Chapter 26
Preparing teachers to teach conditional probability: a didactic situation based on the Monty Hall problem

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Abstract In this paper we reflect on the need for a better preparation of teachers to teach conditional probability, a topic in which a wide variety of misconceptions have been described. We also suggest a way to organize didactical situations that serves to improve the teachers' mathematical and pedagogical knowledge related to this topic. As an example, we analyze an activity based on the Monty Hall problem, for which there is a wide availability of Internet simulators and resources. We describe the mathematical objects underlying the correct solutions and common incorrect solution approaches. We conclude by analyzing the didactic suitability of this problem in the training of teachers.

1 Introduction

Conditional probability is included in the secondary education curricula (e.g., MEC 2007) because of its relevance in daily life and in the applications of statistics. This concept allows us to change our degree of belief about chance events as we acquire new information; however, extensive research literature suggests the existence of incorrect intuitions in its application (e.g., Falk 1986; Gras and Totohasina 1995; Díaz et al. 2010).

Moreover, several authors (e.g., Franklin and Mewborn 2006) suggest that few current programs train mathematics teachers adequately to teach statistics and probability and that misconceptions related to conditional probability may be shared by the prospective teachers who consequently may be unable to assess and correct these incorrect reasoning in their own students. As stated by Ball, Hill and Bass (2005, p.14) teachers' mathematical knowledge "is central to their capacity to use instructional materials wisely, to assess students' progress, and to make sound judgments about the presentation, emphasis, and sequencing of a topic" (Ball et al. 2005, p.14).

1.1 Components in Teachers' Knowledge

Many authors have analyzed the nature of knowledge needed by teachers to achieve effective teaching outcomes. For example, Ball and her colleagues (Ball et al. 2001; Hill et al. 2008) developed the notion of "mathematical knowledge for teaching" (MKT) in which they distinguished six main categories:

- *Common content knowledge (CCK),* or the mathematical knowledge shared by most educated adults, which includes basic skills and general knowledge of the subject.
- *Specialized content knowledge (SCK),* the particular way in which teachers master the subject matter that supports their activity in planning and handling classes and in assessing students' knowledge.
- *Knowledge in the mathematical horizon,* a view of the larger mathematical landscape, that teaching requires.
- *Knowledge of content and students (KCS),* including knowledge about common student conceptions and misconceptions, and their strategies to solve specific mathematics tasks.
- *Knowledge of content and teaching (KCT),* needed in the design of instruction, including how to choose examples and representations, and how to guide students towards accurate mathematical ideas.
- *Knowledge of the content and the curriculum:* how this content is included in the school curriculum.

Given the variety of knowledge needed by teachers and the scarce time available in their initial preparation, we need activities that serve to simultaneously increase several components in teachers' knowledge (Batanero et al. 2005). One possibility is using some paradoxical problems that would be posed to the prospective teachers, and explored with the help of simulation. Batanero et al. (2005) suggest that in helping teachers overcome some of their biases it may be helpful for teachers to first confront their wrong intuitions in probabilistic problems and then conduct experiments via simulation that provide counter evidence. After the teachers reach their solutions (which will likely include some wrong reasoning), they are then involved in a didactic analysis of the problem. This includes an analysis of the correct solutions as well as common incorrect approaches. This analysis may serve to help teachers acquire new knowledge of students' strategies and difficulties and of methods to help students overcome these difficulties. Below we reflect on some didactic features of simulation and then present an example of such a situation, based on the Monty Hall problem.

1.2 The role of simulation in teaching probability

In the perspective of "stochastics teaching", a sub-domain of mathematics comprising probability and statistics (Burrill and Biehler 2011), probability is not only the bases of statistics, but also is the mathematical branch that models nondeterministic relationships, random phenomena, and decisions under uncertainty. In this perspective, Heitele (1975) included simulation in his list of fundamental stochastic ideas because of its role, similar to that played by isomorphism in other mathematical areas. Through simulation we put two random experiments in correspondence in order to indirectly study one of them; in this way we overcome the difficulty of solving some probability problems by analytical or combinatorial methods. It is also possible to substitute formal proofs by more intuitive reasoning. Three additional positive points in simulation according to Biehler (1991) are: (1) the representations used may help students think through concrete models before being able to generalize to abstract probabilistic spaces; (2) most simulators include the data processing capability and facilitate the estimation of probability; and (3) the students must first construct a coherent understanding of the target situation as well a model of it before doing the computation. Biehler (1997) expands on this analysis and other roles of technology in the teaching of statistics, suggesting that simulation can serve to explore abstract probabilistic objects by creating virtual microworlds where students can experiment with the different variables involved. Having reflected on the didactic interest of simulation, we now analyze a didactic situation based on the Monty Hall problem and directed to increase the didactic knowledge of teachers.

2 A didactic situation based on the Monty Hall problem

As it is widely known, this problem arose from the TV game show "Let's Make a Deal". It was popular between 1963 and 1986 on American television and received its name from the game host, Monty Hall (Bohl et al. 1995). A controversy developed, which is still covered in both the popular media and academic circles regarding the decision faced by every contestant in a wide literature (e.g., Selvin

1975a; 1975b; Gillman 1992; Rosenhause 2009; Shaugnessy and Dick 1991; Eisenhauer 2000; Krauss and Wang 2003; Rosenhouse 2009; Borovcnik 2012). One possible formulation of the problem is reproduced below:

> Suppose you are on a game show and are given the choice to select one of three doors. Behind one door there is a car and behind each of the other two doors there is a goat. Once you select a door, say No. 1 (which is closed), the host, who knows what is behind each door, opens another door (say No. 3), which contains a goat. You are now given the option of changing your selection to door No. 2 or sticking with door No. 1. What would you do?

In courses we imagine for teachers, the activity would start by asking the participants to solve the problem and decide whether the contestant should change the door or not, as well as to justify their responses with a probabilistic argument. The teacher's instructor would encourage them to figure out what kind of player is most likely to win the car, the one who changes the door initially selected or the one who sticks to the initial choice. If teachers fail to provide an analytical solution or give a wrong solution (which is common), they would be given the opportunity to simulate the game, perhaps using one of the applets available on the Internet, to get some experimental data that may help them find the correct solution. After some solutions have been provided, the class would conduct a didactic analysis, led by the instructor. Firstly, some possible correct solutions at different levels of formalization would be analyzed. Then the teachers would be asked to identify the mathematical objects used in each correct solution. Finally the incorrect reasoning behind the wrong solutions would be identified. Below we summarize these analyses.

2.1 Some solutions

When confronted with this problem, the majority of subjects (event those with statistical training) choose not to switch doors (Shaughnessy and Dick 1991). Apparently, the information about the open door with the goat can be used to eliminate this door and, since there were a priori equal probabilities for all the doors to contain the car, the remaining doors are viewed as equally likely. It is, however, possible to get to the correct analysis in an intuitive way (Solution 1). Before analyzing some solutions, we remind that the game host never picks a door randomly to show the contestant after they select the door. The host always shows a door with a goat behind it.

Since there are two doors with no prize and one door with a prize, the probability of choosing the door with the prize with no other information available is 1/3. If we do not change the initial solution, we just have 1/3 chance of winning and 2/3 of losing the prize. However, suppose after selecting a door and learning which of the two remaining doors does not contain the prize, we change our initial choice. Then the probability of winning is equal to the probability of having initially selected the door with no prize, which is 2/3. It is therefore advantageous to change the door.

In another solution (Solution 2) we use the compound experiment and a tree diagram (Figure 26.1) to help teachers visualize the situation. Let's first consider the experiment "door containing the prize" (each door has 1/3 probability). Next, a second experiment, consists of the door chosen by the contestant (again there is 1/3 probability of choosing each door). These two experiments are independent. The third experiment is the door opened by the host, which depends on the two previous experiments (Figure 26.1) because the host never opens the door with the car.

To compute the probability of winning when we do not change the door, we add the probabilities along all branches (choosing door 1 if the car is in behind door 1, choosing door 2, if the car is behind door 2 and to choosing door 3 if the car is behind that door). Each of these compound events has a probability of 1/9, so that the chances of winning are 1/3.

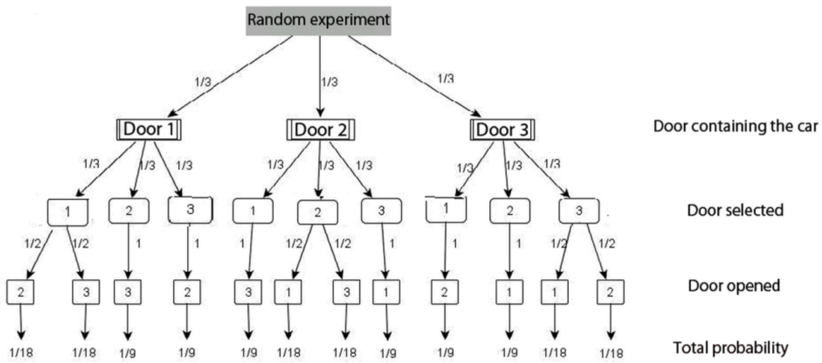

Fig. 26.1 Tree diagram showing the game structure

Suppose we change the door. If we choose a door with a goat, the host will open the door with the other goat. We change to the door with the car and win. For example, if the car is behind door 1, and we chose door 2, the host opens door 3 and we can only switch to door 1, which is the one with the car. This event has a probability of 1/9. The same reasoning would apply to the other doors.

If we choose the door with the car, the host shows us one of the two doors with the goats. For example, if the car is behind door 1 and we choose door 1, the host opens either door 2 or 3, each with probability 1/18, totaling 1/9. If we change the door, we lose with probability 1/9. Since there are three doors, the total probability of winning if we change is 2/3 and the probability of losing would be 1/3.

2.2 Formal solution

The above solutions may be approached more formally using, mathematical symbols and the properties of probability. For example we can use the following notation:

- C: The player selects the door containing the car.
- G: The player selects a door containing a goat.
- W: The player wins the car.

We compute P(W) for two different types of player, the one who changes the door and the one who maintains the initial selection.

Since $(C \cap G) = \emptyset$, $W = (W \cap C) \cup (W \cap G)$, and we can then apply the addition rule; since {C, G} is a partition of Ω, the sample space of the experiment, therefore:

$$P(W) = P((W \cap C) \cup (W \cap G)) = P(W \cap C) + P(W \cap G)$$
$$= P(W|C)P(C) + P(W|G)P(G)$$

By applying the Laplace rule: *P(C)* = 1/3 and *P(G)* = 2/3 because there is a car and two goats. Finally, we compute the probability of winning for each player:

- If the player does not change his/her initial selection *P(W|G) =0* and *P(W|C) =1*. Therefore, *P(W) =1/3.*
- If the player changes the door, then *P(W|C) =0* and *P(W|G) =1.* Therefore, *P(G) =2/3.*

2.3 Empirical solution

By working with an applet or some other simulation tool, the data one collects can make the consequences of the two strategies (stick or switch) visible. If the teacher initially believed that it switching or sticking did not matter, the experimental results of a sufficiently large trial will provide contradictory evidence. This will, ideally, produce cognitive conflict that will motivate one of the intuitive analyses shown above.

Given that the results are random, we should repeat the game a considerable number of times (at least 100) so that the results are reasonably close to the expected percentage; this is easy since the computer allows a quick simulation of a large number of experiments. However, even though the applet or simulation tool provides supporting evidence in the form of data, to understand the results the teacher educator will still need to prompt the teachers to an analysis that is consistent with the results.

3 Mathematical objects involved in the problem

After having presented some possible intuitive, experimental and formal solutions for the Monty Hall problem, the teachers would be asked to analyze the mathematical components involved in each solution. We can discuss with the teachers the different categories of these components and how they are linked to various mathematical practices (Godino et al. 2007):

- *Problem-situation:* Applications, exercises, problems, actions that induce a mathematical activity. In the situation described in this paper, the problem is deciding the best strategy (in the sense of giving the highest probability of winning) in the Monty Hall game. *Language*: Since mathematical objects are immaterial, we use different representations of them; for example, in the solutions described we used tree diagrams, symbols, words. In the experimental solution we might also use iconic language or other graphs, depending on the simulator used.
- *Concepts:* When solving mathematical problems, we use mathematical concepts whose definitions need to be known and recalled. In the Monty Hall problems we use the ideas of randomness, sample space, event, simple, conditional, and, joint probability, independence.
- *Properties* of the concepts or relations between different concepts. In the solutions described for the Monty Hall problems, we had to remember that the sum of all the probabilities in the sample space adds one; in the experimental solution, students should grasp an intuitive idea of "convergence", by understanding the tendency of relative frequencies towards an underlying probability and that larger samples are more reliable than small samples.
- *Procedures:* operations, algorithms, rules. In some solutions we used Laplace's rule, the product and sum of probability rules, listing of events, construction of tree diagram or arithmetic operations.
- *Arguments:* Reasoning or proofs used to validate or explain the properties, procedures or solution to problems.

In Table 26.1 we summarize some of the mathematical objects involved in each of the solutions described in section 2. Depending on the solution provided, the solver may use more or less complex mathematical objects. Formal solutions use symbolic language are, which in general is complex to understand; an intuitive understanding of convergence and the difference between probability and relative frequency only appear in the experimental solution. The intuitive solution could still be simpler if we work only with natural frequencies instead of using probabilities and the more complex formal solution with the explicit use of the Bayes' theorem (Krauss and Wang 2003). Consequently, both the game and the type of solution reached determine the mathematical work in the classroom. This makes it possible to work at various levels of difficulty, depending on the type of student and their prior knowledge.

Table 26.1 Configurations of mathematical objects in five solutions to the Monty Hall problem

Type	Mathematical objects in the situation	Meaning in the situation	Int. Sol.1	Int. Sol.2	Emp. Sol.	Formal Sol
Pr.	Changing the door or not	Finding the best strategy	x	x	x	x
Language	Verbal	Explaining the situation	x	x	x	x
	Graphical	Tree diagram	x	x		
	Symbolic	Events and probabilities				x
	Numeric	Probabilities of each event	x	x		x
	Numeric	Relative frequencies			x	
	Iconic	Icons in the applet			x	
Concepts	Events; sample space	Door number; Wining/not	x	x	x	x
	Compound experiment	Composition of experiments	x	x	x	x
	Compound sample space	Cartesian product	x	x	x	x
	Relative frequency	N. of successes / n. of trials			x	
	Convergence	Frequency tends towards probability			x	
	Events union	Joining elements of two events *A* and *B*				
	Impossible event	Intersection of an event and its complementary				
	Classical probability	Proportion of favourable to possible cases	x	x		x
	Frequentist probability	Frequency limit			x	
	Conditional probability	Probability of one event conditioned on another event	x	x	x	x
Properties	Probability axioms	Formal rules				x
	Sum rule	Probability of winning the car	x	x		x
	Product rule	Joint probability; dependence	x	x		x
	Total probability theorem	Application to the situation				
	Relationship between simple and conditional probability	Restriction of sample space	x	x		x
	Frequency converges towards probability	Law of large numbers (empirical)			x	
Procedures	Intuitive calculus of probability	Applying intuitive rules	x	x		
	Formal calculus of probability	Apply formal rules				x
	Computing probability from frequencies	Estimating probability from frequency			x	
	Graphical representation	Building tree diagram	x	x		
Arg	Deductive reasoning	Proving the solution	x	x		x
	Empirical reasoning	Comparing strategies			x	

4 Students' possible difficulties

The complexity of the Monty Hall problem shown in the above analysis of mathematical objects is also reflected in the extensive literature describing wrong solutions. For example, in Granberg and Brown's (1995) experimental study only 3% of participants correctly chose to change doors.

Analysis of incorrect reasoning should also be part of the teachers' knowledge of content (probability). Consequently, when working with the Monty Hall problem in a course directed at teachers, the wrong solutions provided by the teachers themselves may serve to organize a discussion that help teachers increase their KCS. Below we describe some of usual incorrect reasoning (more examples can be found in Krauss and Wang 2003).

4.1 Assuming independence of experiments

An incorrect solution may be produced when the person does not perceive the dependence of successive actions (how the door opened by the host depend on the door selected by the contestant). Either the solver does not understand the structure of the compound event or attributes an incorrect property (independence) to these events. To be specific, the solver believes that it makes no difference whether you switch or stick reasoning (correctly) that the host showing them what is behind another door could not influence the probability (1/3) that the person selected the correct door in the first place. One way to explain this reasoning error is to say that the person fails to realize that it is possible to condition an event A by another event B that happens after A, and that this conditioning can change the initial probability of event A. Relevant new knowledge can, and should, affect the probabilities we assign to events. Otherwise, we would never subject ourselves to medical tests, which basically allow us to revise the probability that we have a certain condition.

This reasoning is explained by the "fallacy of the temporal axis" described by Falk (1986), where people mistakenly believe that a current piece of information (the door shown by the host) cannot affect the probability for an event that occurred before it (probability of the door containing the prize). This fallacy is partly caused by confusion between conditioning and causation.

From the point of view of probability, if an event A is the strict cause of an event B, whenever A happens, B will also happen, so that $P(B|A) = 1$. If an event A causes another event B, then B is dependent on A, but the opposite is not always true, since an event B can depend on another event A, and still neither of them is the strict cause of the other (Tversky and Kahneman 1982; Falk 1986; Díaz et al. 2010).

4.2 Incorrect enumeration of the sample space

Another source of potential errors in the Monty Hall problem is an incomplete enumeration of the sample space in one or more of the events involved. There is a failure in describing sample space which is specific in the conditioning situation, a failure which is, according to Gras and Totohasina (1995), frequent in solving conditional probability problems. We may intuitively believe that, after a door with no prize is opened, this door should no longer be taken into account in computing the probabilities. Accordingly, there are only two possible remaining doors (events), the door containing the prize and the one that does not. Therefore, the probability is 1/2 for each door and it does not matter whether you stick or switch.

The failure appears in not considering as relevant the door opened by the host (second step in the game); this door will depend on the contestant selection (first step in the game); therefore the sample space in the second stage is different, depending on the outcome of the first stage.

- If the player, in the first step, chooses the door containing the car (with a probability of 1/3) then the host in the second step can open either of the remaining two doors. Consequently, the sample space for the selection of those two doors by the host in the second step has two possibilities each with probability of being opened as 1/2.
- But if the player chooses a goat in the first step (with a probability of 2/3), in the second step the host only has the option of opening the remaining door containing a goat. In that case, the sample space in the second step has a single element.

4.3 Incorrect assignment of probabilities

Once the host opens a door, people are inclined to believe that this door is irrelevant, so the probabilities to win are equal ½ for the two remaining doors. This reasoning is incorrect as the three doors can split into two subsets (Granberg and Brown 1995): (1) Subset 1: the door selected by the contestant, and (2) Subset 2: the two doors not selected. People incorrectly apply the sum of the probabilities rule: although the opening of the door by the host does not affect the original choice (the probability that the prize is in subsets 1and 2 are still 1/3 and 2/3) it affects the other two non-selected doors. Once a door is opened and shown to have a goat, that door has a probability of 0 to have a car behind it. As the probability that the subset 2 contains the car is equal to 2/3 in the initial experiment and given that there is now only one door left in subset 2, the probability of that door having the prize becomes 2/3. That is, the probability of 1/3 associated with the opened door is entirely transferred to the door which was neither chosen nor opened by the host, because the host cannot open the door initially chosen. Added to the initial probability of 1/3, the probability that the unselected and unopened door has the car is now 2/3.

4.4 Incorrect interpretation of convergence

If the solver obtains, as a result of a series of simulations, a solution that is contrary to the correct solution of the problem (due to chance), these empirical results may reaffirm the solver's belief that switching or sticking makes no difference. He will be reassured in his "belief in the law of small numbers" (Tversky and Kahneman 1982), by which people believe that even small samples are representative of the underlying probabilities.

This possibility is greater when the number of experiments made with the applet is small, since the convergence of the relative frequencies to the underlying probability is achieved in the long term, but not in small series of trials. The teacher educator is responsible for encouraging the solver to increase the sample size and organize a discussion around the effect of sample size on the reliability of results.

Another difficulty is that not all students would agree that the best strategy is that would give the highest probability in the long run. Some may argue that you only play the game once, and that looking at what happens with multiple simulations is not relevant to a single decision. By this reasoning, they might still think that it is indifferent whether you switch or stick.

5 Final remarks

In this paper we have suggested a possible situation with a first stage where the teachers are asked to solve the Monty Hall problem. This problem, based on a paradox by Bertrand (1888), has a counterintuitive solution, and can be used either in teacher training or teaching of conditional probability to students. Its solution illustrates some basic fundamental ideas (Heitele 1975), including that of probability, sample space, addition and product rules, compound experiment, independence, sampling and convergence.

In courses directed to teachers, and once the problem is posed, the teachers´ educator will provide some time to teachers until they reach a possible solution. Then, he will organize a debate of the correct and incorrect solutions, until the majority come to a consensus on the solution. Throughout the debate the teachers´ educator would help the teachers analyze the causes of errors and would provide a summary of what was learned in the activity. In the case of resistance to the correct solution, the solutions can be compared with the empirical evidence produced by the applet or simulation tool so that teachers eventually realize their erroneous intuitions and revise them. Since the Monty Hall problem could also be posed to high school or university students, the teachers would also acquire new knowledge about instructional approaches to teach probability.

Similar activities may be based on other paradoxical problems (see, for example, the situation described in Batanero et al. 2004). These situations fulfill the criteria of didactic suitability proposed by Godino, Wilhelmi, and Bencomo (2005):

- Epistemic or mathematics suitability: This involves the amount and quality of mathematical content in the situation that we wish to teach. The Monty Hall problem is rich in this content, as shown in the analysis of the latent mathematical objects. Note that the epistemic suitability will depend on the type of solution you expect and not just on the problem. In general, formal solutions have greater suitability in a university and teacher training course; intuitive solutions may be sufficient at secondary school level.
- Cognitive suitability is the degree to which the situation is appropriate for students and whether it is likely to provoke learning.
- Interactional suitability: This is the degree to which the organization of learning allows for the identification and resolution of the participant´s conflicts. This will depend on how the teachers´ educator organizes the classroom work and the discussion of the solutions.
- Media suitability: This concerns the availability and adequacy of the resources needed for the teaching-learning process. Since not many resources are needed in the Monty Hall problem, it rates high on this dimension.
- Emotional suitability: This concerns the interest and motivation of students in the situation. In the Monty Hall problem, interest and motivation are high as indicated by their eagerness in playing the game, in arguing about the correct solution and in working to understand the solution.

Finally, a didactic analysis, similar to the one described in this paper, serves to increase the teachers' knowledge and awareness of probability, teaching probability, and the difficulties students are likely to have. The process could be improved if solutions given by actual students were available so that teachers could evaluate these solutions to detect the errors described.

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