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Björling Representation for spacelike surfaces with $H = cK$ in $\mathbf{L}^3$

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Abstract

In this work we study the Björling problem for linear Weingarten spacelike surfaces of maximal type in the 3-dimensional Lorentz-Minkowski space, i.e. spacelike surfaces whose mean and Gaussian curvature are related by $H = cK$ for some $c \in \mathbf{R}$.

Keywords: Björling problem, linear Weingarten surfaces, maximal surfaces, Lorentz-Minkowski space

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1. Introduction

A linear Weingarten spacelike surface of maximal type (in short, an LWM-spacelike surface) in the 3-dimensional Lorentz-Minkowski space $\mathbf{L}^3$ is a spacelike surface whose mean curvature is proportional to its Gaussian curvature.

In the previous paper [1], the first two authors described a conformal representation for LWM-spacelike surfaces, and used it in order to prove the existence of complete examples and to study its geometric behaviour. The representation actually extends the one for maximal surfaces in $\mathbf{L}^3$, i.e. for the spacelike surfaces with $H = 0$ in $\mathbf{L}^3$, obtained by McNertney and Kobayashi.

In this work we consider an initial value problem for LWM-spacelike surfaces, which consists on the following: given a regular analytic spacelike curve

β in $\mathbf{L}^3$ together with a unit timelike analytic vector field V parallel along β, can one find all LWM-spacelike surfaces in $\mathbf{L}^3$ that span this configuration?

This problem has been motivated by the Björling problem for maximal surfaces in $\mathbf{L}^3$ studied in [2]. Some other research regarding the Björling problem for maximal surfaces can be found in [3, 4]

The goal of the present paper is to characterize when the initial data β, V of the above problem can actually span an LWM-spacelike surface, and to construct in such case the only solution to the Björling problem in terms of the initial data.

2. Preliminaries

Let $\mathbf{L}^3$ be the 3-dimensional *Lorentz-Minkowski space*, that is, the real vector space $\mathbf{R}^3$ endowed with the Lorentzian metric tensor $\langle , \rangle = dx_1^2 + dx_2^2 - dx_3^2$, where (x_1, x_2, x_3) are the canonical coordinates of $\mathbf{R}^3$. An immersion $\psi : M^2 \longrightarrow \mathbf{L}^3$ of a 2-dimensional connected manifold M is said to be a *spacelike surface* if the induced metric via ψ is a Riemannian metric on M, which, as usual, is also denoted by $\langle , \rangle$.

It is well-known that such a surface is orientable. Thus, we can choose a unit timelike normal vector field N globally defined on M. Observe that, up to a symmetry of $\mathbf{L}^3$, we can suppose that the image of N lies on $\mathbf{H}^2_+ = \{x \in \mathbf{H}^2 : x_3 > 0\}$. We shall call N the *unit normal* of ψ. Let us introduce complex coordinates in $\mathbf{H}^2_+$ using the usual stereographic projection $\pi : \mathbf{H}^2_+ \longrightarrow \mathbb{D}$ from the hyperbolic plane $\mathbf{H}^2_+$ onto the unit disk $\mathbb{D}$ given by

$$\pi(x_1, x_2, x_3) = \frac{x_1 - ix_2}{1 + x_3},$$

with inverse map

$$\pi^{-1}(z) = \left(\frac{z + \bar{z}}{1 - |z|^2}, i\,\frac{z - \bar{z}}{1 - |z|^2}, \frac{1 + |z|^2}{1 - |z|^2} \right).$$

We will refer to $g = \pi \circ N$ as the *Gauss map* of the surface.

Let $H = -\text{trace}(A)/2$ and $K = -\det(A)$ denote the mean and Gaussian curvatures of M respectively, where $A : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ stands for the shape operator of M in $\mathbf{L}^3$ associated to N, given by $A = -dN$. Then, we will say that $\psi : M^2 \longrightarrow \mathbf{L}^3$ is a *linear Weingarten spacelike surface of maximal type*, in short, an LWM-spacelike surface, if there exist $c \in \mathbf{R}$ such that $H = cK$. This condition is equivalent to the existence of $a, b \in \mathbf{R}$, $a \neq 0$, satisfying $-2aH + bK = 0$. We will adopt this last notation to follow the one in [1].

As it can be seen in [1], on such a surface the Gaussian curvature is either always negative or always non negative. Moreover, the symmetric tensor σ on M for the immersion ψ

$$\sigma(X,Y) = a\langle X,Y\rangle - b\langle AX,Y\rangle, \qquad X,Y \in \mathfrak{X}(M),$$

is positive definite (reversing orientation if necessary). Hence, we will choose N so that σ is a Riemannian metric.

If we consider M as a Riemann surface with the conformal structure induced by σ, then $g = \pi \circ N$ is a conformal map from M into $\mathbb{D}$, and

$$\triangle^{\sigma}(2a\psi + bN) = 0.$$

These two facts are the basis of a conformal representation for LWM-spacelike surfaces, obtained in [1]:

Theorem 1 ([1]) *Let $\psi : M^2 \longrightarrow \mathbf{L}^3$ be an LWM-spacelike surface such that $-2aH + bK = 0$, $a \neq 0$, and let us consider on M the conformal structure induced by σ. Then there exists a function $\phi : M \longrightarrow \mathbf{C}^3$ such that the immersion can be recovered as*

$$\psi = -\frac{b}{2a}\pi^{-1}(g) + \frac{1}{a}\,\mathrm{Re}\int \phi(\zeta)d\zeta. \tag{1}$$

Here $g : M \longrightarrow \mathbb{D}$ is its Gauss map and:

- *If $K \geq 0$ then*
$$\phi = (2a\psi + bN)_z \tag{2}$$
and both g and ϕ are holomorphic;

- *If $K < 0$ then*
$$\phi = (2a\psi + bN)_{\overline{z}}$$
and both g and ϕ are anti-holomorphic,

being z a conformal parameter on M and N the unit normal of ψ.

We refer the readers to [1] for the details.

3. Björling Representation

Let $\psi : \Omega \subset \mathbf{R}^2 \longrightarrow \mathbf{L}^3$ be an LWM-spacelike surface and $z = s + it$ be a local conformal parameter on Ω with respect to σ. First, we will study the case $K \geq 0$.

Since $\langle a\psi_t + bN_t, \psi_s\rangle = \sigma(\psi_t, \psi_s) = 0$, it can be easily obtained that $a\psi_t + bN_t = -aN \times \psi_s$. Using this equality and (2), we get that $\phi + bN_z = a\psi_s + bN_s + iaN \times \psi_s$ and so

$$\phi = \frac{1}{2}\left(a\left(2\psi_s + i(N \times \psi_s - \psi_t)\right) + bN_s\right). \tag{3}$$

On the other hand, as $\langle a\psi_s + bN_s, \psi_t\rangle = \sigma(\psi_s, \psi_t) = 0$ it follows from a straightforward computation that $\psi_t = \psi_s \times N + (b/a)N_s \times N$, which jointly with (3) allow us to obtain

$$\phi = \frac{1}{2}\left(a\left(2\psi_s + i(2N \times \psi_s - \frac{b}{a}N_s \times N)\right) + bN_s\right). \tag{4}$$

Let us define $\beta(s) = \psi(s, 0)$, $V(s) = N(s, 0)$ on a real interval $I \subset \Omega$. Observe that $\beta(s)$ and $V(s)$ are real analytic functions, because so is ψ. Let us choose any simply connected open set Δ containing I over which we can define holomorphic extensions $\beta(z)$, $V(z)$ of β, V. Then, formula (4) can be written on the curve $\beta(s)$ as

$$\phi(s,0) = \frac{1}{2}\left(a\left(2\beta'(s) + i(2V(s) \times \beta'(s) - \frac{b}{a}V'(s) \times V(s))\right) + bV'(s)\right),$$

and by analytic extension one has

$$\phi(z) = a\left(\beta'(z) + iV(z) \times \beta'(z)\right) + \frac{b}{2}\left(V'(z) + iV(z) \times V'(z)\right).$$

Finally, since

$$N(z) = \pi^{-1}\left(\frac{V_1(z) - iV_2(z)}{1 + V_3(z)}\right)$$

we get from (1) that

$$\begin{aligned}\psi(z) \;&=\; \mathrm{Re}\beta(z) + \frac{b}{2a}\left(\mathrm{Re}V(z) - \pi^{-1}\left(\frac{V_1(z) - iV_2(z)}{1 + V_3(z)}\right)\right) \\ &\quad -\mathrm{Im}\int_{s_o}^{z}\left(V(\omega) \times \beta'(\omega) + \frac{b}{2a}V(\omega) \times V'(\omega)\right)d\omega.\end{aligned} \tag{5}$$

As it can be checked, this formula agrees with the one for maximal surfaces in [2] when we take $a = 1$, $b = 0$.

The case $K < 0$ is analogous, resulting

$$\begin{aligned}\psi(z) \;&=\; \mathrm{Re}\beta(z) + \frac{b}{2a}\left(\mathrm{Re}V(z) - \pi^{-1}\left(\frac{\overline{V_1(z)} - i\overline{V_2(z)}}{1 + \overline{V_3(z)}}\right)\right) \\ &\quad -\mathrm{Im}\int_{s_o}^{z}\left(V(\omega) \times \beta'(\omega) + \frac{b}{2a}V(\omega) \times V'(\omega)\right)d\omega.\end{aligned} \tag{6}$$

The following lemma, whose proof is a simple exercise, shows that the geometry of the surface along the curve $\beta(s)$ can be expressed in terms of $\beta(s)$, $V(s)$.

Lemma 2 *Let $\psi : \Omega \subset \mathbf{R}^2 \longrightarrow \mathbf{L}^3$ be an LWM-spacelike surface and let us consider on Ω the conformal structure induced by σ. Let us define $\beta(s) = \psi(s,0)$, $V(s) = N(s,0)$ on a real interval $I \subset \Omega$. Then*

i) $aD(s) + b\langle \beta', V' \rangle \neq 0$ *for all* $s \in I$.

ii) $$K|_{\beta(s)} = \frac{a}{D(s)} \left(\frac{\det(\beta', V, V')^2 + \langle \beta', V' \rangle^2}{aD(s) + b\langle \beta', V' \rangle} \right)$$

iii) $$H|_{\beta(s)} = \frac{b}{2D(s)} \left(\frac{\det(\beta', V, V')^2 + \langle \beta', V' \rangle^2}{aD(s) + b\langle \beta', V' \rangle} \right)$$

where

$$\begin{aligned} D(s)^2 &= \det \langle , \rangle|_{\beta(s)} \\ &= \langle \beta', \beta' \rangle^2 + \frac{b^2}{a^2} \langle \beta' \times V', \beta' \times V' \rangle + \frac{2b}{a} \langle \beta', \beta' \rangle \langle \beta', V' \rangle. \end{aligned}$$

A pair made up of a regular analytic spacelike curve $\beta(s) : I \longrightarrow \mathbf{L}^3$ and an analytic unit vector field $V(s) : I \longrightarrow \mathbf{H}^2_+$ such that $\langle \beta', V \rangle = 0$ will be called a pair of *Björling data*. The previous Lemma shows that one cannot expect in general for prescribed Björling data the existence of an LWM-spacelike surface spanning such configuration.

Taking this into account, we can now formulate the *Björling problem* for LWM-spacelike surfaces in $\mathbf{L}^3$.

> Let $\beta : I \longrightarrow \mathbf{L}^3$ and $V : I \longrightarrow \mathbf{L}^3$ be a pair of Björling data such that for some $a \neq 0$, $b \in \mathbf{R}$ the condition $aD(s) + b\langle \beta', V' \rangle \neq 0$ holds for all $s \in I$. Determine all LWM-spacelike surfaces with $-2aH + bK = 0$ that contain $\beta(s)$, and whose unit normal along $\beta(s)$ is given by $V(s)$.

Any pair of Björling data in the above conditions will be called *admissible.*

Our main result is the following, where we assure the existence and uniqueness of the solution to Björling problem. Observe that the sign of the Gaussian curvature of the solution is given by the pair of curves β, V and the sign of a, as follows from Lemma 2.

Theorem 3 *Let $\beta(s), V(s)$ be admissible Björling data. There exists a unique solution to Björling problem for LWM-spacelike surfaces in $\mathbf{L}^3$ with the initial data $\beta(s), V(s)$. This unique solution can be constructed in a neighbourhood of the curve as follows:*

- *if $a(aD(s) + b\langle\beta', V'\rangle) > 0$, the map $\psi : \Omega \longrightarrow \mathbf{L}^3$ given by (5) is the only solution to Björling problem, and has non-negative Gaussian curvature;*
- *if $a(aD(s) + b\langle\beta', V'\rangle) < 0$, the map $\psi : \Omega \longrightarrow \mathbf{L}^3$ given by (6) is the only solution to Björling problem, and has negative Gaussian curvature.*

Here $\Omega \subseteq \mathbf{C}$ is a sufficiently small simply connected open set containing I over which β, V admit holomorphic extensions $\beta(z)$, $V(z)$.

Proof: The uniqueness result follows from the computations used to derive the formulas (5) and (6). The existence is a straightforward computation.

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