrm{Area}(\partial C(a,R))} = 1 - 2\frac{\mathrm{Area}(D^{\mathrm{Top}}(a,R))}{\mathrm{Area}(\partial C(a,R))} - \frac{\mathrm{Area}(S(a,R))}{\mathrm{Area}(\partial C(a,R))}. \tag{3.57}$$

We now distinguish between the unimodular and non-unimodular cases. In the non-unimodular case we have $\operatorname{trace}(A) > 0$ and (3.54), (3.56), (3.57) imply

$$\sup_{R,a>1} \operatorname{trace}(A) \frac{\operatorname{Volume}(C(a,R))}{\operatorname{Area}(\partial C(a,R))} = 1,$$

which gives the desired inequality $\operatorname{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R}) \leq \operatorname{trace}(A)$.

It remains to prove that when $\operatorname{trace}(A) = 0$, then $\operatorname{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R}) = 0$. Using that $\operatorname{Area}(D^{\mathrm{Top}}(a,R))$ does not depend on a and (3.57) we have

$$\operatorname{Area}(\partial C(a,R)) = 2\operatorname{Area}(D(R)) + \operatorname{Area}(S(a,R)).$$

Dividing the last equation by $\operatorname{Area}(D(R))$ and applying the second equality in (3.56), we conclude that

$$\lim_{R\to\infty} \frac{\operatorname{Area}(\partial C(a,R))}{\operatorname{Area}(D(R))} = 2. \tag{3.58}$$

Applying the coarea formula to the z-coordinate (recall that $\nabla z = \partial_z$ is unitary in the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$), we have

$$\operatorname{Volume}(C(a,R)) = \int_0^a \operatorname{Area}(C(a,R) \cap \{z = t\})\, dt = a \operatorname{Area}(D(R)),$$

and so, by (3.58),

$$\lim_{R\to\infty} \frac{\operatorname{Volume}(C(a,R))}{\operatorname{Area}(\partial C(a,R))} = \frac{a}{2}.$$

Hence letting a go to ∞,

$$\inf_{a,R>1} \frac{\operatorname{Area}(\partial C(a,R))}{\operatorname{Volume}(C(a,R))} = 0,$$

which clearly implies $\operatorname{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R}) = 0$. □

Let X be a metric Lie group diffeomorphic to $\mathbb{R}^3$. We have already defined the critical mean curvature $H(X)$ for X, see (3.49), and the Cheeger constant $\operatorname{Ch}(X)$ studied in this section. We next consider another interesting geometric constant associated to X. Let

$$I(X) = \inf\{\text{mean curvatures of isoperimetric surfaces in } X\}.$$

We next relate these geometric constants.

Proposition 3.33. *Let X be a three-dimensional metric Lie group which is diffeomorphic to $\mathbb{R}^3$. Then,*

$$H(X) \leq I(X) \leq \frac{1}{2}\operatorname{Ch}(X).$$

Furthermore, if X is a metric semidirect product, then the three constants $H(X)$, $I(X)$ and $\frac{1}{2}\operatorname{Ch}(X)$ coincide.

Proof. The fact that $H(X) \leq I(X)$ follows directly from their definitions. The argument to prove that $2\,I(X) \leq \operatorname{Ch}(X)$ is very similar to the proof of the inequality $\operatorname{trace}(A) \leq \operatorname{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R})$ in Theorem 3.32: just exchange the number $\operatorname{trace}(A)$ by $2\,I(X)$ and follow the same arguments to produce a point $(V_1, I(V_1))$ in the $(\mathcal{V}, \mathcal{A})$-plane which lies in the graph $G(I)$ of the isoperimetric profile of X, such that the slope of $G(I)$ at $(V_1, I(V_1))$ is strictly smaller than $2\,I(X)$. This implies that the mean curvature of $\partial\Omega_1$ is strictly smaller than $I(X)$, which is impossible by definition of $I(X)$.

In the particular case that $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ for some $A \in \mathcal{M}_2(\mathbb{R})$, then Theorem 3.32 gives that $\frac{1}{2}\mathrm{Ch}(X)$ is the mean curvature H_0 of each of the leaves of the foliation $\{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ of X. Now the inequality $H_0 \leq H(X)$ follows directly from the mean curvature comparison principle. $\square$

Remark 3.34. Some of the results in this section have been recently improved upon by Meeks, Mira, Pérez and Ros. In [**MIMPRa**] it is shown that if X is diffeomorphic to $\mathbb{R}^3$, then $H(X) = I(X) = \frac{1}{2}\mathrm{Ch}(X)$, which improves the result stated above in Proposition 3.33. In the same paper it is also shown that given any sequence $\mathcal{D}_n$ of isoperimetric domains, each with volume greater than n, then:

(1) $\frac{\mathrm{Area}(\partial \mathcal{D}_n)}{\mathrm{Volume}(\mathcal{D}_n)} > \mathrm{Ch}(X)$ for each n, and $\lim_{n\to\infty} \frac{\mathrm{Area}(\partial \mathcal{D}_n)}{\mathrm{Volume}(\mathcal{D}_n)}$ exists and is equal to $\mathrm{Ch}(X)$.
(2) If H_n is the mean curvature of $\partial \mathcal{D}_n$, then $H_n > H(X)$, $\lim_{n\to\infty} H_n$ exists and is equal to $H(X)$.
(3) If R_n is the radius[15] of $\mathcal{D}_n$, then $\lim_{n\to\infty} R_n = \infty$.

4. Open problems and unsolved conjectures for H-surfaces in three-dimensional metric Lie groups.

We finish this excursion on surface theory in three-dimensional metric Lie groups by discussing a number of outstanding problems and conjectures. In the statement of most of these conjectures we have listed the principal researchers to whom the given conjecture might be attributed and/or those individuals who have made important progress in its solution.

In all of the conjectures below, X will denote a simply-connected, three-dimensional metric Lie group.

In reference to the following open problems and conjectures, the reader should note that Meeks, Mira, Pérez and Ros are in the final stages of completing two papers [**MIMPRa, MIMPRb**] that solve several of these conjectures. Their work should give complete solutions to Conjectures 4.1, 4.3 and 4.8. Their claimed results would also demonstrate that every H-sphere in X has index one (see the first statement of Conjecture 4.2) and that whenever X is diffeomorphic to $\mathbb{R}^3$, then X contains an $H(X)$-surface which is an entire Killing graph (this result implies that the last statement in Conjecture 4.7 and the statement (2b) in Conjecture 4.9 both hold). In the case that X is diffeomorphic to $\mathbb{R}^3$, it is shown in [**MIMPRa**] that when the volumes of isoperimetric domains in X go to infinity, then their radii[15] also go to infinity and the mean curvatures of their boundaries converge to $H(X)$; it then follows that item (1) of Conjecture 4.10 holds. We expect that by the time these notes are published, the papers [**MIMPRa, MIMPRb**] will be available and consequentially, some parts of this section on open problems should be updated by the reader to include these new results.

The first four of the conjectures below were mentioned earlier in the manuscript; see Conjectures 3.29 and 3.31. These first four conjectures are motivated by the results described in Corollary 3.28 and Theorems 3.15, 3.24 and 3.30. We start by restating Conjecture 4.1, of which Corollary 3.28 is a partial answer.

[15] The radius of a compact Riemannian manifold M with boundary is the maximum distance of points in M to its boundary.

CONJECTURE 4.1 (Hopf Uniqueness Conjecture, Meeks-Mira-Pérez-Ros). *For every $H \geq 0$, any two H-spheres immersed in X differ by a left translation of X.*

Recall that every immersed index-one H-sphere $\Sigma \looparrowright X$ has nullity three (see Corollary 3.26), and that Σ has index one provided that it is weakly stable (3.45). The next conjecture claims that this index property does not need the hypothesis on weak stability, and that weak stability holds whenever X is non-compact.

CONJECTURE 4.2 (Index-one Conjecture, Meeks-Mira-Pérez-Ros). *Every H-sphere in X has index one. Furthermore, when X is diffeomorphic to $\mathbb{R}^3$, then every H-sphere in X is weakly stable.*

Note that by Theorem 3.30, the first statement in Conjecture 4.2 holds in the case X is SU(2) with a left invariant metric. Also note that the hypothesis that X is diffeomorphic to $\mathbb{R}^3$ in the second statement of Conjecture 4.2 is necessary since the second statement fails to hold in certain Berger spheres, see Torralbo and Urbano [**TU09**]. By Corollary 3.28, the validity of the first statement in Conjecture 4.2 implies Conjecture 4.1 holds as well.

Hopf [**Hop89**] proved that the moduli space of non-congruent H-spheres in $\mathbb{R}^3$ is the interval $(0, \infty)$ (parametrized by their mean curvatures H) and all of these H-spheres are embedded and weakly stable, hence of index one; these results and arguments of Hopf readily extend to the case of $\mathbb{H}^3$ with the interval being $(1, \infty)$ and $\mathbb{S}^3$ with interval $[0, \infty)$, both $\mathbb{H}^3$ and $\mathbb{S}^3$ endowed with their standard metrics; see Chern [**Che70**]. By Theorem 3.30, if X is a metric Lie group diffeomorphic to $\mathbb{S}^3$, then the moduli space of non-congruent H-spheres in X is the interval $[0, \infty)$, again parametrized by their mean curvatures H. However, Torralbo [**Tor10**] proved that some H-spheres fail to be embedded in certain Berger spheres. These results motivate the next two conjectures. Recall that $H(X)$ is the critical mean curvature of X defined in (3.49).

CONJECTURE 4.3 (Hopf Moduli Space Conjecture, Meeks-Mira-Pérez-Ros). *When X is diffeomorphic to $\mathbb{R}^3$, then the moduli space of non-congruent H-spheres in X is the interval $(H(X), \infty)$, which is parametrized by their mean curvatures H. In particular, every H-sphere in X is Alexandrov embedded and $H(X)$ is the infimum of the mean curvatures of H-spheres in X.*

The results of Abresch and Rosenberg [**AR04, AR05**] and previous classification results for rotationally symmetric H-spheres demonstrate that Conjecture 4.3 holds when X is some $\mathbb{E}(\kappa, \tau)$-space. More recent work of Daniel and Mira [**DM08**] and of Meeks [**MI**] imply that Conjecture 4.3 (and the other first five conjectures in our listing here) holds for Sol_3 with its standard metric.

CONJECTURE 4.4 (Hopf Embeddedness Conjecture, Meeks-Mira-Pérez-Ros). *When X is diffeomorphic to $\mathbb{R}^3$, then H-spheres in X are embedded.*

We recall that this manuscript contains some new results towards the solution of the last conjecture, see Theorem 3.11 and Corollary 3.28.

The next conjecture is known to hold in the flat $\mathbb{R}^3$ as proved by Alexandrov [**Ale56**] and subsequently extended to $\mathbb{H}^3$ and to a hemisphere of $\mathbb{S}^3$.

CONJECTURE 4.5 (Alexandrov Uniqueness Conjecture). *If X is diffeomorphic to $\mathbb{R}^3$, then the only compact, Alexandrov embedded H-surfaces in X are topologically spheres.*

In the case $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ where A is a diagonal matrix, there exist two orthogonal foliations of X by planes of reflectional symmetry, as is the case of Sol_3 with its standard metric. By using the Alexandrov reflection method, the last conjecture is known to hold in this special case; see [**EGR09**] for details.

Although we do not state it as a conjecture, it is generally believed that for any value of $H > H(X)$ and $g \in \mathbb{N}$, there exist compact, genus-g, immersed, non-Alexandrov embedded H-surfaces in X, as is the case in classical $\mathbb{R}^3$ setting (Wente [**Wen86**] and Kapouleas [**Kap91**]).

Conjecture 4.6 (Stability Conjecture for SU(2), Meeks-Pérez-Ros). *If X is diffeomorphic to $\mathbb{S}^3$, then X contains no stable complete H-surfaces, for any value of $H \geq 0$.*

Conjecture 4.6 is known to hold when the metric Lie group X is in one of the following two cases:

- X is a Berger sphere with non-negative scalar curvature (see item (5) of Corollary 9.6 in Meeks, Pérez and Ros [**MIPR08**]).
- X is SU(2) endowed with a left invariant metric of positive scalar curvature (by item (1) of Theorem 2.13 in [**MIPR08**], a complete stable H-surface Σ in X must be compact, in fact must be topologically a two-sphere or a projective plane; hence one could find a right invariant Killing field on X which is not tangent to Σ at some point of Σ, thereby inducing a Jacobi function which changes sign on Σ, a contradiction).

It is also proved in [**MIPR08**] that if Y is a three-sphere with a Riemannian metric (not necessarily a left invariant metric) such that it admits no stable complete minimal surfaces, then for each integer $g \in \mathbb{N} \cup \{0\}$, the space of compact embedded minimal surfaces of genus g in Y is compact, a result which is known to hold for Riemannian metrics on $\mathbb{S}^3$ with positive Ricci curvature (Choi and Schoen [**CS85**]).

Conjecture 4.7 (Stability Conjecture, Meeks-Mira-Pérez-Ros). *Suppose X is diffeomorphic to $\mathbb{R}^3$. Then*

$$H(X) = \sup\{\textit{mean curvatures of complete stable } H\textit{-surfaces in } X\}.$$

Furthermore, there always exists a properly embedded, complete, stable $H(X)$-surface in X.

By the work in [**MIMPRb**], the validity of the first statement in Conjecture 4.7 would imply Conjecture 4.1 and the first statement in Conjecture 4.2 (essentially, this is because if a sequence of index-one spheres $S_{H_n} \looparrowright X$ with $H_n \searrow H_\infty \geq 0$ have areas diverging to infinity, the one can produce an appropriate limit of left translations of S_{H_n} which is a stable H_∞-surface in X, which in turn implies that $H_\infty = H(X)$ and this is enough to conclude both Conjecture 4.1 and the first statement in Conjecture 4.2, see the sketch of proof of Theorem 3.30). Note that the second statement of Conjecture 4.7 holds whenever $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, since $\mathbb{R}^2 \rtimes_A \{0\}$ is a properly embedded, stable $H(X)$-surface.

Conjecture 4.8 (Cheeger Constant Conjecture, Meeks-Mira-Pérez-Ros). *If X is diffeomorphic to $\mathbb{R}^3$, then $\mathrm{Ch}(X) = 2\,H(X)$.*

By Proposition 3.33, if X is of the form $\mathbb{R}^2 \rtimes_A \mathbb{R}$, then $\mathrm{Ch}(X) = \mathrm{trace}(A) = 2\,H(X)$, and so Conjecture 4.8 is known to hold except when X is isomorphic to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. It is also known to hold in the case of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with an $\mathbb{E}(\kappa,\tau)$-metric, since by Theorem 2.14, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with such a metric is isometric as a Riemannian manifold to the non-unimodular group $\mathbb{H}^2 \times \mathbb{R}$ with some left invariant metric, and hence it is isometric to some $\mathbb{R}^2 \rtimes_A \mathbb{R}$ (Theorem 2.14). Therefore, it remains to prove Conjecture 4.8 in the case of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ equipped with a left invariant metric whose isometry group is three-dimensional.

CONJECTURE 4.9 (CMC Product Foliation Conjecture, Meeks-Mira-Pérez-Ros).

(1) If X is diffeomorphic to $\mathbb{R}^3$, then given $p \in X$ there exists a smooth product foliation of $X - \{p\}$ by spheres of constant mean curvature.

(2) Let $\mathcal{F}$ be a CMC foliation of X, i.e., a foliation all whose leaves have constant mean curvature (possibly varying from leaf to leaf). Then:
  (a) $\mathcal{F}$ is a product foliation by topological planes.
  (b) The mean curvature of the leaves of $\mathcal{F}$ is at most $H(X)$.

Since spheres of radius R in $\mathbb{R}^3$ or in $\mathbb{H}^3$ have constant mean curvature, item (1) of the above conjecture holds in these spaces. By results of Meeks [**MI88**] and of Meeks, Pérez and Ros [**MIPR08**], items (2a-2b) of the above conjecture are also known to hold when X is $\mathbb{R}^3$ or $\mathbb{H}^3$.

More generally, by work of Meeks, Pérez and Ros [**MIPR08**] on CMC foliations $\mathcal{F}$ of complete, homogeneously regular Riemannian three-manifolds with a given bound on the absolute sectional curvature (not necessarily a metric Lie group), the supremum Δ of the mean curvature of the leaves of $\mathcal{F}$ is uniformly bounded independently of the choice of the CMC foliation $\mathcal{F}$. In the case of a simply-connected, three-dimensional metric Lie group X, the related supremum $\Delta(\mathrm{X})$ can be proven to be achieved by a complete stable H-surface with $H = \Delta(X)$. Hence, items (2a-2b) of Conjecture 4.9 would follow from the validity of Conjecture 4.7.

Regarding item (2) of Conjecture 4.9, note that there are no CMC foliations of X when X is not diffeomorphic to $\mathbb{R}^3$; to see this, suppose $\mathcal{F}$ is a CMC foliation of a metric Lie group diffeomorphic to $\mathbb{S}^3$. Novikov [**Nov65**] proved that any foliation of $\mathbb{S}^3$ by surfaces has a Reeb component C, which is topologically a solid doughnut with a boundary torus leaf ∂C and the other leaves of $\mathcal{F}$ in C all have ∂C as their limits sets. Hence, all of leaves of $\mathcal{F}$ in C have the same mean curvature as ∂C. By the Stable Limit Leaf Theorem for H-laminations, ∂C is stable. But an embedded compact, two-sided H-surface in SU(2) is never stable, since some right invariant Killing field induces a Jacobi function which changes sign on the surface; see Theorem 4.18 below.

Suppose for the moment that item (1) in Conjecture 4.9 holds and we will point out some important consequences. Suppose $\mathcal{F}_p$ is a smooth CMC product foliation of $X - \{p\}$ by spheres, p being a point in X. Parametrize the space of leaves of $\mathcal{F}_p$ by their mean curvature; this can be done by the maximum principle for H-surfaces, which shows that the spheres in $\mathcal{F}_p$ decrease their positive mean curvatures at the same time that the volume of the enclosed balls by these spheres increases. Thus, the mean curvature parameter for the leaves of $\mathcal{F}_p$ decreases from ∞ (at p) to some value $H_0 \geq 0$. The following argument shows that $H_0 = H(X)$ *and every compact H-surface in X satisfies $H > H(X)$:* Otherwise there exists a compact, possibly non-embedded surface S in X such that the maximum value of the absolute mean curvature function of S is less than or equal to H_0. Since S is compact, then S is contained in the ball enclosed by some leaf Σ of $\mathcal{F}_p$. By left translating S until touching Σ a first time, we obtain a contradiction to the usual comparison principle for the mean curvature, which finishes the argument. With this property in mind, we now list some consequences of item (1) in Conjecture 4.9.

(1) All leaves of $\mathcal{F}_p$ are weakly stable. To see this, first note that all of the spheres in $\mathcal{F}_p$ have index one (since the leaves of $\mathcal{F}_p$ bounding balls of small volume have this property and as the volume increases, the multiplicity of zero as an eigenvalue of the Jacobi operator of the corresponding sphere cannot exceed three by Cheng's theorem [**Che76**]). Also note that every function ϕ in the nullity of a leaf Σ of $\mathcal{F}_p$ is induced by a right invariant Killing field on X, and hence, $\int_\Sigma \phi = 0$ by the Divergence Theorem applied to the ball enclosed by Σ. In this situation, Koiso [**Koi02**] proved that the weak stability of Σ is characterized by the non-negativity of the integral $\int_\Sigma u$, where u is any smooth function on Σ such that $Lu = 1$ on Σ (see also Souam [**Sou10**]). Since the leaves of $\mathcal{F}_p$ can be parameterized by their mean curvatures, the corresponding normal part u of the variational field satisfies $u > 0$ on Σ, $Lu = 1$ and $\int_\Sigma u > 0$. Therefore, Σ is weakly stable.
(2) The leaves of $\mathcal{F}_p$ are the unique H-spheres in X (up to left translations), by Corollary 3.28.

If additionally the Alexandrov Uniqueness Conjecture 4.5 holds, then the constant mean curvature spheres in $\mathcal{F}_p$ are the unique (up to left translations) compact H-surfaces in X which bound regions. Since the volume of these regions is determined by the boundary spheres, one would have the validity of the next conjecture.

Conjecture 4.10 (Isoperimetric Domains Conjecture, Meeks-Mira-Pérez-Ros). *Suppose X is diffeomorphic to $\mathbb{R}^3$. Then:*

(1) $H(X) = \inf\{$mean curvatures of isoperimetric surfaces in $X\}$.
(2) Isoperimetric surfaces in X are spheres.
(3) For each fixed volume V_0, solutions to the isoperimetric problem in X for volume V_0 are unique up to left translations in X.

Recall by Proposition 3.33 that in the case the metric Lie group X is of the form $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some matrix $A \in \mathcal{M}_2(\mathbb{R})$, then item (1) in the previous conjecture is known to hold.

The next conjecture is motivated by the isoperimetric inequality of White [**Whi09**] and applications of it by Meeks, Mira, Pérez and Ros [**MIMPRa**]. In particular, these authors prove that the next conjecture holds when the surface Σ is minimal and has connected boundary.

Conjecture 4.11 (Isoperimetric Inequality Conjecture, Meeks-Mira-Pérez-Ros). *Suppose that X is diffeomorphic to $\mathbb{R}^3$. Given $L_0 > 0$, there exists $C(L_0) > 0$ such that for any compact immersed surface Σ in X with absolute mean curvature function bounded from above by $\mathrm{Ch}(X)$ and whose boundary has length at most L_0, then*

$$\mathrm{Area}(\Sigma) \leq C(L_0)\,\mathrm{Length}(\partial\Sigma).$$

The next conjecture exemplifies another aspect of the special role that the critical mean curvature $H(X)$ of X might play in the geometry of H-surfaces in X.

Conjecture 4.12 (Stability Conjecture, Meeks-Mira-Pérez-Ros).
A complete stable H-surface in X with $H = H(X)$ is a graph with respect to some Killing field V projection, i.e., the projection of X to the quotient space of integral curves of V. In particular, if $H(X) = 0$, then any complete stable minimal surface

Σ in X is a leaf of a minimal foliation of X and so, Σ is actually homologically area-minimizing in X.

The previous conjecture is closely related to the next conjecture, which in turn is closely tied to recent work of Daniel, Meeks and Rosenberg [**DIRa, DIRb**] on halfspace-type theorems in simply-connected, three-dimensional metric semidirect products.

CONJECTURE 4.13 (Strong-Halfspace Conjecture in Nil_3, Daniel-Meeks-Rosenberg). *A complete stable minimal surface in Nil_3 is a graph with respect to the Riemannian submersion $\Pi\colon \mathrm{Nil}_3 \to \mathbb{R}^2$ or it is a vertical plane $\Pi^{-1}(l)$, where l is a line in $\mathbb{R}^2$. In particular, by the results in* [**DIRb**], *any two properly immersed disjoint minimal surfaces in Nil_3 are parallel vertical planes or they are entire graphs F_1, F_2 over $\mathbb{R}^2$, where F_2 is a vertical translation of F_1.*

CONJECTURE 4.14 (Positive Injectivity Radius, Meeks-Pérez-Tinaglia). *A complete embedded H-surface of finite topology in X has positive injectivity radius. Furthermore, the same conclusion holds when $H \leq H(X)$ under the weaker assumption of finite genus.*

Conjecture 4.14 is motivated by the partial result of Meeks and Pérez [**MIPa**] that the injectivity radius of a complete, embedded *minimal* surface of finite topology in a homogeneous three-manifold is positive. A related result of Meeks and Peréz [**MIPa**] when $H = 0$ and of Meeks and Tinaglia when $H > 0$, is that if Y is a complete locally homogeneous three-manifold and Σ is a complete embedded H-surface in Y with finite topology, then the injectivity radius function of Σ is bounded on compact domains in Y. Meeks and Tinaglia (unpublished) have also shown that the first statement of Conjecture 4.14 holds for complete embedded H-surfaces of finite topology in metric Lie groups X with four or six-dimensional isometry group.

CONJECTURE 4.15 (Bounded Curvature Conjecture, Meeks-Pérez-Tinaglia). *A complete embedded H-surface of finite topology in X with $H \geq H(X)$ or with $H > 0$ has bounded second fundamental form. Furthermore, the same conclusion holds when $H = H(X)$ under the weaker assumption of finite genus.*

The previous two conjectures are related as follows. Curvature estimates of Meeks and Tinaglia [**MITc**] for embedded H-disks imply that every complete embedded H-surface with $H > 0$ in a homogeneously regular three-manifold has bounded second fundamental form if and only if it has positive injectivity radius. Hence, if Conjecture 4.14 holds, then a complete embedded H-surface of finite topology in X with $H > 0$ has bounded curvature.

CONJECTURE 4.16 (Calabi-Yau Properness Problem, Meeks-Pérez-Tinaglia). *A complete, connected, embedded H-surface of positive injectivity radius in X with $H \geq H(X)$ is always proper.*

In the classical setting of $X = \mathbb{R}^3$, where $H(X) = 0$, Conjecture 4.16 was proved by Meeks and Rosenberg [**MIR06**] for the case $H = 0$. This result was based on work of Colding and Minicozzi [**CMI08**] who demonstrated that complete embedded finite topology minimal surfaces in $\mathbb{R}^3$ are proper, thereby proving what is usually referred to as the *classical embedded Calabi-Yau problem* for minimal surfaces of finite topology. Recently, Meeks and Tinaglia [**MITa**] proved

Conjecture 4.16 in the case $X = \mathbb{R}^3$ and $H > 0$, which completes the proof of the conjecture in the classical setting.

As we have already mentioned, Meeks and Pérez [**MIPa**] have shown that every complete embedded minimal surface M of finite topology in X has positive injectivity radius; hence M would be proper whenever $H(X) = 0$ and Conjecture 4.16 holds for X. Meeks and Tinaglia [**MITb**] have shown that any complete embedded H-surface M in a complete three-manifold Y with constant sectional curvature -1 is proper provided that $H \geq 1$ and M has injectivity radius function bounded away from zero on compact domains of in Y; they also proved that any complete, embedded, finite topology H-surface in such a Y has bounded second fundamental form. In particular, for $X = \mathbb{H}^3$ with its usual metric, an annular end of any complete, embedded, finite topology H-surface in X with $H \geq H(X) = 1$ is asymptotic to an annulus of revolution by the classical results of Korevaar, Kusner, Meeks and Solomon [**KKIS92**] when $H > 1$ and of Collin, Hauswirth and Rosenberg [**CHR01**] when $H = 1$.

A key step in proving Conjecture 4.16 might be the validity of Conjecture 4.12, since by the work of Meeks and Rosenberg [**MIR06**] and of Meeks and Tinaglia [**MITc**], if Σ is a complete, connected, embedded non-proper H-surface of positive injectivity radius in X with $H \geq H(X)$, then the closure of Σ has the structure of a weak H-lamination with at least one limit leaf and the two-sided cover of every limit leaf of such a weak H-lamination is stable [**MIPR08, MIPR10**].

The next conjecture is motivated by the classical results of Meeks and Yau [**MIY92**] and of Frohman and Meeks [**FI08**] on the topological uniqueness of minimal surfaces in $\mathbb{R}^3$ and partial unpublished results by Meeks. By modifications of the arguments in these papers, this conjecture might follow from the validity of Conjecture 4.12.

CONJECTURE 4.17 (Topological Uniqueness Conjecture, Meeks). *If M_1, M_2 are two diffeomorphic, connected, complete embedded H-surfaces of finite topology in X with $H = H(X)$, then there exists a diffeomorphism $f\colon X \to X$ such that $f(M_1) = M_2$.*

We recall that Lawson [**Law70**] proved a beautiful unknottedness result for minimal surfaces in $\mathbb{S}^3$ equipped with a metric of positive Ricci curvature. He demonstrated that whenever M_1, M_2 are compact, embedded, diffeomorphic minimal surfaces in such a Riemannian three-sphere, then M_1 and M_2 are ambiently isotopic. His result was generalized by Meeks, Simon and Yau [**MISY82**] to the case of metrics of non-negative scalar curvature on $\mathbb{S}^3$. The work in these papers and the validity of Conjecture 4.6 would prove that this unknottedness result would hold for any X diffeomorphic to $\mathbb{S}^3$ since the results in [**MISY82**] imply that the failure of Conjecture 4.17 to hold produces a compact, embedded stable two-sided minimal surface in X, which is ruled out by the next theorem. The reader should note that the two-sided hypothesis in the next theorem is necessary because there exist stable minimal projective planes in $\mathrm{SU}(2)/\mathbb{Z}_2$ for any left invariant metric and many flat three-tori admit stable non-orientable minimal surfaces of genus 3.

THEOREM 4.18. *Suppose that Y is a three-dimensional (non-necessarily simply-connected) metric Lie group. The compact, orientable stable H-surfaces in Y are precisely the left cosets of compact two-dimensional subgroups of Y and furthermore, all such subgroups are tori which are normal subgroups of Y. In particular, the* *existence of such compact stable H-surfaces in Y implies that the fundamental group of Y contains a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

Before proving Theorem 4.18, we give the following corollary of the previous discussion.

COROLLARY 4.19 (Unknottedness Theorem for Minimal Surfaces in SU(2)). *Let X be isomorphic to* SU(2) *and let M_1, M_2 be two compact, diffeomorphic embedded minimal surfaces in X. Then M_1 is ambiently isotopic to M_2.*

THE PROOF OF THEOREM 4.18. Let M be a compact, two-dimensional subgroup of Y. Then the left cosets of M give rise to an H-foliation of Y (here H is the constant mean curvature of M) and so M admits a positive Jacobi function, which implies that M is stable.

Now suppose that M is a compact, immersed, two-sided stable H surface in Y. After a left translation suppose that $e \in M$. Let V_1, V_2 be a pair of linearly independent, right invariant vector fields in Y which are tangent to M at e. Since Y and M are both orientable and M is stable, these two Killing fields must be everywhere tangent to M. It follows that M is a two-dimensional subgroup of X and so M is a torus.

It remains to prove that M is a normal subgroup of Y. To do this, it suffices to check that the right cosets of M near M are also left cosets of M. Note that given $a \in Y$, the right coset Ma lies at constant distance from M. Since M is compact, there exists an element $a \in Y - \{e\}$ sufficiently close to e so that Ma is a small normal graph over M that lies in a product neighborhood of M foliated by left cosets. If Ma is not one of these left cosets in this foliated neighborhood, then we can choose this neighborhood to be the smallest one with distinct boundary left cosets, so assume that the second possibility holds. In this case Ma is tangent to the two boundary surfaces b_1M, b_2M for some $b_1, b_2 \in Y$. Since right cosets are left cosets of a conjugate subgroup, it follows that Ma has constant mean curvature. The mean curvature comparison principle applied to the points of intersection of Ma with b_1M and of b_2M shows that the mean curvature of Ma is equal to the constant mean curvature of b_1M which is equal to the value of the constant mean curvature of b_2M. Therefore, by the maximum principle for H-surfaces, $b_1M = Ma = b_2M$, which is a contradiction. The theorem is now proved. □

The next conjecture is motivated by the classical case of $X = \mathbb{R}^3$, where it was proved by Meeks [**MI88**], and in the case of $X = \mathbb{H}^3$ with its standard constant -1 curvature metric, where it was proved by Korevaar, Kusner, Meeks and Solomon [**KKIS92**]. We also mention the case of $\mathbb{H}^2 \times \mathbb{R}$ which was tackled by Espinar, Gálvez and Rosenberg [**EGR09**].

CONJECTURE 4.20 (One-end / Two-ends Conjecture, Meeks-Tinaglia).
Suppose that M is a connected, non-compact, properly embedded H-surface of finite topology in X with $H > H(X)$. Then:
(1) M has more than one end.
(2) If M has two ends, then M is an annulus.

The previous conjecture also motivates the next one.

CONJECTURE 4.21 (Topological Existence Conjecture, Meeks).
Suppose X is diffeomorphic to $\mathbb{R}^3$. Then for every $H > H(X)$, X admits connected

properly embedded H-surfaces of every possible orientable topology, except for connected finite genus surfaces with one end or connected finite positive genus surfaces with 2 ends which it never admits.

Conjecture 4.21 is probably known in the classical settings of $X = \mathbb{R}^3$ and $\mathbb{H}^3$ but the authors do not have a reference of this result for either of these two ambient spaces. For the non-existence results alluded to in this conjecture in these classical settings see [**KKIS92, KKS89, MI88, MITb**]. The existence part of the conjecture should follow from gluing constructions applied to infinite collections of non-transversely intersecting embedded H-spheres appropriately placed in X, as in the constructions of Kapouleas [**Kap90**] in the case of $X = \mathbb{R}^3$.

The intent of the next conjecture is to generalize some of the classical results for complete embedded H-surfaces with $H \geq H(X)$ of finite topology in $X = \mathbb{R}^3$ or $X = \mathbb{H}^3$. First of all, we recall that a complete embedded H-surface Σ in X with finite topology is properly embedded in the following particular cases:

(1) When $X = \mathbb{R}^3$ and $H = H(X) = 0$ (Colding and Minicozzi).
(2) When $X = \mathbb{R}^3$ and $H > 0$ (Meeks and Tinaglia).
(3) When $X = \mathbb{H}^3$ and $H \geq H(X) = 1$ (Meeks and Tinaglia).

Then, in the above cases for X and for $H > H(X)$, the classical results of Korevaar, Kusner, Meeks and Solomon [**KKIS92, KKS89, MI88**] for properly embedded H-surfaces of finite topology give a solution to the next conjecture.

Conjecture 4.22 (Annulus Moduli Space / Asymptotic Conjecture, Große Brauckmann-Kusner-Meeks). *Suppose X is diffeomorphic to $\mathbb{R}^3$ and $H > H(X)$.*

(1) Let $\mathcal{A}(X)$ be the space of non-congruent, complete embedded H-annuli in X. Then, $\mathcal{A}(X)$ is path-connected.
(2) If the dimension of the isometry group of X is greater than three, then every annulus in $\mathcal{A}(X)$ is periodic and stays at bounded distance from a geodesic of X.
(3) Suppose that M is a complete embedded H-surface with finite topology in X. Then, every end of M is asymptotic to the end of an annulus in $\mathcal{A}(X)$.

We end our discussion of open problems in X with the following generalization of the classical properly embedded Calabi-Yau problem in $\mathbb{R}^3$, which can be found in [**FnI**] and [**MIPb, MIP11**]. Variations of this conjecture can be attributed to many people but in the formulation below, it is primarily due to Martín, Meeks, Nadirashvili, Pérez and Ros and their related work.

Conjecture 4.23 (Embedded Calabi-Yau Problem). *Suppose X is diffeomorphic to $\mathbb{R}^3$ and Σ is a connected, non-compact surface. A necessary and sufficient condition for Σ to be diffeomorphic to some complete, embedded bounded minimal surface in X is that every end of Σ has infinite genus.*

In the case of $X = \mathbb{R}^3$ with its usual metric, the non-existence implication in the last conjecture was proved by Colding and Minicozzi [**CMI08**] for complete embedded minimal surfaces with finite topology; also see the related more general results of Meeks and Rosenberg [**MIR06**] and of Meeks, Peréz and Ros [**MIPRa**]. The non-existence implication in the last conjecture should follow from the next general conjecture.

CONJECTURE 4.24 (Finite Genus Lamination Closure Conjecture, Meeks-Pérez). *Suppose that M is a complete, embedded, non-compact minimal surface with compact boundary in a complete Riemannian three-manifold Y. Then either $\overline{M} - \partial M$ is a minimal lamination of $Y - \partial M$, or the limit set*[16] *$L(M) - \partial M$ of M is a minimal lamination of $X - \partial M$ with every leaf in $L(M) - \partial M$ being stable after passing to any orientable cover.*

The reason that the above conjecture should give the non-existence implication in Conjecture 4.23 is that it should be the case that every metric Lie group X diffeomorphic to $\mathbb{R}^3$ admits a product H-foliation $\mathcal{F}$ for some $H \geq 0$. The existence of such a foliation $\mathcal{F}$ of X and the maximum principle would imply that X cannot admit a minimal lamination contained in a bounded set of X. Trivially, any X which can be expressed as a metric semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ admits such a product H-foliation, namely, the collection of planes $\{\mathbb{R}^2 \rtimes_A \{t\} \mid t \in \mathbb{R}\}$ where $H = \frac{1}{2}\text{trace}(A)$ is the mean curvature of these planes.

References

[AdCT07] H. Alencar, M. do Carmo, and R. Tribuzy, *A theorem of Hopf and the Cauchy-Riemann inequality*, Comm. Anal. Geom. **15** (2007), no. 2, 283–298, MR2344324, Zbl 1134.53031.

[Ale56] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large I*, Vestnik Leningrad Univ. Math. **11** (1956), no. 19, 5–17, MR0150706.

[AR04] U. Abresch and H. Rosenberg, *A Hopf differential for constant mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$*, Acta Math. **193** (2004), no. 2, 141–174, MR2134864 (2006h:53003), Zbl 1078.53053.

[AR05] ———, *Generalized Hopf differentials*, Mat. Contemp. **28** (2005), 1–28, MR2195187, Zbl 1118.53036.

[BP86] C. Bavard and P. Pansu, *Sur le volume minimal de $\mathbb{R}^2$*, Ann. scient. Éc. Norm. Sup. **19** (1986), no. 4, 479–490, MR0875084 (88b:53048), Zbl 0611.53038.

[Che70] Shiing Shen Chern, *On the minimal immersions of the two-sphere in a space of constant curvature*, Problems in analysis (Lectures at the Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, N.J., 1969), Princeton Univ. Press, Princeton, N.J., 1970, pp. 27–40. MR 0362151 (50 #14593)

[Che76] S.Y. Cheng, *Eigenfunctions and nodal sets*, Comment. Math. Helv. **51** (1976), no. 1, 43–55.

[CHR01] P. Collin, L. Hauswirth, and H. Rosenberg, *The geometry of finite topology Bryant surfaces*, Ann. of Math. **153** (2001), no. 3, 623–659, MR1836284 (2002j:53012), Zbl 1066.53019.

[CMI08] T. H. Colding and W. P. Minicozzi II, *The Calabi-Yau conjectures for embedded surfaces*, Ann. of Math. **167** (2008), 211–243, MR2373154, Zbl 1142.53012.

[CS85] H. I. Choi and R. Schoen, *The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature*, Invent. Math. **81** (1985), 387–394, MR0807063, Zbl 0577.53044.

[Dan07] B. Daniel, *Isometric immersions into 3-dimensional homogeneous manifolds*, Comment. Math. Helv. **82** (2007), no. 1, 87–131, MR2296059, Zbl 1123.53029.

[DIRa] B. Daniel, W. H. Meeks III, and H. Rosenberg, *Half-space theorems and the embedded Calabi-Yau problem in Lie groups*, Preprint available at arXiv:1005.3963.

[DIRb] ———, *Half-space theorems for minimal surfaces in Nil_3 and Sol_3*, J. Differential Geom. **88**, no. 1, 41–59, Preprint available at arXiv:1005.3963.

[16]Let M be a complete, embedded surface in a three-manifold Y. A point $p \in Y$ is a *limit point* of M if there exists a sequence $\{p_n\}_n \subset M$ which diverges to infinity in M with respect to the intrinsic Riemannian topology on M but converges in Y to p as $n \to \infty$. In the statement of Conjecture 4.24, $L(M)$ denotes the set of all limit points of M in Y, which is a closed subset of Y satisfying $\overline{M} - M \subset L(M)$.

[DM] B. Daniel and P. Mira, *Constant mean curvature surfaces in homogeneous manifolds*, Preprint.

[DM08] ———, *Existence and uniqueness of constant mean curvature spheres in* Sol_3, Preprint available at arXiv:0812.3059, 2008.

[EGR09] J. M. Espinar, J.A. Galvez, and H. Rosenberg, *Complete surfaces with positive extrinsic curvature in product spaces*, Comment. Math. Helv. **84** (2009), 351–386.

[FI08] C. Frohman and W. H. Meeks III, *The topological classification of minimal surfaces in* $\mathbb{R}^3$, Ann. of Math. **167** (2008), no. 3, 681–700, MR2415385, Zbl 1168.49038.

[FnI] L. Ferrer, F. Martín, and W. H. Meeks III, *The existence of proper minimal surfaces of arbitrary topological type*, Preprint available at arXiv.org/abs/0903.4194.

[Gal88] S. Gallot, *Inégalités isopérimetriques et analytiques sur les variétés Riemannienes*, Asterisque **163–164** (1988), 31–91.

[Hop89] H. Hopf, *Differential geometry in the large*, Lecture Notes in Math., vol. 1000, Springer-Verlag, 1989, MR1013786, Zbl 0669.53001.

[III89] F. F. Hoke III, *Lie groups that are closed at infinity*, Trans. Amer. Math. Soc. **313** (1989), no. 2, 721–735, MRj:58108.

[Jac62] N. Jacobson, *Lie Algebras*, J. Wiley and Sons, N.Y., 1962, MR0143793 (26 #1345).

[Kap90] N. Kapouleas, *Complete constant mean curvature surfaces in Euclidean three space*, Ann. of Math. **131** (1990), 239–330, MR1043269, Zbl 0699.53007.

[Kap91] ———, *Compact constant mean curvature surfaces in Euclidean three space*, J. Differential Geometry **33** (1991), 683–715, MR1100207, Zbl 0727.53063.

[KKIS92] N. Korevaar, R. Kusner, W. H. Meeks III, and B. Solomon, *Constant mean curvature surfaces in hyperbolic space*, American J. of Math. **114** (1992), 1–43, MR1147718, Zbl 0757.53032.

[KKS89] N. Korevaar, R. Kusner, and B. Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom. **30** (1989), 465–503, MR1010168, Zbl 0726.53007.

[Koi02] M. Koiso, *Deformation and stablity of surfaces of constant mean curvature*, Tôhoku Mat. J. **54** (2002), no. 1, 145–159, MR1878932 (2003j:58021), Zbl 1010.58007.

[Kow90] O. Kowalski, *Counterexample to the second "Singer's Theorem"*, A.. Global Anal. Geom. **8** (1990), 211–214, MR1088512 (92b:53057), Zbl 0736.53047.

[Law70] H. B. Lawson, *The unknottedness of minimal embeddings*, Invent. Math. **11** (1970), 183–187, MR0287447, Zbl 0205.52002.

[LR89] F. J. López and A. Ros, *Complete minimal surfaces of index one and stable constant mean curvature surfaces*, Comment. Math. Helv. **64** (1989), 34–53, MR0982560, Zbl 0679.53047.

[LT93] F. Lastaria and F. Tricerri, *Curvature-orbits and locally homogeneous Riemannian manifolds*, Ann. Mat. Pura Appl. (4) **165** (1993), 121–131, MR1271415 (95b:53063), Zbl 0804.53072.

[MI] W. H. Meeks III, *Constant mean curvature spheres in* Sol_3, Preprint.

[MI88] ———, *The topology and geometry of embedded surfaces of constant mean curvature*, J. of Differential Geom. **27** (1988), 539–552, MR0940118 (89h:53025), Zbl 0617.53007.

[Mil76] J. W. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Mathematics **21** (1976), 293–329, MR0425012, Zbl 0341.53030.

[MIMPRa] W. H. Meeks III, P. Mira, J. Pérez, and A. Ros, *The Cheeger constant of a homogeneous three-manifold*, Work in progress.

[MIMPRb] ———, *Constant mean curvature spheres in homogeneous* 3*-manifolds*, Work in progress.

[MIPa] W. H. Meeks III and J. Pérez, *Classical minimal surfaces in homogeneous three-manifolds*, Work in progress.

[MIPb] ———, *A survey on classical minimal surface theory*, Preprint available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[MIP11] ———, *The classical theory of minimal surfaces*, Bulletin of the AMS **48** (2011), 325–407, Preprint available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[MIPRa] W. H. Meeks III, J. Pérez, and A. Ros, *The embedded Calabi-Yau conjectures for finite genus*, Preprint available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[MIPRb] ———, *Local removable singularity theorems for minimal and* H*-laminations*, Preprint available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[MIPR08] ———, *Stable constant mean curvature surfaces*, Handbook of Geometrical Analysis, vol. 1, International Press, edited by Lizhen Ji, Peter Li, Richard Schoen and Leon Simon, ISBN: 978-1-57146-130-8, 2008, MR2483369, Zbl 1154.53009, pp. 301–380.

[MIPR10] ———, *Limit leaves of an H lamination are stable*, J. Differential Geometry (2010), no. 1, 179–189, Preprint available at arXiv:0801.4345 and at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[MIR06] W. H. Meeks III and H. Rosenberg, *The minimal lamination closure theorem*, Duke Math. Journal **133** (2006), no. 3, 467–497, MR2228460, Zbl 1098.53007.

[MISY82] W. H. Meeks III, L. Simon, and S. T. Yau, *The existence of embedded minimal surfaces, exotic spheres and positive Ricci curvature*, Ann. of Math. **116** (1982), 221–259, MR0678484, Zbl 0521.53007.

[MITa] W. H. Meeks III and G. Tinaglia, *Curvature estimates for constant mean curvature surfaces*, Work in progress.

[MITb] ———, *Embedded Calabi-Yau problem in hyperbolic 3-manifolds*, Work in progress.

[MITc] ———, *H-surfaces in locally homogeneous three-manifolds*, Work in progress.

[MIT10] ———, *The CMC dynamics theorem in* $\mathbb{R}^3$, J. of Differential Geom. **85** (2010), 141–173, Preprint available at arXive 0805.1427.

[MIY92] W. H. Meeks III and S. T. Yau, *The topological uniqueness of complete minimal surfaces of finite topological type*, Topology **31** (1992), no. 2, 305–316, MR1167172, Zbl 0761.53006.

[MPR11] M. Manzano, J. Pérez, and M. M. Rodríguez, *Parabolic stable surfaces with constant mean curvature*, Calc. Var. Partial Differential Equations **42** (2011), 137–152, Zbl pre05941954.

[Nov65] S. P. Novikov, *The topology of foliations*, Trudy Moskov. Mat. Obšč. **14** (1965), 248–278, MR0200938, Zbl 0247.57006.

[Pat96] V. Patrangenaru, *Classifying 3 and 4 dimensional homogeneous Riemannian manifolds by Cartan triples*, Pacific J. of Math. **173** (1996), no. 2, 511–532, MR1394403, Zbl 0866.53035.

[PS04] N. Peyerimhoff and E. Samiou, *The Cheeger constant of simply connected, solvable Lie groups*, Proc. Amer. Math. Soc. **132** (2004), no. 5, 1525–1529, MR2053361, Zbl 1045.53033.

[Ros02] W. Rossman, *Lower bounds for Morse index of constant mean curvature tori*, Bull. London Math. Soc **34** (2002), no. 5, 599–609, MR1912882, Zbl 1039.53014.

[Ros05] A. Ros, *The isoperimetric problem*, Global theory of minimal surfaces, American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005, MR2167260, Zbl 1125.49034, pp. 175–209.

[Sim85] L. Simon, *Isolated singularities of extrema of geometric variatational problems*, Harmonic mappings and minimal immersions, vol. 1161, Springer Lecture Notes in Math., 1985, MR0821971, Zbl 0583.49028, pp. 206–277.

[Sou10] R. Souam, *On stable constant mean curvature surfaces in* $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$, Trans. Am. Math. Soc. **362** (2010), no. 6, 2845–2857, MR2592938, Zbl pre05718350.

[Tor10] F. Torralbo, *Rotationally invariant constant mean curvature surfaces in homogeneous 3-manifolds*, Differential Geometry and its Applications **28** (2010), 523–607, Zbl 1196.53040.

[TU09] F. Torralbo and F. Urbano, *Compact stable constant mean curvature surfaces in the Berger spheres*, To appear in Indiana Univ. Math. Journal, preprint available at arXiv:0906.1439, 2009.

[War83] F. Warner, *Foundations on differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, 1983.

[Wen86] H. Wente, *Counterexample to a conjecture of H. Hopf*, Pacific J. of Math. **121** (1986), 193–243, MR0815044, Zbl 0586.53003.

[Whi09] B. White, *Which ambient spaces admit isoperimetric inequalities for submanifolds?*, J. Differential Geom. **83** (2009), 213–228, MR2545035, Zbl 1179.53061.

Mathematics Department, University of Massachusetts, Amherst, MA 01003
E-mail address: profmeeks@gmail.com

Department of Geometry and Topology, University of Granada, 18001 Granada, Spain
E-mail address: jperez@ugr.es