SUITABILITY CRITERIA OF A MATHEMATICAL INSTRUCTION PROCESS

A teaching experience of the function notion[1]

Juan D. Godino[2], Miguel R. Wilhelmi[3], Delisa Bencomo[4]

Summary

The final objective of didactic research is to find "suitable" devices for teaching and learning of notions, processes and meanings of mathematical objects. In this way, an objective for mathematics education should be describing and assessing effectiveness of mathematical instruction; it is also necessary to determine rules for the improvement of design and implementation of teaching and learning processes of specific mathematical contents. These implications for teaching do not have normative or technician character (obtaining of a listing of prescriptions "to execute"), but explanatory. In this article we analyse the suitability of an instruction process on the function notion with university students according to three dimensions: epistemic, cognitive and instructional.

1. DIDACTIC PROBLEM AND CONCEPTUAL FRAMEWORK

Didactic engineering (Artigue, 1989) has a double objective: critical interventions in the didactic systems (the didactic knowledge scientifically based delimits the action) and contrasting the theoretical proposals elaborated. In this way, the didactic engineering seeks to *a priori* control the implementation of teaching projects. In a second phase, called analysis *a posteriori*, the analysis a priori is compared to the effective realization to rejects or not the hypotheses on which it is based. This comparison is carried out distinguishing three dimensions (cognitive, epistemic and instructional) and, of course, taking into account the specific objectives of the investigation.

The final intention of the didactic research is to find "good" devices for teaching and learning notions, processes and meanings of mathematical objects, keeping in mind the institutional restrictions of the epistemic, cognitive and instructional. Didactic engineering articulates the researchers' productions with the action needed in teaching processes, allowing the evolution of an explanatory didactics toward a normative or technical didactics (supported in a theory and experimentally contrasted). This evolution is complex and expensive, of course. Nevertheless, the application of the technical products is also influenced by teachers' mathematical and didactical training, who ultimately should implement these resources.

Therefore, it is necessary to evaluate the educational practice of teachers and determine criteria to improve the design and implementation of teaching and learning processes of specific mathematical contents. These implications for teaching do not have normative or technical character (obtaining a listing of prescriptions "to execute"), but explanatory

[1] *Mediterranean Journal for Research in Mathematics Education* (to appear)

[2] The University of Granada, Spain. E-mail: jgodino@ugr.es
[3] The Public University of Navarre, Spain. E-mail: miguelr.wilhelmi@unavarra.es
[4] The National Experimental University of Guiana, Venezuela. E-mail: dbencomo@uneg.edu.ve

(determination of approaches for the evaluation of the viability and adaptation to a teaching project).

The objective of this article is, in certain sense, inverse to that of didactic engineering: starting from effective realizations, to evaluate the suitability and relevancy of a mathematical instruction process. Based on the Godino's (2003) theoretical proposal we intend to evaluate the suitability of a mathematical instruction process according to three dimensions:

1. *Epistemic suitability*: adaptation among the implemented institutional meaning and the reference meaning, in particular, it would suppose the elaboration of a viable didactic transposition (able to adapt the implemented meaning to the intended meaning) and pertinent (able to adapt the intended meaning to the reference meaning).
2. *Cognitive suitability*: the "learning material" is in the students' ZPD (zone potential development) (Vygotski, 1934); in other words, the gap between the implemented institutional meanings and the initial personal meanings is the maximum that it is possible to tackle by students taking into account the students' cognitive restrictions and the available resources (time, teacher' support, …).
3. *Instructional suitability*: capacity of didactical configurations and trajectories so that teachers and students identify most potential semiotic conflicts (*a priori*), real conflicts (during the instruction process) and residual conflicts (*a posteriori*), to solve these conflicts by meaning negotiation (using the available resources that determine institutional restrictions of mathematical and didactic character).

We will apply and develop these notions to the analysis of the suitability of an instructional process concerning the function notion, implemented by a teacher with a group of university students of a first engineering course; in short, we wonder, to what extent the observed instruction process is suitable? The relevance of this case is based on its representativeness for a more general type of instructional behaviour implemented according to a "naïf-constructivism" approach in math education and the elaboration of the theoretical notions mentioned.

The article is structured in the following way: in section 2 we describe the observed instructional process; in section 3, we determine the types of institutional meanings associated to the function notion; in section 4, we discuss the suitability of the instructional process and, finally, in section 5, we make a brief synthesis of the study carried out and we stand out some implications concerning teacher training.

2. THE INSTRUCTIONAL PROCESS

The objective of the observed teaching is that students remember, interpret and formalize the definitions of correspondence, function, range, domain and types of functions, applying them in a situation that involves knowledge from Physics: the vertical throwing of a ball with an initial speed. It is supposed that students have previously studied the definitions of these notions and it is accepted that the mathematical task is an "application exercise." Implicitly, the teacher presupposes that students are able to interpret these definitions, carrying out a *disjunctive generalization* (Tall, 1991, p.12) and, this way, to identify the components of the parabolic function proposed and to use the meaning learned as a tool to solve the task. In table 1 we include part of the questions posed to the students.

Table 1: Some questions posed to the students

A ball is thrown directly up with a speed V_o so that its height, t seconds later, is $y(t) = v_0 \cdot t - g \cdot t^2 / 2$ meters, where g is the gravity acceleration. If the ball is thrown with a speed of $32\ m/s$ and $g = 10 m/s^2$ (approx.): 1. Determine the maximum height that reaches the ball, drawing the graph of $y(t)$. 2. Is $y(t)$ a correspondence or a function? 3. If it is a function, which is it domain, co-domain and range? 4. Is $y(t)$ an injective, surjective or bijective function?

The teacher organized the instructional process splitting the class in four students' teams, assigning to each one of them a part of the task. A student of each group explained to the class the solution; the teacher completed or corrected the student's explanation. The *implemented didactic trajectory*, that is to say, the sequence of ways of managing the meanings implemented concerning a specific mathematical object, includes, therefore, configurations of cooperative type, dialogical, and magisterial (Godino, 2003, pp.202–204).

To work the proposed questions the teacher devoted three classes of 45 minutes. In table 2 we present a brief description of the activities carried out in each class.

Table 2: Summary of activities carried out in class

Class	Description
1	The teacher gave the task instructions and he drew the parts of the problem that should be presented in class by each one of the groups. Almost all the time was devoted to the resolution of the first question. The class concluded with the graphical representation of the expression $y(t) = v_0 \cdot t - g \cdot t^2 / 2$.
2	This class begins with the student's presentation of the second problem item; the teacher presents an example of function using the graphical representation of the mathematical expression, $x = (y-a)^2 - b;\ a,b > 0$, and he went on working the second item by a series of questions and answers. A group of students presented also the fourth item and the teacher exposed the composition of functions using Venn's diagrams. The teacher concluded the class with another series of questions and answers in order to clarify the injective, surjective and bijective notions.
3	To begin this class the students exposed the third item. The teacher, dissatisfied with the students' explanations, proposed a new task: to determine what is a correspondence, what is a function and what types of functions there exist (in particular, to determine if the function $y(t) = v_0 \cdot t - g \cdot t^2 / 2$ is surjective or injecive).

The analysis of the instructional process described is dependent from the didactical theory used, as it is in general every fact or didactical phenomenon. The suitability criteria are based on the meaning notion and on the meaning types identified (institutional and personal). This way, it is necessary to determine what we means with the expression "meaning of a mathematical object" and what types of meanings we identify. In the following section we carry out a more engaged description of the instructional process, adopting the point of view of the onto-semiotic approach to mathematics knowledge and instruction (Godino, Batanero and Roa, to appear).

3. TYPES OF INSTITUTIONAL MEANINGS

Godino (2003) identifies the "system of operative and discursive practices" with the meaning that an institution assigns to a mathematical object, settling down, therefore, a correspondence among the system of practices and the expression of the mathematical object. To analyse the meanings of a mathematical object, Godino (2003) propose to distinguish four types of institutional meanings that he designates as *reference*, *intended*, *implemented* and *evaluated* institutional meaning.

3.1 INSTITUTIONAL MEANINGS

When a research plans an instructional process on a mathematical object for a group of students, he/she begins defining "what this object is for the mathematical and didactical institutions." He/she will study, therefore, the corresponding mathematical texts, the curricular orientations, and in general what "the experts" consider they are the inherent operative and discursive practices to that object, fixed as instructive objective. The teacher will also use his previously acquired personal knowledge. With all these things, he/she will build a system of practices that consider as the institutional reference meaning for the object.

Starting from the reference meaning, the teacher selects, orders, and defines the specific part that he/she goes to propose to his/her students during an instructional process. He/she will takes into account the available time, the previous students' knowledge and the available instructional tools. The system of practices that he/she plans on a mathematical object for a certain instructional process we name *intended institutional meaning*.

As result of teacher and students' interactions the intended institutional meaning changes in fact, so that finally the system of practices implemented can differ regarding those planned. With the purpose of introducing as research object these processes of change in the institutional meanings it is needed to speak of the *implemented meaning*, as the system of practices that indeed it takes place in the mathematics class, which will serve as immediate reference for the students' study and the learning assessment.

A fourth type of institutional meaning is involved in the assessment process. Teacher selects a collection of tasks or questions that he/she includes in the assessment tests and observation grills of the cognitive trajectories. These items will be a sample (it is expected to be representative) of the implemented meaning. This way it takes shape the *evaluated institutional meaning*.

In the case of our teaching experience, the intended meaning of the function notion is identified with the system of practices in the context of set theory, that is, as the essentially discursive manipulations of a triplet $(A, B, y = f(x))$, where A represents the initial set; B, the final set; and, finally, $y = f(x)$ the rule of correspondence. In this setting, the prototypical definition of function (emergent of the system of practices and explicit reference thereof) is enunciated in terms of set theory in the following way:

> Given two sets A and B, not empties, a function from A to B is a correspondence $f \subseteq A \times B$ that fulfills:
>
> (i) $(a, b), (a, b') \in f \Rightarrow b = b'$
>
> (ii) $\forall a \in A, \exists\, b \in B\ ((a, b) \in f)$

In our case the teaching has the objective of formalization the function notion (intended meaning). For it, the teacher proposes an extra-mathematical modelling situation (see table 1), from which he builds the actual implemented (or taught) meaning. The analysis of the instructional process shows in what extent there is a fundamental distortion among the intended and implemented meanings, and how this conditions the meaning students are able to learn and the type of problems they potentially could deal with concerning the formal notion of function.

3.2. REFERENCE MEANING OF THE FUNCTION NOTION

When we are interested in the teaching and learning of any mathematical notion we cannot limit ourselves to explain the more general possible definition; any definition of a mathematical concept is as the top of the iceberg of a system of operative and discursive practices, relative to diverse use contexts and reference frameworks that constitute its origin and reason of being. In the onto-semiotic approach to mathematical knowledge (Godino, 2002) it is assumed that such systems of practices constitute "the meaning of the object"; and since such practices are relative to each institutional context it is derived a relativity and plurality of objects and meanings, where the usual mathematical culture identifies one object and meaning. The adoption of a plural and relativist ontology for mathematics education is useful to describe and understand the processes of didactic transposition and the social and personal construction of mathematical knowledge.

The function concept is a good example to show the diversity of systems of practices and use contexts, progressively wider, in which we can show the plurality of derived meanings of each subsystem of practices (Biehler, 1997). The reconstruction of the "function meaning" is a first necessary step to be able to understand the actual implemented teaching processes and to elaborate criteria for their assessment and improvement. Several authors have been interested in this reconstruction from a historical and epistemological point of view (Youschkevitch, 1976; Sierpinska, 1992). In particular, Ruiz (1998) made a systematic study and characterized seven "epistemological conceptions" of the function object, which she described using the conceptual triplet by Vergnaud (1990) (situations, invariants and representations). We prefer to interpret such "conceptions" in terms of subsystems of institutional practices linked to specific contexts of use, and of emergent objects (types of problems, actions, language, concepts, properties and arguments); each one of these epistemic configurations model partial aspects of the *comprehensive meaning*

(Wilhelmi, Godino and Lacasta, 2004) of the object function, which will play the role of reference meaning in a specific investigation.

The comprehensive meaning represents the objective frame of the institutional meanings on which every teaching project should be elaborated. Any institution, explicitly or not, determines a priori, in parallel to the notions, processes and intended meanings, a *didactical configuration*, that is, specific management guideline of the institutional meanings associated to the notions that the institution wants to introduce or develop, of course, taken into account the available human resources, materials and time. The balance among these dimensions (mathematical and didactics) is a necessary condition for an instructional process can be suitable. In the following section we will show in what sense the observed instruccional process is not suitable.

4. SUITABILITY OF A PROCESS OF MATHEMATICAL INSTRUCTION

To assess the suitability and appropriateness of a process of mathematics study we have to take into account three dimensions: *epistemic* (relative to institutional meanings), *cognitive* (relative to personal meanings) and *instructional* (relative to teacher-students' interventions, the readiness and use of material resources and time assigned).

As we have said, an instructional process is suitable from the epistemic viewpoint if the implemented meaning is faithful to the intended meaning and this, in turn, is faithful to reference meaning. In many occasions, in a process of mathematical study, it is possible to identify some fundamental mismatch among the reference, intended and implemented institutional meanings that they have not been a priori foreseen as constituent of the instructional process and that represent unfortunate didactical decisions. We call *epistemic conflicts* all these mismatch, which condition the study process and the students' learning.

The cognitive suitability is achieved when the gap between the implemented institutional meaning and the initial personal meanings is the maximum that students could deal with, taken into account the cognitive and resources restrictions. In occasions, this gap is excessive and it causes the students' cognitive failed-adaptations that cannot be tackle without partial or total modifications of the intended meanings, or the means assigned for its development. We name *cognitive conflicts* to these mismatch.

Finally, a study process is suitable from the instructional viewpoint, when the teacher and the students can, first, to identify semiotic conflicts and, second, to solve these conflicts with meaning negotiations. There is an *instructional conflict* when the teacher and students are incapable to identify a semiotic conflict or, in the event of identifying it, they don't have the necessary mathematical-didactic resources to solve them.

Evaluating the suitability of a mathematical instruction process requires to have detailed information of teaching and learning facts and reference elements that allow to emit statements about adaptation, relevancy or effectiveness corresponding to each dimension. One of the objectives of the stochastic modelling of an instructional process and their corresponding states by Godino (2003) is helping to identify epistemic, cognitive and instructional conflicts, which could cause the mismatch between the design of the instructional process and its implementation. The identification of these conflicts and their description allows to emit judgments on the suitability of a mathematical instruction process.

We are aware that evaluating the suitability of a instructional process requires to collect a complex of information on the state and evolution of the different components and dimensions that define it. It is necessary, therefore, to use diverse methods and observation techniques, recording and measure of data (questionnaires, interviews, audio-visual recordings, etc.), and to determine the students' cognitive states in different moments of the process. In the case we use as illustrative example we have the general program of the theme and recommended text books (institutional meaning of local reference) and the guide of tasks to carry out (intended institutional meaning); we also have the audio-visual recording of the development of three classes (implemented institutional meaning). Basing on this material, we will show some aspects of the implemented didactic trajectory and a partial evaluation of the different suitabilities.

4.1. Epistemic suitability

The observation of the teaching process described in Section 2 allows us to characterize the system of operative and discursive practices really implemented concerning the mathematical object *function*. The comparison of these practices with the reference meaning of this object allows us to identify diverse mismatch and to formulate hypothesis about the suitability of this process, mainly for its epistemic facet. In the epistemic trajectory we can distinguished sequences in which exist epistemic conflicts that it doesn't obey to a priori established teacher' interventions and whose objective could be that the students overcame a *cognitive or epistemological obstacle*.

According to the specificity of the epistemic conflict, regarding the system of operative and discursive practices relative to the mathematical object, we classify the conflicts in *general* and *specific*. We have a general epistemic conflict when it refers to a mathematical process (definition, proof, interpretation, etc.) not specific of the class of problems from which the object emerges. Otherwise, we call specific to that epistemic conflict.

The identification of a conflict, general or specific, supposes the observation of a mismatch fundamental between two praxemic entities (problems or actions), between three discursive entities (concepts, properties or arguments), or between two language games that are introduced or developed in two related institutional frames. These mismatches are identified in the use (action), the construction (action-arguments) and the communication (language-argument) of notions, statements and problems. In fact, the notions, statements, language, arguments, actions and problems (as constituent entities of the institutional and personal meanings) they are the empirical entities that allow us to make operative the suitability criteria and, therefore, to assess an instructional process.

This way, the conflicts can be classified, at least theoretically, in 120 different types (not disjoints) taken into account: the adaptations of institutional meanings (referential-intended; intended-implemented) (2 types), the specificity of the conflicts with relationship to the mathematical object of study (general, specific) (2), the entity involved (6) and the dual facet (5). Therefore, to identify a conflict it is necessary to give four features: level of process, specificity, component and facet.

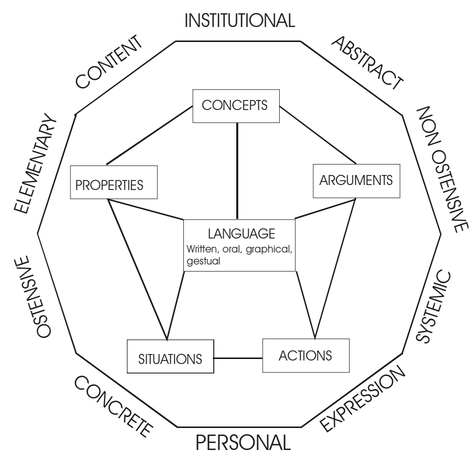


Figure 2: Components and facets of mathematical knowledge (Godino, 2003, p.146)

Next we will identify some epistemic conflicts in several segment of the transcription of the teaching experience.

Conflict E1: Intended-implemented, specific, problem, concrete-abstract

The main epistemic conflict that we find is relative to the election and formulation of the mathematical task posed to the students to learn the notion of function. Question 1 poses the mathematical modelling of a physical problem (forecast the space known the time) by means of a function given by an algebraic formula. The following questions ask to determine if the correspondence defined by the formula is or not a function, its domain, co-domain, range and type of function (injective, surjective, bijective).

> Is $y(t)$ a correspondence or a function? If it is a function, which is it domain, co-domain and range?
> Is $y(t)$ an injective, surjective or bijective function?

These questions refer to a problem of formal or discursive nature that is very different from the modelling problem. The "reason of being" of the system of objects involved in the set-theory model is the description, generalization and structuring of mathematical knowledge, unaware to the practice of functional modelling. Putting together question 1 and 2 creates an abrupt rupture among the different epistemic configurations associated to the object function.

Considering the classification of the conflicts introduced, this conflict is identified by the four features: *intended-implemented* (since it takes place in the adjustment institutional meaning process), *specific* (because it is specific to the task asked), *problem* (the main primary entity involved is the propose situation) and *concrete-abstract* (the quadratic function are representative of the generic notion of function; the lecturer accepts, implicitly, that the transference from the particular case to the formal notion of function is transparent).

Conflict 2: Referential-intended, general, concept, elementary-systemic

The set-theory configuration requires making explicit what are the initial and final sets whose elements correspond each other, so that if some of these sets, or the correspondence criterion change, one has a different function. The physical problem represented with the algebraic formula involves two real number intervals that can be taken as "initial sets, or domains of definition of the function", [0, 3.2] (time until the ball reaches the maximum height), or [0, 6.4] (time until it returns to the floor; the set image is the interval [0, 51.2] in both cases. The nature of the independent variable, time, makes that the values that it can take are positive real numbers, but the algebraic formula that establishes the correspondence criterion is valid for all real number. The final set (co-domain) of the function $y(t) = v_0 \cdot t - g \cdot t^2 / 2$ is totally indefinite, and therefore, it does lack sense to ask if "the" function (which?) is or not surjective.

> *T: Then, look, this function d, relates me a positive set of real numbers, it has to be used including the zero, and where will go that set?; to what number set will go? Integer again, integer and the 27.5; positive or negative? And the zero, will you leave it away? There you have this function, the function height relates the set of real numbers including zero with the set of real numbers and zero.*

In mathematical terms the quadratic formula $y(t) = v_0 \cdot t - g \cdot t^2 / 2$, with v_0 = 32 m/sec and g = 10, is valid for every real number t. From the point of view of the physical modelling the variable t varies between 0 and 3.2 seconds (interval in which the maximum height is reached), included both end and not from the positive numbers to the positive numbers, as it was settled down in the class. This fact could have been used to explain the difference between *range* and *final set* and to introduce or develop the notion of surjective function. In a general way, the physical model acts as "distractor in the instructional project", that is, it takes the intended meanings away from those implemented.

This conflict is identified by the four features: *referential-intended* (the teacher does not settle down a clear distinction between the function as "model-instrument" and as "intra-mathematical object" that it conditions all the discourse), *general* (all model must be interpreted in terms of the referential system; it is always necessary to value the relevance of a solution of a problem[5]), *concept* (domain, image, …) and *elementary-systemic* (the quadratic formula that models the situation as isolated mathematical object or as an element of the formal definition of function in Set Theory).

Conflict 3: Referential-intended, general, property, elementary-systemic

> *T: A correspondence. This cannot be a function. This is a correspondence.*

Every function is a correspondence; in the same way that every square is a rectangle or that every sequence is a function. The mathematical practice tends to identify with the name the characteristic that discriminates to an object within a wider class. This way, it is forced in the language the exclusion of families of objects contained in more large classes: "it is a function, a particular type of relation", "it is a square, not a rectangle", "it is a succession, not a function", etc.

The conflict is identified by the four features: *referential-intended* (the intented meaning establishes the following categorical statement: the set of the functions and the set of the correspondences are disjoint); *general* (a class of identified objects —functions— is excluded of the objects of the reference universe —correspondences); *property* (to identify a class of objects —functions— which belong to a more large class —correspondences— it

[5] For example, in a "problem of ages" in which the procedure leads to an equation of second degree with one positive integer solution and another negative integer, it is needed to indicate that second it is not valid.

is used only one necessary characteristic —if $f(a) = b$ and $f(a) = b'$, $b = b'$—, that it is not sufficient); *elementary-systemic* (the function is an example of a more larger class denominated correspondence).

Conflict 4: Intended- implemented, general, language, expression-content

> *T:… Now the key question is, besides a correspondence, what is it needed so that it is a function? What is necessary to add it so that it becomes, or it constitutes a function?*

The question that the teacher poses suggests that the function definition is the union of conditions and not, as it is indeed, an intersection (Winicki-Landman & Leikin, 2000).

The conflict is identified by the four features: *intended- implemented*, *general* (given two sets A and B, if the set of the objects of A is defined by all the properties that define the elements of B and an additional one, then $A \subseteq B$); *language* (the teacher tries that the students assume the formalization process of the function notion; for that reason, he avoids to use a formalized language which give "clues" and, instead, he uses colloquial and confused expressions); *expression-content* (the register used by the teacher does not adapt to the theoretical development of the function notion).

The last conflict that we will show is similarly identified to the previous conflict since it refers an abusive use of the language too; nevertheless, in this occasion, it is specific of the function notion.

Conflict 4: Intended- implemented, specific, language, expression-content

> *T: And that breaks the continuity of a function?*

The teacher use the expression "continuity of the function" as synonym of the "basic condition that discriminates the particular type of correspondence called function" (for each x exists an unique y)"; this suppose a rupture in the use of the language that conditions the formalization of the function notion: are all the functions continuous?, a "discontinuous" function is the "connection of two functions"?

The described epistemic conflicts, that they have their origin in teacher's cognitive conflict on the "meaning of the function notion", have had serious consequences in the development of the study process. Particularly, the epistemic conflicts cause cognitive conflicts in the students that are not solved.

5.2. Cognitive suitability

In the onto-semiotic approach to mathematics cognition (Godino, 2002; 2003) the notion of personal meaning is introduced to designate the student's knowledge. These meanings are conceived, like the institutional meanings, as those "systems of operative and discursive practices" that students are able to carry out concerning a certain type of problems. The personal meanings are progressively built along the instructional process, starting from some initial meanings at the beginning of the process, and reaching some certain final meanings (achieved or learned). With the information recorded in the teaching experience, the cognitive suitability, that is, the proximity between the meanings implemented regarding to students' initial personal meanings can only be analysed by means of the students' interventions.

In the following dialogue we can observe that students remember the general set-theory definition of function, but they have many difficulties to interpret and applied them to the cases of functions involved in the modelling situation.

> T:… The key question now is, besides a correspondence, what is it needed so that it is a function? What is necessary to add it so it becomes or makes up a function?
>
> S2: That each element in the initial set has an element in the final set. That each element in the initial set doesn't have two elements in the final set.
>
> […]
>
> T:… Did you understand what she said? Now we ask: Then, that, [pointing out to the graph on the blackboard] is that a function? Isn't? There they said no. Is there some element in the initial set that has two images, or three or four?
>
> S3: Yes. For example, in *y,* I put the value zero in x and it gives me zero; in x I put six point four, also.
>
> T: And that means that it is not function? Why is not a function?
>
> S2: Well, … Yes, it is function, but apart from that it has another function.
>
> T: Another function?
>
> S2: Yes.
>
> T: Why do you say that it is not a function?
>
> S2: I have not said that it is not function, I cannot explain it, but it fulfils two types of functions.
>
> T: Two types of functions or two types of applications?
>
> S2: Through that graph one can know if it is surjective, because apart from that the definition is fulfilled, each element has one and only one image in the range, it also fulfils that one element has two images.
>
> P: There one element has two images?
>
> S2: Yes.
>
> T: Which element does it have two images?
>
> S2: No, none.
>
> S4: Teacher, one element of the initial set has two elements of the final set; for example, the zero and the six point four… There it is seen that two elements of the initial set fall in the same element of the final set.
>
> T: And that breaks the continuity of a function?
>
> S2: Not.
>
> T: […] How is the function definition? … To each element of the initial set it corresponds one and only one element of the final set. That doesn't mean that three elements of the initial set can have the same image.

We observe that the teacher refers to the drawing traced on the blackboard (an ostensive object) as if it were the object function (whose nature is not ostensive); we find this incorrect identification a possible explanation of the cognitive conflicts that students manifest. The teacher see the graph as a transparent, elementary entity, when in fact it constitutes a system of rules that it is necessary to explicit and to share so that the question, *is that a function?*, acquires sense. The teacher is not aware of the difficulties that presents the pursuit of a general rule (the function definition), an intensive entity, of the potentially limitless variety of particular situations in which it can be applied (extensive entities).

Another instructional segment in which we find cognitive conflicts that were not solved during the observed classes refers to the co-domain, range and surjective notion. The institutional meanings implemented for these notions were not concordant with the reference meaning. It was not fixed in the classroom a final set that allows speaking of co-domain and their coincidence or not with the image set.

> T:.. What is the co-domain?
>
> S2: [Pointed out to the x axis of the graph]
>
> T: Co-co-co-domain!!
>
> S2: If the range would come being the set "y", final set, the co-domain would have to be the reflection of both set, that is,…
>
> T: What the co-domain and range are the same ones?
>
> S2: No, no, it would have to be the path running the ball, since it leaves until it arrives.

These isolated incidents don't allow us, however, to evaluate the cognitive suitability of the instruction process in terms of proximity of the students' zone of potential development. It would be necessary to make a detailed pursuit of the students to determine if the explanations given by the teacher were actually effective. This pursuit could be made using tests, interviews, etc.

5.3. INSTRUCTIONAL SUITABILITY

The degree of instructional suitability takes into account the possibilities that the didactic configurations and its sequence along the didactic trajectory offer to identify potential semiotic conflicts and solving them by means of meanings negotiation (using the resources and time available). In our case we have observed some important facts that allow us to assess this dimension in the teaching experience. We next describe some observed "instructional conflicts".

A general pedagogical problem

The teacher assigns each one of the seven items of the task to a different student groups, in such a way each group, formed by 5 students, assumes solving one of these questions and present it to the class. This way of organise the classroom work imply that students who not try to solve some of the questions will have difficulties to follow the explanations of other classmates. The result is that the students' presentation become magisterial configurations to the majority of students, conducted in this case by "non-expert teachers" (the student who represents each group).

Didactical intended and implemented configurations

The teacher outlines a dialogical didactic configuration. Starting from a previous students' personal study the teacher outlines a dialogue based on the context of a physical situation whose aim is developing the function notion. The role that the teacher attributes to the students in the construction and communication of knowledge is crucial: he supposes the students will be able to identify the intended objects in the modelling situation and “de-contextualise” them to build the intended function meaning. With other words, the teacher tries to negotiate a constructivist learning: students, through the situation and in interaction with the teacher, should be able to evolve the personal meaning attributed to the function notion (as result of the personal study) and to obtain a faithful adaptation to the intended institutional meaning.

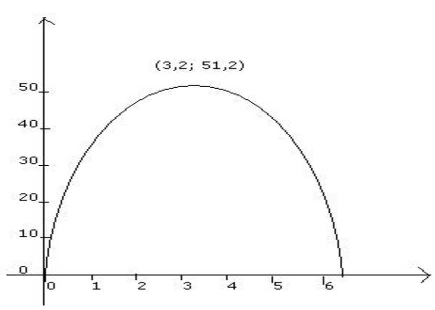
However, the effective didactic configuration cannot be considered dialogical. Most students have not previously carried out the personal study. The teacher is aware of it, but it doesn't modify the design of the intended instructional process. Little by little, the dialogical character of the implemented didactic configuration is diluted in a “fictitious mayeutic” in which the teacher takes in charge the formulation and the validation. This way, behind the effective dialogue, the learning session hides a magisterial didactic configuration, assumed by the professor without thinking. The teacher actually interprets many of the students’ answers as constituents of the intended meaning. The institutionalisation is then in a land “of anybody”, where the previously acquired students’ personal meanings, the contextualized meanings that the teacher introduces (related to the modelling situation) and the intended institutional meaning that the teacher states in some passages (and that he considers transparent in terms of the modelling situation), are coupled without success.

Evidently, this distortion among the intended and effective didactical configurations is the origin of conflicts. In short, a prototypical didactic problem emerges when the teacher is not aware, in the course of the effective teaching process, of these discrepancies, and he has the illusion that the process is developing in the terms that he had a priori settled down. In the following section we will show in what sense the instructional process observed is not suitable.

The role of regulation (institutionalisation)

By the end of the first class the first team wrote in the blackboard a table of values to graph the function, including negative value for the time.

T	***y*(*t*)**
–2	–84
–1	–37
0	0
1	12
2	48



The student marks in the graph the *t* values in which the maximum height is reached and the value of that height (3.2, 51.2). But just when the student starts to explain the solution, the teacher interrupts her and poses new questions aliens to the work previously carried out:

T: What is the independent variable?; and the dependent variable?

The institutionalisation (regulation) phase of solving the modelling task is aborted, and instead the teacher focused the attention in two concepts whose interest is merely formal.

He miss the opportunity to discuss the student's cognitive conflict about the use of negative values for the time, and to identify the real number intervals that constitute the domain and range of the function that models the Physics problem posed.

Transparency illusion and *Topaze' effect* (Brousseau, 1998, pp.52–53)

As we have commented, the contextualised dialogue that the lecturer raises, little by little, concludes into a "fictitious mayeutic" in which the teacher assumes the formulation and the validation:

> *P: Is that a function?… Sure, not? There, they said not. Is there some element of the initial set that has two images… or three or four?*

That is, he explicitly uses, in a rhetorical question, the main characteristic that discriminates functions from correspondences (*Topaze' effect*). Even more, the teacher explicitly gives a correspondence that is not function:

> *P: What it is not correct… Here you have a not-function. Look at this! This is a counter-example of function… it is the same graph but turned.*

The institutionalisation is then in a "land of anybody", where the students' personal meanings, previously acquired, are confronted with the institutional meanings that the teacher introduces using the physical modelling, and the intended institutional meaning that the teacher explains in some passages (and that he considers transparent in terms of the modelling situation).

The teacher continued with a series of questions and answers; with this activity he attempts, without success, to institutionalise the function notion, domain, range, etc., the teacher leaves his objective and he attributes the failure to that the students didn't fulfilled the pedagogical contract of carrying out the requested personal study:

> *P: But if you don't inquiry…*

In conclusion, we can consider the didactical configuration as "situated dialogue", which is suitable in itself. This configuration allows the teacher to identify certain semiotic conflicts. Nevertheless, the teacher, with his interventions, was not able to negotiate the meaning of the objects involved in the proposed situation, in such a way that the instructional process allows to adapt the learned personal meanings to the institutional intended meaning. For it, the teacher could have used diverse means or resources as helping devices to the study: means of presentation of the information (slides, overhead projector, etc.), calculation and graph devices (calculators, computers). In addition, he should have modified "step by step" the didactic configuration intended and not "to blame" the students for not carried out the pedagogical contract:

> *P: But if you do not find out…*

6. FINAL REMARKS AND IMPLICATIONS

A formal knowledge of the function notion, focused on the discursive component, is not enough from the educational point of view; the design of instructional tasks and the implementation of a suitable didactical trajectory require that teachers have a deep knowledge of the different meanings of mathematical objects.

The analysis we have carried out of the teaching and learning process of the function notion has shown the utility and relevancy of the theoretical tools applied. The notion of *didactical suitability*, and their three main dimensions, epistemic, cognitive and instructional (Godino, 2003; Bencomo, Godino and Wilhelmi, 2004)— allow us to focus the attention of didactical analysis in the interactions among the institutional and personal meanings, in the context of an educational project. In this article we have developed mainly the notion of epistemic suitability; we have described the reference meaning of the function notion and some elements of the implemented meaning in the teaching experience. The comparison between both meanings has allowed us to identify the agreements and mismatches (epistemic conflicts) between both, and therefore, to evaluate the degree of epistemic suitability.

We can evaluate the epistemic suitability of the study process observed as inadequate, taking into account the epistemic conflicts identified. The mathematical task posed can be clearly improved (it would be better not to introduce the complication of the V_0 parameter, neither the composition of functions). Nevertheless, we find appropriate to begin with prediction questions, as primary motivation of the function, using the graphic language ("*To determine the maximum height that reaches the ball, building the graph of y(t)*"). But the development of the study process had many critical points when the discrimination of the formal model of function (correspondence among sets), has been approached, and the relationships among the formal model with the tabular, graphical and analytical models.

On the other hand, the didactic configuration implemented, of constructivist-mayeutic type, has not clearly allowed to solve the semiotic conflicts appeared along the instruction process. The teacher has initially given to students the responsibility to read the definitions, to interpret them and apply to an example that has been complex (the teacher doesn't even interpret appropriately the role of the algebraic criterion of a correspondence). Students have remembered the definitions by heart, but they confuse the terms; the teacher has to invest a long time correcting errors, even formulating inadequate definitions that cause confusion and unnecessary instructional segments, which could be avoided by means of explicit regulative teacher's interventions (magisterial interventions).

The epistemological analysis of mathematical objects, carried out with an approach and conceptual tools appropriated, should be an essential objective in teacher's mathematical training. We have seen how such an elementary mathematical object and "seemingly well-known", as the function, has posed many complications as much to the teacher as to the students. Most of these complications are derived from an insufficient knowledge, on the part of the teacher, of what we can denominate "the archaeology of the object function"; the teacher produced a chaotic overlapping of the different models, loss the control of the instructional process and adopted unfortunate didactical decisions.

The observed facts can be considered as manifestations of a general phenomenon on the management of the didactical time and the "topo-genesis" (distribution of responsibility between teacher and students concerning presenting or building mathematical knowledge). The descriptions and interpretations we have carried out of these facts, using some notions of the onto-semiotic approach to mathematics cognition (Godino, 2003), have consequences for teachers' training. It is necessary that teachers plan teaching keeping in mind the intended institutional meanings, adopting for these meaning a wide viewpoint, not reduced to the discursive aspects (epistemic suitability). It is also necessary to design and to implement a didactical trajectory that take into account the initial students' knowledge (cognitive suitability), to identify and solve the semiotic conflicts that appear in any teaching-learning process, using the necessary material resources including time (instructional suitability). These suitabilities should be integrated taken into account the interactions among thereof, which requires to speak of the *didactical suitability* as the systemic criterion of relevancy (adaptation to the teaching project) of an instructional process, and whose empirical indicator can be the adaptation among the students' personal meanings and the intended institutional meanings.

Acknowledgement

This research work has been carried out in the frame of the projects: MCYT-BSO2002-02452, Resolution #1109/2003, October 13 (UPNA), and MCYT-HA2002-0069.

REFERENCES

Aliprantis, C. D. & Burkinshaw, O. (1999). *Problems in real analysis. A Workbook with solutions*, 2nd Edition. San Diego, CA: Academic Press.

Artigue, M. (1989). Ingenierie didactique. *Recherches en Didactique des Mathématiques*, 9 (3): 281–308.

Beardon, A. F. (1997). *Limits. A new approach to Real Analysis*. New York: Springer-Verlag.

Biehler, R. (1997). Reconstruction of meaning as a didactical task –the concept of function as an example. BACOMET IV, May 1997.

Brousseau, G. (1997). *Theory of didactical situation in mathematics*. Dordrecht: Kluwer.

Godino, J. D. (2002). Un enfoque ontológico y semiótico de la cognición matemática. *Recherches en Didactique des Mathématiques,* 22 (2/3): 237–284.

Godino, J. D. (2003). *Teoría de las funciones semióticas. Un enfoque ontológico y semiótico de la cognición e instrucción matemática*. Departamento de Didáctica de la Matemática. Universidad de Granada. (URL: http://www.ugr.es/local/jgodino).

Tall, D. (1997). Functions and calculus. En A. J. Bishop et al (Eds.), *International Handbook of Mathematics Education* (pp. 289–325). Dordrecht: Kluwer.

Tall, D. (1991). Advanced mathematical thinking.

Ross, K. A. (1980). *Elementary analysis: the theory of calculus*. New York: Springer-Verlag, 2000.

Ruiz, L. (1998). *La noción de función: Análisis epistemológico y didáctico*. Jaén, ESP: Publicaciones de la Universidad de Jaén.

Sierpinska, A. (1992). On understanding the notion of function. En G. Harel and E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy.* MAA Notes # 25 (pp. 3–58). Washington, DC: Mathematical Association of America.

Vergnaud, G. (1990). La théorie des champs conceptuels. *Recherches en Didactiques des Mathématiques*, 10 (2/3): 133–170.

Vygotski, L.S. (1934). *El desarrollo de los procesos psicológicos superiores*, 2ª edición. Barcelona, ESP: Crítica-Grijalbo, 1989.

Youschkevitch, A. P. (1976). The concept of function up to the middle of the 19th century. *Archive for History of Exact Sciences*, 16: 39–85.

Wilhelmi, M. R., Godino, J. D. y Lacasta, E. (2004). Configuraciones epistémicas asociadas a la noción de igualdad de números reales. Departamento de Didáctica de la Matemática. Universidad de Granada. (URL: http://www.ugr.es/local/jgodino).

Wilhelmi, M. R., Bencomo, D. y Godino, J. D. (2004). Criterios de idoneidad de un proceso de instrucción matemática. *XVI Simposio Iberoamericano de Enseñanza de la Matemática*. Castellón: Universitat Jaume I y Real Sociedad Matemática Española.