The emergence of objects from mathematical practices[1]

Vicenç Font, Juan D. Godino & Jesús Gallardo

Abstract. The nature of mathematical objects, their various types, the way in which they are formed, and how they participate in mathematical activity are all questions of interest for philosophy and mathematics education. Teaching in schools is usually based, implicitly or explicitly, on a descriptive/realist view of mathematics, an approach which is not free from potential conflicts. After analysing why this view is so often taken and pointing out the problems raised by realism in mathematics this paper discusses a number of philosophical alternatives in relation to the nature of mathematical objects. Having briefly described the educational and philosophical problem regarding the origin and nature of these objects we then present the main characteristics of a pragmatic and anthropological semiotic approach to them, one which may serve as the outline of a philosophy of mathematics developed from the point of view of mathematics education. This approach is able to explain from a non-realist position how mathematical objects emerge from mathematical practices.

Keywords. Classroom discourse; Conventional *versus* realist mathematics; Epistemology; Mathematical objects; Onto-semiotics

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V. Font
Departament de Didàctica de les Ciències Experimentals i la Matemàtica, Facultat de Formació del Professorat, Universitat de Barcelona, Campus Vall d'Hebron, Passeig de la Vall d'Hebrón, 171, 08035. Barcelona, Spain
e-mail: vfont@ub.es

J. D. Godino
Departamento de Didáctica de la Matemática, Facultad de Educación, Universidad de Granada, 18071 Granada, Spain
e-mail: jgodino@ugr.es

J. Gallardo
Departamento de Didáctica de las Matemáticas, Didáctica de las Ciencias Sociales y de las Ciencias Experimentales. Facultad de Ciencias de la Educación, Universidad de Málaga. Campus de Teatinos. 29071 Málaga. Spain.
e-mail: gallardoromero@telefonica.net

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1. Introduction

The ontological status of mathematical objects, or to put it another way, the form in which they exist, is a problem belonging to the philosophy of mathematics and one which since the time of Plato has generated an enormous amount of literature (Rozov, 1989, p. 105). However, interest in this question has now spread beyond the confines of these fields and has become an object of study for research in mathematics education. Thus, as highlighted by Radford (2008), "any didactic theory, at one moment or another (unless it voluntarily wants to confine itself to a kind of naïve position), must clarify its ontological and epistemological position" (p. 221).

There are several reasons why it is important to clarify these positions. Indeed, the ontology and epistemology of mathematical entities lie at the heart of a number of fundamental questions related to the study of key issues in mathematics education. For example, they are closely related to the central problem of the meaning of mathematical symbols, as in questions such as "What is the ontological status of these entities? Where do they come from? How can one get hold of them (or construct them)?" (Sfard, 2000, p. 43). Above all, however, understanding the nature of mathematical objects is especially relevant when it comes to studying and tackling the ontological and epistemological conflicts and difficulties that arise in mathematics classrooms, for example, those that derive from attributing *existence* to mathematical objects.

In this context the present paper aims to explore the nature of mathematical objects by analysing their ways of being and their existence in mathematical practices. Specifically, it seeks to explain how mathematical objects emerge from mathematical practices and, ultimately, to provide an operative response to the basic question of how mathematical knowledge is constructed within schools.

The three questions addressed by the paper are as follows:

1. Why are students usually offered a descriptive/realist view of mathematics in the classroom?
2. Is the realist view the most appropriate when it comes to explaining the nature of mathematical objects?
3. Is it possible, in the classroom, to offer a non-realist explanation of how mathematical objects emerge from mathematical practices?

In addressing these questions we first explore why mathematics is presented to students in a realist way. Then, in Section 3, the paper considers whether teaching practices should actually end up offering this view, a discussion that leads us to propose the philosophical and didactic problem raised by realism in mathematics. Section 4 summarizes anti-realist positions as philosophical alternatives to realism. In Section 5 we take a more detailed look at the basic notion of mathematical practice, including examples of classroom activities that help to clarify the notion of mathematical practices and the theoretical ideas which are put forward in the following section. Thus, in Section 6 we present a semiotic ontology for an educational mathematics philosophy, contrasting it with traditional approaches within the philosophy of mathematics. Specifically, the paper develops the proposed ontology of mathematical objects, their types and the resulting configurations, as well as specifying their ways of 'being' in mathematical practice, their forms of existence, and their participation in mathematical activity. Basing on these assumptions, we will try to explain how mathematical practices can produce a referent that, implicitly or explicitly, is considered to be a mathematical object, and which is apparently independent of the language used to describe it. Finally, in Section 7, we compare our proposal with other theoretical frameworks in which the notion of mathematical object plays a key role. The paper concludes with some final reflections about the usefulness of developing an ontology based on the emergence of mathematical objects from mathematical practices.

2. The descriptive/realist view of mathematics

In this section it is argued that the way in which mathematics is taught in schools leads students to develop, albeit implicitly, a realist view of the nature of mathematical objects. Our aim here is to answer the following question: Why, explicitly or otherwise, is a descriptive/realist view of mathematics suggested during the teaching process? This view assumes that mathematical statements are a description of reality, and that the mathematical objects described by such statements form part of this reality. In the teaching process this 'reality' to which mathematical objects belong is located at an intermediate point between what, in the philosophy of mathematics, are referred to as Platonic and empiricist positions, although depending on the teaching process considered, one may observe a clear preference for one or the other of these two points of view, for example, in contextualized teaching or realist mathematics.

The following paragraph from a textbook illustrates how the descriptive/realist view of mathematics may be present in classroom discourse:

> Complex numbers are of enormous interest. It has been shown that among complex numbers every polynomial equation has at least one root, and may have as many roots as the degree of the polynomial. This is known as the fundamental theorem of algebra. Complex numbers have many applications in other sciences, such as in calculations for AC circuits, the formulation of quantum mechanics and even in aerodynamics. You have just discovered the existence of complex numbers... (Barceló, Bujosa, Cañadilla, Fargas, & Font, 2002, p. 81)

In what follows, we consider five arguments that, explicitly or otherwise, lead students to view mathematical objects as having real existence.

2.1. The Objectivity of Mathematics

It is immediately apparent that the textbook paragraph makes use of a personal/institutional duality, that is, students are invited to see how the results they have obtained are already known and form part of established mathematical knowledge. In the first three lines, prior to speaking about the existence of complex numbers, the author refers to the important findings associated with them, what might be called known representations, definitions and properties of complex numbers. This is an example of a mathematics discourse which gives students the message that mathematics is a 'certain', 'true' or 'objective' science. We thus face the epistemological problem of explaining the generality and objectivity of mathematical propositions.

The classroom discourse usually explains the objectivity of mathematics, explicitly or otherwise, by suggesting that while mathematics is the result of the problem-solving activity carried out by different human societies, its truth or objectivity does not depend on the people who have developed it. In a way this leads students toward what in philosophy is regarded as realism in terms of truth value or epistemological realism (Shapiro, 2000). This type of realism can be formulated as follows: Regardless of whether mathematical objects exist independently of people and of the language through which they are known, the truth value of mathematical statements is objective and independent of the people who make such statements.

The step from epistemological to ontological realism is related to the assumption that in order for mathematical knowledge to be objective and general, its objects must also be regarded as such. The epistemological question regarding the objectivity and generality of mathematical knowledge therefore leads to the postulation of mathematical objects that are also objective and general.

2.2. The Predictive Success of Sciences that Make Use of Mathematics

The textbook paragraph also contains a discourse about the applications of complex numbers to reality. The predictive success of those sciences that make use of mathematics is used to argue in favour of the existence of mathematical objects. This argument strengthens, above all, the idea of ontological realism. The relationship between mathematics and reality is present, to varying degrees, in almost all teaching processes. Indeed, certain processes of mathematics teaching and learning include a descriptive discourse about mathematics that facilitates, explicitly or otherwise, the emergence of mathematical objects as if they existed in reality. This is the case, for example, of some teaching processes that propose a contextualized or realist approach to mathematics.

2.3. Differentiation Between Ostensive and Non-ostensive Objects

As discussed in Font, Godino, Planas, and Acevedo (2010) it is possible in mathematics discourse (a) to talk about ostensive objects representing non-ostensive objects that do not exist, for example, we can write that $f'(a)$ does not exist because the graph of f has a pointed form at $x = a$; and (b) to differentiate the mathematical object from one of its representations. Duval (1995, 2006) has pointed out the importance of the different representations and transformations between representations in students' understanding of the mathematical object as something different from its representation. Both aspects lead students to interpret mathematical objects as being different from their ostensive representations. When produced within the mathematics classroom this type of discourse leads us to infer the *existence* of the object as something independent of its representation.

2.4. The Object Metaphor in Teachers' Discourse

As described in Font, Godino, et al. (2010) we consider that the object metaphor plays a relevant role in understanding the existence of mathematical objects as pre-existing objects.

The object metaphor is a conceptual metaphor that has its origins in our experiences with physical objects and enables events, activities, emotions and ideas, etc. to be interpreted as if they were real entities with properties. For example, Font, Bolite, and Acevedo (2010) analysed different classroom episodes in which the graphical representation of functions was being taught, and found that the object metaphor was always present in teachers' discourse because the mathematical entities were presented here as 'objects with properties'.

At all events, it is common in mathematics discourse to use certain metaphorical expressions[2] of this conceptual metaphor, ones which suggest that mathematical objects are not constructed but, rather, are discovered as pre-existing objects. For example, words such as 'describe' or 'find', etc. are often used, while the textbook paragraph cited above contains the word 'discovered'.

2.5. Simplicity, Intentionality and Reification

In everyday life it is useful to assume that the different experiences which one has of an object, for example, a chair, are due to there being an object called a chair which is the cause of these experiences. Just as postulating objects such as chairs is a useful fiction, regardless of whether they actually exist, it is also useful to postulate the existence of mathematical objects. Their postulation is justified on the basis of the practical benefits, especially as regards simplifying the mathematical theory which is being studied. Indeed, it is highly convenient to consider that there exists a mathematical object that is represented by different representations, which can be defined by various equivalent definitions, or which has properties, etc.

In addition to the convenience and simplicity that derive from postulating the existence of mathematical objects, there are also philosophical reasons for doing so. Here we will mention only the philosophical argument, introduced into modern philosophy by Brentano, which states the intentionality of thought. Basically, intentionality means that the activity of the mind refers to, indicates or contains an object. According to this point of view, therefore, the subject assigns an intentional content to representations, definitions, properties, etc. This assignation is what leads us to consider that language represents 'something', that definitions define 'something', that properties are the properties of 'something', etc. In other words, an intentional content is associated with the representations, definitions, properties, etc., and this content is regarded as an object.

The process by which we assume, or state linguistically, that there is an object with various properties or various representations, etc. is known as reification (Quine, 1990). This process is described by Quine for objects such as chairs, trees, etc., as well as for abstract objects in several works, it being frequently linked with notions related to the psychology of

[2] Conceptual metaphors enable metaphorical expressions to be grouped together. A metaphorical expression, on the other hand, is a particular case of a conceptual metaphor. For example, the conceptual metaphor "the graph is a path" appears in classroom discourse through expressions such as "the function *passes* through the coordinate origin" or "if *before* point M the function is ascending and *after* it is descending then we have a maximum." The teacher is unlikely to say to students that "the graph is a path" but, rather, will use metaphorical expressions that suggest this.

learning. The process of reification has also been studied in the area of mathematics education (see, for example, Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1996; Sfard, 1991).

Figure 1 summarizes the relevant arguments leading to a descriptive/realist view in school mathematics.

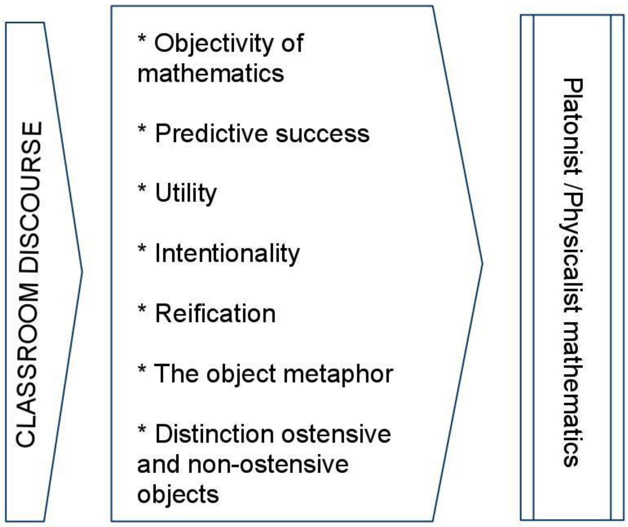

Figure 1. Arguments Leading to a Descriptive/Realist View in School Mathematics

3. The problem of realism

Having argued in the previous section that mathematics teaching leads students to develop, explicitly or implicitly, a realist view of mathematics it is now necessary to ask whether this is a desirable outcome. In principle, one might answer affirmatively, since such a view corresponds to a philosophical position on the nature of mathematical objects that is supported by well-founded epistemological theories within the philosophy of mathematics. Specifically, the descriptive/realist position considers that mathematics are neither invented nor constructed, but rather discovered. Consequently, mathematical theories describe pre-existing mathematical objects. The two most representative approaches in this regard are Platonism and empirical realism.

Mathematical Platonism can be defined as the conjunction of the following three theses: (a) Existence: there are mathematical objects, mathematical sentences and theories provide true descriptions of such objects; (b) abstractness: mathematical objects are abstracts, that is, non-spatiotemporal entities; and (c) independence: mathematical objects are independent of intelligent agents and their language, thought and practices. Furthermore, according to the Platonists, abstract objects are wholly non-physical, non-mental, non-spatial, non-temporal and non-causal (Linnebo, 2009).

Empirical realism shares with Platonism the view that mathematics is the description of objects that exist independently of people and of the language used to represent these objects.

However, rather than situating them beyond space and time, empirical realism locates such objects within a spatio-temporal world. The main perspectives in this regard are physicalism, holistic empiricism and radical empiricism.

Physicalism views mathematical sentences and theories as being about ordinary physical objects. John Stuart Mill, for instance, regarded mathematics as simply a very general natural science. For the holistic empiricist (Quine, 1969) the existence of mathematical objects is based on the argument of indispensability: Their existence is indispensable for justifying the existence of the objects dealt with by physics, and for the latter's predictive success. This is an indirect argument, since the existence of mathematical objects is inferred from the predictive success of those sciences which make use of mathematics. Radical empiricism (Maddy, 1990; Tymoczko, 1991) can be considered as a development of Mill's physicalism, since it assumes that mathematical propositions are ultimately based on our sense perceptions and on the inductive generalizations made from them. Some authors associated with this view, such as Maddy, argue that certain mathematical objects (for example, certain sets) are perceivable spatio-temporal objects in the same way as ordinary physical objects are.

The different varieties of realism in mathematics pose a common epistemological problem that can be stated in the form of a question: If mathematical objects are independent of people, how can we gain access to them? (Benacerraf, 1973). Radical empiricists go as far as to state that we actually perceive such objects, whereas the less radical holistic empiricists respond to the above question by claiming that access is gained through an indirect mechanism. The Platonists, for their part, are left to postulate some kind of bridge (for example, intuition) between the spatio-temporal world and the Platonic world. At all events, none of these responses offers an entirely satisfaction explanation of how we gain access to mathematical objects, and neither do they adequately explain mathematical activity. These limitations suggest that the realist view is perhaps not the most suitable for explaining the processes through which mathematics is learnt. In this regard, Ernest (1998) has highlighted the negative consequences that Platonism and mathematical realism, as well as the foundationalist and absolutist positions, may have for mathematics education.

4. Philosophical explanations that offer an alternative to realism

In our opinion the realist view of mathematics can be broken down into two main arguments. The first states that there are objects to which mathematical activity refers, that is, those objects we describe when we do maths. The second states, more strongly, that these objects exist independently of people and of the language used to describe them. By breaking down the realist view in this way it is possible to explain how plausible philosophical alternatives to realism have developed. These alternative views seek to show that mathematical objects do not exist independently of people and of the language used to describe them, thereby refuting the second argument of mathematical realism. Furthermore, they also respond to the first argument by offering a plausible explanation of how the objects to which mathematical activity refers may develop. Specifically, they argue that what is developed is a fictional object that serves as a reference for mathematical activity, or alternatively that mathematics is a social construction which may be useful in describing the world of physical objects, etc.

This non-descriptive perspective on mathematics, which we will here refer to as anti-realist, is shared by a number of different theoretical approaches, notably psychologism, constructivism, conventionalism and nominalism.

Psychologism states that mathematical sentences and theories are about mental objects. Probably the most common version of this view holds that numbers are akin to ideas in our heads, and that ordinary mathematical sentences like “3 is prime” provide descriptions of these ideas (Balaguer, 2008). A point of view that shares common ground with psychologism is constructivist intuitionism, whose origins lie in the philosophy of Kant. The principle of construction or constructibility, which forms the basis of mathematical intuitionism, states that mathematics is the study of certain types of mental constructions. For intuitionism, mathematical objects are entities that the mind produces based on (a) the fundamental intuition of natural objects, and (b) the use of methods of effective construction. All that exists are natural objects and that which can be constructed effectively from them.

Conventionalism has been described as “the view that a priori truths, logical axioms, or scientific laws have no absolute validity but are disguised conventions representing one of a number of possible alternatives” (Norton, 1997, p. 121), and one of its historical antecedents would be formalism. In the conventionalist view the essential feature of logical and mathematical systems is not their descriptive nature, which does not mean they cannot be applied to the study of the physical world, but rather their constructive or constitutive nature, in the sense that the meaning of logical or mathematical signs is determined by the rules of inference and axioms. Among the different approaches that may be considered as conventionalist, the most relevant to the present paper is that of Wittgenstein (1978), especially in terms of the following three arguments. Firstly, Wittgenstein states that mathematical propositions are rules (of a grammatical kind) governing the use of a certain kind of sign; for example, the mathematical statement $6^2 = 36$ provides a rule which, in empirical statements, allows us to write 6^2 instead of 36. The fact that mathematical statements are written in the form of declarative sentences is not in itself sufficient to reject the argument that such statements actually express rules, since despite taking the grammatical form of declarative sentences they are *used as rules*. The other two arguments of relevance here are that the meaning of mathematical expressions is determined (and exhausted) by the rules governing the use of such expressions, and that these rules are conventional in nature.

As regards the nominalist approach, here there is no need to speak of the existence of mathematical objects, nor of abstract or any other type of object for that matter, except in terms of the signs we use to do mathematics. In a nominalist reconstruction of mathematics, concrete entities take up the role played by abstract entities in Platonist accounts of mathematics. With respect to the entities alleged by some to be universals or abstract objects (e.g. properties, numbers or propositions), nominalism proposes two general options: (a) to deny the existence of the alleged entities in question, and (b) to accept the existence of these entities but to argue that they are particular or concrete (Rodriguez-Pereyra, 2008).

At all events, a detailed examination of the relationship between these traditional approaches to the philosophy of mathematics and mathematics education is beyond the scope of the present paper. Rather, our aim is to find a way of overcoming the ontological and epistemological problem of realism by proposing, from the perspective of mathematics education, a mathematical ontology that is compatible with non-realist positions such as conventionalism and constructivism. Specifically, the goal is to develop a semiotic ontology for an educational mathematics philosophy that can explain how mathematical practices may produce a referent that is also, explicitly or implicitly, considered to be a mathematical object, and which exists apparently as an independent object of the language used to describe it. As will be shown in the following sections, the application of anthropological and semiotic postulates to mathematical practice and to the objects involved in it enables us to develop a useful framework for describing mathematical activity, both that of professionals and that which takes place in schools.

In the mathematical ontology described in the following sections, mathematical objects emerge from the practices performed by people within particular contexts, communities, cultures, or institutions, and since such practices depend on the available artifacts and linguistic tools, mathematical objects depend on language and culture.

5. mathematical practices and emergence

Our ontological proposal is derived from mathematical practice, this being the basic context in which individuals gain their experience and in which mathematical objects emerge. Consequently, the object here acquires a status derived from the practice that precedes it.

One way of conceptualizing mathematical practices is to regard them as the combination of an operative practice, through which mathematical texts are read and produced, and a discursive practice, which enables reflection upon the operative practice. Both these practices would be able to be recognized as mathematical by an expert observer. This way of understanding mathematical practice requires consideration of the personal and institutional facets, among which complex dialectical relationships are established and whose study is essential for mathematics education.

At all events, it is important to distinguish between human behaviour, understood as the apparent and observable behaviour of individuals, and practice, in the sense that purposeful human action has a meaning for both the individual performing it and the person who interprets it. In the case of mathematical practices, however, their meaning is determined by the function that a given practice serves in problem-solving processes, or in communicating the solution to another person, validating the solution and generalizing it to other contexts and types of problem. It should be emphasized that this way of understanding the meaning of mathematical practices implies that they be considered as rule-governed.

The notion of emergence is often used when studying complex systems, whether of a physical, biological or some other nature, although it is also employed in cognitive science, ontology and epistemology (Bunge, 2003; O'Connor & Wong, 2006). It is usually considered that emergent entities (properties or substances) 'arise' out of more fundamental entities and yet are 'novel' or 'irreducible' with respect to them. In the context of the present ontological proposal the idea of emergence is useful for describing how mathematical objects are constituted, although the notion will largely be used metaphorically and in its 'weak' sense (Bedau, 1997).

In order to make it easier to understand the theoretical notions being proposed and their usefulness in terms of explaining the emergence of mathematical objects in the teaching context, we will present a number of examples of mathematical activity generated from problematic tasks carried out (or proposed) in classrooms.

Activity 1 (6-year-olds pupils, primary education)

The children are asked to add 14 + 27. To this end they are given base-ten blocks so that they can perform the actions shown in Figure 2, after which the teacher explains the figure on the board. This task forms part of a teaching process whose methodology passes through the stages of manipulation, graphical representation and symbolic representation, and whose objective is to enable children to perform the 'carrying over' of natural two-digit numbers.

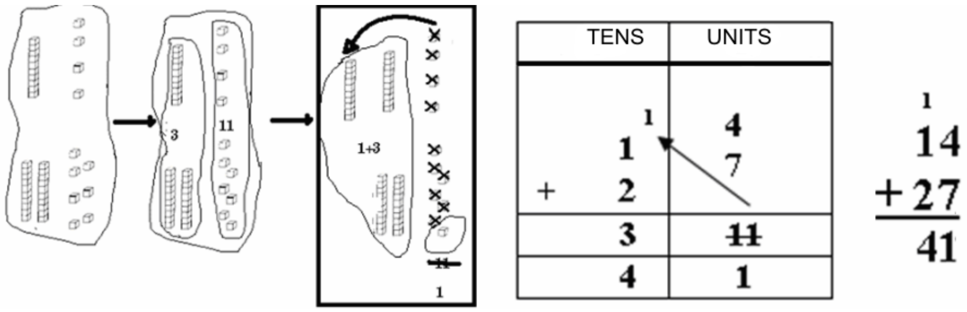

Figure 2. Example of a Mathematical Activity

What emerges from this activity is the property that 14 + 27 is equal to 41 and a procedure for adding two-digit numbers. This emergence is achieved by using material that can be handled (base-ten blocks) and through different representations on the board (graphical, symbolic), and it is also assumed that the pupils have a certain knowledge about the meaning of natural numbers and sums, etc. For example, it is assumed that they are already able to perform certain procedures such as addition without carry-over and mental arithmetic, etc. The question is: How do pupils become convinced that 14 + 27 = 41? The answer is by means of the evidence provided by their senses and through certain implicit processes of generalization and idealization. The only thing about which the pupils have empirical evidence is that 14 blocks + 27 blocks make 41 blocks. However, in the class the blocks are used as generic objects and it is implicitly accepted that 14 objects + 27 objects make 41 objects (physical operation of adding objects). Hence they conclude that 14 + 27 = 41 is true (mathematical operation with numbers). In this activity, therefore, there has been a shift from a physical plane (operations with physical objects) to the mathematical plane (operations with mathematical objects). This activity, together with others, also serves to demonstrate the emergence of a procedure for adding two-digit numbers.

In summary, the two things that emerge in the classroom through this activity are (a) properties, which are implicitly assumed to be properties of mathematical objects, in this case, natural numbers; and (b) procedures, which are implicitly assumed to be rules about how to use these mathematical objects.

Activity 2

The second example that we will use to explain our proposed mathematical ontology involves the following series of problems, which are related to one another and which were set in different school contexts.

Activity 2A (12-year-olds pupils, compulsory secondary education)

We firstly propose the following task to the students: Here you have this chess boar. How many smaller or equal-sized chess boards you can build? The solution to this problem requires that pupils apply an organized system of actions or practices (see Figure 3): writing (numbers, table), operations (addition, multiplication), argumentation, etc. and doing so implies the interaction of several ingredients (natural numbers, square of a number, sum, etc.). The conclusion they reach (which emerges) is that the following proposition is true: The number of boards is equal to 12 + 22 +… + 82. This is a result that was not known and which, implicitly, is presented as a result in the world of things, it tells us about the number of boards that could be made.

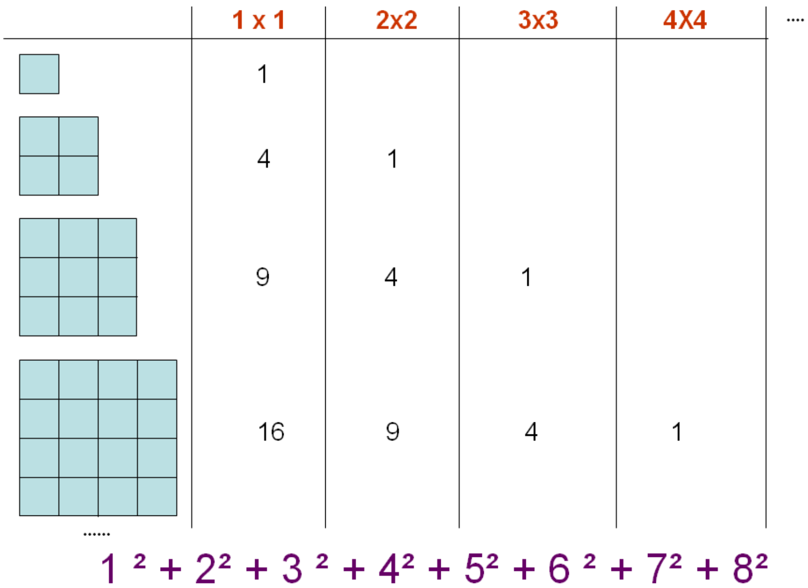

Figure 3. A Student’s Solution

Activity 2B (16-year-olds students in a problem-solving workshop; these were bright students in compulsory education who had a certain mastery of algebra)

Now, the students are given the following task: Find the maximum number of squares of any size that can be built inside a square of side n. (Suggestion: Look for simpler examples and find a regular pattern). Justify your answer. What emerges in this case is the following general statement: For any square of side n, the number of squares of any size that can be formed is 12 + 22 +… + n2. It should be noted that here, implicitly, this statement is presented as a property in the world of mathematical objects (squares in this case).

Activity 2C (continuous education course for secondary school teachers)

In Figure 4 we present the third proposed task. This problem is designed so that in step (*c)* the teachers state the property $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ and justify it on the basis of their answers in step (*b*). This activity leads to the emergence, above all, of properties of natural numbers that are unknown to the participating teachers.

(a) Using the following pieces (below), construct a cube of Side 4.

(b) Describe the pieces which are necessary to construct (in the same way) a cube of Side 5, one of Side 8 and another of side n.

(c) In the case of the cube with side n, can the quadrangular pieces be used to construct this if there are $\frac{n(n+1)(2n+1)}{6}$ *cubes of Side 1? Justify your answer.*

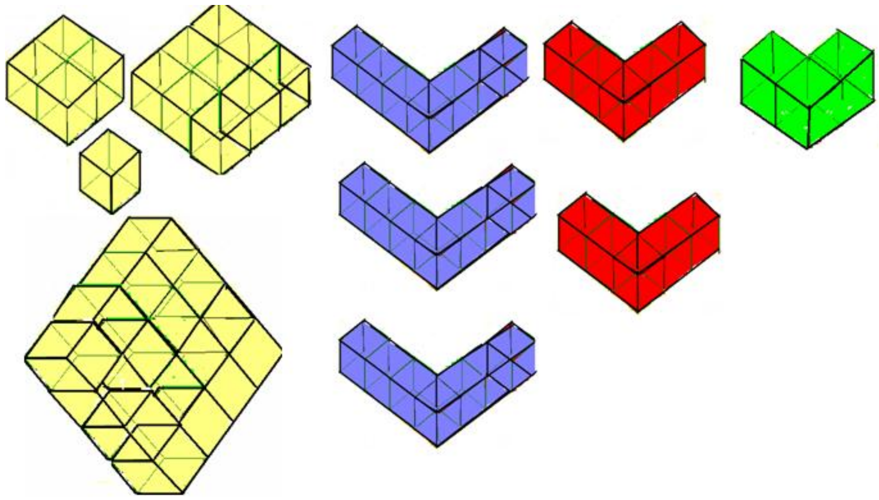

Figure 4. Pieces of a cube of side 4

Activity 2D (first-year mathematics undergraduates)

In the last task we ask the students to prove the following property using complete induction: $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. In this case the property of the numbers is already known. The proof by empirical induction which the students performed led to the emergence of an argument that enables us to ensure the validity of the property.

The notion that mathematical objects emerge from practices helps us to understand that the construction of these objects is progressive and dependent on the institutional framework and the artefacts used in these practices. From the point of view of those who take part in such activities, the appearance of objects has a certain novel aspect to it. For example, in Activity 2C the property $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ is initially unknown, is then something which is conjectured on the basis of the actions performed in the three steps, and finally, a deductive proof of its validity can be developed for any natural number, for example, by using the principle of mathematical induction (Activity 2D).

6. A semiotic ontology for an educational mathematics philosophy

In this section we describe a mathematical ontology that seeks to avoid the philosophical problems of realism in mathematics, and its negative consequences as regards mathematics education. The view taken of the philosophy of mathematics is a broad one and is not limited to classical problems about the origin and justification of professional mathematical knowledge. Rather, the aim is to develop an ontological and epistemological model that is useful for analysing the mathematical activity that takes place in educational institutions. It is assumed that mathematics is a human activity (anthropological postulate) and that the entities involved in this activity come or emerge from the actions and discourse through which they are expressed and communicated (semiotic postulate). This model has been progressively developed in several papers by Godino and colleagues (Godino & Batanero, 1998; Godino, Batanero, & Font, 2007) and is referred to as the 'onto-semiotic approach' to knowledge and mathematics instruction. The objective in this paper is to examine in greater detail and clarify the nature of different types of objects and their emergence through mathematical practices. We also seek to locate the onto-semiotic model within the framework of traditional approaches to the philosophy of mathematics.

6.1 The Objects Involved in Mathematical Practices

In ordinary language the word object is used to refer to things that are material, tangible or real, but this is not the case in philosophy. In philosophical language, an object is a basic and universal metaphysical category, a synonym for entity, thing or something that can be individualized. In the philosophy of mathematics the term mathematical object usually refers to abstract objects such as classes, propositions or relationships. However, in our proposed ontology, and in accordance with symbolic interactionism (Blumer, 1969; Cobb & Bauersfeld, 1995), we use object in a wide sense to mean any entity which is involved in some way in mathematical practice or activity and which can be separated or individualized. For example, in Activity 2B (Section 5) the following objects, among others, are involved:

- Concepts/definitions of natural number, square of a natural number, first consecutive natural numbers, sum, square, etc.
- Properties of arithmetic operations (as distributive) and the proposition: The number of squares is $1^2 + 2^2 + \ldots + n^2$.
- Ways of expressing the concepts/definitions and properties (geometric and algebraic).
- An argument. For example, one pupil represented a particular but indeterminate square of side k, which enabled him to justify, by means of reasoning in terms of generic elements, that the number of squares of side k is $(n-k+1)(n-k+1)$.

In the onto-semiotic approach (OSA), *being* a mathematical object is equivalent to *being involved* somehow in mathematical practices. The expression *mathematical object* is a metaphor that projects some features found in the source domain (physical reality) to the target domain (mathematics). We remark the possibility that physical objects be separated from other "objects". Therefore, initially, all we can "separate" and "individualize" in mathematics is considered an object, for example, a concept, a property, a representation, a procedure, etc. According to Lakoff and Johnson (1980) this is an ontological metaphor that "allows us to pick up parts of our experience and treat them as discrete entities or substances of a uniform kind" (p. 25).

We recognize that this conception of mathematical object is very broad (or weak), since in fact "everything" involved in mathematical practice should be considered a mathematical object. However, we think this broad use of the term mathematical object is useful at the beginning, because at this moment we assume that mathematical objects are not just the abstract concepts, but any entity or thing to which we refer, or of which we speak, no matter if it is real or imaginary, whenever it is involved in some way in mathematical activity.

This initial "weakness" of the notion of object should be reinforced by a "strong theory" that would categorize the various types of objects of interest to describe mathematical activity. The main aim in this paper is to develop this theory. To achieve this aim the ideas of being involved in mathematical practice and existing, taking into account different types of existence, are very useful.

In the next section we will describe the types of objects that are involved in mathematical practice and their different forms of existence.

6.2. Configurations of Objects Involved in the Systems of Mathematical Practices

An analysis of Activities 1 and 2 in Section 5 reveals a first type of object that is involved in mathematical practices, problems, concepts/definitions, propositions, etc., which we will refer to here as *primary objects* (see Godino et al., 2007).

If we consider, for example, the objects involved when carrying out and evaluating the practice that enables a problem to be solved (e.g. Activities 2A-2D in Section 5), then we can see the use of languages, both verbal and symbolic. These languages are the ostensive part of a series of concepts/definitions, propositions and procedures that are involved in the elaboration of arguments whose purpose is to decide whether the simple actions of which the practice is composed, and the practice itself as a compound action, are satisfactory.

The following typology of primary mathematical objects[3] can therefore be proposed:

- *Linguistic elements*: Terms, expressions, notations, graphs, etc. in their various registers (written, oral, gestural, etc.).
- *Situations/problems*: Extra-mathematical applications, tasks, exercises, examples, etc.
- *Concepts/definitions:* Introduced by means of definitions or descriptions, explicit or otherwise (straight line, point, number, mean, function, etc.).
- *Propositions:* Statements about concepts, etc.
- *Procedures*: Algorithms, operations, techniques of calculation, etc.
- *Arguments*: Statements used to validate or explain the propositions and procedures, whether deductive or of another kind.

[3] Some authors have argued that some of what are here called primary objects should be considered as objects. For example, Quine proposes referring to *propositional objects* rather than propositions (Quine, 1969). Another example can be found in some versions of nominalism, which consider that signs are the only mathematical object which exists.

The six types of primary entities proposed here extend the traditional distinction between conceptual and procedural entities, the latter being considered insufficient when it comes to describing the objects that are involved in, and emerge from, mathematical activity[4]. The situations/problems are the origin or raison d'être of the activity, the language represents the remaining entities and serves as a tool for action, and the arguments justify the procedures and propositions which relate the concepts among one another. Considering an entity as primary is a relative rather than absolute matter, since we are dealing with functional entities that are relative to the language games, institutional frameworks, communities of practices and contexts of use, in which they participate. They also have a recursive nature, in the sense that each object, depending on the level of analysis, may be composed of entities of the remaining types, for example, an argument may bring into play concepts, propositions, procedures, etc.

Primary objects are related to one another and form configurations, which can be defined as networks of objects that are involved in, and emerge from, systems of practices (Godino et al., 2007). These configurations may be socio-epistemic (networks of institutional objects) or cognitive (networks of personal objects).

It should be noted that by considering the components of these configurations as primary objects, we are engaging in a process of reification, in the sense that notions which would normally be regarded as abstract are conceived of as objects. However, this position is consistent with usual mathematical practice, which also engages in reification. In the mathematics classroom this process is facilitated by, among other factors, the use of the object metaphor within the teaching discourse (Font, Bolite, et al., 2010).

The key question that now needs to be addressed is: What is the nature of primary objects? Problems, linguistic elements and arguments do not face the problem of appearing to be of a different nature than they actually are, but this is not the case of the other primary objects (procedures, concepts/definitions and propositions).

As regards procedures, it is clear that they are rules, even though in many cases they are formulated as propositions, for example, the quotient rule for the derivative of functions is stated in the form of a proposition, or even as definitions, since, for example, if we define the perpendicular bisector of a segment as the perpendicular that passes through the mid-point, then we are implicitly stating a construction procedure. The object nature which proves to be particularly problematic is that of concepts/definitions and propositions since, in addition to the fact that ostensive representations are normally assumed to represent ideal mathematical objects, it is usually assumed that definitions and propositions refer to mathematical objects which exist in some form or another.

In our ontological proposal, and in line with Wittgenstein's philosophy of mathematics (Baker & Hacker, 1985; Bloor, 1983; Wittgenstein, 1978), concepts/definitions and

[4] Depending on the text analysed, some of the elements in this configuration may be missing; this would be the case of epistemic configurations of axiomatic texts.

propositions are regarded as 'grammatical' rules of a certain kind. From this point of view, mathematical statements are rules (of a grammatical kind) governing the use of certain types of signs, since that is precisely how they are used, as rules. They do not describe properties of mathematical objects with any kind of existence that is independent of the people who wish to know about them or the language through which they are known, even if this may appear to be the case.

Let us consider the two primary objects that emerged in Activity 1 (Section 5) in order to justify that they are, by nature, rules. On the one hand, we saw how the 'carrying over' procedure for two-digit natural numbers emerged, and here its status as a rule is clear. On the other hand, there is the proposition $14 + 27 = 41$. This statement was presented to the pupils as a generalization of operations that can be performed with objects and, implicitly, a process of idealization led them to consider that this was a property of natural numbers, thereby obscuring its status as a rule. However, in this activity the objects employed are only used as positive examples in which the proposition is fulfilled, since were a counter-example to be presented this could not invalidate the proposition as it would be rejected. It is not possible to refute empirically the mathematical statement $14 + 27 = 41$, since in the event that some empirical evidence led us to conclude that $14 + 27$ does not equal 41, the statement itself would then be used to argue that an error had been made in gathering this evidence.

This is a conventionalist position that is opposed to the realist argument in the philosophy of mathematics: Mathematical statements do not describe any type of reality (neither ideal nor natural) which exists a priori of the mathematician's constructive activity. Therefore, concepts/definitions, propositions and procedures of epistemic configurations exist in the same way as conventional rules do. From this perspective, mathematical 'truth', 'certainty' or 'necessity' simply means 'agreeing' with the result of following a rule that forms part of a language game which operates within certain social practices. It is not an agreement of arbitrary opinions, but rather an agreement of practices that are subject to rules. However, this does not mean that the socio-epistemic configuration of primary mathematical objects is the result of arbitrary, whimsical or even ideologically motivated decisions and choices.

The above view can be regarded as an intermediate position between the nominalist and realist viewpoints. On the one hand, it disagrees with those nominalist positions that deny the existence of abstract objects such as classes, propositions or relationships, since it is assumed that socio-epistemic configurations of primary objects are used and emerge within classroom mathematical practices. On the other hand, by considering that the definitions and propositions of socio-epistemic configurations have the status of rules, because that is precisely how they are used, our approach also differs from those realist positions which consider that mathematical objects exist and that they function as a referent for the configuration as a whole, their existence being independent of people and the language used to describe them.

This conventionalist position is supported by Wittgenstein's philosophy of mathematics and is also compatible with, or at least does not contradict, other approaches in mathematics education that are based on the same philosophy, for example, the theory of *commognition*

(Sfard, 2008), as well as, more generally, all those approaches which consider that mathematics discourse and its objects are mutually constitutive. It is also consistent with the notion of praxeology proposed by the anthropological theory of the didactic (Chevallard, 1992). Finally, the proposed ontology of primary objects and their characterization as rules is, in our view, also compatible with the notion of mathematical object in Radford's cultural semiotics and, moreover, renders that approach more operative: "Mathematical objects are fixed patterns of reflexive activity incrusted in the ever changing world of social practice mediated by artefacts" (Radford, 2008, p. 222).

6.3. Ways of 'Being' of Primary Mathematical Objects

The notion of *language game* (Wittgenstein, 1953) plays a key role in our onto-semiotic approach, since, together with that of *institution*, it is considered to be one of the contextual elements that relativize the ways in which primary mathematical objects can *be* or *exist*.

The notion of language game in Wittgenstein's system of aphorisms characterizes the pragmatic-anthropological position on the meaning of words and linguistic expressions: Language game is meant to highlight the fact that the speaking of language is part of an activity, or of a way of life. Each word has meaning insofar as it has a use in a particular language game, but outside the language game there is no meaning. For example, in Activities 2A-2D of Section 5 we can see how the "proof" is immersed in various language games that correspond to the institutional context in question. Activities 2A and 2B, for instance, are immersed in a language game corresponding to basic arithmetic, one in which pupils participate during their first years at school. Activity 2C, however, implies a sharp shift away from this language game and forms part of another one that corresponds to algebra. Finally, the proof by mathematical induction (Activity 2D) implies participation in the rules, customs and agreements associated with the "way of life" of the professional mathematics community. In this case the language game is one of universal quantifiers, corresponding to the logic of propositions.

Depending on the language game in which they are involved the primary mathematical objects that form part of mathematical practices, and those which emerge from them, may be considered in terms of how they participate, and the different ways of doing so may be grouped into dual facets or dimensions (Godino et al., 2007), as we will see below.

Expression/content

Primary mathematical objects require a sign that enunciates (or accompanies) them. However, being an object or a sign is relative, it is a distinction based on a temporary relationship rather than on substance, since what is a sign at one moment may become an object at another, or vice-versa. Subjects may therefore identify the sign with the object or differentiate between the two, according to what is most appropriate.

The possibility of differentiating between sign and object enables a given individual to establish a semiotic function between two objects (something for something). In this relationship it is considered that one of the objects is an expression that is related with a content (the other object).

One way in which primary mathematical objects can be regarded as 'being' in mathematical practices is therefore related to the expression/content duality. They may be participating as representations or as represented objects and, depending on the language game, they may shift from being representations to being represented objects. For example, Font, Bolite, et al. (2010, p. 17) discuss the case of a textbook which, having stated that functions can be represented in four different ways and that it is important not to confuse the function with one of its representations, then proceeds, in the following paragraph, to identify the function with the symbolic representation, by literally stating "Given the function $f(x) = 1/x$ …".

If we then ask how the expression is related to the content, we come up against the problem of classifying representations as internal or external. Consider, for example, the word "triangle" written on a sheet of paper. We normally consider this expression to be related to the mathematical object triangle (institutional content) through the subject's concept (personal content). Furthermore, both the subject's concept and its expression are considered to be representations; the written word triangle is an external representation and the subject's concept an internal (or mental) representation.

The classification into mental (or internal) and external representations is by no means transparent. Indeed, the ambiguity of the internal/external classification has been pointed out by different researchers (Kaput, 1998; Sáenz-Ludlow, 2002). Mathematical objects are represented in books, on boards, etc. by mathematical sign systems and using materials that form part of the real world; since it is also assumed that the subject is related to this real world by mental representations, a conclusion is that what is considered to be external is also, in a way, internal.

In agreement with Godino and Font (2010), we consider that the internal/external duality does not adequately account for the institutional dimension of mathematical knowledge, since it confuses, to some extent, mathematical objects with the ostensive resources that serve as the basis for the creation or emergence of institutional entities. In addition to being problematic, the internal/external classification is not very operative, and we therefore propose that it be replaced with two more useful dualities (or ways of being). These dualities are personal/institutional and ostensive/non-ostensive.

Personal/institutional

Mathematical cognition must contemplate both the personal and institutional facets, between which complex dialectical relationships are established and whose study is essential for mathematics education. *Personal cognition* is the result of the individual subject's thought and action with respect to a certain class of problems, whereas *institutional cognition* is the result of the dialogue, agreement and rules that emerge from a group of individuals who form a practice community[5].

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[5] The notion of institutional cognition is associated with socio-cultural approaches to the construction of knowledge. If mathematical objects do not exist independently of people, but rather are produced intersubjectively through dialogue and convention, then this process by which objects are created constitutes a kind or form of cognition, namely institutional cognition.

The personal/institutional duality provides a way of thinking about how primary mathematical objects can be regarded as having ‘being’ in mathematical practices. Such objects may participate as either personal or institutional objects and, depending on the language game, they may shift from being personal to institutional.

The personal/institutional dialectic is essential in teaching processes, whose aim is to enable students to take on board the institutional objects (i.e. learning). Indeed, the mathematics classroom makes use of a language game that leads students to distinguish between personal objects and institutional objects. This game is related to the need to justify mathematical statements, although what is understood by proof or justification may differ depending on the institutional context in question, as can be seen in Activities 2A-2D. In the classroom, personal objects may be described using a first-person discourse, whereas institutional objects require a third-person discourse since they are considered to be objective. The main outcome of this language game is that mathematics is presented as an objective science.

The personal/institutional duality leads us to consider configurations of institutional primary objects and configurations of personal primary objects, and to problematize the relationship between them. Our main interest in this paper lies with the former, and in relation to the latter we merely wish to highlight two points. Firstly, these two types of configurations are not identified with one another, but rather are regarded as being composed of similar elements. In the former case this means configurations that are shared within a practice community (the institution), whereas the latter are, in principle, personal to the individual subject. Secondly, the construct of configurations occupies a common ground with other constructs used in research into the teaching of mathematics (conception, schema, belief, intuitive rules, etc.).

The ontology proposed in this paper considers mathematics as a series of intersubjective social constructions, and it is necessary for these constructions to be conventional, a point of view that overlaps with social constructivism (Ernest, 1998).

The notion of cognitive configuration (personal objects network) is a tool for describing the objects involved and emerging from personal practices, and therefore, to describe the subject’s knowledge, understanding and skills at some point in the learning process. Learning can be described in terms of students’ construction of a cognitive configurations network, with progressive levels of complexity, in line with the institutional intended configurations.

This description is consistent with the theory of objectification (Radford, 2008), which posits that learning is the social process of objectification. This process is conceived as the subjective awareness of the cultural object: "Learning does not consist in constructing or reconstructing a piece of knowledge. It is a matter of actively and imaginatively endowing the conceptual objects that the student finds in his/her culture with meaning" (Radford, 2008, p. 223). We think that the cultural object corresponds to the institutional object, while the personal object accounts for the outcome of the objectification process.

Ostensive/non-ostensive

Objects outside mathematics, such as oranges, trees, etc. are considered to be particular and to have a material existence. This type of existence means that they are ostensive in the sense that they can be shown directly to another person. This ostensive nature is shared by the material representations used in mathematics.

Another way in which primary objects can be regarded as 'being' in mathematical practices is therefore related to the ostensive/non-ostensive duality. These two forms must be considered in an inter-subjective sense: Something that can be shown directly to another person, versus something that cannot itself be shown directly and must therefore be complemented by another something that can be shown directly. To put it another way, mathematical ostensive objects have a characteristic akin to things like oranges, tables, etc. that is, they have real existence in time and space. By contrast, this kind of existence is not attributed to non-ostensive objects, which are usually considered to have an ideal existence. One problem in the philosophy of mathematics is to define the nature of this ideal existence.

In the ostensive category, one must, as a minimum, consider the material representations (for example, $f'(x)$ on the blackboard) and material examples (for example, three oranges) that are normally considered as extra-mathematical. We also believe it useful to include in this category the so-called instruments that form part of mathematical practices (for example, the compass). However, there are several reasons why, in mathematical discourse, a distinction is made, whether implicitly or explicitly, between ostensive representations and non-ostensive objects. Here we will focus on just two, the first being that in mathematics discourse it is possible to talk about ostensive objects representing non-ostensive objects that do not exist. The second reason is that there are different representations which are regarded as representations of the same mathematical object.

The type of existence that can be ascribed to non-ostensive objects is a key question in the philosophy of mathematics. In addition to Platonism, which argues that these objects exist in a different world to that of their material representations, a number of different theoretical approaches have been proposed as regards the possible objects: (a) the Meinongian approach, which attempts to construct a general theory of objects other than ordinary, concrete, existing objects; (b) possibilist realism; and (c) actualist representationism. Possibilist realism takes non-actual[6] possible objects to be (real, genuine) objects, that is, their metaphysical status is taken to be on a par with that of actual objects. By contrast, in actualist representationism, existence is conceptually prior to actual existence. When possibilist realists assert that "non-actual possible objects exist", the word exist has the same linguistic meaning as when actualists state that "actual objects exist" (Yagisawa, 2009). Other approaches worthy of mention in this context are the fictionalism that derives from the ontology of the philosopher Roman Ingarden (Błaszczyk, 2005; Smith 1975), and the nominalist fictionalism of Field (1980, 1989). Fictionalism argues that we can think about mathematical objects in the same

[6] The terms ostensive and non-ostensive can be considered as equivalent, respectively, to the terms actual and non-actual.

way as we do about a fictional character in a novel, in other words, the number 13 has the same ontological status as a fictional character.

As argued in Section 2, our ontological approach considers that in the context of classroom mathematics a language game is developed which leads students to regard ostensive material representations as being different from both the thoughts which people have when using them and the objects they represent. Moreover, both people's thoughts and these objects are regarded as having an existence that is different from the material form. For example, the ostensive text $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ can be considered as an ostensive representation or as an objective non-ostensive proposition represented by this text, and both these possibilities can be considered as being different from the mental process that occurs in the subject who thinks about this proposition.

Our ontological proposal therefore contemplates two kinds of existence for non-ostensive objects. The first corresponds to personal non-ostensive objects, which are located within the subject, whereas the second corresponds to the rules for manipulating ostensive objects that are employed in the context of problem-solving practices, and which we therefore locate within the configurations of institutional primary objects. Consequently, their form of existence is not independent of people, whether considered individually or socially.

This refinement of the internal/external duality by means of the ostensive/non-ostensive and personal/institutional dualities is a theoretical tool that enables us to clarify and differentiate between processes and objects which, despite being very different, are often presented simply in terms of the internal/external classification. For example, the mathematical notion of derivative function would be considered as external according to the duality internal/external, as would be the symbol $f'(x)$ written in ink in a mathematical text. Yet if we apply jointly the dualities ostensive/non-ostensive and personal/institutional, then the former example would be regarded as a non-ostensive institutional object, while the latter would be an institutional ostensive object.

Having postulated that one way in which primary objects participate in mathematical practice is in the form of non-ostensive objects, the problem to be addressed is as follows: In mathematical discourse a language game is produced that leads to the emergence of an object, one that is not considered to be ostensive, cannot be identified with any of the primary objects of the configuration, and which, moreover, is considered to be the referent for that configuration when considered as a whole. According to our ontological proposal, this latter type of non-ostensive object, which serves as the referent for the configuration as a whole, has a virtual existence, since mathematical practices, in fact, make use of socio-epistemic configurations which are rules of convention resulting from a process of intersubjective construction. This approach therefore assumes, in part, a certain degree of fictionalism as regards the kind of existence which can be ascribed to these objective non-ostensive objects that cannot be identified with some of the primary objects contained within the socio-epistemic configurations, an existence made possible by, among other things, the human capacity to engage in processes of idealization on the basis of ostensive objects. In the first section we saw how this fiction emerges and how it promotes a realist view of mathematics among pupils.

In the philosophy of mathematics it is also usual to consider the duality concrete/abstract, which is often regarded as equivalent to the duality particular/universal. In our proposal, however, it is considered important to make a clear distinction between these two dualities, since, in our view, a failure to do so render the particular and abstract categories invisible. An exemplar such as “a tiger” is considered to be an object in space-time, whereas in mathematics, particular objects are usually regarded as ideal. This issue cannot be addressed simply in terms of particular vs. general, but requires use of the ostensive/non-ostensive duality. As argued by Font and Contreras (2008), it is not possible to identify processes of generalization with those of idealization (as illustrated, for example, in Activity 1 of Section 5). Neither, therefore, can the dualities concrete/abstract and particular/universal be identified with one another, and thus we replace them with ostensive/non-ostensive and extensive/intensive.

Extensive/intensive

We constantly seek to break reality down, in some way or another, into a multiplicity of identifiable and discriminable objects, to which we refer by means of singular and general terms (this chair, a table, etc.). This also occurs when we analyse mathematical practices (the letter x on the board, the function $f(x) = 3x +2$, etc.). The extensive/intensive facet (exemplar/type; particular/general) acts upon these objects.

Hence, another way in which primary mathematical objects can be regarded as ‘being’ in mathematical practices is related to this extensive/intensive duality. Such objects may be participating as particular or general objects and, depending on the language game, they may shift from being particular to general or vice-versa. For example, in the problem posed in Activity 2B of Section 5 it is obvious that certain propositions are considered as particular cases of more general propositions.

The extensive/intensive duality can be used to explain one of the basic characteristics of mathematical activity: the use of generic elements (Font & Contreras, 2008). Specifically, this duality enables attention to be focused on the dialectic between the particular and the general, which is a key issue in the construction and application of mathematical knowledge. According to widespread usage a universal is something that can be instantiated by different entities, and the distinction between particulars and universals can thus be made in terms of a relationship of instantiation: We can say that something is a universal if and only if it can be instantiated by more than one entity whether by particulars or universals, otherwise it is a particular. Therefore, while both particulars and universals can instantiate entities, only universals can be instantiated. If *whiteness* is a universal then every white thing is an instance of it (Rodriguez-Pereyra, 2008).

In our proposed ontology, intensive objects correspond to those collections or sets of entities, of whatever nature, which are produced either extensively, by enumerating the elements when this is possible, or intensively, by formulating the rule or property that characterizes the membership of a class or type of objects. The elements of these intensive objects may be other intensives, which in this case will be regarded as extensive objects. In other words, our distinction between extensive and intensive is a relative one that depends on the context or language game in which the objects participate. Therefore, in addition to not identifying the particular/universal duality with the concrete/abstract one, we also interpret the extensive/intensive duality in a broader sense than is usually the case for the distinction between the particular and the universal.

Although the extensive/intensive duality can be applied to a single primary object of an epistemic configuration, we believe that a useful approach as regards mathematics education is to apply it to the configurations of primary objects. For example, if a term is considered to be a derivative or a natural number, then this derivative (or natural number) will have participated in many different mathematical practices throughout the history of mathematics. This set of practices can be grouped into different sub-sets of practices that are carried out due to the activation of certain socio-epistemic configurations, some of which may be regarded as reorganizations or generalizations of previous ones. By applying the extensive/intensive duality to epistemic configurations it is possible to develop, in its institutional facet, the idea behind Radford's cultural semiotics, that is, the mathematical object as an entity stratified in layers of generality: "First, the conceptual object is not a monolithic or homogenous object. It is an object made up of layers of generality" (Radford, 2008, p. 226).

Unitary/systemic

When one studies a new topic a systemic presentation is made, since what one studies are socio-epistemic configurations and the practices that these configurations enable. However, when a new topic begins, the previously studied configuration and the practices it enables are considered as a whole. When addition and subtraction are studied in the final levels of primary education, the decimal number system (tens, hundreds, etc.) is regarded as something known and, consequently, as consisting of unitary (elemental) entities. In the first year of school, however, these same objects have to be considered systemically in order to be learned.

Both socio-epistemic configurations and the primary objects of which they consist may be considered from a unitary perspective. However, we may, at times, be interested in adopting a systemic perspective on them, for example, considering what can be done with them or with the parts of which they consist. Consequently, another way in which primary mathematical objects and socio-epistemic configurations can be regarded as 'being' in mathematical practices is related to the unitary/systemic duality. They may participate as unitary objects or as a system, and, depending on the language game, they may shift from being unitary objects to being a system.

As a summary of the above, Figure 5 shows the types of objects that are involved in mathematical practices and the different ways in which they may participate therein. This constitutes an initial proposal for the ontological dimension of a philosophy of mathematics education.

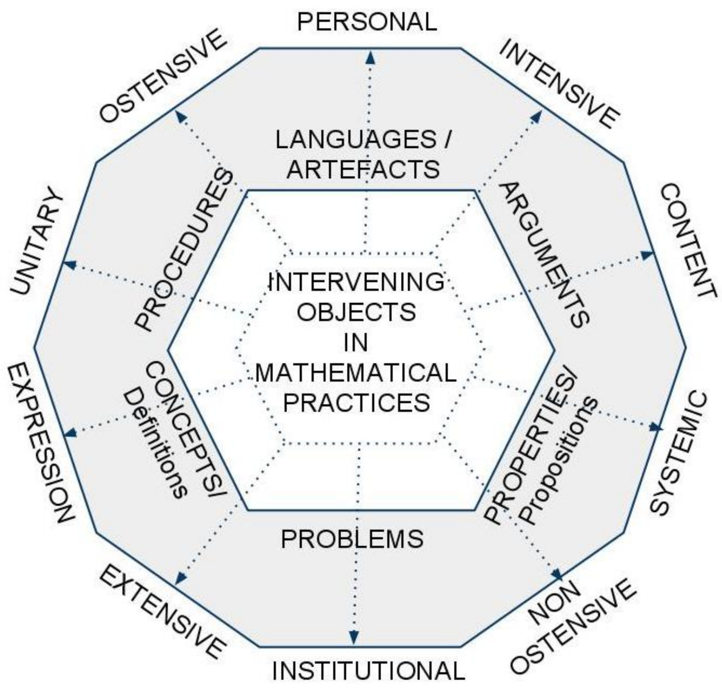

Figure 5. Ontology for an Educational Mathematics Philosophy

Both the dualities and the configurations of primary objects may be analysed from the process/product perspective. The objects of a configuration (problems, definitions, propositions, procedures and arguments) emerge through the respective mathematical processes of communication, problematization, definition, enunciation, development of procedures (algorithms, routines, etc.) and argumentation. For their part, the dualities give rise to the following *cognitive/epistemic processes:* institutionalization–personalization; generalization – particularization; analysis/ decomposition – synthesis/ reification; materialization/ concretion– idealization/abstraction; expression/representation–signification. In Activities 1 and 2 of Section 5 we discussed the role that some of these processes play in the emergence of primary objects. However, a more detailed exploration of the typology of processes illustrated in Figure 5 is beyond the scope of the present paper.

6.4. Singularity and Plurality of the Mathematical Object

The ontology that has been developed above can now be used to answer the third question posed in this paper: Is it possible to offer a non-realist explanation of how mathematical objects emerge from mathematical practices?

Firstly, it should be noted that it is not difficult to explain the emergence of properties or definitions, etc. As pointed out in the examples used in Section 5, the new primary objects appear as a result of mathematical practices and become institutional primary objects due, among other things, to processes of institutionalization. In order to explain how objects emerge from socio-epistemic configurations a useful metaphor is that of “climbing stairs”. When we climb stairs we have to stand on one foot as we move, but that foot then moves progressively to a higher stair. Mathematical practice can be considered as climbing stairs. The stair on which we stand in order to carry out the practice is an already-known configuration of primary objects, whereas the higher stair which we then reach, as a consequence of the practice carried out, is a new configuration of objects.

This explanation has been illustrated with two examples in Section 5. However, in these two examples we have not addressed the specific emergence of linguistic terms. With regard to that emergence we could solve the above objection: How can linguistic terms be considered emerging objects since they are constitutive of the practice from which primary objects emerge? The following definition of derivative function, taken from a Spanish textbook for 16–17-year-old students, is an example of an institutionalization process from which the definition of derivative, as well as different representations of the derivative function, emerge:

> Given a function *y = f(x)*, we can consider a new function $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ that associates at each point of the domain of *f* its derivative *f'(x)* when it exists. This function is the derivative function of *y = f(x)* and is represented with *f'(x)* or *y'*.

Two objects are institutionalized, the terms *f'(x)* and *y'*, which implicitly appear as objects referring to other objects.

Moreover, the history of algebra suggest that an essential step in its development was the passage from considering mathematical signs as objects that represent other objects to consider them as objects on which actions can be performed.

Despite having explained how primary objects emerge from socio-epistemic configurations, we still have to explain the emergence, in the mathematics classroom, of the object that is considered to be the referent for the epistemic configuration as a whole. This mathematical object is considered, implicitly or explicitly, to be defined, represented and described, etc. by the primary objects of socio-epistemic configurations, and, moreover, these can be defined and represented in different ways, with different properties, etc., in other words, it serves as a referent for several different socio-epistemic configurations.

What, in the Platonic or empiricist approach to philosophy, would be considered a mathematical object that existed independently of people (for example, the limit or the derivative) is, in the ontology proposed here, regarded as a virtual or fictitious object that emerges from the different ways of, globally (i.e. holistically), seeing, speaking about, or operating on all the objects of the socio-epistemic configuration(s). In other words[7], this object would be the content that, explicitly or otherwise, is globally referred to by the pair: mathematical practices and the socio-epistemic configuration which activates them. Let us consider the case of the derivative function defined as the limit of rates of variation. This definition forms part of a socio-epistemic configuration in which there are properties that are assumed to belong to this object called the derivative, and procedures that are understood to be those which enable the object to be used. Furthermore, when studying the derivative, one studies, in addition to the socio-epistemic configuration limit, the socio-epistemic

[7] In phenomenological terms this would be the intentional object.

configuration slope of the tangent line, which leads to the understanding that the derivative can be defined and represented in various ways. The result is that one considers there to be an object, called the derivative, which serves as the global referent for one or several socio-epistemic configurations.

The emergence of this virtual or fictitious object can be explained by the combined effect of different dualities. The unitary/systemic duality enables the configuration(s) to be considered as an object. The expression/content duality allows the object to be duplicated by considering the representation and the object represented. As regards the represented object, the ostensive/non-ostensive duality enables it to be considered as non-ostensive, whereas according to the personal/institutional duality it is objective.

In the context of mathematical activity this virtual or fictitious object *presents itself* in the form of an epistemic configuration, and it is therefore this configuration which determines what can be done with the object. This statement implies that the point of view being proposed here regarding the emergence of mathematical objects has aspects in common with structuralist perspectives (Resnik, 1988; Shapiro, 1997), which consider that the substance of mathematics is not individual mathematical objects but, rather, the structures through which they are constituted.

Furthermore, it should be taken into account that this object emerges over time from various systems of different practices. A given object may be considered as singular for reasons of simplicity (for example, the whole number), but, in each subset of practices, the configuration of objects in which it presents itself varies (different constructions of the whole number) and, therefore, different practices become possible (Godino, Font, Wilhelmi, & Lurduy, 2011).

In a way this object can be considered to be one and many at the same time. On the one hand, it can be regarded in a unitary way as emerging from various systems of different practices and from the configurations that activate them. In this case it would be the object associated, as the global referent, with the pairs of mathematical practices and the socio-epistemic configurations that activate them; this association would explain how it is possible to consider that the object can be defined in different, equivalent ways, or that it can be represented by different representations, etc. On the other hand, we can consider that, in each configuration, the object associated as the global referent of that configuration is different, even in the event that it is possible to establish an isomorphism between two structures[8].

7. The OSA ontology within the current debates in mathematics education

The reflection on mathematical objects is present in most mathematics education research programs, more explicitly in some of them. Research programs in which mathematical objects play a central role have very different theoretical basis, ranging from cultural semiotic

[8] According to this latter interpretation the point of view being expressed here regarding the emergence of mathematical objects would have certain commonalities with the way in which phenomenology characterizes such objects (Błaszczyk, 2005; Smith 1975).

positions (Radford, 2006), cognitivist approaches (APOS Theory), or frameworks based on cognitive linguistics (Lakoff & Núñez, 2000).

The theory of knowledge objectification suggests that mathematical objects are historically being generated along the individuals' mathematical activity. More precisely, "mathematical objects are fixed patterns of reflexive activity incrusted in the ever changing world of social practice mediated by artefacts" (Radford, 2008, p. 222). We believe that the onto-semiotic approach complements this notion of object in three aspects. On one hand, the ontology explained in the previous sections conceives the nature of mathematical objects as rules and, on the other hand, it proposes a typology for these rules (the primary objects of configurations). It also suggests that the elements of each configuration and of the whole configurations can be related through semiotic functions, which in some cases relate extensive with intensive objects.

While the extensive-intensive duality can be applied to the primary objects, in our opinion, its application to configurations of objects is still more useful for mathematics education. For example, let us consider the term derivative. Derivatives have been used throughout the history of mathematics in many different mathematical practices. This set of practices can be partitioned into different subsets of practices carried out by activating certain epistemic configurations, some of which can be considered as reorganizations and generalizations of the above. These last two aspects help to operationalize the statement that makes the theory of objectification (Radford, 2008, p. 226): "The conceptual object is not a monolithic or homogenous object. It is an object made up of layers of generality." As stated Santi (2011):

> Although the cultural semiotic approach and the onto-semiotic approach have different systems of principles, they are complementary theories that can be integrated to investigate the issue of meaning. This integration is accomplished by the social and cultural understanding of mathematical objects they share: a fixed pattern of reflexive activity stratified in layers of generality and objects with a regulatory nature whose rules determine different types of objects (configurations of primary entities), respectively. (Santi, 2011, pp. 306-307)

In the APOS theory (Asiala et al., 1996; Dubinsky & McDonald, 2001) encapsulation and de-encapsulation of objects play an important role. APOS theory begins with actions and moves through processes to objects. These are then integrated into schema which can themselves become objects. In this theory basically two uses of the term object is made. An object is considered as the result of the encapsulation process, or the result of the thematization scheme. This way of conceptualizing the emergence of objects in the APOS highlights partial aspects of the complex process that according to the OSA brings up the students' personal mathematical objects from mathematical practices carried out in the classroom. The OSA helps explain relevant aspects of the complexity of both mechanisms, revealing that they produce the emergence of different objects (primary objects in encapsulation) and a global reference (in the thematization).

The notion of object also plays an important role in the Cognitive Science of Mathematics (CSM). Lakoff and Núñez (2000) state that the mathematical structures that people build should be looked for in everyday cognitive processes, such as image schemas and metaphorical thinking. People use these basic schemas, called image schemas, to make sense of their experiences in abstract domains through metaphorical mappings. These authors claim that metaphors create a conceptual relationship between the source domain and the target domain. They distinguish between two types of conceptual metaphors in relation to mathematics: (a) grounding metaphors that relate a source domain outside mathematics with a target domain inside mathematics; and (b) linking metaphors with both source and target domains within mathematics.

Grounding metaphors include ontological metaphors, where we find the object metaphor. This is a conceptual metaphor that originates in our experiences with physical objects, and enables us to interpret events, activities, emotions and ideas as if they were real entities with properties. This metaphor is combined with other ontological, classical metaphors such as that of the "container" and that of the "part-whole". Their combination leads to the interpretation of ideas and concepts, etc. as part of other entities and constituted by them.

We believe that Lakoff and Núñez's methodology of "mathematical idea analysis" is very important to explain the emergence of mathematical objects, but it is insufficient to describe adequately this emergence and the nature of mathematical objects. This limitation was pointed out by various authors in the discussions that followed the publication of Lakoff and Núñez's book (e.g. Sinclair & Schiralli, 2003).

The way in which the OSA explains the emergence of mathematical objects does not only extends and improves the explanation offered by the CSM, but it also clarifies one central process considered by the latter, namely the metaphorical process (Malaspina & Font, 2010). The epistemic/cognitive configurations explain and make precise the structure that is projected onto the conceptual metaphors. There is a source domain with an epistemic/cognitive configuration structure (no matter whether the adopted point of view is institutional or personal) and which projects itself onto a target domain with the same structure (epistemic/cognitive configuration).

A more detailed analysis of the concordances, complementarities and possible links between these theories on the mathematical object (and others, such as the Sfard (1991)'s theory of reification) and the model proposed by the OSA should be addressed in other papers.

8. Final considerations

This paper has illustrated the interpretative potential of our ontology of mathematical objects and has offered an explanation of how the latter emerge from mathematical practices. This theoretical framework enables us to understand how, in the classroom, students come to be convinced, explicitly or otherwise, that there are mathematical objects which exist independently of people (in the Platonic world or in nature) and which are defined or represented by some of the primary objects of socio-epistemic configurations.

When performing mathematical practices we make reference to an already-known socio-epistemic configuration of primary objects and, as a result, access a new configuration of primary objects in which one (or more) of these objects was previously unknown. This leads to the emergence of the object that serves as the global referent. This emergence is due to the combined effect of different dualities, as explained in the previous section. If to this we add the discourse about the objectivity of mathematics, its application to the real world and the use of certain words (for example, discovered), then we have a plausible explanation of how, in the classroom, there emerges the descriptive/realist view of mathematics which considers (a) that mathematical propositions describe properties of mathematical objects, and (b) that these objects have a certain kind of existence that is independent of the people who encounter them and the language through which they are known. This view is hard to avoid since the reasons why it is adopted are always operating, albeit subtly. More than a consciously assumed philosophical position we are dealing here with an implicit way of understanding mathematical objects.

Without going into detail it is also worth noting that this descriptive/realist view of mathematics is not only generated by students but also by prospective mathematics teachers in the initial stages of their training, many of whom continue to take this view after qualifying. Although there is no direct relationship between conceptions of mathematics and ways of teaching, the descriptive/realist view does, in certain cases, influence the type of mathematics that is taught and the criteria used by teachers to assess students' performance (Morgan & Watson, 2002).

We consider the notion of emergence useful to describe not only mathematical activity at the individual level (learning processes) but also the progressive social construction of mathematical knowledge from a historical and epistemological point of view. Moreover, when applied to mathematics as a science, this notion explains the latter as a social construction, whereas when applied to individuals, it does not necessarily involve a constructivist perspective of teaching.

By assuming a first-level emergence of primary objects (understood as conventional rules), we partly adopt a kind of constructivism. Our assumption is that mathematics does not describe a certain reality pre-existing to the mathematician's activity; these rules are created by human minds and therefore are the result of mental constructions. However, its conventional acceptance involves the existence of social institutionalization processes. This conception of emergence can be applied to both the subjects' cognitive configurations and to the shared epistemic configurations within a community of practice (institution).

This leads us to share some features of constructivist theories that conceives construction as follows: The person, who has been formed within a community (institution) and that, therefore, is sharing an intersubjectivity, builds a personal cognitive configuration from his/her actions on the physical and social environment. This configuration can be represented in the material world by different systems of signs subject to certain rules (syntactic, semantic and pragmatic) conveyed by the language and agreed by intersubjectivity (epistemic configuration).

Opting for emergence rather than the notion of construction therefore enables us to occupy a common ground with both social constructivism (Ernest, 1998) and Radford's cultural semiotics (Radford, 2006). Indeed, although the explanation offered in this paper regarding the emergence of mathematical objects has its own specific features, it also shares or at least does not contradict a number of principles associated with other theories of how these objects emerge from mathematical activity. These theories would include activity theory (Engeström, 1987) as applied to mathematics (e.g. Jurdak, 2006), Radford's theory of knowledge objectification (Radford, 2008), the anthropological theory of didactics (Chevallard, 1992), social constructivism (Ernest, 1998) and socio-epistemology (Cantoral, Farfán, Lezama, & Martínez-Sierra, 2006). It is also compatible with, or at least does not contradict, the explanation offered by the theory of *commognition* (Sfard, 2008) and, more generally, with all those approaches which consider that mathematical discourse and its objects are mutually constitutive.

This paper has developed a conventionalist position that, in ontological terms, occupies a middle ground between nominalism and realism. With respect to the nature of mathematical objects our approach shares with other theoretical models the idea that their emergence is intra-discursive, even though this statement says nothing about their nature. We therefore believe it is necessary to go one step further and state that they are basically conventional–normative entities, in other words, and as Wittgenstein proposed, they should be regarded as the grammatical rules we follow when working with the languages we use to express our worlds. Our way of conceiving of the emergence of mathematical objects enables us to understand how such objects, regarded as emerging from a system of practices, can be considered as singular. Yet it also shows that, in each subset of practices, the socio-epistemic configuration in which the object in question presents itself varies and, therefore, different practices become possible.

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