

⑥ a) $F^{\mu\nu} \rightarrow \Lambda^\mu_\rho \Lambda^\nu_\sigma F^{\rho\sigma}$

$$X^\mu_\rho = \left[e^{-\frac{i}{2} \omega_{\alpha\beta} J^{\alpha\beta}} \right]^\mu_\rho \quad \left\| \quad (J^{\alpha\beta})^\mu_\rho = i(g^{\alpha\mu} \delta^\beta_\rho - g^{\beta\mu} \delta^\alpha_\rho) \right.$$

$$\begin{aligned} \Rightarrow \boxed{\delta F^{\mu\nu}} &= -\frac{i}{2} \left[\omega_{\alpha\beta} (J^{\alpha\beta})^\mu_\rho F^{\rho\nu} + \omega_{\alpha\beta} (J^{\alpha\beta})^\nu_\sigma F^{\mu\sigma} \right] \\ &= \frac{1}{2} \omega_{\alpha\beta} \left[(g^{\alpha\mu} \delta^\beta_\rho - g^{\beta\mu} \delta^\alpha_\rho) F^{\rho\nu} + (g^{\alpha\nu} \delta^\beta_\sigma - g^{\beta\nu} \delta^\alpha_\sigma) F^{\mu\sigma} \right] \\ &= \frac{1}{2} \left[\underbrace{\omega^\mu_\rho F^{\rho\nu}} - \underbrace{\omega_\rho^\mu F^{\rho\nu}} + \underbrace{\omega^\nu_\sigma F^{\mu\sigma}} - \underbrace{\omega_\sigma^\nu F^{\mu\sigma}} \right] \\ &= \boxed{\omega^\mu_\rho F^{\rho\nu} + \omega^\nu_\sigma F^{\mu\sigma}} \end{aligned}$$

b) $F_{0i} = -E^i, F_{ij} = -\epsilon^{ijk} B^k \iff E^i = -F_{0i}, B^i = -\frac{1}{2} \epsilon^{ijk} F_{jk}$

$$\begin{aligned} \delta E^i &= -\delta F_{0i} = -\omega^0_\rho F^{\rho i} - \omega_i_\sigma F^{0\sigma} \\ &= -\omega^0_j F^{ji} - \omega_{ij} F^{0j} = \omega^j F^{ji} + \omega^j F^{0j} \\ &= -\eta^j \epsilon^{ijk} B^k - \epsilon^{ijk} \theta^k E^j \\ \omega^j &= \eta^j \\ \omega^j &= \epsilon^{ijk} \theta^k \end{aligned}$$

$$= \epsilon^{ijk} \eta^j B^k + \epsilon^{ijk} \theta^j E^k \Rightarrow \boxed{\delta \vec{E} = \vec{\eta} \times \vec{B} + \vec{\theta} \times \vec{E}}$$

$$\begin{aligned} \delta B^i &= -\frac{1}{2} \epsilon^{ijk} \delta F_{jk} = -\frac{1}{2} \epsilon^{ijk} (\omega^j_\rho F^{\rho k} + \omega^k_\sigma F^{j\sigma}) \\ &= -\frac{1}{2} \epsilon^{ijk} (\omega^j_0 F^{0k} + \omega^j_l F^{lk} + \omega^k_0 F^{j0} + \omega^k_l F^{jl}) \\ &= -\frac{1}{2} \epsilon^{ijk} (+\eta^j E^k + \omega^j_l \epsilon^{lkn} B^n + (j \leftrightarrow k)) \\ &= -\epsilon^{ijk} (+\eta^j E^k + \theta^k B^j) \Rightarrow \boxed{\delta \vec{B} = -\vec{\eta} \times \vec{E} + \vec{\theta} \times \vec{B}} \end{aligned}$$

$$-\frac{1}{2} \epsilon^{ijk} F_{jk} = +\frac{1}{2} \epsilon^{ijk} \epsilon^{jkl} B^l = +\delta^{il} B^l = B^i \quad \left\| \quad \begin{aligned} \omega^j_l \epsilon^{lkn} B^n &= \epsilon^{jlm} \theta^m \epsilon^{lkn} B^n \\ &= -\theta^m B^n (\delta^{jkm} - \delta^{jnm}) \end{aligned} \right.$$

$$(7) (J^{\mu\nu})^\alpha_\beta = i(g^{\mu\alpha}\delta^\nu_\beta - g^{\nu\alpha}\delta^\mu_\beta)$$

$$[J^{\mu\nu}, J^{\rho\sigma}]^\alpha_\beta = (J^{\mu\nu})^\alpha_\gamma (J^{\rho\sigma})^\gamma_\beta - (J^{\rho\sigma})^\alpha_\gamma (J^{\mu\nu})^\gamma_\beta$$

$$(J^{\mu\nu})^\alpha_\gamma (J^{\rho\sigma})^\gamma_\beta = - (g^{\mu\alpha}\delta^\nu_\gamma - g^{\nu\alpha}\delta^\mu_\gamma) (g^{\rho\gamma}\delta^\sigma_\beta - g^{\sigma\gamma}\delta^\rho_\beta)$$

$$= - \underline{g^{\mu\alpha}} \underline{g^{\rho\nu}} \delta^\sigma_\beta + \underline{g^{\mu\alpha}} \underline{g^{\sigma\nu}} \delta^\rho_\beta + \underline{g^{\nu\alpha}} \underline{g^{\rho\mu}} \delta^\sigma_\beta - \underline{g^{\nu\alpha}} \underline{g^{\sigma\mu}} \delta^\rho_\beta$$

$$(J^{\rho\sigma})^\alpha_\gamma (J^{\mu\nu})^\gamma_\beta = - (g^{\rho\alpha}\delta^\sigma_\gamma - g^{\sigma\alpha}\delta^\rho_\gamma) (g^{\mu\gamma}\delta^\nu_\beta - g^{\nu\gamma}\delta^\mu_\beta)$$

$$= - \underline{g^{\rho\alpha}} \underline{g^{\mu\sigma}} \delta^\nu_\beta + \underline{g^{\rho\alpha}} \underline{g^{\nu\sigma}} \delta^\mu_\beta + \underline{g^{\sigma\alpha}} \underline{g^{\rho\mu}} \delta^\nu_\beta - \underline{g^{\sigma\alpha}} \underline{g^{\rho\mu}} \delta^\mu_\beta$$

$$\Rightarrow [J^{\mu\nu}, J^{\rho\sigma}]^\alpha_\beta = -g^{\nu\rho} (g^{\mu\alpha}\delta^\nu_\beta - g^{\sigma\alpha}\delta^\mu_\beta) + g^{\mu\rho} (g^{\nu\alpha}\delta^\sigma_\beta - g^{\sigma\alpha}\delta^\nu_\beta)$$

$$- g^{\nu\sigma} (g^{\rho\alpha}\delta^\mu_\beta - g^{\mu\alpha}\delta^\rho_\beta) + g^{\mu\sigma} (g^{\rho\alpha}\delta^\nu_\beta - g^{\nu\alpha}\delta^\rho_\beta)$$

$$= i [g^{\nu\rho} (J^{\mu\sigma})^\alpha_\beta - g^{\mu\rho} (J^{\nu\sigma})^\alpha_\beta + g^{\mu\sigma} (J^{\rho\nu})^\alpha_\beta - g^{\nu\sigma} (J^{\rho\mu})^\alpha_\beta]$$

$$\Rightarrow [J^{\mu\nu}, J^{\rho\sigma}] = i (g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\rho\mu} + g^{\mu\sigma} J^{\rho\nu})$$