Height estimates for surfaces with positive constant mean curvature in $\mathbf{M}^2\times\mathbf{R}$

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Abstract. We obtain height estimates for compact embedded surfaces with positive constant mean curvature in a Riemannian product space $\mathbb{M}^2 \times \mathbb{R}$ and boundary on a slice. We prove that these estimates are optimal for the homogeneous spaces $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ and we characterize the surfaces for which these bounds are achieved. We also give some geometric properties on properly embedded surfaces without boundary.

1 Introduction

The existence of height estimates for a wide class of surfaces in a 3-dimensional ambient space reveals, in general, important properties on the geometric behaviour of these surfaces, as well as existence and uniqueness results (see for instance [H], [HLR], [KKMS], [KKS], [R] and [RS]).

E. Heinz [H] showed that a compact graph with positive constant mean curvature H in $\mathbb{R}^3$ and boundary on a plane can reach at most a height $1/H$ from the plane. Actually, this estimate is optimal because it is attained by the hemisphere of radius $1/H$. As a consequence, a compact embedded surface with constant mean curvature $H \neq 0$ and boundary on a plane is at most a distance $2/H$ from that plane.

An optimal bound was also obtained for graphs and for compact embedded surfaces in the hyperbolic 3-space $\mathbb{H}^3$ with non zero constant mean curvature and boundary on a plane by N. Korevaar, R. Kusner, W. Meeks and B. Solomon [KKMS].

Later, H. Rosenberg [R] exhibited an optimal bound for surfaces with positive constant Gauss curvature in $\mathbb{R}^3$ and $\mathbb{H}^3$ (see also [GM]). In fact, he demonstrated the existence of optimal height estimates for hypersurfaces with a positive constant symmetric function of curvature in the Euclidean and hyperbolic n-space.

[1]The first author is partially supported by Junta de Comunidades de Castilla-La Mancha, Grant No PAI-05-034. The authors are partially supported by MEC-FEDER, Grant No MTM2007-65249

These estimates in the Euclidean 3-space were generalized by H. Rosenberg and R. Sa Earp [RS] (see also [GMM]). They obtained an optimal height estimate for a large class of Weingarten surfaces in $\mathbb{R}^3$.

On the other hand, for the product space $\mathbb{M}^2 \times \mathbb{R}$ of a Riemannian surface $\mathbb{M}^2$ and the real line $\mathbb{R}$, height estimates have recently been exhibited by D. Hoffman, J.H.S. de Lira and H. Rosenberg [HLR] for a surface with positive constant mean curvature H and boundary on a slice (see also [CR]). However, these estimates are not optimal and do not work for every expected value of H, see [HLR, Remark 1].

In this paper, we obtain adjusted height estimates for graphs and compact embedded surfaces with positive constant mean curvature in $\mathbb{M}^2 \times \mathbb{R}$ and boundary on a slice, in such a way that they are optimal when $\mathbb{M}^2$ is a space form, that is, for the homogeneous spaces $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

We also show that if these bounds are reached for a graph Σ on a domain $\Omega \subseteq \mathbb{M}^2$, then Ω has constant Gauss curvature and the Abresch-Rosenberg differential [AR] must vanish identically. In particular, Σ must be a hemisphere of a complete rotational sphere with constant mean curvature if $\mathbb{M}^2 \times \mathbb{R}$ is a homogeneous space. Moreover, these estimates are valid for all $H > 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$ as it was expected (see [HLR, Remark 1]).

Finally, using Alexandrov reflection principle for surfaces with constant mean curvature and following the ideas given in [HLR], we get some geometric properties of properly embedded surfaces without boundary.

2 Main Results

Throughout this paper we will deal with a 3-dimensional ambient space $\mathbb{M}^2 \times \mathbb{R}$ given by the product of a Riemannian surface without boundary $\mathbb{M}^2$ and the real line $\mathbb{R}$. In particular, the homogeneous spaces $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ are contained in this family of manifolds.

Let us denote by $g_{\mathbb{M}^2}$ the metric of $\mathbb{M}^2$. Then the metric of $\mathbb{M}^2 \times \mathbb{R}$ is given by $\langle\, ,\, \rangle = g_{\mathbb{M}^2} + dt^2$.

Let us consider now a surface S and an immersion $\psi : S \longrightarrow \mathbb{M}^2 \times \mathbb{R}$ of mean curvature H and Gauss map N. If we take a conformal parameter z for the induced metric on S via ψ, then the first and second fundamental forms can be written, respectively, as

$$\begin{aligned} I &= \lambda\,|dz|^2 \\ II &= p\,dz^2 + \lambda\,H\,|dz|^2 + \overline{p}\,d\bar{z}^2, \end{aligned} \tag{2.1}$$

where $p\,dz^2 = \langle -\nabla_{\frac{\partial}{\partial z}} N, \frac{\partial}{\partial z}\rangle\, dz^2$ is the Hopf differential of ψ and ∇ is the Levi-Civita connection of $\mathbb{M}^2 \times \mathbb{R}$.

Let $\pi_1 : \mathbb{M}^2 \times \mathbb{R} \longrightarrow \mathbb{M}^2$ and $\pi_2 : \mathbb{M}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ be the usual projections. If we denote by $h : S \longrightarrow \mathbb{R}$ the height function, that is, $h(z) = \pi_2(\psi(z))$, and $\nu = \langle N, \frac{\partial}{\partial t}\rangle$, then we have:

Lemma 2.1. *Given an immersion $\psi : S \longrightarrow \mathbb{M}^2 \times \mathbb{R}$, the following equations must be satisfied:*

$$
\begin{aligned}
|h_z|^2 &= \frac{1}{4}\,\lambda\,(1-\nu^2) && (2.2)\\
h_{zz} &= \frac{\lambda_z}{\lambda}\,h_z + p\,\nu && (2.3)\\
h_{z\bar z} &= \frac{1}{2}\,\lambda\,H\,\nu && (2.4)\\
\nu_z &= -H\,h_z - \frac{2}{\lambda}\,p\,h_{\bar z} && (2.5)\\
p_{\bar z} &= \frac{\lambda}{2}\,(H_z + k\,\nu\,h_z) && (2.6)
\end{aligned}
$$

where $k(z)$ stands for the Gauss curvature of $\mathbb{M}^2$ at $\pi_1(\psi(z))$.

Proof. Let us write

$$\frac{\partial}{\partial t} = T + \nu\,N$$

where T is a tangent vector field on S. Since $\frac{\partial}{\partial t}$ is the gradient in $\mathbb{M}^2 \times \mathbb{R}$ of the function t, it follows that T is nothing but the gradient of h on S.

Thus, from (2.1), one gets $T = \frac{2}{\lambda}(h_{\bar z}\frac{\partial}{\partial z} + h_z\frac{\partial}{\partial \bar z})$ and so

$$1 = \langle\frac{\partial}{\partial t},\frac{\partial}{\partial t}\rangle = \langle T,T\rangle + \nu^2 = \frac{4\,|h_z|^2}{\lambda} + \nu^2,$$

that is, (2.2) holds.

On the other hand, from (2.1) we have

$$
\begin{aligned}
\nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z} &= \frac{\lambda_z}{\lambda}\,\frac{\partial}{\partial z} + p\,N\\
\nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial \bar z} &= \frac{1}{2}\,\lambda\,H\,N && (2.7)\\
-\nabla_{\frac{\partial}{\partial z}} N &= H\,\frac{\partial}{\partial z} + \frac{2}{\lambda}\,p\,\frac{\partial}{\partial \bar z}.
\end{aligned}
$$

The scalar product of these equalities with $\frac{\partial}{\partial t}$ gives us (2.3), (2.4) and (2.5), respectively.

Finally, from (2.7) we get

$$\langle\nabla_{\frac{\partial}{\partial \bar z}}\nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z} - \nabla_{\frac{\partial}{\partial z}}\nabla_{\frac{\partial}{\partial \bar z}}\frac{\partial}{\partial z}, N\rangle \;=\; p_{\bar z} - \frac{1}{2}\,\lambda\,H_z.$$

Hence, using the relationship between the curvature tensors of a product manifold (see, for instance, [O, p. 210]), the Codazzi equation becomes

$$\frac{1}{2}\,\lambda\,k\,\nu\,h_z \;=\; p_{\bar z} - \frac{1}{2}\,\lambda\,H_z,$$

that is, (2.6) holds. □

It should be observed that the equations given in Lemma 2.1 are the integrability equations in the case of a surface in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, see [FM] and [D].

Now, in order to obtain some height estimates, let us first describe the only topological spheres with constant mean curvature which can be embedded in the homogeneous product spaces.

Let us consider the 2-dimensional hyperbolic space of curvature $c < 0$

$$\mathbb{H}^2(c) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \, -x_1^2 + x_2^2 + x_3^2 = \frac{1}{c}, \; x_1 > 0\}$$

with the metric induced by the quadratic form $-dx_1^2 + dx_2^2 + dx_3^2$. Let us also consider the 2-dimensional sphere of curvature $c > 0$

$$\mathbb{S}^2(c) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \, x_1^2 + x_2^2 + x_3^2 = \frac{1}{c}\}$$

with the metric induced by the standard Riemannian metric of $\mathbb{R}^3$.

It is well-known that the only topological sphere with constant mean curvature $H > 0$ which can be immersed in $\mathbb{R}^3 \equiv \mathbb{R}^2 \times \mathbb{R}$ is the totally umbilical sphere of radius $1/H$.

Regarding $\mathbb{H}^2(c) \times \mathbb{R}$ and $\mathbb{S}^2(c) \times \mathbb{R}$, it was shown by Abresch and Rosenberg [AR] that the only such surfaces are the Hsiang and Pedrosa spheres which, up to congruences, can be parametrized as follows:

- For $\mathbb{H}^2(c) \times \mathbb{R}$, given $H_0 > 1/2$, let us take the revolution surface which results when we turn the curve $\alpha(t) = \frac{1}{\sqrt{-c}}(\cosh k(t), \sinh k(t), 0, h(t))$, $-1 \leq t \leq 1$, around the axis $\{(\frac{1}{\sqrt{-c}}, 0, 0)\} \times \mathbb{R}$, with
$$k(t) = 2 \operatorname{arcsinh}\left(\sqrt{\frac{1 - t^2}{4H_0^2 - 1}}\right), \qquad h(t) = \frac{4\,H_0}{\sqrt{4\,H_0^2 - 1}} \operatorname{arcsin}\left(\frac{t}{2\,H_0}\right).$$
This surface has constant mean curvature $H = \sqrt{-c}\,H_0$ and the height difference between its upper point and its lower point is
$$\frac{8\,H}{\sqrt{-4\,c\,H^2 - c^2}} \operatorname{arcsin}\left(\frac{\sqrt{-c}}{2\,H}\right).$$

- With regard to $\mathbb{S}^2(c) \times \mathbb{R}$, given $H_0 > 0$, let us take the revolution surface which results when we turn the curve $\alpha(t) = \frac{1}{\sqrt{c}}(-\cos k(t), \sin k(t), 0, h(t))$, $-1 \leq t \leq 1$, around the axis $\{(\frac{1}{\sqrt{c}}, 0, 0)\} \times \mathbb{R}$, with
$$k(t) = 2 \arctan\left(\frac{2\,H_0}{\sqrt{1 - t^2}}\right), \qquad h(t) = \frac{4\,H_0}{\sqrt{4\,H_0^2 + 1}} \operatorname{arcsinh}\left(\frac{t}{\sqrt{1 + 4\,H_0^2 - t^2}}\right).$$
This sphere has constant mean curvature $H = \sqrt{c}\,H_0$ and the height difference between its upper point and its lower point is
$$\frac{8\,H}{\sqrt{4\,c\,H^2 + c^2}} \operatorname{arcsinh}\left(\frac{\sqrt{c}}{2\,H}\right).$$

Observe that these surfaces are characterized by the fact that the Abresch-Rosenberg differential

$$Q\,dz^2 = (2\,H\,p - c\,h_z^2)\,dz^2,$$

where z is a conformal parameter, vanishes identically on them [AR]. That is, an immersion with constant mean curvature $H > 0$ ($H > \sqrt{-c}/2$ when $c < 0$) in a homogeneous product space such that $Q \equiv 0$ must be a piece of one of the complete spheres described above.

Bearing that in mind, we can establish the following optimal estimate for the maximum height that a surface with constant mean curvature can rise on a slice $\mathbb{M}^2 \times \{t_0\}$.

Theorem 2.1. *Let $\Sigma \subseteq \mathbb{M}^2 \times \mathbb{R}$ be a compact graph on a set $\Omega \subseteq \mathbb{M}^2$, with constant mean curvature $H > 0$ and whose boundary is contained on the slice $\mathbb{M}^2 \times \{0\}$. Let c be the minimum of the Gauss curvature on $\Omega \subseteq \mathbb{M}^2$. Then the maximum height that Σ can rise on $\mathbb{M}^2 \times \{0\}$ is*

$$\begin{array}{ll} \dfrac{4\,H}{\sqrt{-4\,c\,H^2 - c^2}}\,\arcsin\left(\dfrac{\sqrt{-c}}{2\,H}\right) & if \quad c<0 \quad \textit{and} \quad H > \dfrac{\sqrt{-c}}{2}, \\ \dfrac{1}{H} & if \quad c = 0, \\ \dfrac{4\,H}{\sqrt{4\,c\,H^2 + c^2}}\,\operatorname{arcsinh}\left(\dfrac{\sqrt{c}}{2\,H}\right) & if \quad c>0. \end{array}$$

Moreover, if the equality holds, then Ω has constant Gauss curvature c and the Abresch-Rosenberg differential vanishes identically on Σ. In particular, Σ must be a hemisphere of a complete example described above if $\mathbb{M}^2 \times \mathbb{R}$ is a homogeneous space.

Proof. We can assume, without loss of generality, that Σ lies over the slice $\mathbb{M}^2 \times \{0\}$ and so $\nu \leq 0$ everywhere. Moreover, in order to simplify the proof, we will suppose that c is $-1, 0$ or 1. To do that it is enough to consider, if $c \neq 0$, the new metric on $\mathbb{M}^2 \times \mathbb{R}$ given by the quadratic form $|c|\,g_{\mathbb{M}^2} + dt^2$ and the surface $\Sigma' = \{(x, \sqrt{|c|}\,t) \in \mathbb{M}^2 \times \mathbb{R} :\ (x,t) \in \Sigma\}$ which has constant mean curvature $H/\sqrt{|c|}$.

By differentiating (2.5) with respect to $\bar{z}$ and using (2.3), (2.4) and (2.6), one gets

$$\nu_{z\bar{z}} \;=\; -k\,\nu\,|h_z|^2 - \frac{2}{\lambda}\,|p|^2\,\nu - \frac{H^2}{2}\,\lambda\,\nu.$$

Then, from (2.2),

$$\nu_{z\bar{z}} \;=\; -\frac{\lambda\,\nu}{4}\left(k(1-\nu^2) + \frac{8\,|p|^2}{\lambda^2} + 2\,H^2\right). \tag{2.8}$$

In addition, from (2.5)

$$|\nu_z|^2 \;=\; \frac{4\,|p|^2\,|h_z|^2}{\lambda^2} + H^2\,|h_z|^2 + \frac{2\,H}{\lambda}(p\,h_{\bar{z}}^2 + \overline{p}\,h_z^2),$$

and taking into account that

$$|Q|^2 \;=\; 4\,H^2\,|p|^2 + |h_z|^4 - 2\,c\,H(p\,h_{\bar{z}}^2 + \overline{p}\,h_z^2),$$

we obtain, using also (2.2), that

$$|\nu_z|^2 \;=\; \left(\frac{|p|^2}{\lambda}+\frac{H^2\,\lambda}{4}\right)(1-\nu^2)+\frac{c}{\lambda}\,\left(4\,H^2\,|p|^2+\frac{\lambda^2}{16}(1-\nu^2)^2-|Q|^2\right) \tag{2.9}$$

when $c\neq 0$.

Now, let us define ϕ as the map on Σ given by

$$\phi \;=\; h+g(\nu), \tag{2.10}$$

where the function g will be chosen later. Then we have

$$\phi_{z\bar z} \;=\; h_{z\bar z}+g'(\nu)\nu_{z\bar z}+g''(\nu)|\nu_z|^2. \tag{2.11}$$

Let us distinguish the cases $c=0$ and $c\neq 0$ separately. First, if $c=0$ we take $g(s)=s/H$. Thus, one gets from (2.4), (2.8), (2.11) and $k\geq 0$

$$\phi_{z\bar z} \;=\; -\frac{\nu}{H}\left(\frac{2\,|p|^2}{\lambda}+k\,|h_z|^2\right)\;\geq\;0. \tag{2.12}$$

On the other hand, if $c\neq 0$, from (2.4), (2.8), (2.9) and (2.11)

$$\begin{aligned}\phi_{z\bar z} \;=\;& -\frac{c}{\lambda}|Q|^2\,g''(\nu)+\frac{|p|^2}{\lambda}\left((1-\nu^2+4\,c\,H^2)g''(\nu)-2\nu\,g'(\nu)\right)+\\ &+\frac{\lambda}{16}\left(8\,H\,\nu-(2\,H^2+k(1-\nu^2))4\,\nu\,g'(\nu)+(4\,H^2\,(1-\nu^2)+c\,(1-\nu^2)^2)g''(\nu)\right).\end{aligned} \tag{2.13}$$

We choose g to make the coefficient of the $|p|^2$ term vanish and $g'>0$. Hence, the derivative of g is only determined up to a positive constant m_0

$$g'(\nu) \;=\; \frac{m_0}{4\,H^2+c\,(1-\nu^2)}$$

and so we have

$$\begin{aligned} g(\nu) \;&=\; \frac{m_0}{\sqrt{4\,H^2-1}}\,\mathrm{arcsin}\left(\frac{\nu}{\sqrt{4\,H^2-1+\nu^2}}\right) \quad \text{if} \quad c=-1,\\ g(\nu) \;&=\; \frac{m_0}{\sqrt{4\,H^2+1}}\,\mathrm{arcsinh}\left(\frac{\nu}{\sqrt{4\,H^2+1-\nu^2}}\right) \quad \text{if} \quad c=1.\end{aligned}$$

By using $k\geq c$, we get from (2.13)

$$\phi_{z\bar z} \;\geq\; -\frac{8\,H\,\nu\,|Q|^2}{\lambda\,(4\,H^2+c(1-\nu^2))^2}+\frac{\lambda\,\nu}{8}\,(4\,H-m_0). \tag{2.14}$$

Therefore, taking $m_0=4\,H$ one has $\phi_{z\bar z}\geq 0$.

Consequently, for every value of c we have that $\phi \le 0$ on the boundary of our surface $\partial\Sigma$ and, since $\phi_{z\bar z} \ge 0$, it follows that the Laplacian of ϕ verifies that $\Delta\phi \ge 0$ on Σ. Therefore, $\phi \le 0$ everywhere on Σ. Thus, by using that g is strictly increasing one has

$$h \;\le\; -g(\nu) \le\; -g(-1),$$

as we wanted to prove.

Finally, observe that if the maximum height is attained at a point then ϕ vanishes identically on Σ. But, from (2.2) and (2.4), $|h_z|$ cannot vanish on an open set of Σ, or equivalently $\nu \not\equiv 1$ on an open set, because $H > 0$. Thus, it follows from (2.12) and (2.14) that $k \equiv c$ and $Q \equiv 0$. □

Remark 2.1. *In* [HLR] *height estimates were given for all $H > \sqrt{-c/2}$ when $c < 0$. Here, we have sharp bounds for all $H > \sqrt{-c}/2$ and $c < 0$.*

In addition, the requirement $H > \sqrt{-c}/2$ when $c < 0$ is essential, because there exists a revolution surface in $\mathbb{H}^2(c) \times \mathbb{R}$ with $H = \sqrt{-c}/2$ which is a graph on $\mathbb{H}^2$ such that the height function attains a minimum but not a maximum [AR].

As a standard consequence of the Alexandrov reflection principle for surfaces of constant mean curvature with respect to the slices $\mathbb{M}^2 \times \{t_0\}$, we have the following corollaries (see [HLR]).

Corollary 2.1. *Let c be the infimum of the Gauss curvature on $\mathbb{M}^2$ and $\Sigma \subseteq \mathbb{M}^2 \times \mathbb{R}$ an embedded compact surface with constant mean curvature $H > 0$ ($H > \sqrt{-c}/2$ if $c < 0$) and boundary contained on the slice $\mathbb{M}^2 \times \{t_0\}$. Then the maximum height that Σ can attain on $\mathbb{M}^2 \times \{t_0\}$ is*

$$\begin{array}{rl} \dfrac{8\,H}{\sqrt{-4\,c\,H^2 - c^2}}\ \operatorname{arcsin}\left(\dfrac{\sqrt{-c}}{2\,H}\right) & if \quad c < 0, \\ \dfrac{2}{H} & if \quad c = 0, \\ \dfrac{8\,H}{\sqrt{4\,c\,H^2 + c^2}}\operatorname{arcsinh}\left(\dfrac{\sqrt{c}}{2\,H}\right) & if \quad c > 0. \end{array} \tag{2.15}$$

Corollary 2.2. *Let c be the infimum of the Gauss curvature on $\mathbb{M}^2$ and $\Sigma \subseteq \mathbb{M}^2 \times \mathbb{R}$ a properly embedded surface without boundary and with constant mean curvature $H > 0$ ($H > \sqrt{-c}/2$ if $c < 0$). Then*

- *If Σ is compact, the height difference between its upper point and lower point is less than or equal to the one given by* (2.15).
- *If Σ is not compact and $\mathbb{M}^2$ is compact, Σ must have at least one top end and one bottom end.*

These corollaries are an improvement of the corresponding ones given by Hoffman, de Lira and Rosenberg [HLR], but their proofs are analogous to those.

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