SOME RIGIDITY RESULTS FOR COMPACT SPACELIKE SURFACES IN THE 3-DIMENSIONAL DE SITTER SPACE

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In this work we prove that the only compact spacelike surfaces in the 3-dimensional de Sitter space with a constant principal curvature are the totally umbilical round spheres. We also characterize the totally umbilical round spheres of the 3-dimensional de Sitter space as the only compact linear Weingarten spacelike surfaces. As a consequence, we show that the only compact linear Weingarten surfaces in the 3-dimensional hyperbolic space with non-degenerate second fundamental form are the totally umbilical round spheres.

1 Introduction

Spacelike surfaces in the de Sitter space $\mathbf{S}_1^3$ have been of increasing interest in the recent years from different points of view. That interest is motivated, in part, by the fact that they exhibit nice Bernstein-type properties. For instance, Ramanathan [8] proved that every compact spacelike surface in $\mathbf{S}_1^3$ with constant mean curvature is totally umbilical. This result was generalized to hypersurfaces of any dimension by Montiel [7]. On the other hand, Li [6] obtained the same conclusion when the compact spacelike surface has constant Gaussian curvature. More recently, the first author jointly Romero [3] have proved that the totally umbilical round spheres are the only compact spacelike surfaces in the de Sitter space such that the Gaussian curvature of the second fundamental form is constant. As another kind of Bernstein-type property for such surfaces, Alías [4] proved that two compact spacelike surfaces in the de Sitter space for which there exists an isometry preserving their mean curvature functions are necessarily congruent. In particular, there exists no compact spacelike Bonnet surface in the de Sitter space.

In this work we characterize the totally umbilical round spheres in $\mathbf{S}_1^3$ from certain conditions on their curvatures. First, in Section 3, we study the case of compact spacelike surfaces in $\mathbf{S}_1^3$ with a constant principal curvature, proving that the only compact spacelike surfaces in $\mathbf{S}^3_1$ with a constant principal curvature are the totally umbilical round spheres (Theorem 1).

In Section 4, as a natural generalization of Ramanathan and Li results, we are able to characterize the totally umbilical round spheres of $\mathbf{S}^3_1$ as the only compact linear Weingarten spacelike surfaces (Theorem 3). It is worth pointing out that the corresponding result for compact linear Weingarten surfaces in 3-dimensional space forms turns out to be false. In fact, Umehara and Yamada [10] constructed examples of tori with constant mean curvature in such ambient spaces (see also [5], [11]). Indeed, we need the additional hypothesis of topological sphere (see [1]). However, our Theorem 3 allows us to characterize the totally umbilical round spheres of the 3-dimensional hyperbolic space $\mathbf{H}^3$ as the only compact linear Weingarten surfaces with non-degenerate second fundamental form (Theorem 4). Actually, this result is the best possible, as the tori of Umehara and Yamada become plain.

2 Preliminaries

Let $\mathbf{L}^4$ be the 4-dimensional *Lorentz-Minkowski space*, that is, the real vector space $\mathbf{R}^4$ endowed with the Lorentzian metric tensor $\langle,\rangle$ given by

$$\langle,\rangle = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2,$$

where (x_1, x_2, x_3, x_4) are the canonical coordinates of $\mathbf{R}^4$. The 3-dimensional unitary *de Sitter space* is given as the following hyperquadric of $\mathbf{L}^4$,

$$\mathbf{S}^3_1 = \{x \in \mathbf{L}^4 : \langle x, x\rangle = 1\}.$$

As it is well known, $\mathbf{S}^3_1$ inherits from $\mathbf{L}^4$ a time-orientable Lorentzian metric which makes it the standard model of a Lorentzian space of constant sectional curvature one. A smooth immersion $\psi : M^2 \longrightarrow \mathbf{S}^3_1 \subset \mathbf{L}^4$ of a 2-dimensional connected manifold M is said to be a *spacelike surface* if the induced metric via ψ is a Riemannian metric on M, which, as usual, is also denoted by $\langle,\rangle$. The time-orientation of $\mathbf{S}^3_1$ allows us to choose a timelike unit normal field N globally defined on M, tangent to $\mathbf{S}^3_1$, and hence we may assume that M is oriented by N.

We will denote by $H = -\mathrm{trace}(A)/2$ the mean curvature of M, where A stands for the shape operator of M in $\mathbf{S}^3_1$ associated to N. The choice of the sign $-$ in our definition of H is motivated by the fact that, in that case, the mean curvature vector is given by $\vec{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in M$ if and only if $\vec{H}(p)$ is in the time-orientation determined by $N(p)$. On the other hand, the Gaussian curvature of M is given by $K = 1 - \det(A)$.

We will say that a spacelike surface $\psi : M^2 \longrightarrow \mathbf{S}^3_1 \subset \mathbf{L}^4$ is a linear Weingarten surface if there exist constants $\alpha, \beta, \mu \in \mathbf{R}$ such that

$$\alpha H + \beta K = \mu$$

being α and β not both zero.

Finally, recall that every compact spacelike surface in $\mathbf{S}^3_1$ is diffeomorphic to a 2-sphere (see, for instance, [4]).

3 Surfaces with a constant principal curvature

It was proved by Shiohama and Takagi [9] that the only compact surfaces of genus zero with a constant principal curvature in the 3-dimensional Euclidean space are the totally umbilical round spheres. Following their ideas, we have the following result, where the hypothesis of genus zero can be removed:

Theorem 1 *The only compact spacelike surfaces in $\mathbf{S}^3_1$ with a constant principal curvature are the totally umbilical round spheres.*

Proof: Let $\psi : M^2 \longrightarrow \mathbf{S}^3_1 \subset \mathbf{L}^4$ be a compact spacelike surface in $\mathbf{S}^3_1$ with a constant principal curvature $\lambda_1 = R \geq 0$ (up to a change of orientation). If there exists a non umbilical point $p \in M$, then we can consider local parameters (u, v) in a neighborhood U of p without umbilical points, such that

$$\begin{aligned} \langle d\psi, d\psi \rangle &= E\ du^2 + G\ dv^2 \\ \langle d\psi, -dN \rangle &= RE\ du^2 + \lambda_2 G\ dv^2 \end{aligned}$$

where the principal curvature $\lambda_2 \neq R$. Then, the structure equations are given by

$$\begin{aligned} \psi_{uu} &= \frac{E_u}{2E}\ \psi_u - \frac{E_v}{2G}\ \psi_v - REN - E\psi \\ \psi_{uv} &= \frac{E_v}{2E}\ \psi_u + \frac{G_u}{2G}\ \psi_v \\ \psi_{vv} &= -\frac{G_u}{2E}\ \psi_u + \frac{G_v}{2G}\ \psi_v - \lambda_2 GN - G\psi \\ N_u &= -R\ \psi_u \\ N_v &= -\lambda_2\ \psi_v \end{aligned}$$

and the Mainardi-Codazzi equations for the immersion ψ are

$$(R-\lambda_2)\frac{E_v}{2E}=0$$
$$(R-\lambda_2)\frac{G_u}{2G}+(R-\lambda_2)_u=0$$

Since $\lambda_2\neq R$, the coefficient E does not depend on v, that is, $E=E(u)$. If we consider the new parameters

$$x=\int\sqrt{E(u)}\;du,\qquad y=v$$

the structure equations become

$$\begin{aligned}
\psi_{xx}&=-RN-\psi\\
\psi_{xy}&=\frac{G_x}{2G}\;\psi_y\\
\psi_{yy}&=-\frac{G_x}{2}\;\psi_x+\frac{G_y}{2G}\;\psi_y-\lambda_2GN-G\psi\\
N_x&=-R\;\psi_x\\
N_y&=-\lambda_2\;\psi_y
\end{aligned}\tag{1}$$

and the Mainardi-Codazzi equation is

$$(R-\lambda_2)\frac{G_x}{2G}+(R-\lambda_2)_x=0,\tag{2}$$

whence the Gauss equation results

$$\left(\frac{G_x}{2G}\right)_x+\left(\frac{G_x}{2G}\right)^2=R\lambda_2-1=R(\lambda_2-R)+R^2-1.\tag{3}$$

Thus, if we take

$$\varphi=\frac{1}{R-\lambda_2}$$

we obtain from (2) and (3) that

$$\varphi_x=\frac{G_x}{2G}\;\varphi$$

$$\varphi_{xx}=\left(\left(\frac{G_x}{2G}\right)_x+\left(\frac{G_x}{2G}\right)^2\right)\;\varphi=-R+(R^2-1)\varphi\tag{4}$$

Let γ_q be the maximal integral curve passing through a point $q = \psi(x_o, y_o) \in U$ for the principal curvature R. Then, from (1) it follows that $\gamma_q(t) = \psi(x_o + t, y_o)$ satisfies

$$(\gamma_q)_{tt} = -R(N \circ \gamma_q) - \gamma_q$$

$$(N \circ \gamma_q)_t = -R(\gamma_q)_t$$

so that γ_q is a geodesic curve, which is a solution of the differential equation

$$(\gamma_q)_{tt} - (R^2 - 1)\gamma_q = Rv_o$$

for a constant vector $v_o \in \mathbf{L}^4$. Therefore, γ_q is given by

$$\gamma_q = \cosh(\sqrt{R^2-1}\ t)v_1 + \sinh(\sqrt{R^2-1}\ t)v_2 - \frac{R}{R^2-1}\ v_o \tag{5}$$

when $R > 1$,

$$\gamma_q = v_1 + tv_2 + \frac{1}{2}\ t^2 v_o \tag{6}$$

when $R = 1$ and

$$\gamma_q = \cos(\sqrt{1-R^2}\ t)v_1 + \sin(\sqrt{1-R^2}\ t)v_2 - \frac{R}{R^2-1}\ v_o \tag{7}$$

when $0 \leq R < 1$, for suitable vectors $v_1, v_2 \in \mathbf{L}^4$.

From (4), the principal curvature λ_2 can be calculated on γ_q as

$$R - \lambda_2 = \left(a\cosh(\sqrt{R^2-1}\ t) + b\sinh(\sqrt{R^2-1}\ t) + \frac{R}{R^2-1}\right)^{-1} \tag{8}$$

when $R > 1$,

$$R - \lambda_2 = \left(a + bt - \frac{1}{2}\ t^2\right)^{-1} \tag{9}$$

when $R = 1$ and

$$R - \lambda_2 = \left(a\cos(\sqrt{1-R^2}\ t) + b\sin(\sqrt{1-R^2}\ t) + \frac{R}{R^2-1}\right)^{-1} \tag{10}$$

when $0 \leq R < 1$, for real constant a, b.

Hence, if $\gamma_q(t_1)$ is the first umbilical point on γ_q, we obtain from (8), (9), (10) and the continuity of λ_2 that

$$0 = R - \lambda_2(\gamma_q(t_1)) = \lim_{t \to t_1} R - \lambda_2(\gamma_q(t)) \neq 0$$

which is a contradiction. Therefore, there is not any umbilical point on γ_q. Moreover, since M is complete it follows that the geodesic γ_q is defined for all $t \in \mathbf{R}$, so that from the compactness of M the cases (5) and (6) are not possible, that is, necessarily $0 \leq R < 1$. Moreover $R \neq 0$, because in that case

$$a \cos(\sqrt{1-R^2}\ t) + b \sin(\sqrt{1-R^2}\ t) = 0$$

for some $t \in \mathbf{R}$, which contradicts the continuity of λ_2.

Let $\widetilde{U}$ be the connected component of non umbilical points containing p. Note that $\widetilde{U}$ is an open set, and from the above reasoning, can be parametrized by $(x, y) \in (-\infty, \infty) \times (\alpha, \beta)$ for certain $\alpha, \beta \in \mathbf{R}$, $\alpha < \beta$.

Let us suppose that there exists an umbilical point $\widetilde{q} \in \partial\widetilde{U}$. Then there exists a sequence of points $q_n = \psi(x_n, y_n) \in \widetilde{U}$ tending to $\widetilde{q}$. Therefore the sequence of geodesics γ_n passing through q_n associated to the principal curvature R converges to a geodesic $\gamma_{\widetilde{q}}$ passing through $\widetilde{q}$ which is also a line of curvature for the eigenvalue R.

Now, from the above argument, it is sufficient to prove that there exists a non umbilical point on $\gamma_{\widetilde{q}}$. In fact, from (10), we are able to choose a point $p_n \in \gamma_n$ such that $\lambda_2(p_n) = 1/R \neq R$. Finally, since M is compact, there exists a subsequence $\{p_k\}$ of $\{p_n\}$ converging to a non umbilical point $\widetilde{p} \in \gamma_{\widetilde{q}}$.

Consequently M is umbilically free, which is not possible because M is a topological sphere. Therefore M must be a totally umbilical round sphere. ■

Remark 2 Observe that we have not assumed that the principal curvatures λ_1, λ_2 *are necessarily ordered*, but

$$(\lambda_1 - R)(\lambda_2 - R) = 0.$$

■

4 Linear Weingarten Surfaces

The following Theorem generalizes the results of Ramanathan and Li about constant mean curvature and constant Gaussian curvature respectively:

Theorem 3 *The only compact linear Weingarten spacelike surfaces in* $\mathbf{S}^3_1$ *are the totally umbilical round spheres.*

Proof: Let $\psi : M^2 \longrightarrow \mathbf{S}^3_1 \subset \mathbf{L}^4$ be a compact linear Weingarten spacelike surface in the de Sitter space $\mathbf{S}^3_1$. Then we can choose constants $a, b, c \in \mathbf{R}$ such that

$$-2aH + b(K-1) = c, \tag{11}$$

being a and b not both zero. Let us consider the symmetric tensor on M

$$\sigma(X,Y) = a\langle X,Y\rangle - b\langle AX,Y\rangle, \qquad X,Y \in \mathcal{X}(M).$$

It can be easily seen that $\det\sigma = (a^2 - bc)\det(\langle,\rangle)$ where $\langle,\rangle$ is the induced metric on M, thus σ is non-degenerate if and only if $a^2 - bc \neq 0$. Moreover, it is not possible that $a^2 - bc < 0$, because M admits no Lorentzian metric since M is a topological sphere. Hence, we must distinguish the following two cases:

1) If $a^2 - bc > 0$, σ defines a Riemannian metric on M if a suitable time-orientation is chosen on M. Let us consider (u,v) local isothermal parameters for the induced metric $\langle,\rangle$, and let

$$\begin{aligned} \langle d\psi, d\psi\rangle &= E(du^2 + dv^2) \\ \langle d\psi, dN\rangle &= e\,du^2 + 2f\,dudv + g\,dv^2 \end{aligned}$$

be the first and second fundamental forms. Then, the structure equations are given by

$$\begin{aligned} \psi_{uu} &= \frac{E_u}{2E}\,\psi_u - \frac{E_v}{2E}\,\psi_v + eN - E\psi \\ \psi_{uv} &= \frac{E_v}{2E}\,\psi_u + \frac{E_u}{2E}\,\psi_v + fN \\ \psi_{vv} &= -\frac{E_u}{2E}\,\psi_u + \frac{E_v}{2E}\,\psi_v + gN - E\psi \\ N_u &= \frac{e}{E}\,\psi_u + \frac{f}{E}\,\psi_v \\ N_v &= \frac{f}{E}\,\psi_u + \frac{g}{E}\,\psi_v \end{aligned}$$

and the Mainardi-Codazzi equations for the immersion ψ are

$$\begin{aligned} e_v - f_u &= HE_v \\ g_u - f_v &= HE_u \end{aligned} \tag{12}$$

Then we can obtain from a simple computation that the Laplacian $\triangle^\sigma$ with respect to the metric σ of a smooth function $h : M \longrightarrow \mathbf{R}$ is given by

$$\begin{aligned} \big((a^2 - bc)E\big)\,\triangle^\sigma h = {} & \frac{aE - bg}{E}\,h_{uu} + 2\,\frac{bf}{E}\,h_{uv} + \frac{aE - be}{E}\,h_{vv} \\ & -b\left(\left(\left(\frac{g}{E}\right)_u - \left(\frac{f}{E}\right)_v\right)h_u + \left(\left(\frac{e}{E}\right)_v - \left(\frac{f}{E}\right)_u\right)h_v\right). \end{aligned}$$

In particular

$$\big((a^2 - bc)E\big)\,\langle\triangle^\sigma\psi, \psi_u\rangle = \frac{b(e - g)}{2E}\,E_u + \frac{bf}{E}\,E_v - bE\left(\left(\frac{g}{E}\right)_u - \left(\frac{f}{E}\right)_v\right)$$

and

$$\begin{aligned}\big((a^2-bc)E\big)\,\langle\triangle^\sigma N,\psi_u\rangle = {} & \frac{aE-bg}{E}\left(e_u-\frac{1}{2E}(eE_u-fE_v)\right)\\ & +2\,\frac{bf}{E}\left(e_v-\frac{1}{2E}(eE_v+fE_u)\right)\\ & +\frac{aE-be}{E}\left(f_v-\frac{1}{2E}(fE_v+gE_u)\right)\\ & -b\left(\left(\left(\frac{g}{E}\right)_u-\left(\frac{f}{E}\right)_v\right)e+\left(\left(\frac{e}{E}\right)_v-\left(\frac{f}{E}\right)_u\right)f\right)\end{aligned}$$

Then, using (11) and the Codazzi-Mainardi equations (12), it can be shown that

$$\langle\triangle^\sigma\psi,\psi_u\rangle = 0 = \langle\triangle^\sigma N,\psi_u\rangle$$

and analogously

$$\langle\triangle^\sigma\psi,\psi_v\rangle = 0 = \langle\triangle^\sigma N,\psi_v\rangle.$$

Hence, if M is considered as a Riemann surface with the conformal structure induced by σ, we get that

$$\frac{\partial}{\partial\bar{z}}\langle\psi_z,\psi_z\rangle = 0 = \frac{\partial}{\partial\bar{z}}\langle N_z,\psi_z\rangle,$$

for any conformal parameter z on M, that is, the 2-forms $\langle\psi_z,\psi_z\rangle dz^2$ and $\langle N_z,\psi_z\rangle dz^2$ are holomorphic and consequently they vanish identically. Therefore, the first and second fundamental forms of M are conformal, namely, M is a totally umbilical round sphere.

2) If $a^2-bc=0$, then the principal curvatures λ_1,λ_2 associated to A verify that

$$(a-b\lambda_1)(a-b\lambda_2)=0$$

on M. Then, from Theorem 1 and Remark 2, M must be a totally umbilical round sphere. ■

As a consequence we have:

Theorem 4 *The only compact linear Weingarten spacelike surfaces in the 3-dimensional hyperbolic space* $\mathbf{H}^3$ *with non-degenerate second fundamental form are the totally umbilical round spheres.*

Proof: If $\psi : M\longrightarrow\mathbf{H}^3\subset\mathbf{L}^4$ is a surface in the hyperbolic space $\mathbf{H}^3$ with non-degenerate second fundamental form, then its Gauss map $N : M\longrightarrow\mathbf{S}^3_1$ is a spacelike surface in the de Sitter space. Furthermore, the shape operators A_ψ

and A_N of the immersions ψ and N, respectively, satisfy $A_\psi \circ A_N = I_2$, so that $\psi : M \longrightarrow \mathbf{H}^3$ is a linear Weingarten surface in $\mathbf{H}^3$ if and only if $N : M \longrightarrow \mathbf{S}^3_1$ is a linear Weingarten surface in $\mathbf{S}^3_1$. Hence, our assertion is a consequence of Theorem 3. ■

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