SEMIOTIC FUNCTIONS IN TEACHING AND LEARNING MATHEMATICS [1]

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1. Introduction

We describe a theoretical model which aims to articulate different approaches to ontological, epistemological and cognitive issues involved in teaching and learning mathematics from a semiotic and anthropological perspective. Mathematical activity is characterised by the use of semiotic functions among four types of entities - ostensive entities (notations, external representations), extensive (situation-problems), intensive (ideas, abstractions) and actuative entities (subject's actions), which can play either the role of expression or content.

We distinguish four types of semiotic functions (and therefore types of meanings) according to the different nature of the content plane: ostensive, extensive, intensive, and actuative semiotic functions, whose specific characteristic should be kept in mind in teaching and learning mathematics. Taking into account the simple or compound nature of its content, we differentiate between elementary and systemic meaning. This distinction is necessary to analyse the interpretative processes being at stake in the various types of situations in the study of mathematics.

To analyse the acts and processes of understanding in mathematics, as well as their enhancement and assessment within the school, we should also distinguish between personal and institutional meanings, depending whether the practices (interpretations, semiotic functions) are idiosyncratic to an individual or shared by a community of interpreters. The theoretical model outlined is exemplified by analysing some semiotic processes in the study of whole numbers.

2. The Relationships Between Semiotics and Epistemology in Mathematics Education

Growing interest has emerged for the use of semiotic notions for studying teaching and learning processes within the mathematics education research community. Examples of this interest are some projects presented at the PME Conferences (Ernest 1993; Vile and Lerman 1996), and those carried out by Bauersfeld and colleagues (Cobb and Bauersfeld 1995) from the perspective of symbolic interactionism, which considers the notions of meaning and negotiation of meanings to be central to mathematics education. We also highlight the published research concerning the influence of representation systems (Duval 1993), symbolisation (Pimm 1995), and language (Ellerton and Clarkson 1996) in mathematics teaching and learning, and the investigations about understanding mathematics (Sierpinska 1994; Godino 1996), which cannot avoid the issues of meaning.

This interest is a natural consequence of the essential role played by expressive means in thinking processes. This role is highlighted by Vygotski (1934), who considers the meaning

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of a word to be the analysis unit of psychic activity and by Cassirer (1964: 27) for which "the sign is not a mere incidental cover of thought, but is its essential and necessary organ".

Even more attention is needed on the notion of meaning and on the relationships between the semiotic and epistemological components involved in mathematical activity, that is, on the nature and type of objects whose meanings are involved therein. "What we variously understand by 'understanding' and mean by 'meaning' is far from obvious or clear,despite these being two central terms in any discussion of the learning and teaching of mathematics at whatever level" (Pimm 1995:3).

We think it is necessary to carry out a more extensive and in-depth analysis of the dialectical relationships between thought (mathematical ideas), mathematical language (sign systems) and problem-situations. In fact we would have to progress by developing a specific semiotics to study the interpretation processes of mathematical sign systems involved at the heart of didactic systems. These issues are central to other disciplines (such as semiotics, epistemology and psychology), although we realise that a clear answer to them has not yet been found. The solutions proposed are diverse, incompatible, or difficult to integrate, as it is shown by comparing the approaches suggested by Peirce (1965), Saussure (1915) and Wittgenstein (1953).

In mathematical work, symbols (signifiants) convey or substitute conceptual entities (meanings). Even when this is also important, the crucial point in mathematics instruction processes is not just mastering the syntax of the mathematical symbolic language, but mastering its semantics. That is to say, understanding the nature of mathematical concepts and propositions and their relationship to contexts and problem-situations. Furthermore, it is necessary to elaborate theoretical models to link the semiotic dimension in mathematics education (their syntactic, semantic and pragmatic aspects), to the epistemological, psychological and sociocultural dimensions. These models require taking, amongst others, the following elements and assumptions into account:

- Diversity of objects involved in mathematical activity, both on the plane of expression, and in that of content (conceptual, notational, and situation-problems).

- Diversity of acts and processes of semiosis involved in different types of objects and ways of producing signs.

- Diversity of contexts and circumstances that shape and relativise semiotic processes.

In Godino and Batanero (1994), and (1998), we developed a theory regarding the meaning of mathematical objects, from pragmatic and anthropological assumptions, and considering such objects as signifiants of the "system of practices carried out by people when faced with a certain class of situation-problems". We attributed a diachronic and evolutionary character to this meaning, depending on institutional contexts and personal circumstances. This notion of meaning could be useful for describing certain interpretative processes, particularly in the stages for designing, developing and evaluating teaching and learning plans for mathematical contents.

Beyond the interpretation of conceptual entities required in the communicative processes carried out in mathematics education, the expressive means and problematic situations themselves give rise to interpretations by the message receivers at a given time and under certain circumstances. There, meaning has a synchronous and local character: It is the content that the speaker of an expression refers to, or the content that the receiver interprets. In other words, what one means, or what the other understands.

We, therefore, need a more complete semiotic model available to take into account the network of objects and interpretative processes intervening in the teaching and learning of mathematics. Moreover, this model will allow us to confront different epistemological approaches to mathematics education; in particular the prevailing conceptualist-idealist and the anthropological approaches.

In this paper we intend to outline a semiotical-epistemological model which is useful for this purpose. Our theoretical model adopts and reinterprets some elements from Hjemslev's (1943) language theory, such as semiotic function, sign, expression and content, from the epistemological triangle (Ogden and Richard 1923, Steinbring 1997) and from the Vergnaud's (1982) conceptual triplet.

We postulate the existence of four types of elementary entities that shall be called ostensive, extensive, intensive and actuative entities, each one of which can play either the role of expression or content within the corresponding semiotic functions. The typology of meaning which is elaborated, depending on the diverse nature of semiotic functions content and its conditioning factors, might serve to describe and explain some relevant phenomena linked to the diffusion of mathematical knowledge.

The paper is structured in the following way:

- Elementary entities put at stake in mathematical activity.
- Semiotic functions and their types.
- Differentiation between elementary and systemic, and personal and institutional meanings.
- Application of the model to identify semiotic acts and processes in the study of whole numbers.

3. Basic Elements Involved in Mathematical Activity

The relationships between the signs used to codify knowledge and the contexts that are needed to establish their meaning has been modelled by several authors using triangular schemes. Amongst these schemes, we highlight those proposed by Frege, Peirce, Ogden and Richards, as well as their interpretation by Steinbring (1997), which he calls the epistemological triangle. The elements included by Steinbring in this triangle are concept, sign/symbol and object/reference context. Vergnaud (1982: 36) himself considers a concept as a triplet formed by the "set of situations that make the concept meaningful, the set of invariants that constitute the concept, and the set of symbolic representations used to represent the concept, its properties and the situations it refers to".

Having been inspired by these authors, we shall outline a theoretical model, where the following types of elementary entities will be considered:

- *Ostensives*, that is, every material representation used in mathematical activity (terms, expressions, symbols, tables, graphs, in general, notational entities)
- *Extensives*, that is, problem-situations, applications; in general, phenomenological entities inducing mathematical activities.
- *Intensives*, mathematical ideas, abstractions (concepts, propositions, procedures, and theories, i. e., mathematical generalisations)

The genesis of mathematical knowledge is produced as a consequence of the subjects' activity when faced with problematic situations and when using the ostensive and intensive elements available. We include in our model a fourth category, due to the relevance of the subjects' action in this genesis:

- *Actuative entities*, the subjects' actions when faced with situations or tasks (describing, operating, arguing, generalising).

In mathematical work, generalisations and problem situations are given by notations that describe their characteristic properties. Both entities are inseparable from the ostensives, in which they are embodied but with which they are not identifiable; we consider that mathematics cannot be reduced to the language with which it is expressed.

Ostensive entities can be letter or number chains, graphs, diagrams, or even physical objects. These notational systems frequently play the role of "representation systems", which means, they replace something else or some aspect of another entity. However, in our model, this representational role is not exclusive to this class of objects (mathematical ideas and situations can also be signs of other entities).

On the other hand, in some circunstances, an ostensive, intensive or actuative element can play the role of the extensive element. For example, the "number two" is an intensive element when it emerges from the action of counting collections of two objects and an extensive element when it is considered as an example of the concept of number. This is due to the fact that in mathematical activity complex concepts are built from other previous concepts.

From a conceptualist- idealist perspective, mathematical generalisations and extensive entities are conceived in the same way that Freudenthal (1982) describes the *noumena* and *phainomena*. Mathematical objects are noumena for this author, that is, objects of thought (ideas), such as numbers. Mathematical concepts, in general mathematical structures serve to organise phenomena of concrete and mathematical worlds.

In the anthropological approach to mathematics sustained by Wittgenstein it is avoid speaking about the concepts as ideal entities. Considering mathematical objects to be something real which the mathematician discovers and considering propositions as descriptions of its properties is a source of much philosophical confusion (Baker & Hacker 1985: 273). It is suggested instead to speak of tasks, practices, habits, techniques and rules to transform mathematical symbols and expressions.

In our semiotical-epistemological model, habit, rules and techniques-, which are different from their respective linguistic expressions - are intensive entities. Practices, conceived as typified actions, will also be considered intensive entities. Tasks requiring the subjects' action (either routine or problematic) are extensive entities, although the type of tasks are intensive entities.

In addition, the semiotic and epistemological model outlined requires another primitive element to describe and explain interpretation and communication processes in mathematics education. This is the notion of *pragmatic context* that we conceive in a very general way as the set of external and internal linguistic factors sustaining and determining mathematical activity, and, therefore, the form, suitability and meaning of objects involved therein. We can also describe it as the frame or scenario where mathematical activity is developed, and which is characterized by,

- its interpretive (conventions, rules) and instrumental (technological tools) elements;
- its internal organisation, that is, the systemic nature of the relationships among its elements;

- its association to expressive systems requiring mutual translations.

4. Semiotics Functions and their Types

In mathematical work we usually take some objects to represent others, especially abstract objects, and a correspondence, frequently implicit, exists between the representative and the objects represented. "There are words, symbols or other *conventional* mechanisms that *mean* or express something, represent or symbolise something else besides them, and make it *publicly understandable*" (Searle 1997: 76).

We believe that Hjelmslev's language theory could be useful to describe mathematical activity. Starting from the text as data, this linguistic theory conceives text analysis as an exhaustive deductive progression from the class to components and as an identification of the dependencies among the different parts, its components and the text as a whole. The basic principle is that "the object under examination as well as its components only exist in virtue of their mutual dependencies; the whole object under examination can only be defined as the total sum of these dependencies" (Hjelmslev 1943: 40).

Key notions in Hjemslev's theory are those of sign function- which Eco designs as semiotic function-, expression and content. Those dependencies in which one part designs or denotes some other thing are outstanding cases of dependencies within the components of a text. Therefore, the first part (expression plane) functions or represents the second (content plane), that is, it points to a content which is beyond the expression. It is necessary to take into account the existence of other operational or actuative dependencies among a text's different components in addition to these representational dependencies.

In this theory, and in accordance with Saussure's proposals, the word "sign" is not applied to the expression but to the entity generated by the connection between an expression and a content. Expression and content are functives between which the sign function establishes a dependence: "a function can not be conceived without its terminals, and the terminals are just the final points of the function, and, therefore, they are inconceivable without it. (Hjelmslev 1943: 75).

According to Eco "there is a semiotic function when an expression and a content are in correlation" (Eco 1979: 83). Such a correlation is conventionally established, though this does not imply arbitrariness, but it is coextensive to a cultural link. In his book "*Semiotic and epistemology of language*", Eco emphasises that "the semiotic correlation should not be understood as substituting identical by identical, as blind equivalence. On the contrary, the sign is always open to something different. When clarifying the sign that somebody interprets, there is no interpreter who does not slightly displace its limits" (Eco 1984: 72). The idea of sign is not reduced to the plane of expression, but is conceived as the pair (signifiant, meaning). Furthermore, the relationship between both functives is not reduced to mere equivalence. The sign, therefore, is not just mere correspondence between expression and content; it is not something that stands for some thing else, but it implies someone who makes a possible interpretation.

We think that semiotic function could be conceived, at least metaphorically, as a "correspondence between sets", involving three components:

- an expression plane (the initial object, frequently considered as the sign)
- a content plane (the final object, considered as the meaning of the sign, i.e., what is represented, or meant, what the speaker refers to).

- a criteria or correspondence rule, this is, an interpretative code regulating the correlation between the expression and content planes, establishing the aspect or character of the content referred by the expression.

Any subset of the four types of primary elements considered can play the role of initial or final objects in semiotic functions, which are frequently given by just one of their three components, being the other two implicitly established. Speaking of meaning also supposes that there is an expression and an interpretative code.

The four types of primary entities considered (ostensive, extensive, intensive and actuative entities) can play both the roles of expression and content in semiotic functions. Hence, different types of such functions are applicable. Though some of these functions can be clearly interpreted as specific cognitive processes (generalisation, symbolisation, abstraction), in this work we will classify and characterise these functions regarding the plane of content (meaning); therefore these types are reduced to the four described below.

(1) Ostensive meaning: Let us call a semiotic function ostensive when the final object (its content), is an ostensive object. This type of function is the characteristic use of signs to name world objects and states, to indicate real things, to say that there is something and that thing is built in a given manner. The following examples show this type of meaning:

- When a particular collection of five things are represented by the numeral '5'.
- The symbol Pn (or n!) represents the product n (n-1) (n-2) ... 1
- In the phrase, "In the histogram in figure x, which is the absolute frequency of the modal interval?" The word 'histogram' refers to another ostensive object that is shown in the figure. Also the expressions 'absolute frequency' and 'modal interval' refer to ostensive observable objects in the figure (a number labelled on the ordinates axis, an identifiable interval on the abscissas axis.
- The expression 'multiplication table' refers to a specific arrangement of numerals.

(2) *Extensive meaning*: A semiotic function is extensive when the final object is a situation - problem or a phenomenology, as in the following examples:

- As a rule, the verbal, graphical or mixed description of a problem-situation. Such a description is a different object of the situation itself.
- The simulation of phenomena (i.e., it is possible to represent a variety of probabilistic problems with urn models)

(3) *Intensive meaning*: A semiotic function is intensive when its content is an intensive object, as in the following examples:

- In the definitions of a concept, for example, "an angle is a pair of rays with the same origin", the word 'angle' refers to an abstract object.
- In expressions such as, "Let μ be the mathematical expectation of a random variable ξ", or "Let f (x) be a continuous function ".

The notations μ, ξ, f(x), or the expressions 'mathematical expectation', 'random variable' and 'continuous function', refer to mathematical generalisations.

(4) Actuative meaning: A semiotic function is actuative when its content is a subject's action. In any calculus process dependencies among the different parts of an operational or actuative sequence are established. For example, the expression (2/3)/(12) is pointing (i.e., it means) to multiplying 12 by 2 and dividing its results by 3".

Every extensive, intensive and actuative semiotic function can be interpreted, furthermore, as a ostensive function and vice-versa, since situations-problems, abstractions and actions are textually fixed. We consider, however, very important to distinguish between ostensive, extensive, intensive and actuative semiotic functions to achieve mathematical understanding. This understanding is always partial and can be improved to the extent to which subjects enrich the system of semiotic functions they are able to use for a given class of problem-situations or for a conceptual structure.

In Figure 1 the semiotical and epistemological model we have proposed to describe mathematical activity is represented.

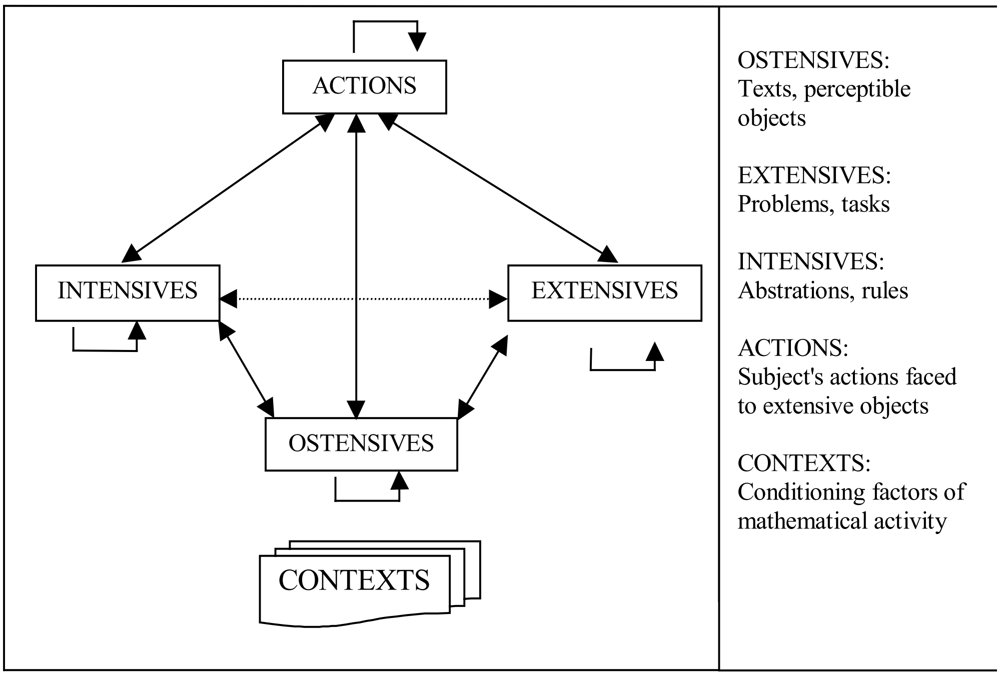

Figure 1: Semiotical-epistemological model of mathematical activity

5. Elementary and Systemic Meanings

To study educational processes, we distinguish other types of semiotics functions, depending on different contextual factors. Taking into account the uniform or multiform nature of the correspondences we call them elementary or systemic meanings.

An elementary meaning is produced when a semiotic act (interpretation / understanding) relates an expression to a specific content within some specific spatial and temporal circumstances: It is the content that the emitter of an expression refers to, or the content that the receiver interprets. In other words, what one means, or what the other understands. Examples of this use of the word 'meaning' are the deictic signs, rigid designations (Eco 1990) where the content is indicated by gestures, indications or proper names. The content of the semiotic function is a precise object, which may be determined without ambiguity in the spatial and temporal circumstances fixed.

The semiotic processes involved in building mathematical concepts, establishing and validating mathematics propositions, and, as a rule, in problem solving processes, yield systemic meanings. In this case, the semiotic function establishes the correspondence between a mathematical object and the system of practices which originates such an object (Godino and Batanero 1994; 1996). This systemic meaning of a mathematical object is conceived as a composed and organised entity (system), according to Putnam (1975) among others. The structural elements of this systemic meaning would be the notations or mathematical registers (ostensive elements), the problem situations (extensive elements), the definitions and statements of characteristic properties (intensive elements) and the actions a subject should carry out when faced with the given tasks (actuative elements). The semiotic functions would be the relationships that can be established between these elements.

We can define systemic meanings as organised and intentional lattices of elementary semiotic functions. They could also be understood as the chains of interconnected interpretations that are induced by the need of understanding an object in the more complete possible way in the given circumstances.

6. Personal and Institutional Meanings

The systemic meaning of an object has a theoretical nature and tries to explain the complexity of semiosis acts and processes, but it is not fully describable. Practice systems differ substantially according to the institutional and personal contexts where problems are solved. These contexts determine the types of cultural instruments available and the interpretations shared, and therefore the types of practices involved.

"Even when, from a general semiotic viewpoint, the encyclopaedia could be conceived as global competence, from a sociosemiotic view is interesting to determine the various degrees of possession of the encyclopaedia, or rather the partial encyclopaedias (within a group, sect, class, ethnic groups, etc" (Eco 1990: 134).

Due to these characteristics of the systemic meanings, we consider it necessary to distinguish between *institutional meanings* and *personal meanings*, depending on whether practices are socially shared, or just idiosyncratic actions or manifestations of an individual. In the second case, when the subject tries to solve certain classes of problems, he builds a personal meaning of mathematical objects. When this subject enters into a given institution (for example, the school) he/she might acquire practices very different from those admitted for some objects within the institution.

A matching process between personal and institutional meanings is gradually produced. The subject has to appropriate the practice systems shared in the institution. But the institution should also adapt itself to the cognitive possibilities and interests of its potential members. The types of institutions interested by a specific class of mathematical problems might be conceived as communities of interpreters sharing some specific cultural instruments and constitute a first factor for conditioning the systemic meanings of mathematical objects.

The institutional relativity of meanings can be extended to any context or interpretive framework (temporary, technological, conceptual, and others.)

7. Semiotic Acts and Processes in the Study of Numbers

In this section, and to give examples of the theoretical concepts described, we apply the semiotic model outlined to the analysis of some semiotic acts and processes involved in the study of whole numbers.

7.1. Elementary Meanings: The First Encounter with Numbers

The first encounter with numbers for most children, is produced at pre-school age, when their parents teach them the series of words 'one', 'two', 'three' to count small collections of objects: hand fingers, marbles, sweets. Afterwards, they will find school tasks close to those reproduced in Figure 2, which have been taken from a first grade book in elementary school.

The text is intended to make the child recognise and write the numerals 1, 2, 3, ..., at the same time as different collections of objects represented are assigned to the corresponding numerical symbol. From the drawing of a head, a sun, a cat an arrow points to the symbol 1. As an exercise, drawing 1 beside a flower and a moon is implicitly requested. A similar method is used for teaching the number 2, its form, writing, and use.

In the tasks proposed, we can identify the four types of entities and semiotic functions which characterise mathematical activity, according to our semiotical-epistemological model:

Notations (ostensives), problem-situation (extensives), abstractions (intensives) and subject's actions (actuatives).

In fact, the concrete object drawings (head, sun, cat, flowers, eyes) are iconic representations of such objects; the meaning of the icons is the corresponding concrete object; the schemes implicitly suggest answering the question, "How many objects are there?", so that the reference context is not just made up by mere concrete objects, but rather the problem-situation of computing the cardinal (extensive meaning). It is important to observe that, in the task proposed by the book, the physical objects are not in fact present. Therefore,

the immediate reference context to which numerical symbols refer is a world of ostensive representation (textual). This fact implies additional semiotic complexity, and hence, interpretative effort by the child. In this series of tasks, we also identify two operative invariants (generalisations):

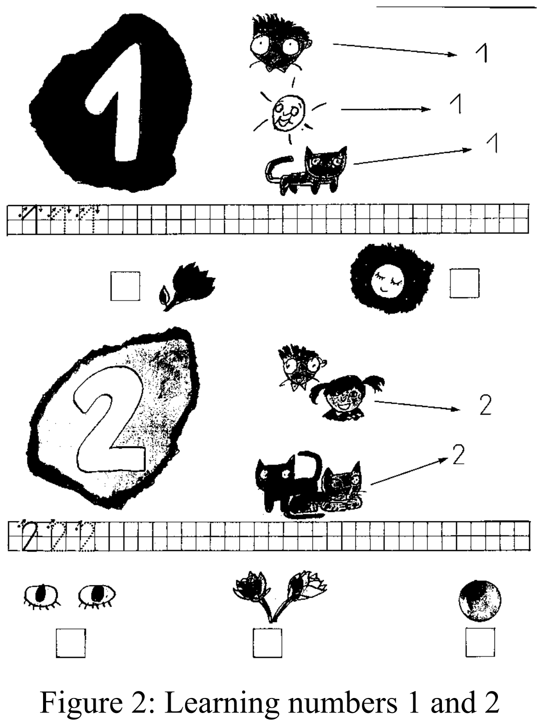

Figure 2: Learning numbers 1 and 2

- the same symbol, '1', and the word-number 'one' (when the teacher presents the task) are associated to various drawings of unitary collections;

- the symbol, '2', and the number-word 'two' are associated to different pairs of objects. The mental objects (or better, the logical entities), one and two, are implicitly evoked as from the first teaching levels. The tasks aim, in psychological terms, is the progressive construction of the objects, number one, two in the child's mind. In anthropological terms it is the child's acquisition of the habit of naming any collections by using the series of number-words, and the series of numerical-symbols (numerals).

We also identify in Figure 2 the entities we have called actuatives. The children are asked:

- to draw the numerals '1' and '2' according the given pattern;

- to write the symbols '1' and '2' in the corresponding empty squares.

We also identify the following semiotic functions (acts and interpretation processes):

I1: The drawing of concrete objects is implicitly related to the concrete objects (extensional meaning) they represent.

I2: The object collections are interrelated with the numerical symbols, '1', '2' (notational meaning).

I3: The number-words 'one', 'two', are phonetically and implicitly associated with the numerical symbols, '1', '2' (notational type).

I4: Each iconic collection, and its associated numerical symbol, is implicitly interrelated with the corresponding mathematical concepts of oneness and twoness (intensional semiotic function).

I5: The guidelines to draw the numerals are suggesting: "make a drawing similar to these by sketching them as indicated by the arrows (actuative meaning).

I6: The placing of the empty square near one or two drawn objects suggests the order: "Write the corresponding numeral" (actuative meaning).

The interpretation of these semiotic functions requires specific codes and conventions which should be known and interpreted by the receiver of the message (the child), to successfully complete the tasks and to progressively acquire the notion of number. Thus, for example, the matching of an icon (or group of icons) with a numerical symbol, is done in two different ways:

- using an arrow (→) which starts from one icon and points at the corresponding numeral;

- using an empty box placed near the icons, in which it is (implicitly) suggested the numeral be put. Hence, the box is a sign playing the role of unknown to communicate the order, "put the number suited to this collection into the box".

We point out that the spaces left between the different icons play a fundamental role, since they inform which objects should be counted in each case. The writing guides for learning to draw the numerals '1' and '2' are also full of symbolisms, which they are difficult to decode without the teacher's assistance.

Our analysis shows the multiplicity of codes for whose recognition the children will require a systematical teacher's assistance. This supports Solomon' s thesis (1989: 160), that "knowing number should be reconceptualized as involving entering into the social practices of number use", and not as an issue of individual construction of the necessary and sufficient logical structures for understanding numerical concepts.

7.2. Systemic Meanings of Numbers

In the previous section we have shown examples of notational (i.e., ostensive), extensivel, intensive and actuative elementary meanings involved in the study of numbers at elementary school. The organised set of these elementary meanings would correspond to what we call a number systemic meaning, which would be personal or institutional depending on whether we take a particular subject (a child) or an institution (community of interpreters) as a reference.

The systemic meaning of numbers within a given teaching level (school institution) is determined by curricular documents, school textbooks, and by the teachers' own preparation of their lessons on a mathematical topic. For example, in the "Curriculum and evaluation standards for school mathematics" (NCTM 1989), we find several standards that jointly describe the meaning of numbers at primary education level. In particular, the Standard 6 (K-4 grades), "numerical sense and numeration", includes the following concepts and skills:

- construct number meanings through real-world experiences and the use of physical materials;
- understand our numeration system by relating counting, grouping, and place-value concepts;
- develop number sense;
- interpret the multiple uses of numbers encountered in real world.

It is suggested that children should understand numbers when they give sense to the daily uses of numbers to quantify, identify positions, identify a specific object in a collection, and measure. Children with good number sense are characterised by: (1) have well-understood number meanings (this includes the cardinal and ordinal number meanings), (2) have developed multiple relationships between numbers (compositions and decompositions of numbers), (3) recognise the relative magnitudes of numbers, (4) know the relative effect of operations on numbers, and (5) develop referents for measures of common objects and situations in their environment.

We can observe that the meaning of numbers in the Standards, or other curricular documents, is described in an encyclopaedic or systemic form, since it refers to a complex of notational, situational, intensional and actuative elements. Numerical competence will be achieved through carrying out an organised practice system of progressive complexity throughout a prolonged period of time, which is extended beyond primary teaching.

The institutional systemic meaning of numbers described in the curricular documents, will later be interpreted by the textbook authors and by the teachers themselves when designing their didactic interventions, to select the practices they consider most appropriate to their institutional circumstances. These practices will finally be carried out by the pupils, and will determine the personal meanings that these pupils progressively build.

At primary school, numbers are some "special symbols", 1, 2, 3, ... associated to collections of objects, to count, order, and name them. Children may also carry out activities with concrete materials (rods, toothpicks, multibase blocks, abaci) which constitute more primitive numbering systems than the place-value decimal numbering system, privileged by mathematical culture due to its efficiency. It is not rare, therefore, that if we ask a child, "What are numbers?", he/she will answer, at best: "They are symbols, 0, 1, 2, 3,..., invented by man to count and compare quantities. These symbols form the set of numbers".

At elementary school, numbers are neither "the cardinal of finite sets", nor "the common property to all finite sets mutually coordinable". Few people (children, adults, even teachers) would provide such a description of numbers; however, they handle numbers, know to use them effectively for counting and ordering. For these people, numbers have a different meaning from that shared by professional mathematicians.

Even for professional mathematicians the descriptions of numbers may vary substantially. In Cantorian mathematics, whole numbers "are the elements of the quotient set determined on the set of finite sets by the relationship of equivalence of coordinability between sets". However, for Peano's mathematics, a totally ordering set will be called whole numbers if it fulfils the following conditions:

- Any successor of an element of N belongs to N.

- Two different elements of N cannot have the same successor.

- There is an element (0) that it is not a successor of any other element in N.

- All subsets of N that contain 0 and contain the successor of each one of their elements coincide with N.

Therefore, there is no single definition of whole number set, but rather various, adapted to problem situations, intentions and semiotic tools available in each particular circumstance. Each definition we may make for whole numbers emerges from a specific practices system, hence, it involves a class of problem situations and specific notational systems. In principle, each phenomenological numerical context (sequence, counting, cardinal, ordinal, measure, label, number writing, computation) can produce an idiosyncratic meaning (or sense) for numbers. Institutional contexts also share idiosyncratic practice systems and, consequently, they determine differentiated meanings.

8. Conclusions and Implications

The idea guiding our work is the conviction that the notion of meaning, in spite of its extraordinary complexity, may still play an essential role to as a basis for research into the didactic of mathematics. We think that an anthropological approach to this discipline, as Chevallard (1992) proposes, complemented with specific attention to semiotic processes, may help us to overcome a certain illusion of transparency about mathematics teaching and learning processes, showing us the multiplicity of codes involved and the diversity of contextual conditioning factors. The construct of *systemic meaning* postulates the complexity of mathematical knowledge by recognising its diachronic and evolutionary nature. This makes us aware of the relevance of semiotic-anthropological analysis of problem fields associated to each knowledge, their structure variables, and the notational systems used, since knowledge emerges from people's actions when faced with problem situations, as mediated by the semiotic tools available.

Hence, it may be useful in curricular design, development and evaluation as a macro-didactic unit of analysis, guiding the search and selection of representative samples of practices characterising mathematical competence.

On the other hand, and taking into account the complex nature of the meaning of mathematical objects, we should often focus attention on specific interpretative processes and its inherent difficulties of the same when analysing student and teacher classroom performances. The construct of *elementary meaning* and the description of its various types permits us to focus attention on the implicit codes which influence acts and processes of understanding in mathematics education. This will be useful for identifying critical points, conditioning factors of semiotic acts and processes in mathematical activity and anticipating didactical actions.

The meaning of a mathematical object has a theoretical nature and cannot be totally and unitarily described. Practices carried out to solve mathematical problems differ substantially according to institutional and personal contexts. For this reason, we introduce *institutional meanings* to distinguish between these different points of view and uses on the same mathematical object. These practices are interpreted in this article as semiotic functions or sequences of semiotic functions, analysing their types, and taking into account the nature of their content. *Personal meanings* are built by the individual subject - what he/she learns, his/her personal relation to the object - and do not just depend on cognitive factors, but rather on the semiotic-anthropological complex in which this relation is developed, that is, on the elements of meaning and the dialectic relationships between them as they are presented.

REFERENCES

Baker, Gordon, and Hacker, Peter Michale Stephan (1985). *Wittgenstein. Rules, Grammar and Necessity. An Analytical Commentary on the Philosophical Investigations* (Vol. 2). Glasgow: Basil Blackwell.

Cassirer, Ernest (1964). *Filosofía de las Formas Simbólicas*. México: Fondo de Cultural Económica, 1971.

Chevallard, Yves (1992). Concepts Fondamentaux de la Didactique: Perspectives Apportées par une Approche Anthropologique. *Recherches en Didactique des Mathématiques*, Vol. 12, nº 1, pp. 73-112.

Cobb, Paul, and Bauersfeld, Heinrich (Eds.) (1995). *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures.* Hillsdale,N.Y.: Lawrence Erlbaum A. P.

Duval, Raimond (1993). Registres de Représentation Sémiotique et Functionnement Cognitive de la Pensée. *Annales de Didactique et de Sciences Cognitives*, 5: 37-65 (IREM de Strasbourg).

Eco, Umberto (1979). *Tratado de Semiótica General*. Barcelona: Lumen.

Eco, Umberto. (1984). *Semiótica y Filosofía del Lenguaje*. Madrid: Lumen, 1990.

Ellerton, Nerida F., and Clarkson, Philip. C. (1996). Language Factors in Mathematics Teaching and Learning. In Alan J. Bishop et al. (Eds.), *International Handbook of Mathematics Education* (pp. 987-1034). Dordrecht: Kluwer A. P.

Ernest, Paul (1993). Mathematical Activity and Rhetoric: A Social Constructivist Account. In I. Hirabasash, N. Nohda, K. Shigematsu, and F. Lin (Eds.), *Proceedings of Seventeenth International Conference for the Psychology of Mathematics Education* (pp. II-238-245). University of Tsukuba. Japan.

Freudenthal, Hans (1982). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: Kluwer A. P.

Godino, Juan D. (1996). Mathematical Concepts, their Meaning, and Understanding. In Luis Puig, and Angel Gutierrez (Eds.), *Proceedings of the 20th Conference of the*

International Group for the Psychology of Mathematics Education (pp. 2-417-424), Universidad de Valencia.

Godino, Juan D., and Batanero, Carmen (1994). Significado Institucional y Personal de los Objetos Matemáticos'. *Recherches en Didactique des Mathématiques*,14 (3): 325-355.

Godino, Juan D., and Batanero, Carmen (1998). Clarifying the Meaning of Mathematical Objects as a Priority Area of Research in Mathematics Education. In Anna Sierpinska, and Jeremy Kilpatrick (Eds.), *Mathematics Education as a Research Domain: A Search for Identity* (pp. 177-195). Dordrecht: Kluwer, A. P.

Hjemslev, Louis (1943). *Prolegómenos a una Teoría del Lenguaje.* Madrid: Gredos, 1971.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standard for School Mathematics*. Reston, VA: NCTM.

Ogden, C. K., and Richard, R. A. (1923). *The Meaning of Meaning*. London: Routledge and Kegan.

Peirce, Charles Sanders. (1965). *Obra Lógico-Semiótica*. Madrid: Taurus, 1987.

Pimm, David (1995). *Symbols and Meanings in School Mathematics*. London: Routledge.

Putnam, Hilary. (1975). El Significado de "Significado". In L. M. Valdés (ed.), *La Busqueda del Significado*. Madrid: Tecnos, 1991.

Saussure, Ferdinand (1915). *Curso de Lingüística General*. Madrid: Alianza, 1991.

Searle, John. R. (1997). *La Construcción de la Realidad Social*. Barcelona: Paidós.

Sierpinska, Anna (1994). *Understanding in Mathematics*. London: The Falmer Press.

Solomon, Yvette. (1989). *The Practice of Mathematics*. London: Routledge.

Steinbring, Heinz (1997). Epistemological Investigation of Classroom Interaction in Elementary Mathematics Teaching. *Educational Studies in Mathematics,* 32: 49-92.

Vergnaud, Gerard. (1982). Cognitive and Development Psychology and Research in Mathematics Education: Some Theoretical and Methodological Issues. *For the Learning of Mathematics* 3, 2: 31-41.

Vile, Adam, and Lerman, Stephen (1996). Semiotics as a Descriptive Framework in Mathematics Domain. In Luis Puig, and Angel Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (pp. 4-395-402), Universitat de Valencia.

Vygotski, Liev Semiónovich (1934). *Pensamiento y Lenguaje*. [Obras escogidas II, pp. 9-287]. Madrid: Visor, 1993.

Wittgenstein, Ludwig. (1953). *Investigaciones Filosóficas*. Barcelona: Crítica.

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