SYSTEMS OF PRACTICES AND CONFIGURATIONS OF OBJECTS AND PROCESSES AS TOOLS FOR THE SEMIOTIC ANALYSIS IN MATHEMATICS EDUCATION[1]

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ABSTRACT

The semiotic approach to mathematics education introduces the notion of "semiotic system" as a tool to describe mathematical activity. It is formed by the set of signs, the production rules of signs and the underlying meaning structures. We consider that this is a fuzzy notion that needs an operative definition to facilitate the description and explanation of teaching and learning phenomena. In this paper we present the notions of system of practices and configuration of objects and processes that develop and complement the notion of semiotic system. We also show in what sense these notions facilitate the description and comprehension of building and communicating mathematical knowledge, by applying them to analyze the semiotic system involved in the study of whole numbers, from the institutional and personal viewpoint.

Key words: onto-semiotic approach, object, meaning, mathematics, natural number, learning, semiotic system

1. INTRODUCTION

Ernest (2006) describes the characteristic features of the semiotic perspective in mathematics education highlighting the new insights that the "science of signs" provides to describe and understand the mathematical learning and communication processes. He exemplifies these contributions by analysing the meaning of numbers, counting and arithmetic operations, from the viewpoint of their historical development, the mathematical formulations of these contents, and the learning issues involved. This theoretical perspective deals with modelling, within a coherent framework, the role of mathematical sign systems, the meanings structure, the mathematical rules and phenomenology that promotes the mathematical activity.

In order to develop an integrative theoretical framework to mathematics education research, and starting from anthropological and semiotic assumptions, Godino and cols. (Godino y Batanero, 1998; Godino, Batanero y Font, 2007) are developing the "onto-semiotic" approach (OSA) to mathematical knowledge and instruction. They propose as a key notion the system of practices, considering a practice as "any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems" (Godino and Batanero, 1998, p. 182).

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Therefore, a practice is conceived in terms of reflective, situated, and intentional actions mediated by material and linguistic resources. The system of practices is introduced as the pragmatic answer to the semiotic question, what does the object O mean? Or the ontological question, what is the mathematical object O?

The notions of “system of practices” and “semiotic system” can be useful for certain comparative studies of global epistemological formations (for example, the numbers used in formal or informal settings). Nevertheless, the analysis of mathematical activity and the emergent objects from the same (concepts, theorems, theories,...) needs an operative definition of such systems (of practices, or semiotic).

In recent research works carried out in the OSA framework (Font, Godino and Contreras, 2008; Font and Contreras, 2008) the notion of configuration of objects and processes has been introduced (in its double version, institutional and personal), to make the notion of system of practice operative and to allow more detailed analyses of mathematical activity. Such configurations are formed by the intervening and emergent objects in a system of practices linked to the solution of a problem (specific configuration), or from a more or less broad class of problem - situations (partial, or global configuration).

In this paper we are going to show that the notions of systems of practices and configuration of objects and processes, along with the notion of semiotic function as the relational entity that connects the different types of objects, develops and makes the notion of semiotic system operative.

In section 2 we present some characteristic notes of the semiotic perspective in mathematics education. In section 3 we show how the mathematical ontology proposed in the OSA might complement the semiotic perspective, in particular the notion of configuration of objects and processes provides an operative definition of the semiotic system construct. In section 4 we apply the notions of system of practices and configuration to analyze some semiotic systems related with the study of whole numbers. In section 5 we apply these notions to analyze the child’s answers to a counting and writing numbers greater than ten task, which involves the difficulties of learning the ten. Our analysis permits the description of the onto-semiotic complexity of the learning process of decimal numeration. We conclude with a synthesis and final reflections, emphasizing the complementariness of the onto-semiotic approach and the semiotic perspectives in mathematics education.

2. CHARACTERISTIC FEATURES OF THE SEMIOTIC PERSPECTIVE

The reasons to use the semiotic viewpoint in understanding the teaching and learning of mathematics are diverse. The semiotic includes all the aspects of the human’s sign construction, the reading and interpretation of signs through the multiple contexts in which this use takes place. Therefore, the use of semiotics to study mathematical activity should not be strange, taking into account the essential role that the use of signs have in mathematics, and their anthropological character (Radford, 2006). We call “mathematical activity” any process of building and communicating mathematical objects, processes and meanings, particularly processes that take place in an educational context. Consequently, it seems justified to study the teaching and learning of school mathematics from the point of view of the science of signs.

The semiotic perspective of mathematical activity is characterized by focusing its attention on the signs and the use of signs, which is an alternative to the psychological perspectives, driven by a primary focus on mental functions and structures. As the sign involves a communicative act, semiotic perspective includes jointly the individual and social dimensions in teaching and learning mathematics.

> "The primary focus in a semiotic perspective is on communicative activity in mathematics utilizing signs. This involves both sign reception and comprehension via listening and reading, and sign production via speaking and writing or sketching" (Ernest, 2006, 69).

We can say that the semiotic perspective goes beyond the limits of cognitive and behavioural psychology because it adopts the sign as the natural and basic unit of analysis. It is interested, not just in the isolated signs but in the systems of mathematical sign, the content, skills, and abilities that can be developed in the educational process. Signs and their use can only be understood as parts of more complex systems: the *semiotic systems*, which include three components (Ernest, 2006):

- a set of signs (S)
- a set of rules of sign production (R)
- a set of relationships between the signs and their meanings embodied in an underlying meaning structure (M).

Ernest (2006, p. 70) considers that the triplet (S, R, M) cannot be established in a precise way because R is at best a fuzzy set and the potential members of M can never be made explicit, let alone represented as definite and well-defined set. Semiotic systems cannot be severed from human understanding and use. They lose meaning once isolated as purely structural systems.

> "Semiotic systems involve signs, rules of sign use and production, and underlying meanings. All of these depend on social practices, and human beings as quintessentially sign using and meaning making creatures can never be eliminated from the picture, even if for some purposes we foreground signs and rules and background people and meanings" (Ernest, 2006, p. 72).

It is clear that for Ernest the semiotic systems are not reduced to the representation of semiotic systems; mathematical activity is not reduced to the manipulation of symbols but there is a world of non-linguistic objects that take part in mathematical activity and emerges from this activity. Since that activity is carried out by people, within communities of practices, the emergent objects are impregnated by "human presence". Semiotic systems depend on human beings and groups who use and produce them; they depend on the language games immersed in social forms of life.

In the case of whole numbers, Ernest questions if they can be described by a unique semiotic system. He considers that this vision is, in general, not appropriate for the semiotic systems, and in particular for the numbers and arithmetic calculation. This is not pertinent from the historical development viewpoint, the foundational studies of mathematics, and even for the subjects' cognitive development. Although there are similarities between the different system representations, and each version might represent, in some sense, the family of diverse semiotic systems about numbers, each one is a contingent result of its social context, and has a specific purpose: "Unity and diversity co-exist simultaneously" (Ernest, 2006, p. 94)

Meanings of numbers change according to historical moments. The systems of numeration of the Egyptians, Chinese, Romans, etc. have substantial differences, not only in the physical appearance of the symbols used, but in the rules of elaboration of numerals (additive, multiplicative, irregular… systems) and the procedures of calculation. Equally, there are substantial differences between the formal number systems elaborated by Frege, Dedekind, Peano, etc., and the number systems used in school mathematics.

From the previous discussion we infer the necessity to elaborate theoretical models that, starting from anthropological postulates for mathematics, help to describe and understand the plurality of meanings of mathematical objects, understood in its double facet, as socio-cultural entities, and personal or mental entities. The notions of system of mathematical practices, configuration of objects and processes and semiotic function, which we describe in the following section, are revealing as operative tools for the analysis of semiotic systems, and generally for didactical analysis.

3. THE ONTO-SEMIOTIC APPROACH TO MATHEMATICAL KNOWLEDGE

The notion of system of practices can be related to the one of semiotic systems. If the systems of practices are complemented with the configurations of objects and processes intervening and emerging in mathematical practices it is even possible to enrich and make operative the notion of semiotic system, as we will try to show in this section.

3.1. Systems of practices and pragmatic meanings

All kinds of performances or expressions (verbal, graphic, etc.), carried out by someone in order to solve mathematics problems, communicate the solution obtained to others, validate it or generalise it to other contexts and problems, are considered to be mathematical practice (Godino and Batanero, 1998). These practices might be idiosyncratic of a person or be shared within an institution. In the study of mathematics, rather than a particular practice to solve a specific problem, it is interesting to consider the systems of practices (operative and discursive) carried out by people when faced with problematic types of situations.

The meaning of a mathematical object (a concept) is conceived in terms of the system of practices in which that object intervenes, playing a relevant role. This implies assuming a pragmatist postulate about mathematical meaning.

The systems of practices (and hence also the meanings) have been categorized in the OSA taking into account diverse points of view. First is the distinction between the personal, or idiosyncratic character of practices (personal practices), and the institutional one (social or shared practices). The semiotic interpretation of the systems of practices allows to discriminate the following types of institutional and personal meanings (Figure 1):

- Institutional meanings. *Implemented*: the system of practices that a teacher effectively implements in a specific teaching experience; *Assessed*: the system of practices that a teacher uses to assess his/her students' learning; *Intended*: the system of practices included in the planning of the study process; *Referential*: the system of practices used as reference to elaborate the intended meaning.

- Personal meanings. *Global*: set of personal practices that the subject is potentially able to carry out related to a specific mathematical object; *Declared*: the personal practices effectively shown in solving assessment tasks and questionnaires, regardless of whether they are correct or incorrect from the institutional point of view; *Achieved*: personal practices that fit the institutional meaning fixed by the teacher.

Learning processes involve the progressive fitting of personal and institutional meanings, as well as the student's appropriation of these institutional meanings; teaching requires the student's participation in the communities of practices that hold institutional meanings (Figure 1).

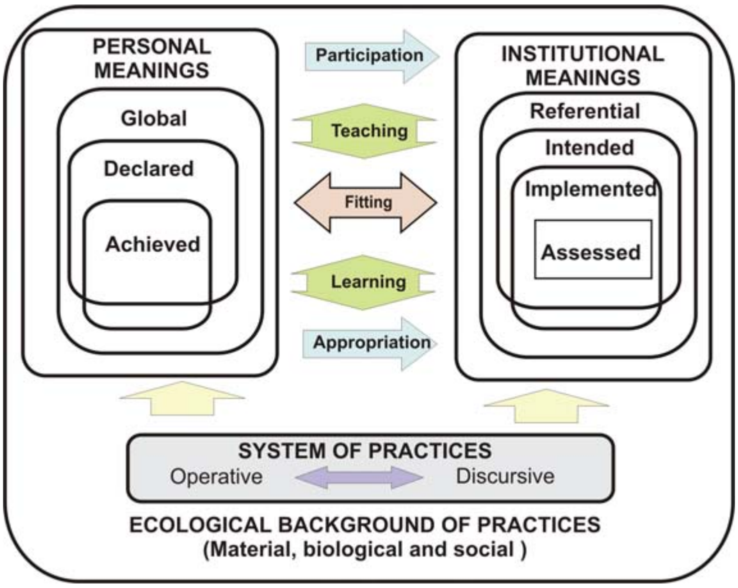

Figure 1: Types of pragmatic meanings

3.2. Configuration of objects and processes

The notion of "system of practices", or the one of "semiotic system", is useful for certain macro-didactical analysis, particularly when we try to compare the particular form adopted by mathematical knowledge in different institutional frameworks, contexts of use or language games. Getting a finer analysis of mathematical activity requires introducing a typology of mathematical objects.

The definition of an object as emergent from the systems of practices, and the typology of primary objects introduced in the OSA, intend to respond to the necessity of describing the systems of practices, in order to compare them and take decisions about the design, development and assessment of mathematics teaching and learning processes.

Emergency of mathematical objects

The onto-semiotic approach assumes the assumptions of the pragmatist epistemology and the objects appear (emerge) from mathematical practices. Such an emergency is a complex phenomenon whose explanation requires considering, at least, two levels of objects coming from mathematical activity. In the first level we have those entities that can be observed in a mathematical text (problems, definitions, propositions, etc.). In a second level we have a typology of objects that emerge from different ways of seeing, talking, operating, etc., about the objects of the first level; we refer to personal or institutional, ostensive or non ostensive, unitary or systemic objects, etc.

First level of objects: Configuration of intervening and emergent objects

For the accomplishment of a mathematical practice and for the interpretation of its results as satisfactory it is necessary to put certain knowledge into practice. If we consider, for example, the knowledge required to pose and solve a system of two equations with two unknowns, we see the use of languages, both verbal and symbolic. These languages are the ostensive part of a set of concepts, propositions and procedures that intervene in the elaboration of arguments necessary to decide whether the simple actions composing the practice, and the practice itself as a compound action, are satisfactory.

Consequently, when an agent carries out and evaluates a mathematical practice it activates a conglomerate formed by situation – problems, languages, concepts, propositions, procedures and arguments, articulated in the configuration of Figure 2 (Font and Godino, 2006, p. 69).

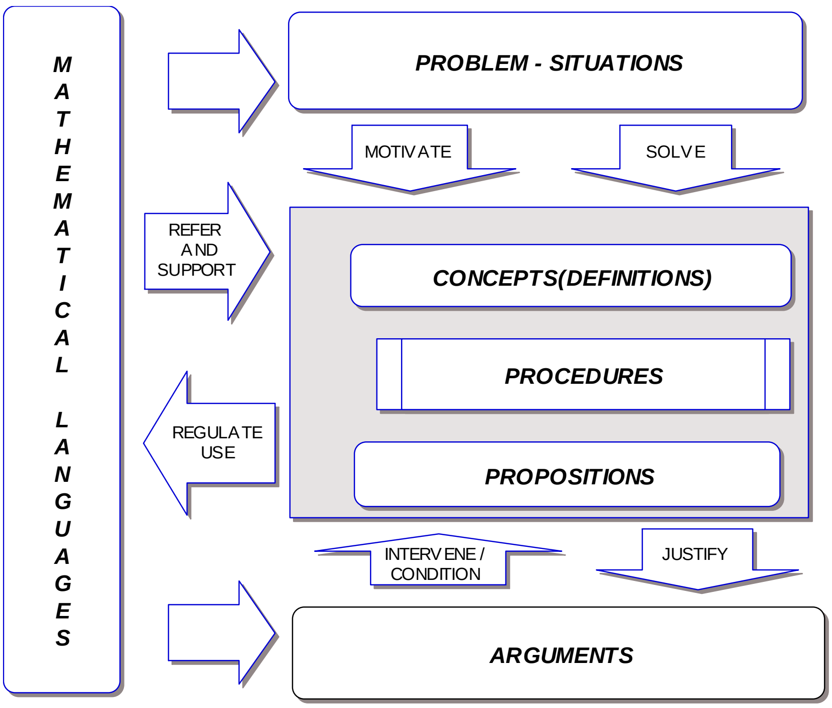

Figure 2: Configuration of primary objects

Therefore, the following types of primary mathematical objects are proposed:
- Language (terms, expressions, notations, graphics);
- Situations (problems, extra or intra-mathematical applications, exercises, etc.);
- Concepts, given by their definitions or descriptions (number, point, straight line, mean, function, etc.);
- Propositions, properties or attributes,
- Procedures (operations, algorithms, techniques);

- Arguments used to validate and explain the propositions and procedures (deductive, inductive, etc.).

The six types of primary entities postulated extend the traditional distinction between conceptual and procedural knowledge, when considering them insufficient to describe the intervening and emergent object in mathematical activity. The situation - problems are the origin or reason of being of the activity; the language represents the remaining entities and serves as an instrument for the action; the arguments justify the procedures and propositions that relate the concepts to each other.

Considering an entity as primary is not an absolute question but it is relative, because they are functional entities and relative to the language games (institutional frameworks, communities of practices and contexts of use) in which they participate; they also have a recursive character, in the sense that each object, depending on the analysis level, can be made up of entities of the remaining types (an argument, for example, may call on concepts, propositions, procedures, etc.)

The primary objects are related to each other forming configurations, defined as the networks of intervening and emergent objects from the systems of practices. These configurations can be socio-epistemic (networks of institutional objects) or cognitive (networks of personal objects).

Second level of objects: Contextual attributes

The notion of *language game* (Wittgenstein, 1953) plays an important role, together with that of institution, in our model. Here we refer to contextual factors to which the meanings of mathematical objects are relative and that attribute a functional nature to them. Mathematical objects intervening in mathematical practices or emerging from them depend on the language game in which they take part, and can be considered from the following dual dimensions or facets (Godino, 2002):

Personal – institutional. Institutional objects emerge from systems of practices shared within an institution, while personal objects emerge from specific practices from a person (Godino and Batanero, 1998, p. 185-6). “Personal cognition” is the result of individual thinking and activity when solving a given class of problems, while “institutional cognition” is the result of dialogue, agreement and regulation within the group of subjects belonging to a community of practices.

Ostensive – non ostensive. Mathematical objects (both at personal or institutional levels) are, in general, non perceptible. However, they are used in public practices through their associated *ostensive* objects (notations, symbols, graphs, etc.). The distinction between ostensive and non-ostensive is relative to the language game in which they take part. Ostensive objects can also be thought, imagined by a subject or be implicit in the mathematical discourse (for example, the multiplication sign in algebraic notation).

Extensive – intensive (example - type). An *extensive* object is used as a particular case (a specific example, i.e., the function $y = 2x+1$), of a more general class (i.e., the family of functions $y=mx+n$), which is an *intensive* object. The extensive / intensive duality is used to explain a basic feature of mathematical activity: the use of generic elements (Contreras and cols, 2005). This duality allows us to focus our attention on the dialectic between the particular and the general, which is a key issue in the construction and application of mathematical knowledge. “Generalization is essential because it is this process that distinguishes mathematical creativity from mechanizable or algorithmic behaviour" (Otte, 2003, p. 187).

Unitary – systemic. In some circumstances mathematical objects are used as unitary entities (they are supposed to be previously known), while in other circumstances they are seen as systems that could be decomposed to be studied. For example, in teaching, addition and subtraction, algorithms, the decimal number system (tens, hundreds,…) is considered as something known, or as unitary entities. These same objects, in first grade, should be dealt with as systemic and complex objects to be learned.

Expression – content. They are the antecedent and consequent of semiotic functions. Mathematical activity is essentially relational, since the different objects described are not isolated, but they are related in mathematical language and activity by means of semiotic functions. Each type of object can play the role of antecedent or consequent (signifier or signified) in the semiotic functions established by a subject (person or institution).

These facets are grouped in pairs that are dually and dialectically complementary. They are considered as attributes applicable to the different primary objects, giving rise to different "versions" of the said objects.

In figure 3 we represent the different theoretical notions that have been briefly described. In the OSA mathematical activity occupies the central position and it is modelled in terms of systems of operative and discursive practices. From these practices the different types of primary mathematical objects which are related to each other forming configurations, emerge. Finally, the objects that take part (or emerge) in mathematical practices, according to the language game in which they participate, can be considered from five facets or dual dimensions, which allow us to introduce the following typology of secondary objects: personal - institutional, ostensive - non ostensive, intensive - extensive, unitary - systemic, expression – content.

Processes

The primary and secondary entities described might be analyzed from the *process-product* perspective, which allows us to introduce the processes indicated in figure 3, into the model. The emergency of primary objects (languages, problems, definitions, propositions, procedures and arguments) takes place by means of the respective mathematical processes of communication, problem posing, definition, enunciation, elaboration of procedures (algorithmization) and argumentation. On the other hand, contextual dualities give rise to the following epistemic and cognitive processes: institutionalization - personalisation; generalization - particularization; splitting – reification, materialization – idealisation, representation – interpretation.

The inclusion in the model of the extensive/intensive, ostensive/non ostensive and unitary/systemic dualities allows us to distinguish the particularization and generalization processes from the materialization and idealization processes, and also to distinguish these from the splitting and reification processes (Font and Contreras, 2008). These are important distinctions because they allow us to carry out a more detailed analysis of each process and their combined presence in mathematical activity, and therefore, to clarify the nature of "mathematical objects", usually conceived merely as ideal or abstract entities.

The use of the term process is very broad and diverse in mathematics education. Usually it is used to refer to instruction, cognitive, meta-cognitive, etc. processes. These are very different processes in which the unique common characteristic, for many of them, might be the consideration of the "time" factor and, to a lesser extent, to be a "sequence of acts in which each element influences the determination of the following". In the OSA it has been chosen to select a list of processes that are considered important in mathematical activity (those included in figure 3) and that, to a certain extent, take part as components of another "hyper" or mega process, such as, for example, problem solving or modelling. It would be necessary to include the meta-cognitive processes involved in carrying out mathematical practices[2].

Figure 3: Configuration of objects and processes

3.3. Meaning as content of semiotic functions

The initial goal of the onto-semiotic approach was to give an answer, which were useful to mathematics education research, to the question, what is the meaning of a concept?, for example, what does the "concept of arithmetic mean", mean? (Godino and Batanero, 1994). A pragmatist – anthropological answer was given to that issue: The meaning of a concept (or "any mathematical object") is the system of practices (operative and discursive) that a person carries out to solve certain types of problems in which the object intervenes. This way a semiotic function is established between the object and the system of practices.

The description of professional or school mathematics activity requires the double language of practices and objects that intervene in the same: there are not practices without objects, nor objects without practices. These two basic categories of entities are complemented with another relational entity: the semiotic function, which connects the objects participating in the practices.

[2] Gusmao's thesis (2006) studied meta-cognitive processes in problem solving using the onto-semiotic approach.

Semiotic functions are the relational tool that facilitates the joint study of manipulating the ostensive objects, and the thinking involved in it, so characteristic of mathematical practices. Figure 3 summarizes the system of objects and processes that could participate as expression, content or criteria of semiotic functions. The notion of representation is radically generalized. Languages (words, symbols, gestures, etc.) are not the only "objects" that can play the role of representation: in accordance with Peirce's semiotic it is assumed that the different types of objects (problems, concepts, proposition, procedures, arguments) might also be expression or content of semiotic functions.

This "onto-semiotic" model is a powerful tool to analyze the mathematical practice and the communication and interpretation processes involved in the same.

3.4. Relationship between configurations and semiotic systems

Figure 3 helps us to give a definition of "semiotic system", which we consider is operative and well adapted to the analysis of teaching and learning processes: *A semiotic system is the system formed by the configuration of intervening and emerging objects in a system of practices, along with the interpretation processes that are established between the same (that is to say, including the network of semiotic functions that relate the constituent objects of the configuration).*

Since the systems of practices depend on the people who carry them out, and on the institutions (communities, cultures,…) where they are shared, the associate semiotic systems will also depend on people and institutions. Practices are linked to the solution of types of problem - situations, which might have a particular, local or global character, and hence the semiotic systems will also have these levels of generality.

The components of a semiotic system that Ernest (2006) considers are some of the elements considered in the configuration of objects and processes activated and emergent in mathematical practices. Indeed, the set of signs (S) is the "language" component of figure 3, when it is considered from its ostensive nature. The set of rules of production of signs (R) are, in our case, the remaining primary entities (definitions, propositions, procedures and arguments). The relations between the signs and their meaning, embodied in an underlying structure of meanings (M), are considered in our case, by the system of objects and processes of the configuration, looked at from the point of view of the expression – content duality. Figure 3 also shows the limitations of the semiotic system construct for the analysis of the inherent complexity of mathematical activity.

3.4. Semiotics conflicts and criteria of didactical suitability

The notion of semiotic conflict has been introduced in the onto-semiotic approach as an explanation of students' errors, difficulties and obstacles in the learning of specific mathematical content, and in general, of difficulties arising in classroom communication. The relativity of the systems of practices (and hence also the meanings) to the institutional frameworks, and the ecological relationships between institutions (dominance, dependence, subordination,...) lead us to consider that the following general definition of semiotic conflict is useful: *It is any disparity or mismatch between the meanings given to an expression (antecedent of a semiotic function) by two subjects (people or institutions) in an interactive communication.* If the disparity refers to institutional meanings we talk about epistemic semiotic conflict, while if the disparity refers to difference between the practices that constitute the personal meaning of a subject we call it cognitive semiotic conflicts. When the disparity is produced between the discursive and operative practices of two different persons in an interactive communication (for example, student – student, or student – teacher) we call it interactional semiotic conflicts.

The semiotic approach to personal and institutional knowledge that the OSA propose, has permitted to introduce the notion of *didactical suitability* for teaching and learning mathematics processes. Godino, Batanero and Font (2007) describe six dimensions in the analysis of these processes and they propose criteria of more or less suitability for each one. Three of these six criteria are based on semiotic concepts:

- *Epistemic suitability*, representativeness of institutional implemented (or intended) meaning as regards the reference meaning previously defined.

- *Cognitive suitability*, extent to which the institutional implemented (or intended) meaning is included in the students' "zone of proximal development" (Vygotski, 1934), and the closeness of personal meanings achieved to implemented (or intended) meaning.

- *Interactive suitability*, extent to which the didactical configurations and trajectories allow to identify and solve semiotic conflicts that might happen during the instructional process.

The system of theoretical notions that composes the onto-semiotic approach is revealing, in diverse research works, as powerful tools for the didactical analysis and instructional intervention. In the following section we apply some of these notions in the analysis of the meanings of whole numbers and the description of some phenomena related to their learning in school.

4. ANALYSIS OF INSTITUTIONAL SEMIOTIC SYSTEMS

In this section we apply the notions of system of practices and configuration of objects and processes to clarify the diverse meanings of whole numbers.

The nature of the whole numbers, and in particular their relation with set theory, is a question that is just as interesting to mathematics as to philosophy of mathematics. But numbers are also essential tools in our daily and professional life, and so the reason why they constitute a subject of essential study in school from the first levels.

We consider it necessary to distinguish between the practical and "informal" uses of numbers (to respond to questions such as, how many items are there? or what place an object occupies?), and the "formal" uses (what the numbers are and how the number systems are constructed?); these last questions belong to the foundations of mathematics as an organized body of knowledge. Within these two broad contexts of use (or institutional frameworks) it is possible to distinguish diverse historical moments at which the questions are tackled with diverse resources and from different approaches, putting into effect specific operative and discursive practices. From a retrospective viewpoint we can identify certain constant features that allow to speak of the "natural number", in singular, but from a local point of view it seems necessary to distinguish between the diverse natural numbers that the primitive peoples and old cultures (Egyptian, Roman, Chinese,…)[3] "handled", and also to distinguish between the numerical practices that are carried out at the moment in primary school, and those that the Logicist mathematicians of XIX century make, or the Hilbertians axiomatic formulations.

Therefore, the understanding of the nature and meaning of numbers requires to adopt an anthropological – socio-cultural vision on mathematics, like the proposal, among other approaches (Chevallard, 1992; Radford, 2006), from the "onto-semiotic approach to mathematical knowledge and instruction".

4.1. Some features of the informal semiotic systems of natural numbers

In order to communicate with other people, and as a means to register for ourselves at other moments, the size or amount of elements of a set can be made using different resources and procedures:

1) In our present western culture the use of the "numerical words" is generalized, one, two, three…, and the Arabic numerals, 1, 2, 3,…These limitless collections of words and symbols are those that our students use when we asked, for example, how many students are there in class? , and they respond, "there are ninety one students", or, they write, "91". To do this they have had only to apply a rigorous procedure of counting, putting in biyective mapping each student of the class with a unique numerical word recited in the established order, and respecting the principles of counting.

2) If we ask students to communicate the counting result without using the "number or symbol words" they might invent other means to express the number of students in the class (or the cardinal of the set formed by all the people in the class). For example:

- The collection of marks ///..., or other symbols, on the sheet of paper, as many elements as the set has.

- A combination of symbols for different partial groupings (*, to indicate ten students, / to express the unity)

As we have freedom to invent symbols and objects as a mean to express the cardinal of sets, that is to say, to respond to the question, how many are there?, the collection of possible numeral systems is unlimited. In principle any limitless collection of objects, whatever its nature may be, could be used as a numeral system: diverse cultures have used sets of little stones, or parts of the human body, etc., as numeral systems to solve this problem.

We see, therefore, that the informal semiotic systems in which the natural numbers are used are characterized by a specific and practical problematic (to describe the cardinal of collections of things), as well as by using particular linguistic resources, procedures, properties, concepts and justifications to solve these empirical or practical problems.

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[3] Rotman (1988) has reached a similar conclusion in his semiotic analysis of mathematical activity, when he asserts that the numbers studied by the Babylonians, Greeks, Romans and present-day mathematicians are different. Nevertheless, we believe that these numbers are similar, because of the phenomenon of regressive appropriation.

4.2. Some features of the formal semiotic systems of natural numbers

The mathematical entities that intervene in problem – situations of counting and arithmetic calculation are formally or structurally analyzed within the internal framework of mathematics. Therefore numbers are not considered as a means to inform the magnitude amounts (numbers of people, or things, role that fulfils in a situation, etc.) and are interpreted, either like elements of one structure characterized according to the set theory, or according to Peano's axioms[4].

In this context of mathematical formalization other questions are posed:

- How should we define numbers?
- How should we define the arithmetic operations starting from the Peano's axioms?
- How should we define arithmetic operations when natural numbers are conceived as cardinal of finite sets?
- What type of algebraic structure does the set N of natural numbers have with the addition operation?

The answer to these questions requires the elaboration of specific linguistic resources, operative techniques (recursion, set operations), concepts (set definitions of addition and subtraction; recursive definitions; algebraic definition of subtraction), properties (semi-group structure with null element for the addition and multiplication) and argumentations (deductive); really, a system of operative and discursive practices with specific features, adapted to the generality and rigor of mathematical work.

In spite of the differences between the informal-empirical and formal meanings of numbers, a fruitful synergy relationship between the same always existed: "Practical requirements have driven notational innovations such as the refinement of place value systems and the introduction of negative number notation. Conceptual developments have underpinned these developments, ensuring that the rules of procedure reflect the underlying meaning structures, as well as developing knowledge of other properties" (Ernest, 2006, p.80).

4.3. Plurality of numbers and meanings

Figure 4 represents the plurality (without looking for thoroughness) of informal and formal meanings of natural numbers. Counting situations have been solved by diverse cultures using different practices and tools, giving rise "to different numbers". These diverse numerical configurations are articulated in new formal contexts of use giving rise to different numerical constructions[5].

It is important to point out that the informal practices do not merely have "a historical" existence. They coexist in time with the scientific formalization in the usual practices at schools and determine the personal progress of meaning. They are not the "lesser of two

[4] There are other formalizations of natural numbers different from those based on sets theory or Peano's axiomatic. For example, Bedoya (2003) introduces and relates Peano, Peirce and Lawvere's axiomatics (using category theory). Oostra (2008) analyzes at length the Peirce's article "On the Logic of Number" asserting that Peirce developed an axiomatic construction of N before Peano.

[5] In figure 4 the "set theory" context refers to the constructions of N based on set coordinability, whereas "axiomatic" refers to Peano's axiomatics (or other equivalents ones)

evils", but landmark necessary in the mental development of children and consubstantial to the processes of didactic transposition.

Numbers are the social answers to the problem of communicating the size or cardinal of sets, to ordering a collection of objects and to analyzing iterative-recurrent processes. But each primitive culture, each form of life gave its own answer to this problem. In principle each society, culture, historical stage, has its own numbers, and their own Arithmetics, which are distinguishable according to the configuration of objects and processes that characterize them. In each configuration there are recursively organized objects, with a first element, and a unique following for each element. These organizations permit the solving of the generic problems of quantification, ordering, iteration and codification.

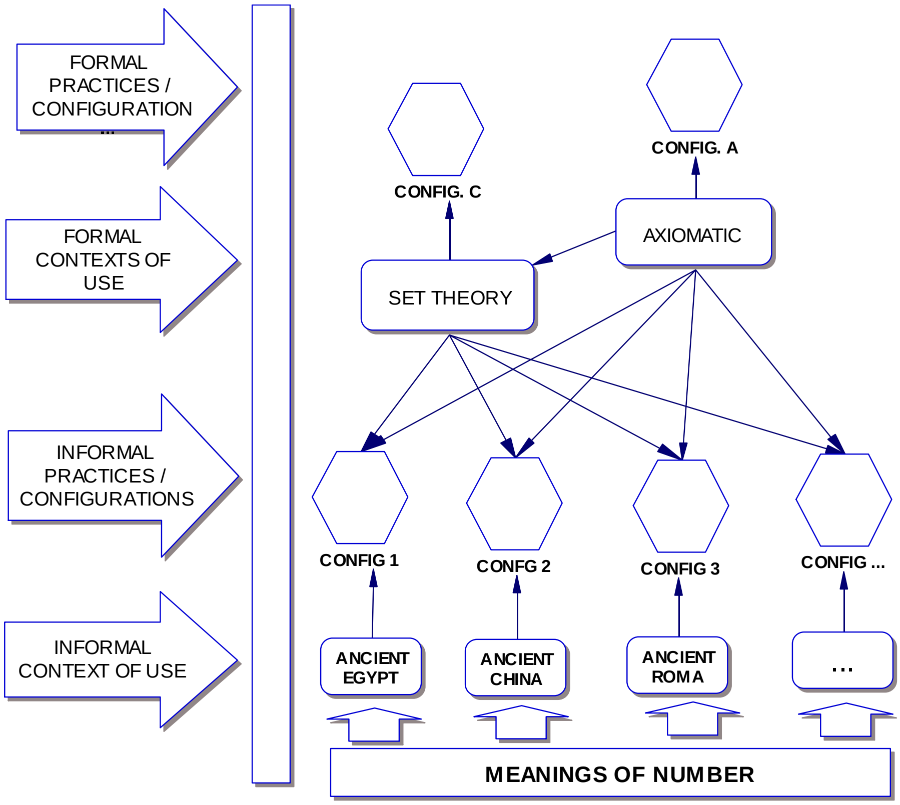


Figure 4: Plurality of number meanings

From the point of view of learning, that is to say, the personal construction of meaning, numbers initially appear in repetitive forms of expression (///..., one, two,…), linked to gestures and body movements (to indicate, separate, walk,…), really in operative representations.

In this process of initial development of numbers a key notion is the one of "following" (immediate successor). Another central principle is the correspondence one by one in the matching of signs to objects. The learning of the morph-syntactic system of natural numbers and counting supposes that the student is able to participate in certain social practices, and in particular that he/she can enunciate or produce pertinent numerical signs. The creative appropriation of these meanings requires several years of intense learning, and supposes, not only the use of numbers for solving practical problems and competence in different systems of calculation and representation, but also knowledge of properties and numerical relations that justifies the procedures and practical applications.

4.4. What are the natural numbers?

What are numbers really, if we called numbers `1, 2, 3…', `one, two, three,… ', or `uno, dos, tres,…', etc. This question is without a doubt difficult to answer, if we consider the strong controversies posed by relevant authors, such as Frege, Russell, Peano, Dedekind, etc., with regard to the different formulations of natural numbers. According to Russell, with the purpose of providing the concept of number with some extension, that is real, we must understand "the number as the number of an amount" and provide an application for the concept thus defined demonstrating the existence of sets of arbitrary cardinal (Otte, 2003, 222). This way the arithmetical intuition is replaced by a set intuition, which is still conflicting.

For Frege, the numbers are perfectly precise objects that exist in a certain ideal world, and his analysis of numbers was developed according to that idea. On the contrary, Dedekind was limited to indicate that all the sets of numbers (either they were in one language or another, denoted with either Arab or Chinese numerals) have the same structure, and that this structure is what characterizes the set of natural numbers. (Ferreirós, 1998, 52).

Benacerraf's work (1983) gave definitive arguments to question the set visions of natural numbers. Benacerraf concludes that numbers cannot be sets, or sets of sets, since very different presentations of the meaning and reference of numerical words, in terms of the set theory, exist. Number 3 is neither more nor less than that object that is preceded by 2 and 1 (and 0, if 0 is considered a natural number), and followed by 4, 5, etc. Or, more precisely, it is an object that is preceded by two (or three) objects in a pre-established order and followed also by infinite ordered numbers, in such a way that two elements defined as "contiguous" will always be like that. What is peculiar to 3 is that it defines that role - not for being a paradigm of any object that it plays, but to represent the relation that any third member of a progression keeps with the rest of the progression.

> "Therefore, numbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an *abstract structure* – and the distinction lies in the fact that the "elements" of the structure have no properties other than those relating them to other "elements" of the same structure" (Benacerraf, 1983, 291).

Once we are aware that, besides Indo-Arabic symbols, 1, 2, 3,..., we can use an infinite variety of "objects" (perceivable, manipulative or mental) to express the size of finite collections of other objects it should be conflicting to say that the natural numbers are, 1, 2, 3… From a philosophical and formal mathematics perspective, the coherent solution will consist of assuming that a natural number is an element of any numeral system, and the set of natural numbers is the class of numeral systems, not a particular numeral system. However, as all numeral systems are characterized by a structure or specific recursive organization (Peano's axioms, for example) we can also say that the set of natural numbers is characterized by the structure of any numeral system. Each particular number will be an element of this system.

5. ANALYSIS OF PERSONAL SEMIOTIC SYSTEMS

The theoretical tools introduced in the onto-semiotic approach (system of practices and configuration of objects and processes) can be used to describe and understand the semiotic systems formed by the students' answers given to specific mathematical tasks.

In this section we illustrated this use by analyzing the answer given by a six year old child to a task of counting and writing numbers greater that ten in the system of decimal numeration (figure 4).

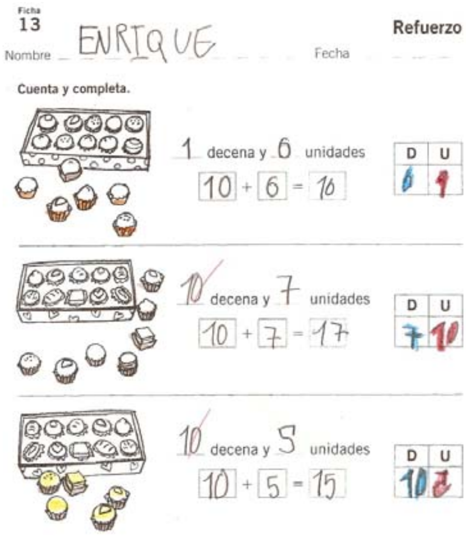


Figure 5: Counting and writing numbers greater than ten

The worksheet asks to count the number of chocolates represented in several drawings and it gives a guide to write the answer in the first task (1 ten is written, and the 6 of the units is suggested with dots that the child has to write over. We can see that the teacher has crossed out the zero of the ten that the boy has written.

5.1. Institutional and personal mathematics system of practices

The teacher expects that the child carry out the following actions:

- Reading and understanding the task;
- Counting the number of chocolates in the picture;
- Writing the results of counting in three different ways:
    - ✓ Ordinary language, 1 ten and the number of units, written in fix places.
    - ✓ Sum of the cardinal of two subsets (chocolates within and outside the box)
    - ✓ Identify the ten and the units of each result and write them in the array *ad hoc.*

D	U

In this case we consider the student's reading and solving the task as a practice. The worksheet shows an incomplete example, since it is a worksheet of "reinforcement" and the child has already done similar activities.

The child counts well, but he shows difficulties with the identification of the tens and the units. The personal meaning does not seem to differentiate between digit and number, in spite of distinguishing between the relative value and absolute value of numbers (1 ten is 10 units). In other words, the operative practice (correct) does not have correlation with the discursive practice. This inconsistency should not be evaluated solely in terms of "the student knows – or does not know".

It is supposed that writing in the table D-U (Tens – Units) requires "solely" the identification of numbers as a "symbol aggregation" and to interpret these according the conventions of the tabular language.

5.2. Interpretation and representation processes

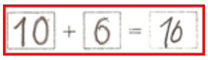
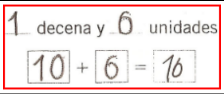
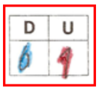
Next we include a detailed analysis of the diverse objects that intervene in the statement and expected solution of the task, and the meanings given to these objects. We use the notion of "semiotic function", relation between an antecedent object (expression, signifier) and another consequent object (content, signified), which is established by the subject that carries out the interpretation process when applying a criterion or rule of correspondence[6].

We classify the network of semiotic functions in four groups, according to the types of objects that take part like expression or antecedent: linguistic objects, concepts, procedures and propositions. The subjects' actions are in the nucleus of the criterion-rule that determine the expression-content relation, that is to say, they are consubstantial to the relational nature of mathematics and to the semiotic functions that allow the description of mathematical activity. The arguments form part of a justification of procedures and proposition. This analysis allows us to understand Enrique's difficulties in terms of the onto-semiotic complexity of the task.

Interpretation of linguistic elements

EXPRESSION (Signifier)	CONTENT (Meaning)	CRITERION/RULE
Count	- Problem: How many chocolates are there drawn in the picture? - Counting procedure - Concept of cardinal, number of items in the collections	- To find "how many are there", you have to count; - To count you have to apply the counting technique
	- Set of object to count, split in two subsets (in and out the box)	- The cardinal of the set union is the sum of the cardinal of disjoint subsets
Fill in	- Write the counting result in the empty spaces, as the sample given, applying the rules of the decimal numeration rules	- Writing rules: divide ten and units; addition in a row; tabular writing
1 ten and 6 units	- Concepts of ten and unit; - 1 ten refers to the amount of chocolates in the	- Conceptual rules - Decimal grouping

[6] Coherently with the OSA semiotic perspective we present a triadic characterization of the data, which are analyzed and systematized by a three column array.

	box; 6 units refers to amount of chocolates out the box.	principle
	- Addition operation - Concepts of addition, sums and result -Polynomial expression of a number; splitting a number in units and tens	- Procedural rule of sum; - Conceptual rules - Representational rules
	1 ten and 6 units can be written as 10+6; and also as 16	- Representational rule
	- U, refers to Unit; D, to Ten (Decena) - In the right cell each digit has its own value; in the left cell each digit has the value of ten	- Representational rules - Principle of positional value of digits

The first vignette establishes the institutional rule to follow: When we have a set with ten objects we say that it forms a ten, and a 1 is written; we call the rest of objects that do not reach ten, units, and their number is written, 6, in the position of the right.

The worksheet displays three different ways to express the same mathematical fact: "1 ten and 6 units"; "10 + 6 = 16"; and the tabular-symbolic register that remembers the positional value of each digit. These three mathematical expressions are also representing a specific situation of the daily world: the cardinal of set union of two subsets of chocolates shown by means of icons. The result of modelling the situation, 16 chocolates, is implicit; the worksheet only expresses the numerical value of the measurement, 16.

The student must understand the meaning (meaning process) of each linguistic element of the text and, mainly, he must understand the text globally. The child's accomplishment of the task shows his difficulties to apply the writing rules of the decimal numeration system in the most elementary case, as it is the writing of the ten as a unit of second order. The student does not see a ten, but ten units. The meaning of symbols D (Decena, Ten) and U (units) does not seem obvious to this student.

Concepts interpretation

EXPRESSION (Signifier)	CONTENT (Meaning)	CRITERION/RULE
Number of elements	- The size of the three sets of chocolates grouped in tens and units: sixteen, seventeen, fifteen	Implicit definition
Ten	- Collection of ten chocolates considered as a unit (complete box) - Second position in the positional decimal writing	- The ten as a container - Writing algorithm of two digit numbers
Unit	- Objects not included in the decimal grouping - First position in the positional decimal writing	- The unit as "remaining" elements -Writing algorithm of two digit numbers
Addition	- Grouping the chocolates inside and outside the box; go on counting	Fix order of the number series
Equality	- Result of the sum operation	The operation and its result are related with the "=" sign

The child counts the number of objects within the box well and knows that ten is written 10, but he does not understand the role (meaning) that the 0 and the 1 have in this writing. Enrique's answers to the requested tasks essentially show the complexity of the notion of unit (of first order) and ten (units of second order): a collection of ten units (chocolates) must be seen unitarily like a new unit, and not like ten units. Likewise, the sample of the activity does not allow us to determine what the meaning that Enrique gives to the rule is, "to add one digit number to 10, it is enough to replace the 0 (of the 10) by this number". In fact, since to the question "how many tens are there", Enrique puts "10", the previous task represents a mere game of symbols for this child, without reference to the ten like "grouping of units".

Interpretation of procedures

Procedures and propositions suppose a "higher level" of mathematical connection. This fact, within the OSA, is taken into account considering that the criterion – rule of the semiotic functions in which the procedures participate require arguments that justify its use and it makes the sentence in which they are inserted, coherent.

PROCEDURE (Antecedent)	USE (Consequent)	CRITERION/RULE (Justification)
Counting technique of the number of elements of a collection	It is used to find the size, o number of elements inside and outside the box	The condition of application are fulfilled (finite collection of objects)
Writing the numbers in the positional system, separating units and tens	Writing 16, 17 and 15	They are numbers greater than the numeration base, and there is an algorithm to write these numbers
Addition operation	It is used to find the total number of chocolates, inside and outside the box	The condition "disjoint collections" is fulfilled

Enrique counts the collections of objects well, but he makes mistakes in the procedure of positional writing of numbers.

Interpretation of propositions

PROPOSITION (Antecedent)	USE (Consequent)	CRITERION/RULE (Justification)
Counting principle: the cardinal number is the ordinal number of the last counted element, …	It is used in the procedure of counting the collections	The condition of application are fulfilled (finite collection of objects)
The cardinal of the union of two disjoint sets is the sum of the cardinal of each set	It is used to calculate the total (16, 17, 15)	The condition "disjoint collections" is fulfilled
There are 16 (17, 15) chocolates	They are the answers to the questions posed	Empirical checking (there are 10 objects in the box and 7 outside, hence I write 10 and 7 according to the model)

Enrique applies the counting principles well because he is able to find the number of chocolates.

5.3. Generalization and particularization processes

In this exercise, the chocolates are intended to be used as generic objects, that is to say, we try to make the students generalize (for the first problem) that 10 objects + 6 objects are 16 objects. With the sequence of the three exercises he should know that, if a ten of objects is joined with a number of objects less than ten, the result is a number of objects equal to 1 followed by the number that represents the units. The student should also learn the general rule of the positional numeration systems for numbers of two digits: "Ten simple units, or first order units, form a unit of higher order and is written to the left like a new unit of higher order". We can observe that the student does not understand this rule.

5.4. Idealization and materialization processes

In this task there is an implicit process of idealisation since, after counting, we have only the empirical evidence that, 10 chocolates + 6 chocolates are 16 chocolates. This is the physical operation of grouping objects. However, when writing the numbers, the chocolates disappear and it is concluded that 10 + 6 are 16, that is to say, we have changed from an operation with physical objects to a mathematical operation with numbers. This process of idealisation is combined with the generalization process previously described,

Chocolates → Any objects → Numbers

The concept of ten is materialized first with a box that contains ten chocolates and later by means of three different notations: 1 ten, 10, "second position to the left" in the writing of numbers. The unit idea is also materialized first by one chocolate and the number of units like a set of chocolates "without container", this number is also materialized with the notation "6" and by the right cell in the table that divides the number into tens and units. The sum is materialized with the fact that there are two disjoint collections (the chocolates inside and outside the box).

5.5. Reification and splitting processes

The reification process of the ten should be achieved first by the presentation of the collection of objects to count divided in two subsets: the box of ten chocolates, and the rest outside the box. While for the ten, the container-contained scheme should be applied, so that the students understand the ten chocolates as a unit of higher order, this scheme is excluded explicitly in the case of the units. This reification is later reinforced with the writing "1 ten…." and by the use of the table that divides the number in tens and units.

The main conflict observed in Enrique's answer is that he does not conceive the 10 objects as a unit (of higher order), he writes 10 instead of a 1 in the cell of the left that separates the number in tens and units. Enrique does not interpret the writing 10 assigned to a collection of ten objects in terms of 0 units of first order and 1 of second order. This fact allows us to affirm that the child has not reified the ten chocolates in one ten of chocolates.

6. FINAL REFLECTIONS

In this paper we have shown that the notions of system of practices, configuration of objects and processes, semiotic function, and semiotic conflict permit the implementation of several levels of detailed analyses for mathematical activity, and consequently get new explanations of phenomena regarding teaching and learning. It is possible to formulate criteria of epistemic, cognitive and interactional suitability in semiotic terms, which reinforces the relevance of the semiotic perspective in mathematics education.

In the case of numbers and Arithmetics, which we have analyzed, the onto-semiotic approach permits the description of the diverse elements that characterize the institutional meaning of numbers (understood as pairs of practices and configurations of objects and processes), and to explain the conflicts of learning in terms of the complexity of objects and meaning involved.

The analysis of the child's answer to the school task of writing numbers greater than ten has allowed the systematic and structured study of the rules involved in the use of numbers: rules of linguistic, conceptual, procedural, propositional and argumentative nature, as well as the associated processes of generalization, idealization and reification. The personal - institutional duality focuses on the analysis from the point of view of teaching and learning, identifying the conflicts between the meaning that the student constructs and the intended institutional meaning. This type of analysis helps to be aware of the complexity of knowledge called on and the difficulty of developing the operative and discursive competences on natural numbers

The emphasis on the ontological aspects that the OSA proposes is compatible with the socio-constructivist and anthropological assumptions taken as starting postulates. The object, considered as coming from a system of practices, can be considered as unique and with a holistic meaning (Wilhelmi, Godino and Lacasta, 2007). However, in each sub-system of practices, the configuration of objects and processes in which the object at issue "appears" is different, and, therefore, different practices are made possible. The systems of practices can be divided up into different classes of more specific practices, made possible by a certain configuration of objects and processes, allowing the distinction between meaning and sense: the senses can be interpreted as partial meanings. This point of view for mathematical objects is closely related to Ernest's position (1998, p. 261); he considers that the social constructivism adopts an approach to mathematical objects that can be described as nominalist, when considering them as objects of conceptual/ linguistic nature.

In the OSA, contrary to a traditional realistic position on the nature and ontological status of mathematical objects - that locates it in the abstract and intangible world of the Forms (Plato) or the World 3 (Popper), or even directly in the empirical world (Maddy, 1990) - we locate it in language games and cultural space of mathematics. The contextual dualities included in Figure 3 (personal - institutional, ostensive – non ostensive, personal - institutional, extensive - intensive), and the analysis of how they become apparent in professional and school mathematical languages allow to explain how the language games lead to conferring a certain type of existence to mathematical objects.

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