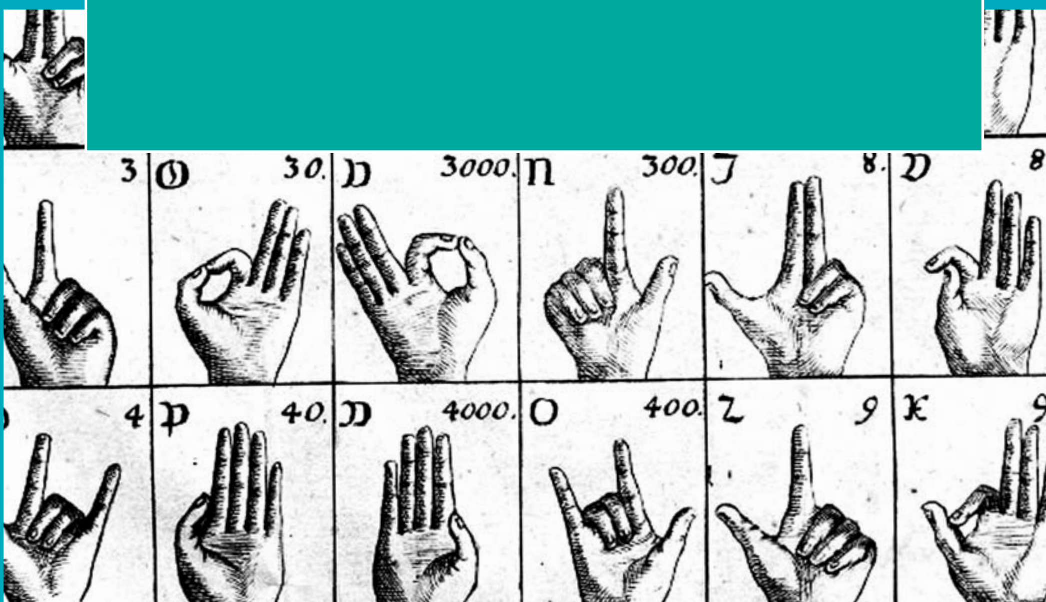
SEMIOTIC PERSPECTIVES IN THE TEACHING AND LEARNING OF MATHEMATICS SERIES

Semiotics in Mathematics Education

Epistemology, History, Classroom, and Culture

Luis Radford, Gert Schubring, and Falk Seeger (Eds.)



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Volume 1

Semiotics in Mathematics Education

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VICENÇ FONT, JUAN D. GODINO AND ANGEL CONTRERAS

FROM REPRESENTATIONS TO ONTO-SEMIOTIC CONFIGURATIONS IN ANALYSING MATHEMATICS TEACHING AND LEARNING PROCESSES

Introduction

In this chapter, we reflect on the key role of semiotic representations in constructing and communicating mathematical knowledge, since they are essential components of mathematical practices. However, other mathematical entities, such as concepts, propositions, procedures and arguments intervene in these practices, in addition to problem-situations whose resolution is the ultimate goal of mathematical activity. Therefore, the socio-epistemic and cognitive analyses of the teaching and learning processes should systematically include the conglomerate of ostensive and non-ostensive objects involved in these processes. In this paper, we present the construct, *onto-semiotic configuration*, which generalizes the notion of representation and moves the focus of research towards the system of objects intervening in and emerging from the mathematical activity. In the third section of this chapter, we include a synthesis of the onto-semiotic approach to mathematics education research developed in several previous publications (D'Amore, Font & Godino, 2007; Godino & Batanero, 1998; Godino, Batanero & Font, 2007; Godino, Batanero & Roa, 2005).

SEMIOTIC REPRESENTATIONS, MATHEMATICAL OBJECTS AND PROCESSES

The essential role played in mathematical activity by different representation registers and, more generally, by material and symbolic "artifacts" is widely recognized regardless of whether the activity is carried out by mathematicians when solving new mathematical problems, or if it concerns the diffusion of mathematical knowledge, that is, the teaching and learning processes. The important role that representations play in learning mathematics can explain the wide number of investigations focusing on this topic in mathematics education research (Goldin, 2002).

Different theoretical frameworks assign a representational role (referring to other entities intervening in mathematical practices), as well as an instrumental role (tools for undertaking the mathematical work) to various types of languages (ordinary, symbolic, graphical, gestural, …). "The part played by signs, or more exactly by semiotic systems of representation, is not only to designate

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mathematical objects or to communicate but also to work on mathematical objects and with them" (Duval, 2006, p. 107).

It is also recognized that mathematics should not be reduced to a language and that mathematical objects should not be confused with any of their possible semiotic representations. Therefore, the socio-epistemic and cognitive analysis of mathematical activity should study the role played by material representations, as well as the role and meaning assigned to the system of non-ostensive objects that intervene and emerge from that activity. This is one of the main objectives assumed by the *"onto-semiotic approach"* to mathematical knowledge and instruction (Godino, 2002; Godino & Batanero, 1998; Godino, Batanero & Font, 2007), where the notion of *configuration* of objects and processes is a tool for jointly analyzing the ostensive and non-ostensive objects that intervene in mathematical practice.

Drawing on previous research (Font & Godino, 2006; Font, Godino & D'Amore, 2007; Godino, Font & Wilhelmi, 2006), in this chapter we introduce the notion of onto-semiotic configuration. An example of application shows its potential utility for overcoming some limitations of the idea of representation in the epistemic and cognitive analyses of mathematical learning.

We first describe a teaching episode that will be used as a context for reflection and a synthesis of some basic notions of the onto-semiotic approach. Secondly, three levels of analysis will be applied to the teaching episode: (1) the identification of representations and the practices they make possible; (2) the description of mathematical configurations; and (3) the description of the socio-epistemic and cognitive processes involved in mathematical activity. The chapter finishes with a synthesis and some conclusions.

A TEACHING EPISODE AS A CONTEXT FOR REFLECTION

As a context for reflection, we shall use a teaching episode in which a group of students (17 years of age) were studying the derivative concept. This teaching episode has been selected from an ethnographic observation carried out by the researchers in ordinary mathematics classrooms, that is, without any influence in the design of such lessons.

Task 1, which follows, was given to the students to solve with the help of the "Cabri" software:

Task 1:

If you move point F in figure 1, you will observe that the parabola $f(x) = x^2$ and the tangent line at point P are represented.

– a) Find the relationship between the segments GO and PF.
– b) Find the relationship between the segments PH and FH.

The students solved this task by working in pairs at the computer lab, where they had figure 1 on the screen. They did not have previous knowledge of the Cabri software. Figure 1 was a construction that changed in real time when the students moved point F (the sole point that could be moved with the mouse). By moving point F and activating the *trace* option, the parabola $f(x) = x^2$ and the tangent line at point P were drawn. The teacher asked the students to find some invariant relations between the segments GO and PF, and also between PH and FH. After a period of time, the following properties were recognized and shared in the classroom: (1) in the parabola $f(x) = x^2$ the tangent line at P cuts the axis of ordinates at a point such that the length of the segment that has for extremes this point and the origin of the coordinates is the ordinate of P, (2) the length of segment PH is twice the length of segment PF.

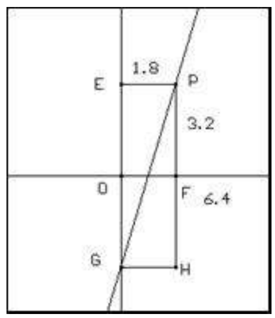

Figure 1. Task 1

They were then asked to use this property to carry out Task 2, as follows:

Task 2:

- a) If $OF = a$, prove that $GH = a$, $PF = a^2$ and $PH = 2a^2$.
- b) If the derivative of the function at a point is the slope of the tangent line, calculate $f'(a)$.
- c) Prove that the derivative of the function $f(x) = x^2$ is $f'(x) = 2x$.

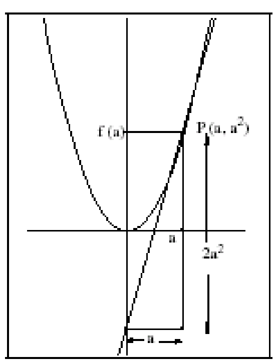

Figure 2. Task 2

The answer given by a student to task 2 is included in Figure 3:

a) $GH = a$ because it is the same distance

$PF = a^2$ *because* the image of a in the function $f(x) = x^2$ is a^2

$PH = 2a^2$ because it is double FP

b) $p = \frac{2a^{\cancel{2}}}{\cancel{a}} = 2a$

$f'(a) = 2a$

c) $a = x$

$p = \frac{2x^{\cancel{2}}}{\cancel{x}} = 2x$

$f'(x) = 2x$

Figure 3. Student's answer

We observe that, with the small letter "*p*" the student indicates the slope of the tangent line.

THE ONTO-SEMIOTIC APPROACH

The onto-semiotic approach to mathematical cognition tackles the problem of meaning and the representation of knowledge by elaborating an explicit mathematical ontology based on anthropological (Bloor, 1983; Chevallard, 1992), semiotic and socio-cultural theoretical frameworks (Ernest, 1998; Presmeg, 1998; Radford, 2006; Sfard, 2000). It assumes a certain socio-epistemic relativity (Cantoral, Farfán, Lezama & Martínez-Sierra, 2006) for mathematical knowledge, since knowledge is considered to be indissolubly linked to the activity in which the subject is involved and is dependent on the institutions and the social context of which it forms a part (Radford, 1997).

In Figure 4, we represent some of the different theoretical notions of the onto-semiotic approach for mathematical knowledge. Here, mathematical activity plays a central role and is modelled in terms of systems of operative and discursive practices. The different types of mathematical objects (problems, languages, concepts, propositions, procedures and arguments) emerge from these practices; these objects are interrelated, forming cognitive or epistemic configurations (hexagon in Figure 1). Lastly, the objects that appear in mathematical practices and those emerging from these practices, depending on the language game in which they participate (Wittgenstein, 1953), might be considered from the five facets of dual dimensions (decagon in Figure 1): personal/institutional, unitary/systemic, expression/content, ostensive/non-ostensive and extensive/intensive. The dualities, as well as the objects can be analysed from a process-product perspective, which leads us to the processes in Figure 4.

The six types of postulated primary objects widen the traditional distinction between conceptual and procedural entities, which we consider insufficient for describing the objects intervening and emerging from mathematical activity. The problem-situations promote and contextualise the activity; languages (symbols, notations, graphics, …) represent the other entities and serve as tools for action; arguments justify the procedures and properties that relate the concepts. These entities have to be considered as functional and relative to the language game (institutional frameworks and use contexts) in which they participate; they also have a recursive character, in the sense that each object may be composed by other entities, depending on the analysis level—for example, arguments, which may involve concepts, properties, operations, etc. The concept, included as a component of the onto-semiotic configurations, is conceived as "concept – definition"; this view is different from that of Vergnaud (1990), who conceives of a concept as the system formed by the "situations, operative invariants and representations".

In the onto-semiotic approach, the intention is not to give a definition of "process" from the beginning, as there are many different types of processes: one can talk of process as a sequence of practices, as cognitive processes, meta-cognitive processes, processes of instruction, processes of change, social processes, etc. These are very different processes and perhaps the only characteristic many of them may have in common is the consideration of the "time" factor and, to a lesser degree, the sequence in which each member takes part in the determination of the following. For this reason, in the onto-semiotic approach, instead of giving a general definition of the process, the selection of a list of processes considered important in mathematical activity is opted for instead (those of Figure 4), without claiming that it includes all the processes implicit in mathematical activity, because, among other reasons, some of the most important of them (for example, the process of understanding, the solving of problems or modelling) are more than just processes and should be considered hyper or mega-processes.

Meaning is a key notion in mathematics education and also in the onto-semiotic approach, where it is conceived of in a very simple, powerful and operative way by means of the "semiotic function" (Eco, 1978; Hjelmslev, 1943/1963): *Meaning is the content of any semiotic function*, that is to say, the content of the correspondences (relations of dependence or function) between an antecedent (expression, signifier) and a consequent (content, signified or meaning), established by a subject (person or institution) according to distinct criteria or a corresponding code. The content of the semiotic functions, and hence the meaning, could be a personal or institutional, unitary or systemic, ostensive or non-ostensive object; it could be a concept – definition, problem –situation, procedure, argument, or a linguistic element. In agreement with Peirce's semiotics, the onto-semiotic approach also assumes that both the expression (antecedent of a semiotic function) and content (consequent) might be any type of entity.

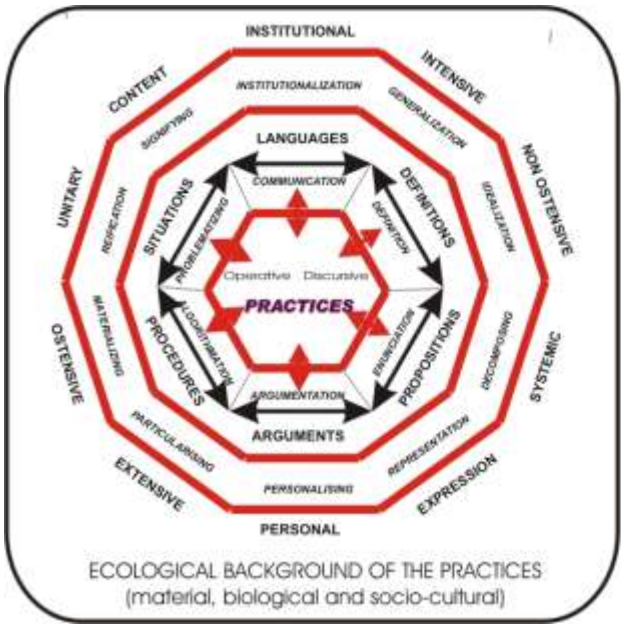

Figure 4. Onto-semiotic configuration for mathematical knowledge

Due to space limitations, we cannot present all the details of the onto-semiotic approach in this chapter. We refer the reader to Godino & Batanero (1998) where the anthropological assumptions are made explicit and to Godino, Batanero & Roa (2005) where the different types of mathematical objects and the dual facets are explained using research on combinatorial reasoning. In Font & Contreras (in press) the materialization-idealization, and particularization–generalization processes are developed and explained.

ANALYSIS OF THE TEACHING EPISODE

In this section, we suggest that the classical dichotomy between external and internal representations is inadequate for analysing the complexity of the epistemological and cognitive phenomena involved in the learning of mathematics, and that the onto-semiotic configuration (Figure 4) allows for a wider and deeper view of such phenomena. Our argument is that the external representations (symbolic, graphical, linguistic, ostensive objects) are inevitable and dialectically accompanied by other non-ostensive mathematical objects and processes. The use of internal representations (conceptions, schemas, competencies) focuses the attention on the individual subject, by dismissing the social context where such non-ostensive objects emerge and are conditioned.

We will apply three levels of analysis to the teaching episode (described in the first section) to show the potential utility of the onto-semiotic approach to mathematical knowledge.

In the first level of analysis, we focus our attention on the linguistic elements (external representations, ostensive objects) and the mathematical practices that these objects make possible. But even in this first step, it is necessary to be aware of the dialectic between ostensive and non-ostensive objects, and the dialectic between the personal (cognitive, internal representations) and the institutional (socio-epistemic) facets. These new categories of analysis are not reducible to the external – internal duality of mathematical knowledge. In a second step, we complete the first level of analysis by making explicit the conglomerate of primary mathematical objects linked to the external representations, and the role played by each of these objects in mathematical teaching and learning processes. The third level of analysis shows new socio-epistemic and cognitive phenomena by focussing attention on the other contextual facets or attributes of mathematical objects and processes. In this chapter, due to space limitations, we restrict the analysis to the materialization– idealization processes (dialectic between ostensive and non-ostensive objects); and particularization–generalization processes (dialectic between extensive and intensive objects) (Figure 4).

Our aim is to show that the application of the onto-semiotic configuration tool offers new analytical insights into the teaching of mathematics that will potentially provide new explanations for the students' difficulties and learning achievements.

First Level of Analysis: Ostensive –Non-Ostensive; Personal - Institutional

It is clear that those representations described in mathematics education bibliography as external (graphics, symbolic expression, etc.) intervene in the teaching episode. Moreover, one student's answer suggests the existence of internal representations related to her answers, usually referred to as conceptions, schemas, competencies, … So, we can say that the student's conception (Figure 3) of the derivative of a function is the slope of the tangent line to the curve, and that she shows a certain competency in algebraic calculations.

In the onto-semiotic approach, the internal/external classification, in addition to being problematic (Kaput, 1998), is considered to be not very operative, and so, we propose converting it into two dualities or contextual attributes, which, in our opinion are more useful. We are referring to the ostensive non-ostensive and personal-institutional dualities. External representations are ostensive (perceptible) objects and internal representations are non-ostensive objects; nevertheless, not all ostensive objects can be considered as internal representations of individual subjects. Mathematical objects, viewed as cultural entities, cannot be reduced to the material representations used in their generation and communication. Moreover, replacing "external – internal representation" by the expressions "ostensive object", "non ostensive object", is not incidental, because these entities have other roles beyond that of representation.

We feel that the internal/external duality does not explain the institutional dimension of mathematics knowledge, thus confusing, to a certain extent, the said objects with the ostensive resources that are used as support for the creation or emergence of institutional entities. The internal/external duality has serious consequences for understanding learning processes, since the role of human activity and social interaction are not adequately modelled in the production of mathematics knowledge and in learning.

The analysis of the students' responses to Task 2 permits us to suppose relevant differences between the mental processes that "occur" in the mind of each student. At this point, we consider it necessary to take the personal-institutional duality into account before first reflecting on mental processes. It is not enough to reflect on the cognitive processes that have (or have not) permitted these students to answer the questions in the two tasks by carrying out a conversion from a graphic representation to a symbolic representation, when they still did not know what the derivative function of $f(x) = x^2$ was. It is necessary to take into account, above all, the *process of instruction* that these students have followed, if we wish to give an explanation of the achieved learning.

The analysis of the responses, even when it detects important differences between different students, enables us to observe that the students apply the same type of practice to calculating the derivative of the function $f(x) = x^2$. The technique used consists of considering a particular point with the tangent drawn (and so its abscissa and ordinate are not considered to be variables). Then, with the manipulation of dynamic computer programs, like *Cabri Géomètre*, the students find a condition that fulfils all the tangent straight lines (in this case: in the parabola f(x) = x2 the tangent line at P cuts the axis of ordinates at a point such that the length of the segment that has for extremes this point and the origin of the coordinates is the ordinate of *P*), and this permits the calculation of its slope. Finally, students should recognize that the condition they have found and the calculation of the slope from which it is obtained are valid for any point, so the point, which was initially considered as a particular point, is then considered as any point.

In order to answer Task 2, in addition to using the graph of the function, the symbolic expression of the graph, $f(x) = x^2$, should be used. So, this technique relates the following ostensive objects:

Graph of $f(x)$ and symbolic expression of $f(x) \Rightarrow$ Symbolic expression $f'(x)$

With this scheme, we symbolise that the starting point of the students' actions for finding a condition that all tangents fulfil is the graph of the function. The symbolic expression of $f(x)$ is necessary for symbolising the condition that fulfils all the slopes of the tangent straight lines, which enables us to deduce the symbolic expression of $f'(x)$. If the students have practiced the calculation of the slope and the geometric meaning of the derivate at a point, they may obtain the symbolic expression of $f'(x)$ without much difficulty.

One of the relevant aspects of framing the representations within the process of instruction is that it enables us to know that, in this instruction process, the teacher opted to include the institutional intended meaning, and also the implemented and the evaluated meanings, along with practices that form part of the historic-epistemological evolution of the derivative object.

The teacher proposes a sequence of activities to the students that do not correspond to the tangent problem or its inverse. They neither deal with the tangent problem, because the tangent has already been constructed, nor with the inverse tangent problem, because the symbolic expression of the function was known. The use of computers facilitates the students' actions and allows them to find the condition fulfilled by all the tangents (using the triangle formed by the ordinate, the tangent and the sub-tangent). These types of constructions help students compute the derivative of functions without using limits, whenever the students have previously studied the geometrical interpretation of the derivative at a point. This method is suggested by the procedure used to construct the tangent and the normal in the period from Descartes to Barrow.

Second Level of Analysis: Configurations of Primary Mathematical Objects

Identifying the diverse objects that intervene in mathematical practices is essential for understanding the semiotic complexity involved and explaining the learning difficulties. Linguistic elements represent others mathematical entities. But problem-situations make conceptual and procedural entities meaningful; arguments justify the propositions and procedures; definitions and propositions underlay the procedures and arguments. In sum, mathematical activity should be described through the network of objects and relationships involved in solving the problems that motivate such activity. "Mathematical activity is essentially directed towards the exploration, construction and analysis of *conceptual relations* and of *systems of relations*. The main objects of theoretical mathematics are relations" (Steinbring, 2007, p. 104). We describe this network of objects and relations as a configuration, which, in the case of computing the derivative of the function $f(x) = x^2$ in the teaching episode, is summarized in Table 1.

Table 1: Socio-epistemic configuration of primary objects to prove $f'(x) = 2x$

LANGUAGES
Verbal: function, derivative *Graphical:* Graph of the function with the tangent line at a point *Symbolic:* $OF = a$, $GH = a$, $PF = a^2$, $PH = 2a^2$, $f'(a)$, $x = a$, (a, a^2), $f(x) = x^2$, $f'(x) = 2x$, ...

PROBLEM - SITUATION	CONCEPTS-DEFINITIONS
- It is an internal mathematics problem: to prove that the derivative of $f(x) = x^2$ is $f'(x) = 2x$.	*Previous* - Graph of a function, coordinate of a point, abscise, ordinate, image, quadratic function, derivative at a point, derivative function, tangent, tangency point, slope, etc. *Emergents*

	- Derivative of the function $f(x) = x^2$.
PROCEDURES	PROPOSITIONS
Previous - Calculation of the slope. *Emergents* 1) Find a condition that fulfills all the tangent lines. 2) Symbolize that condition. 3) Find $f\,'(x)$ from that condition.	*Previous* - The second coordinate of the tangency point is obtained by substituting x by a in the function formula. ... *Emergents* Proposition 1: The tangent line of the function $f(x) = x^2$ at P cuts the axis of ordinates at a point such that the length of the segment that has for extremes this point and the origin of the coordinates is the ordinate of P. Proposition 2: If $OF = a$, then $GH = a$, $PF = a^2$, and $PH = 2a^2$. Proposition 3: The derivative of the function $f(x) = x^2$ is $f\,'(x) = 2x$.

ARGUMENTS
Argument for proposition 1: It is visually observed. Arguments for proposition 2: - Argument 1: GH = OF because they are parallel sides in the rectangle OGHF. - Argument 2: PF = a^2 because the image of a in the function $f(x) = x^2$ is a^2 - Argument 3: FH = OG because they are parallel sides in the rectangle OGHF; OG = PF (Task 1, a). Arguments for proposition 3: The derivative function of $f(x) = x^2$ is the function $f\,'(x) = 2x$. Deductive proof in which the following arguments are used: - Argument 1: The tangent line at P cuts the axis of ordinates at a point such that the length of the segment that has for extremes this point and the origin of the coordinates is the ordinate of P. - Argument 2: (a, a^2) is the tangency point. - Argument 3: The slope of the tangent line is $2a$. - Argument 4: The derivative of $f\,'(a)$ is $2a$ because the derivative is the slope of the tangent line. - Argument 5: The statement about the point (a, a^2) is valid for any other point.

The configuration of objects in Table 1 shows how the problem-situation used in the teaching episode, is the tip of an iceberg that also includes different types of representations, procedures and other mathematical objects. Tasks 1 and 2 place the derivative of $f(x) = x^2$ within a specific context where it is not necessary to use the definition of derivative as the limit of mean variation rates.

Table 1 presents the configuration of primary mathematical objects that the teacher intends to construct in the classroom through a guided teaching process, when solving Tasks 1 and 2 (proving that the derivative of the function $f(x) = x^2$ is $f'(x) = 2x$). We term this conglomerate of objects a "*socio-epistemic configuration*", because it refers to the system of objects and meanings that the teacher wants students to share in the community of practices formed in the classroom. The fact that the mathematical activity was directed at high school students and was based on a specific computer software program determine specific connotations for the linguistic elements, procedures, properties, concepts and arguments, which could be different in other institutional contexts. Put in another words, we recognize some socio-epistemic relativity for mathematical knowledge.

If we apply the onto-semiotic analysis to the student's answer shown in Figure 3, we observe some relevant differences, in particular, in the justification of propositions 2 and 3. That is, the student's *cognitive configuration* shows some agreement with the socio-epistemic configuration, but also some differences, which the teacher should recognize in assessing the achieved learning and making decisions for future implementations of the teaching process.

Third Level of Analysis: Socio-Epistemic and Cognitive Processes

In the final level of analysis, we will focus our attention on the dialectic involved in two pairs of the processes included in the onto-semiotic configuration summarized in Figure 4: materialization–idealization (a complementary look at the ostensive – non-ostensive duality), and the particularization–generalization processes (the extensive–intensive duality).

Processes of materialization – idealization

The ostensive – non-ostensive duality, described in the first level of analysis regarding the dilemma between external and internal representations, permits a complementary insight into the materialization and idealization processes that intervene in building mathematical knowledge. This duality can be applied to each primary entity and, therefore, the materialization – idealization processes take place for the problem – situations, procedures, definitions, etc. We conceive idealization as the generation of non-ostensive objects, while materialization is the linking of ostensive (perceptible) to non-ostensive objects. In both processes, the entities are duplicated, but some students could not recognize this duplication and confused mathematical objects (non-ostensive idealizations) with their related representations (materializations). The distinction between ostensive and non-ostensive is relative to the language game in which they take part. Ostensive objects can also be thought, imagined by a subject or be implicit in the mathematical discourse (for example, the multiplication sign in algebraic notation).

Plato was one of the first thinker to emphasize the relevance of the idealization process, by considering the objects of experience as imperfect copies of mathematical ideas. Since then, the need to take into account the process of idealization in mathematical activity has been noted by many other thinkers. For example, Fischbein (1993) gave a main role to idealization processes in his theory of figural concepts. The idealization process is also emphasized by Kitcher (1984), who assumed empirical and pragmatic origins for mathematics, and adopted a constructivist position by considering mathematics as a science of idealized operations that people are able to carry out on any kind of object. Another example comes from the research program of "embodied cognition" (Lakoff & Núñez, 2000), where a key issue is investigating the way people generate mathematical ideas.

We can recognize processes of idealization and materialization in Tasks 1 and 2 (Figure 5) because the figures (drawn on a sheet of paper or on the computer screen) are materializations of the mathematical objects "graph of the function $f(x) = x^2$" and "tangent line to the function $f(x) = x^2$ at a point".

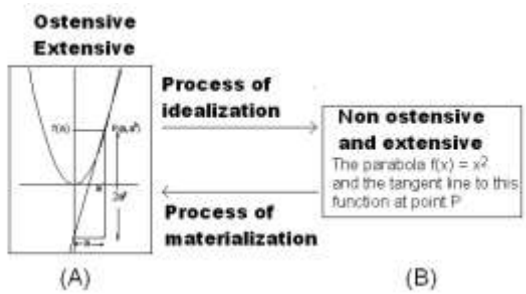

Figure 5. Processes of idealization and materialization

The teacher and students talk about Figure 5(A) as if it were the parabola $f(x) = x^2$ and the tangent line to this function at point *P*. If we look carefully at Figure 5(A) one observes that: (1) the "straight line" is not a straight line, (2) the "straight line" is not the tangent line, (3) the graphic is not a parabola, etc.

It is clear that the teacher hopes the students will go through the same process of idealization of Figure 5(A) drawn on the sheet of paper as he has done. That is to say, Figure 5(A) is an ideal figure, explicitly or implicitly, for the type of discourse the teacher and students makes about it. Figure 5(A), drawn on the sheet of paper, is concrete and ostensive (in the sense that it is drawn with ink and is observable by anyone who is in the classroom) and, as a result of the process of idealization, one has a non-ostensive object (the parabola $f(x) = x^2$ and the tangent line to this function at point *P*) in the sense that one supposes it is a mathematical object that cannot be presented directly. On the other hand, this non-ostensive object is particular, that is, it is the parabola $f(x) = x^2$ and the tangent line to this function at point *P*, and it is not, for example, the parabola $f(x) = 2x^2$ and the tangent line to this function at point *S*. In the onto-semiotic approach, we call this type of "individualized" object an extensive object. Therefore, as a result of the process of idealization, we have moved from an ostensive, which was extensive, to a non-ostensive that continues to be an extensive object.

The other side of the coin is that, to be able to manipulate non-ostensive objects, we need ostensive representations which are the result of a process of materialization (and also of representation). The process of materialization places mathematical knowledge in the "territory of the artefact", (Radford, 2006, p.107), since its products are cultural artefacts that influence and materialize thought.

Processes of particularization and generalization

The generalization processes and the objects emerging from these processes (generalizations) are essential and characteristic of mathematical work. But, these processes are dialectically linked with the respective particular objects that provide meaning to generalizations. The use of examples and particular cases should be the starting point for a meaningful teaching of mathematics.

In the integrative theoretical framework we intend to build, this characteristic of mathematical work can be described and analysed through the extensive – intensive duality (Figure 4). This duality is applied to the different primary entities: a problem, a definition, a procedure, etc., and can be a particular case of a more general problem, definition, procedure, etc. Moreover, the analysis of mathematical practices from the particularization – generalization processes can be complemented with the application of the other contextual dualities, in particular the expression – content (semiotic function) duality, as we will show in this section.

An extensive object is used as a particular case (a specific example, i.e., the function $y = 2x + 1$), of a more general class (i.e., the family of functions $y = mx + n$), which is an intensive object. The terms extensive and intensive are suggested by the two ways of defining a set, by extension (an extensive is one of the members of the set) and by intension (all the elements are considered at the same time). By extensive we understand a particularized object (individualized) and by intensive, a class or set of objects.

The introduction of the extensive/intensive and the expression/content dualities in the onto-semiotic approach can help to clarify the problem of the use of generic elements (Contreras, Font, Luque & Ordóñez, 2005). Expressed differently, the use of the generic element is associated with a complex network of semiotic functions (and therefore, representations) that relate intensive objects with extensive ones. We will show this with the example of the student's response included in Figure 3.

In writing up the questions for Task 2, much attention was paid to the step from the particular to the general. In item *b*) the teacher asks the students to compute the derivative at a particular value "a", and in item *c*), for any value. We assume that the passage from the extensive to intensive has been carefully considered in the design of the tasks. In this problem, the extensive objects "represent" the intensive objects and, hence, the student has to carry out processes of representation and meaning using semiotic functions.

Without entering into a detailed analysis, as is carried out in Contreras, Font, Luque and Ordóñez (2005), in order to calculate the derivative function of $f(x) = x^2$, the students have to identify the following network of semiotic functions:

> Treat separately the variables related by the formula and the graph of $f(x) = x^2$. To do this, it is necessary to understand this function as a process in which other objects, one being x and the other being $f(x)$, intervene. Here, a semiotic function that relates the object $f(x)$ to the object x, is established.
>
> Associate x to the slope of the tangent line at the point of abscise x. This relation can be considered as a semiotic function that relates the object x with the object "slope of the tangent line at the point of abscise x".
>
> Associate the expression that permits us to calculate the slope of the tangent line at the point of abscise x with $f'(x)$. In this case, we have a semiotic function that relates one notation with another different, but equivalent, one.
>
> Consider x as a variable. In this case, we have a semiotic function that relates an object to the class it belongs to.
>
> Understand the function obtained as a particular case of the "derivative function" class. In this case, we have a semiotic function that relates an object to the class it belongs to.

If we look at the Task 2 handed out to the students, we can observe that the sequence of sections aims at making the establishment of these semiotic functions easier. The use of the letter *a*, in question *b* of Task 2, has the role of introducing a specific element into the students' reasoning and so makes step 1, easier. The reason for including the use of the graph and the symbolic notation together for the point of coordinates (a, a^2) is that the teacher wants the students to carry out steps 2 and 3. Steps 4 and 5 are intended to be achieved from question *c*.

This example permits us to shed light on a phenomenon that we consider to be very relevant: the student, in order to carry out the majority of mathematical practices, has to activate a network of complex semiotic functions and the ostensive objects used are determinant, both for reducing or increasing the complexity of this network, or for carrying out the practice correctly. For example, if we had eliminated question *b* and Figure 2 in Task 2, we would still have wanted the students to apply the technique for calculating the derivative function and we would still use graphs (the ones from the previous activity with the computer) and symbolic expressions (question *c*). However, the complexity of the semiotic functions that the students would have had to carry out would have increased considerably and so too the chances of solving the task.

When we use a representation as a generic element in mathematical practices, we are acting on a specific object, but we are situated in a "language game" where we are interested in the object's general characteristics and we disregard its particular aspects. The analysis of the dialogues between teachers and students as regards the use of generic elements is necessary for knowing the details of this language game and the students' difficulties when taking part in it. The knowing and understanding of the rules of this language game are fundamental to making up the network of semiotic functions associated with the practices in which the generic element intervenes.

If we consider the student's answer to Task 2, we can observe that the student is aware of the rules of using the generic element, since she takes into account the calculation of the derivative function. We see that, in the answer to section *c)*, the equivalence "$a = x$", is expressing that the reasoning of sections *a)* and *b)* is valid for any value of *a*. This indicates that the student has entered into the language game that governs the use of generic elements.

SYNTESIS AND CONCLUSIONS

In this chapter, we have seen how the analysis of a teaching episode based only on representations is insufficient for including the many aspects involved in mathematical activity. Moving from analysis in terms of representations to analysis in terms of onto-semiotic configurations (mathematical objects and processes linked to mathematical practices) is necessary for obtaining a better understanding of the complexities of the mathematics teaching and learning processes.

Certainly, representation and interpretation processes are crucial, and should be the focus of attention at a first level of analysis. The figures and algebraic notations used in Tasks 1 and 2 include a complex network of semiotic functions; without these semiotic functions, the mathematical work would be impossible or very difficult to carry out. Nevertheless, a socio-epistemic and cognitive analysis that casts light on the conflicts in teaching and learning requires a systematic look toward the diverse type of objects and processes intervening in the activity. At a second level of analysis, we should focus on the configurations of primary objects (languages, problems, definitions, propositions, procedures and argumentations) and the related primary processes. The third level of socio-epistemic and cognitive analysis should be centered on the contextual attributes and secondary associated processes: personalization – institutionalization; particularization – generalization; materialization – idealization; reification – decomposition.

Regarding the problem of the delimitation between the processes of particularization - generalization and the processes of materialization – idealization, our conclusion is that considering the dual facets in the onto-semiotic approach, especially the ostensive/non-ostensive and extensive/intensive facets, allows one to deal separately with both pairs of processes. This is an important distinction, as it permits a more detailed analysis and consequently a better comprehension, of these processes as well as of their combined presence in mathematical activity.

When we use an ostensive object as a generic element in mathematical practices, we are acting on a particular object, but we situate ourselves in a "language game" in which we are interested in its general characteristics and we disregard its particular aspects. Controlling the rules of this game allows the student to activate a complex network of semiotic functions, which is what produces the understanding of the particular – general dialectic. The knowledge of the network is also useful in explaining the students' difficulties. Therefore, we show how the onto-semiotic approach to mathematical knowledge can help us analyse mathematical texts and thus help us to understand students' learning difficulties.

The use of diverse theoretical frameworks is a feature of present research in mathematics education. These frameworks and methodologies come from different disciplines (epistemology, psychology, sociology, pedagogy, semiotics, ...) and various research paradigms. The plurality of approaches may be inevitable and even productive, but also poses a very important theoretical question: How can we take advantage of so many theoretical results produced by so many researchers in mathematics education? This is the crucial issue tackled by researchers who are interested in developing the onto-semiotic approach to mathematical knowledge and instruction.

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